ON THE UNORIENTED COBORDISM CLASSES OF COMPACT FLAT RIEMANNIAN MANIFOLDS

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Abstract of the Dissertation

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In this thesis we study the unoriented cobordism classes of compact flat Riemannian manifolds. Through the analysis of certain Z_2^k-actions with which these manifolds are naturally endowed we show in III that if the holonomy group of a flat Riemannian manifold, M, is isomorphic to either a finite elementary abelian 2-group, a finite cyclic 2-group, or a finite generalized quaternian group, then M is a boundary. In Iv we employ the Steenrod Algebra to obtain partial results in the case where ϕ is a finite abelian 2-group with no direct factors isomorphic to Z_2. We also provide in III a bound for each finite 2-group, ϕ, on the dimension of a non-bording flat Riemannian manifold whose holonomy is ϕ.

To Mike Ryan, my first teacher

Table of Contents

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CHAPTER I

INTRODUCTION

A flat Riemannian manifold is a compact smooth manifold together with a metric of everywhere vanishing curvature. From this definition it follows by the Weil homomorphism that all of the rational characteristic classes of a compact flat Riemannian manifold are zero. However, the Weil homomorphism gives no information on the Stiefel-Whitney classes of M^n except for W_n since W_n is the mod 2 reduction of the Euler class of the tangent bundle. No known example of a flat Riemannian manifold which fails to bound has been found although many examples with non-trivial Stiefel-Whitney classes have been. (See Vasquez [12]). This thesis provides a partial answer to the conjecture that in fact all compact flat Riemannian manifolds do bound.

This problem is part of the more general problem of the relation between curvature and cobardism. One hopes that concrete formulas for computing Stiefel-Whitney numbers from curvature can be found. As a first step, it is thought that curvature zero implies all Stiefel-Whitney numbers are zero is needed in order for there to be any hope of such a relationship. The theorems in this thesis indicate that this may be true although they do not give a complete answer.

CHAPTER II

PRELIMINARIES

Holonomy. If M is a Riemannian manifold and if p is a point of M then we can consider the linear maps of TM_p into TM_p gotten by parallel translating tangent vectors at p around closed curves at p. The set of all such linear maps, H, is a Lie group known as the holonomy group of M. It is standard that the structure group of the tangent bundle of M can be reduced to H. Further, if H is discrete then the map which associates to each closed curve at p the corresponding element of H, induces a representation, r, of $\pi_1(M)$ in H. Since $M = B(\pi_1(M))$ if we let $\bar{r} : M \to B(H)$, the classifying space of H be the map induced by r, then there is a vector bundle, E, over B(H) such that $\bar{r}^*(E) \approx TM$. E itself is just the induced bundle of the action of H on TM_p. Letting K denote the kernel of r, we have the exact sequence of groups, $0 \to K \to \pi_1(M) \xrightarrow{r} H \to 0$. References for this material are Kobayashi and Nomizu [8] and Wolf [14].

Flat Riemannian Manifolds

Definition: A compact smooth Riemannian manifold is said to be flat if its curvature tensor vanishes everywhere.

Since a compact flat Riemannian manifold, M^n, has a metric of vanishing curvature, the exponential mapping, $\exp_p : TM_p^n \to M^n$ has no singularities so that it is a covering projection from $TM_p \approx R^n \to M^n$. Exp_p is also a local isometry so that if π denotes the group of covering transformations then $\pi \subset M_n$ where M_n is the group of rigid motions of R^n. Since R^n is acyclic, it follows from the exact homotopy sequence of a fibration that M^n is a $K(\pi,1)$ with $\pi \approx \pi_1(M^n)$.

There is an isomorphism, F, of categories which assigns to each homotopy class of $K(\pi,1)$'s its fundamental group, π, and to each homotopy class of continuous maps $\rho : K(\pi,1) \to K(\pi',1)$, the group homomorphism, $\rho_*:\pi \to \pi'$. Further, if $H^*(\pi,G)$ denotes the discrete cohomology of π with co-efficients in G, then the following square commutes.

$$\begin{array}{ccc} H^*(K(\pi,1);G) & \overset{F^*}{\approx} & H^*(\pi;G) \\ \uparrow \rho^* & & \uparrow (\rho_*)^* \\ H^*(K(\pi^1,1);G) & \overset{F^*}{\approx} & H^*(\pi^1;G) \end{array}$$

Thus, questions concerning cohomology of $K(\pi,1)$'s and cohomology of discrete groups are interchangeable. In the case that $K(\pi,1)$ is a flat Riemannian manifold, it is usually easier to decide such questions from the group theoretic standpoint. This partially stems from a general classification of all groups, π, which are isomorphic to fundamental groups of flat Riemannian manifolds (see Wolf [18]) which we now give. The reader will note that this theorem depends upon the fact that the holonomy group of a flat Riemannian manifold is finite and therefore discrete.

Classification Theorem. If π is the fundamental group of a flat Riemannian manifold, then there is an exact sequence, $0 \to Z^n \xrightarrow{i} \pi \xrightarrow{\rho} \phi \to 1$, where Z^n is an n-dimensional lattice called the subgroup of pure translations and where ϕ is a finite group which is isomorphic to the holonomy group of the flat Riemannian manifold, R^n/π. Further, π contains no elements of finite order, and the representation of ϕ on Z^n induced by conjugation is faithful. Conversely, let π be a group with no elements of finite order and satisfying an exact sequence $0 \to Z^n \xrightarrow{i} \pi \xrightarrow{\rho} \phi \to 1$ where ϕ is a finite group acting on Z^n with zero kernel. Then $\exists$ an injective homomorphism $\rho : \pi \to M_n$ such that R^n/π is a compact flat Riemannian manifold.

Definition A. If $0 \to Z^n \to \pi \to \phi \to 1$ is an exact sequence of groups such that Z^n is an n-dimensional lattice, ϕ is a finite group acting faithfully on Z^n, and such that π contains no elements of finite order, then π is called a torsion free Bieberbach group.

It is noteworthy that for a fixed torsion free Bieberbach group, π, any two flat Riemannian manifolds of the form, R^n/π, are affinely equivalent. Said in another way, if r_1, r_2: $\pi \to M_n$ are two faithful representations, then there is an affine motion, A, of R^n such that $Ar_1A^{-1} = r_2$.

We note further that if $0 \to Z^n \to \pi \to \phi \to 1$ is a torsion free Bieberbach group represented in M_n then Z^n may not be represented as the standard lattice. We retain the notation, Z^n, for an arbitrary n-dimensional free abelian group.

Definition B. If $0 \to Z^n \to \pi \to \phi \to 1$ is a torsion free Bieberbach group, then R^n/π is called a ϕ-manifold. So a ϕ-manifold is a compact Riemannian manifold whose holonomy group is isomorphic to ϕ.

Convention. Any element of M_n can be thought of as a pair, (T,σ), where $\sigma \in O(n)$ and T is a vector in R^n. (T_σ,σ) acts on $x \in R^n$ by $x \to \sigma \cdot x + T$. If $\ell \in Z^n \subset \pi$, then the homomorphism $r : \pi \to M_n$ is such that $r(\ell)\cdot x = x + r(\ell)$. The only elements of π which have a matrix part are those which project to a non-identity element of ϕ. Although to each $\sigma \in \phi$, there are infinitely many elements of π which project to σ, we will not make this distinction in our notation. We will denote lifts of σ by (T_σ,σ) understanding that T_σ is not unique. We will also drop the notation, $r(\ell)$, and just write ℓ. We will speak of Z^n rather than $r(Z^n)$.

In this work, only ϕ-manifolds with ϕ at a 2-group will be considered. In trying to decide whether all flat Riemannian manifolds bound, it is sufficient to show that all 2-group-manifolds bound. For let $0 \to Z^n \xrightarrow{i} \pi \to \phi \xrightarrow{\rho} 1$ be an arbitrary torsion free Bieberbach group and let $\pi' = \rho^{-1}(\phi')$

where ϕ' is a subgroup of ϕ. π' is a torsion free Bieberbach group of finite index in π. The inclusion $j : \pi' \to \pi$ corresponds by the above isomorphism of categories to a finite covering projection $R^n/\pi' \to R^n/\pi$ whose degree equals the index of π' in π or equivalently, the index of ϕ' in ϕ. In particular if ϕ' is a Sylow 2-subgroup of ϕ then $i : \pi' \to \pi$ corresponds to an odd fold cover. R^n/π' bounds $\Leftrightarrow$ R^n/π bounds because odd fold covers preserve Stiefel-Whitney numbers. Hence if one can show that all ϕ-manifolds with ϕ a 2-group are boundaries, then it will follow that all compact flat Riemannian manifolds bound. We next provide a list of facts concerning the cohomology of torsion free Bieberbach groups.

1) Since R^n/π is an n-manifold, and since $H^n(\pi,G) \approx H^n(R^n/\pi;G)$, $H^n(\pi;Z_2) \approx Z_2$, $H^n(\pi;Z) \approx Z$ if R^n/π is orientable, $H^n(\pi;Z) \approx Z_2$ if R^n/π is unorientable.

2) $H^{n+1+j}(\pi;G) = 0 \ \forall j \geq 0$

3) The representation σ of ϕ on Z^n induces a homomorphism $r : \phi \to O(n)$ such that the classifying map $c : R^n/\pi \to BO(n)$ of the tangent bundle of R^n/π can be factored as $\overline{r} \circ \rho$ where $\rho : R^n/\pi \to B(\phi)$ corresponds to the projection homomorphism $\rho : \pi \to \phi$ and where $\overline{r}$ is induced by r. This fact will be used heavily in Section 2.

We next give a theorem due to Vasquez [12] which strongly restricts the collection of possible non-bounding flat Riemannian manifolds.

Theorem (Vasquez) Let ϕ be a finite group. There exists an integer $n(\phi)$ such that if R^n/π is a compact flat Riemannian manifold whose holonomy group is ϕ and whose dimension exceeds $n(\phi)$ then R^n/π fibers as a flat k-torus bundle over a lower dimensional compact flat Riemannian manifold, R^{n-k}/π'.

Note that since every torus has a trivial tangent bundle the characteristic algebra of R^n/π is the pull back of the characteristic algebra of R^{n-k}/π'. Since $k \geq 1$, it follows that R^n/π is a boundary. Thus, for each ϕ there only finitely many ϕ-manifolds which are not boundaries. Unfortunately, $n(\phi)$ is difficult to compute for most ϕ.

The proof of this theorem involves the following property of torsion free Bieberbach groups. Vasquez shows that if $n > n(\phi)$ then $\exists$ a non-zero ϕ-submodule, A, of Z^n such that π/A is a torsion free Bieberbach group. The projection homomorphism $\rho : \pi \to \pi/A$ corresponds topologically to the above mentioned fibration.

Involutions

Definition. A homeomorphism of an n-manifold, M^n, whose square is the identity map is called an involution. A group of involutions is a homomorphism of Z_2^k into the group of homeomorphisms of M^n. A group of involutions is also called a Z_2^k-action. A stationary point of a Z_2^k-action is a common fixed point of the involutions determined by the elements of Z_2^k.

Theorem (30.1) Conner and Floyd [5]). If Z_2^k acts differentiably on a closed n-manifold, M^n, without stationary points, then $[M^n]_2 = 0$. (Here $[M^n]_2$ is the unoriented cobordism class of M^n.)

Group Extensions

Definition. Let $0 \to H \to G \to G/H \to 0$ be an exact sequence of groups. Then G is called an extension of H by G/H.

Definition. Let G be an extension of H by G/H. A section of G/H in H is an injective map (not necessarily a homomorphism) s, from G/H into G. The elements, s(g), will also be denoted by the symbols G_g.

For each section we get a cocycle $\alpha \in \text{Hom } G/H(B_2(G/H);H)$ where $B_2(G/H)$ denotes the free G/H-module on the elements of the set $G/H \times G/H$. α is defined by the rule, $\alpha(g,h) = s(g)\cdot s(h)\cdot s(g\cdot h)^{-1}$. In this work we will not need any of the properties of α and thus will not describe them. A general reference for this subject is Cartan and Eilenberg [1]. We merely note that not every element of $\text{Hom } G/H(B_2(G/H);H)$ is a cocycle and that the symbol for those homomorphisms which are cocycles is $Z^2(G/H;H)$. Note that if s is chosen such that

$s(id_{G/H}) = id_G$ and if $\sigma \in G/H$ is an element of order 2, then $\alpha(\sigma,\sigma) = s(\sigma)^2$ since $\alpha(\sigma,\sigma) = s(\sigma)\cdot s(\sigma^2)^{-1}$ $= s(\sigma)^2 s(id_{G/H}) = s(\sigma)^2 id_G = s(\sigma)^2$.

CHAPTER III

TRANSLATIONAL INVOLUTIONS

a) Definitions and Basic Results

Let $\pi \subset M_n$ be a torsion free Bieberbach group whose holonomy group, ϕ, is a 2-group. ϕ a 2-group $\Rightarrow (Z^n/2Z^n)^\phi \neq \{0\}$ (See Sah [8] pg. 77). Let $v \in Z^n$ represent a non-zero element of $(Z^n/2Z^n)^\phi$ and consider the map, $I_v : R^n \to R^n$, defined by $I_v(x) = x + \frac{1}{2}v$. Let $(T_\sigma, \sigma) \in \pi$. Then $(T_\sigma,\sigma)^{-1} \circ I_v \circ (T_\sigma,\sigma)(x)$ $= (T_\sigma,\sigma^{-1}) I_v(\sigma \cdot x + T_\sigma) = (T_\sigma,\sigma)^{-1}(\sigma \cdot x + T_\sigma + \frac{1}{2}v) = (T_\sigma,\sigma)^{-1} \cdot (T_\sigma,\sigma)(x) + \frac{1}{2}\sigma \cdot v$. Thus, $(T_\sigma,\sigma)^{-1} \circ I_v \circ (T_\sigma,\sigma)(x) - I_v(x) = \frac{1}{2}(\sigma \cdot v - v)$. Since v represents an element of $(Z^n/2Z^n)^\phi$, $\frac{1}{2}(\sigma \cdot v - v) \in Z^n$ so that I_v projects to a homeomorphism of R^n/π. This homeomorphism is actually an involution because $I_v^2(x) = x + v$ and $v \in Z^n$ so that $I_v^2(x)$ projects to the identity map.

Definition 1. Let $v \in Z^n$ represent an element of $(Z^n/2Z^n)^\phi$. Then the involution $\bar{I}_v$, of R^n/π induced by I_v is called a translational involution. The set of all translational involutions is called the group of translational involutions of R^n/π.

Note that since any two translational involutions commute and since $(Z^n/2Z^n)^\phi \approx Z_2^k$ for some $k \geq 1$, the group of translational involutions determines a Z_2^k-action on R^n/π. This fact will permit us to use Theorem (30.1) of Conner and Floyd. This theorem implies that if R^n/π is not a boundary, then there must be at least one stationary point of the group of translational involutions. If $x \in R^n$ is a lift of a stationary point of I_v, then $\exists (T_\sigma,\sigma) \in \pi \ni x+\frac{1}{2}v = (T_\sigma,\sigma)\cdot x = \sigma\cdot x+T_\sigma$. Clearly, $(T_\sigma,\sigma)\notin Z^n$ since $\frac{1}{2}v\notin Z^n$. Thus, σ is a non-trivial element of ϕ. Further, σ is of order 2 because $(T_\sigma,\sigma)^2(x) = (T_\sigma,\sigma)(x+\frac{1}{2}v) = x + \frac{1}{2}v + \frac{1}{2}\sigma\cdot v$ and because $\frac{1}{2}v + \frac{1}{2}\sigma\cdot v \in Z^n$. We will need these facts throughout this work. We state them as

Fact Ia. If $x \in R^n$ is a lift of a fixed point of the translational involution, $\overline{I}_v$, then there exists a non-trivial element, σ, of order two in ϕ and a lift, (T_σ,σ), of σ to π such that $I_v(x) = (T_\sigma,\sigma)\cdot x$.

Definition 2. Let $x \in R^n$ be a lift of a stationary point of the translational involution, $\overline{I}_v$, and let $(T_\sigma,\sigma) \in \pi \ni x + \frac{1}{2}v = \sigma\cdot x + T_\sigma$. Then x is said to be fixed with respect to (T_σ,σ) by $\frac{1}{2}v$.

Lemma 1. Suppose $x \in R^n$ projects to a stationary point of the group of translational involutions of R^n/π. Then there is an injective homomorphism $J : (Z^n/2Z^n)^\phi \to \phi$ defined by $J(\overline{v}) = \sigma$ where $x + \frac{1}{2}v = \sigma\cdot x + T_\sigma$ for some $(T_\sigma,\sigma) \in \pi$ and where v represents $\overline{v}$ i.e. x is fixed with respect to (T_σ,σ) by $\frac{1}{2}v$.

Proof. Choose co-ordinates in R^n so that x equals the origin.

J is well-defined if $\exists (T_\sigma, \sigma), (T_\tau, \tau) \in \pi$ such that $\frac{1}{2}v + 0 = (T_\sigma, \sigma)(0) = (T_\tau, \tau)(0)$ then $(T_\sigma, \sigma) = (T_\tau, \tau)$ because π acts as a group of covering transformations on R^n from which it follows that any two elements of π which agree on a single point of R^n are equal. Thus $J(\overline{v})$ is well defined.

J is injective If $J(\overline{v}) = J(\overline{W})$ then $\exists (T_{\sigma,1}, \sigma), (T_{\sigma,2}, \sigma) \ni \frac{1}{2}v + 0 = (T_{\sigma,1}, \sigma)(0)$ and $\frac{1}{2}W + 0 = (T_{\sigma,2}, \sigma)\cdot(0)$. From this we get $\frac{1}{2}v = T_{\sigma,1}$ and $\frac{1}{2}W = T_{\sigma,2}$. But any two lifts of σ to π differ by an element of Z^n. Thus $\frac{1}{2}v = \frac{1}{2}W + \ell$ for some $\ell \in Z^n$. This gives $v = w = 2\ell \Rightarrow \overline{v} = \overline{W}$.

J is a homomorphism Consider two distinct elements $\overline{v}_1, \overline{v}_2$ of $(Z^n/2Z^n)^\phi$ and let $(T_1, \sigma), (T_2, \tau) \in \pi \ni (T_1, \sigma)(0) = \frac{1}{2}v_1$ and $(T_2, \tau)(0) = \frac{1}{2}v_2$, where v_1, v_2 represent $\overline{v}_1, \overline{v}_2$. Now $(T_1, \sigma)\circ(T_2, \tau) = (T_1 + \sigma\cdot T_2, \sigma\tau)$ since if $y \in R^n$ then $(T_1, \sigma)\circ(T_2, \tau)(y) = (T_1, \sigma)(\tau\cdot y + T_2) = T_1 + \sigma\cdot T_2 + \sigma\tau\cdot y = (T_1 + \sigma\cdot T_2, \sigma\tau)(y)$. In particular, $(T_1, \sigma)\circ(T_2, \tau)(0) = T_1 + \sigma\cdot T_2 = \frac{1}{2}v_1 + \frac{1}{2}\sigma\cdot T_2$. Now since $\frac{1}{2}$ represents $\overline{v}_2$, $\sigma\cdot\overline{v}_2 = \overline{v}_2 + 2m$ for some $m \in Z^n$. Consider the element $(T_1 + \sigma\cdot T_2 - m, \sigma\tau) \in \pi$. Then $(T_1 + \sigma\cdot T_2 - m, \sigma\tau) = (\frac{1}{2}v_1 + \frac{1}{2}v_2, \sigma\tau)$ so that $(T_1 + \sigma\cdot T_2 - m, \sigma\tau)(0) = \frac{1}{2}v_1 + \frac{1}{2}v_2$. Thus, $J(\overline{v}_1 + \overline{v}_2) = J(\overline{v}_1)\circ J(\overline{v}_2)$. ∎ (Notice that for each $\overline{v} \in (Z^{n/2}Z^n)^\phi$, $J(\overline{v})$ is of order because $(Z^{n/2}Z^n)^\phi \approx Z_2^k$.)

Example 1: The Klein bottle

The Klein bottle is obtained as the quotient of R^2 by the action of the standard lattice together with the rigid motion,

$((\frac{1}{2},0), \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix})$. Since $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $\phi \approx Z_2$. ϕ acts trivially on $Z^2/2Z^2$ so $(Z^n/2Z^n)^\phi \approx Z_2 \times Z_2$. Thus, by Lemma 1 the intersection of the fixed point sets of the group of translational involutions must be empty. Explicitly the involutions are given by $x \to x + (\frac{1}{2},0)$, $x \to x + (0,\frac{1}{2})$, $x \to x + (\frac{1}{2},\frac{1}{2})$. Note that $x \to x + (0,\frac{1}{2})$ is fixed point free and that modulo Z^2 the fixed point set of $x \to (\frac{1}{2},0) + x$ is represented by the lines $(t,\{0,\frac{1}{2}\})$ $t \in R$. Thus, the fixed point set of $x \to (\frac{1}{2},0) + x$ is two circles.

Example 2. The Hansche-Wendt manifold

π is the subgroup of M_3 generated by the standard lattice in R^3 and the rigid motions

$$((\tfrac{1}{2},0,0), \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}), \quad ((0,\tfrac{1}{2},\tfrac{1}{2}), \begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}).$$

Here $\phi = Z_2 \times Z_2$ while $(Z^n/2Z^n)^\phi \approx Z_2 \times Z_2 \times Z_2$. Thus, the intersection of the fixed point sets of the translational involutions is empty. Unlike the Klein bottle, none of the involutions are fixed point free.

Theorem 1. Let $0 \to Z^n \to \pi \to \phi \to 1$ be a torsion free Bieberbach group with ϕ a 2-group. Let k denote the largest integer such that $\exists$ a subgroup, $G \approx Z_2^k$, of ϕ. Then if $(Z^n/2Z^n)^\phi \approx Z_2^{k+i}$ $i \geq 1$, R^n/π is a boundary.

Proof. $(Z^{n/2}Z^n)^\phi \approx Z_2^{k+i}$ with $i \geq 1 \Rightarrow J$ can not be injective $\Rightarrow$ there does not exist a stationary point of the group of translational involutions. ∎

Corollary 1. Let $0 \to Z^n \to \pi \to \phi \to 1$ be as in Theorem 1. Let $p \in Z \ni \#(\phi) = 2^p$. Then if R^n/π is not a boundary, $n \leq k\cdot 2^p$.

Proof. $\dim_{Z_2}(Z^{n/2}Z^n)^\phi \geq n/2^p$. By Theorem 1 we must have

$$\dim_{Z_2}(Z^{n/2}Z^n)^\phi \leq k \Rightarrow n/2^p \leq k. \quad ∎$$

In general we do not know the relationship between the bound of Corollary 1 and Vasquez's number, $n(\phi)$. Suffice it to say that if a flat manifold, M, fibers as a flat torus bundle over another flat manifold then M has a fixed point free translational involution but it does not follow from this that $k\cdot 2^p \leq n(\phi)$. $k\cdot 2^p$ is probably much too crude a bound to be always less than or equal to $n(\phi)$. It is possible that it is never less than equal to $n(\phi)$. For instance $n(Z_2) = 1$ while $k\cdot 2^p = 2$ since $p = 1$, $k = 1$. Corollary 1 has the virtue of giving an easily computable bound while $n(\phi)$ is generally difficult to determine.

Example 3. Let $\phi = Z_2$. Then $k = 1$, $p = 1 \Rightarrow n \leq 2$. The only example is the flat Klein bottle which is a boundary because it fibers as a circle bundle over a circle. Thus, all

Z_2-manifolds are boundaries.

Example 4. Let ϕ be the dihedral group of order 8. Then $k = 2$, $p = 8$ so that $n \leq 16$.

Corollary 2. Let $0 \to Z^n \to \pi \to \phi \to 1$ be a torsion free Bieberbach group with ϕ a 2-group. Let k denote the largest integer such that $\exists$ a subgroup, $G \approx Z_2^k$, of ϕ. Then if the first βetti number $\beta_1(R^n/\pi)$ is greater than k, R^n/π is a boundary.

Proof. If $\beta_1(R^n/\pi) = m$ then $\exists$ an m-dimensional subgroup of Z^n upon which ϕ acts trivially. (See Wolf [14] pg.). Thus, $\dim_Z(Z^n/2Z^n)^\phi \geq m$. Now $m > k \Rightarrow$ no homomorphism from $(Z^n/2Z^n)^\phi \to \phi$ can be injective so by Theorem 1, R^n/π is a boundary. ∎

Example 5. Let ϕ be the dihedral group of order 2^p. Then $k = 2$ so that any ϕ-manifold of first βetti number greater than 2 is a boundary.

b) A General Lemma - We will need the following repeatedly in this work.

Definition 3. Let Z^n be a ϕ-module and let $\sigma \in \phi$ be an element of order 2. Then σ is said to cancel $v \in Z^n$ if there exists $y \in Z^n$ such that $\sigma \cdot y + y = v$.

Lemma 2. Let $0 \to Z^n \to \pi \to \phi \to 0$ be a torsion free Bieberbach group and let $s = \{\pi_g\}g \in \phi$ be a section of ϕ in π such that $\pi_{id_\phi} = 0$. Let α denote the cocycle in $Z^2(\phi;Z^n)$ determined by s. Then for all elements of order 2, $\sigma \in \phi$, σ does not cancel $\alpha(\sigma,\sigma)$.

Proof. Suppose that $\alpha(\sigma,\sigma) = \sigma\cdot y+y$. Note that since $\sigma^2 = 1$ and $\pi_{id_\phi} = 0$, $\alpha(\sigma,\sigma) = \pi_\sigma\cdot\pi_\sigma$. Consider the element of π, $(-y+T_\sigma,\sigma)$ where $(T_\sigma,\sigma) = \pi_\sigma$. For each $x \in R^n$, $(-y+T_\sigma,\sigma)^2(x)$ $= (-y+T_\sigma,\sigma)(-y+T_\sigma+\sigma\cdot x) = -\sigma\cdot y+\sigma\cdot T_\sigma+x+T_\sigma-y = x-(\sigma\cdot y+y)+\sigma\cdot T_\sigma+T_\sigma$. Now $\sigma\cdot T_\sigma + T_\sigma = \alpha(\sigma,\sigma)$ since $\pi_\sigma^2(x) = (T_\sigma,\sigma)^2(x) = (T_\sigma,\sigma)(\sigma\cdot x+T_\sigma)$ $= x + \sigma\cdot T_\sigma + T_\sigma$. Thus, $(-y+T_\sigma,\sigma)^2(x) = x + \alpha(\sigma,\sigma) - (\sigma\cdot y+y)$. But $\alpha(\sigma,\sigma) - (\sigma\cdot y+y) = 0 \Rightarrow (-y+T_\sigma,\sigma)^2(x) = x$ contradicting the fact that π contains no elements of finite order. ∎

Lemma 3. Let Z^n be a Z_2-module and suppose that the non-identity element, σ, in Z_2 acts trivially on $v \in Z^n$, and that $\sigma\cdot y + y = v + 2x$ for some $y,x \in Z^n$. Then σ cancels, V.

Proof. From $\sigma\cdot y + y = v + 2x$ we have $\sigma\cdot(y+\sigma\cdot y) = \sigma\cdot v + 2\sigma\cdot x \Rightarrow \sigma\cdot y+y$ $= \sigma\cdot v + 2\sigma\cdot x = v + 2\sigma\cdot x$ so that σ acts trivially on x. Thus, $\sigma\cdot(y-x) + (y-x) = \sigma\cdot y + y - 2x = v$. ∎

c) The Cyclic Group of Order 2^p and the Generalized Quaternion Groups

The theorems in this section are essentially corollaries of the following lemma.

Lemma 4. Let $0 \to Z^n \to \pi \to \phi \to 1$ be a torsion free Bieberbach group such that ϕ is a 2-group and such that $(Z^n/2Z^n)^\phi \approx Z_2$. Suppose $\exists e \in Z^n$ representing the non-zero element of $(Z^n/2Z^n)^\phi$, $(T_\sigma,\sigma) \in \pi$, and $x \in R^n$ such that $\sigma\cdot e = e$ and such that x is fixed with respect to (T_σ,σ) by $\frac{1}{2}e$. Then σ is not in the center of ϕ.

Proof. Assume that σ is in the center of ϕ. Up to a change of coordinates in R^n we may assume that x is the origin. Then the equation $\frac{1}{2}e + x = (T_\sigma,\sigma)\cdot x$ becomes $\frac{1}{2}e + 0 = (T_\sigma,\sigma)\cdot 0$ or $\frac{1}{2}e = T_\sigma$. Let $s = \{\pi_g\}$ $g \in \phi$ be a section of ϕ in π chosen by setting $\pi_{id} = id_{R^n}$, $\pi_\sigma = (T_\sigma,\sigma)$, and π_g arbitrary for $g \neq id,\sigma$. Let $\alpha \in Z^2(\phi;Z^n)$ be the cocycle determined by s. Then $\alpha(\sigma,\sigma) = \pi_\sigma^2 = e$ since $\pi_{id} = id_{R^n}$ and since $\pi_\sigma^2(y) = (T_\sigma,\sigma)^2(y)$ $= (T_\sigma,\sigma)(T_\sigma+\sigma\cdot y) = T_\sigma + \sigma\cdot T_\sigma + y = \frac{1}{2}e + \frac{1}{2}\sigma\cdot e + y = y + \frac{1}{2}e + \frac{1}{2}e$ $= e + y$. By Charlap and Vasquez [4], $Z^n \approx_\sigma M \oplus \ell$ where ℓ is the subgroup of Z^n generated over Z by e. Now there must exist a vector, $v \in M$, $\ni \sigma\cdot v = -v$ because ϕ and therefore σ is faithfully represented on Z^n. We may further assume that $v \notin 2Z^n$. Since $(Z^n/2Z^n)^\phi \approx Z_2$ and since e projects to the non-zero element of $(Z^n/2Z^n)^\phi$, it follows from the fact that $v \not\equiv e \bmod 2Z^n$ (since $v \in M$ and $Z^n \approx_\sigma M \oplus \ell$) that v is not ϕ-invariant modulo $2Z^n$. Let $\overline{v}$ be the projection of v to $Z^n/2Z^n$. $\overline{v} \neq 0$ because $v \notin 2Z^n$. Let $L(\overline{v})$ be the ϕ-submodule of $Z^n/2Z^n$ generated by $\overline{v}$. Since ϕ is a 2-group $(L(\overline{v}))^\phi$ contains a non-zero element of $L(\overline{v})$. In other words, $L(\overline{v}) \cap (Z^n/2Z^n)^\phi \neq 0$. Since $L(\overline{v})$ is just the set of all

$Z[\phi]$-linear combinations of $\overline{v}$ where $Z[\phi]$ is the integral group ring of ϕ, $\exists f \in Z[\phi] \ni f\cdot\overline{v} = \overline{e}$. It follows that $f\cdot v = e + 2y$ for some $y \in Z^n$. Since σ is in the center of ϕ, $\sigma\cdot f = f\cdot\sigma \Rightarrow$

$$\sigma\cdot(e+2y) = f(-v) \Rightarrow \sigma\cdot e + 2\sigma\cdot y = -e - 2y \Rightarrow$$

$$2\sigma\cdot y + e = - e - 2y \text{ (because } \sigma\cdot e = e) \Rightarrow$$

$$2e = -2(\sigma\cdot y+y) \Rightarrow e = \sigma\cdot(-y) + (-y) .$$

In otherwords, σ cancels e. But $e = \alpha(\sigma,\sigma)$. This contradicts Lemma 2. ∎

<u>Theorem 2</u>. Let R^n/π be a ϕ-manifold where ϕ is a cyclic 2-group of order 2^p. Then R^n/π is a boundary.

<u>Proof</u>. Since ϕ contains a unique element of order 2, by Theorem 1 we may assume that $(Z^n/2Z^n)^\phi \approx Z_2$. Let $g \in \phi$ be a generator of ϕ, and $\pi_g \in \pi$ a lift of g. Then ${\pi_g}^{2p}$ is an element of $Z^n - 2Z^n$ since otherwise $\frac{1}{2}{\pi_g}^{2p} \in Z^n \Rightarrow$ $(-\frac{1}{2}{\pi_g}^{2p}\circ{\pi_g}^{2p-1})^2(0) = (-\frac{1}{2}{\pi_g}^{2p})\circ{\pi_g}^{2p-1}(-\frac{1}{2}{\pi_g}^{2p}+{\pi_g}^{2p-1}(0))$ $= -{\pi_g}^{2p} + ({\pi_g}^{2p-1})^2(0) = -{\pi_g}^{2p} + {\pi_g}^{2p} = 0$ contradicting the fact that π acts freely on R^n. Now since $g\cdot{\pi_g}^{2p} = {\pi_g}^{-1}{\pi_g}^{2p}\pi_g = {\pi_g}^2$ we see that ${\pi_g}^{2p}$ represents the unique non-zero element of $(Z^n/2Z^n)^\phi$. In particular, we also have that the unique element of order 2 in ϕ, g^{2p-1}, acts trivially on ${\pi_g}^{2p}$. Thus, be Lemma 3, no lift $x \in R^n$ of a fixed point of the translational involution $I_{{\pi_g}^{2p}}$ is fixed with respect to g^{2p-1}. Since g^{2p-1}

is the only element of order 2 in ϕ, Theorem 2 now follows from Lemma 1. ∎

<u>Theorem 3</u>. Let R^n/π be a ϕ-manifold where ϕ is a generalized quaternion group. Then R^n/π is a boundary.

<u>Proof</u>. We will view ϕ as the group with two generators, h, g, subject to the relations $g^{2p-1} = h^2$, $h^4 = 1$, $h^{-1}gh = g^{-1}$. Since g^{2p-1} is the unique element of order 2 in ϕ, g^{2p-1} is in the center of ϕ. Further by Theorem 1 we may assume that $(Z^n/2Z^n)^\phi \approx Z_2$. Let $\pi_g \in \pi$ be a lift of g to π. Then $\pi_g^{2p} \in Z^n - 2Z^n$ and $g\cdot\pi_g^{2p} = \pi_q^{2p}$. If $h\cdot\pi_g^{2p} \equiv \pi_g^{2p}$ nod $2Z^n$, then Lemmas 1 and 3 give that the translational involution, $I_{\pi g^{2p}}$, is fixed point free. Thus we may assume that $h\cdot\pi_q^{2p} \not\equiv \pi_q^{2p}$ mod $2Z^n$. Consider the vector $e = h\cdot\pi_g^{2p} + \pi_g^{2p}$. We claim that ϕ acts trivially on e. Now $g\cdot e = e$ since $g\cdot e = g\cdot(h\cdot\pi_g^{2p}+\pi_g^{2p}) = g\cdot h\cdot\pi_g^{2p} + g\cdot\pi_q^{2p} = hg^{-1}\cdot\pi_q^{2p} + \pi_q^{2p}$ $= h\cdot\pi_q^{2p} + \pi_q^{2p}$. $h\cdot e = e$ because $h\cdot e = h\cdot(h\cdot\pi_q^{2p}+\pi_q^{2p})$ $= h^2\cdot\pi_q^{2p} + h\cdot\pi_q^{2p} = g^{2p-1}\cdot\pi_q^{2p} + h\cdot\pi_q^{2p} = \pi_q^{2p} + h\cdot\pi_q^{2p}$. ϕ acts trivially on e implies that e represents the unique non-zero element of $(Z^n/2Z^n)^\phi$. Since g^{2p-1} is the unique element of order 2 in ϕ and since g^{2p-1} is in the center of ϕ, Theorem 3 follows from Lemmas 1 and 3. ∎

d) Z_2^k - <u>manifolds</u>

We begin with some terminology and some preliminary facts.

Definition 4. Let M be a free abelian group. A subgroup N of M is called a pure subgroup if whenever $m \cdot v \in N$ for some non-zero integer, m, then $v \in N$. If M is a ϕ-module for some group ϕ and if N is a ϕ-submodule then N is called a pure submodule if N is also a pure subgroup.

Fact 1d. Let Q^n be an n-dimensional vector space over Q. Suppose that Q^n is a Z_2^k-module. Then Q^n is Z_2^k-isomorphic to a direct sum of one dimensional submodules.

Proof. This standard fact is derived as follows. Since Q is a field of characteristic zero Q^n is isomorphic to a direct sum of irreducible submodules (see Gorenstien pg. 64). Let S be one of these direct summands and let K be the kernel of the representation of Z_2^k on S. Then Z_2^k/K is cyclic (see Gorenstien pg. 65). Since all cyclic subgroups of Z_2^k are isomorphic to Z_2 or 1, $Z_2^k/K \approx Z_2$ or 1. Therefore S is either an irreducible Z_2-module or an irreducible trivial module. In either case, S is one dimensional. ∎

Fact 2d. Let Z^n be a faithful Z_2^k-module and suppose that Z^n is Z_2^k-isomorphic to a direct sum, $\ell_1 \oplus \cdots \oplus \ell_n$, of one dimensional submodules. Then there exists direct summands $\ell_{j1}, \cdots, \ell_{jk}$ such that Z_2^k is faithfully represented on the k-dimensional submodule, $\ell_{j1} \oplus \cdots \oplus \ell_{jk}$. Furthermore, there exists generators, $\sigma_1, \cdots, \sigma_k$, of Z_2^k such that each σ_i acts by negation on ℓ_{ji} and acts trivially on each of the other direct summands.

Proof. We proceed by induction on k. If $k = 1$ and if $\sigma_1 \in Z_2$ is the unique non-identity element then there is some ℓ_{j1} upon which σ_1 acts by negation. Thus the lemma is true for $k = 1$. Assume inductively that the lemma is true for Z_2^{k-1} and let $Z^n = \ell_1 \oplus \cdots \oplus \ell_n$ be a faithful Z_2^k-module. Z^n faithful implies that there is at least one ℓ_j upon which Z_2^k acts non-trivially. Since ℓ_j is one dimensional the kernel, K., of the action of Z_2^k on ℓ_j is isomorphic to Z_2^{k-1}. By the inductive hypothesis there exists direct summands $\ell_{j1}, \cdots, \ell_{jk-1}$ and generators $\sigma_1, \cdots, \sigma_{k-1}$ of K such that K is faithfully represented on $\ell_{j1} \oplus \cdots \oplus \ell_{jk-1}$ and such that $\forall i = 1, k-1$ σ_i acts trivially on each summand except for ℓ_{ji}. Set $\ell_j = \ell_{jk}$. Then clearly Z_2^k acts faithfully on $\ell_{j1} \oplus \cdots \oplus \ell_{jk-1} \oplus \ell_{jk}$. Let $\sigma \in Z_2^k$ such that σ acts by negation on ℓ_{jk}. Let $i_1, \cdots, i_m$ be a complete list of indices such that σ acts by negation on each ℓ_{ji_p}, $p = 1, m$. Then the element $\sigma_{ji_1} \cdot \cdots \cdot \sigma_{ji_m} \cdot \sigma$ acts trivially on $\ell_{j1}, \cdots, \ell_{jk-1}$ and by negation upon ℓ_{jk}. Set $\sigma_k = \sigma_{ji_1} \cdot \cdots \cdot \sigma_{ji_m} \cdot \sigma$. Then $\sigma_1, \cdots, \sigma_k$ together with the submodule $\ell_{j1} \oplus \cdots \oplus \ell_{jk}$ satisfy the required conditions. ∎

Definition 5. Let Z^n be a Z_2^k-module. A linearlly independent set of vectors, $e_1, \cdots, e_s, V_{s+1}, \ldots, V_n$ of Z^n is called a special set if

1) $\forall \sigma \in Z_2^k$, $\sigma \cdot e_i = \pm e_i$, $i = 1, s$

2) $\forall \sigma \in Z_2^k$, $\sigma \cdot v_i = \pm v_i + W\sigma,i$ where $W\sigma,i \in A$, the subgroup of Z^n generated over Z by $e_1, \cdots, e$.

3) $W\sigma,i \neq 0 \Leftrightarrow W\sigma,i \in A - 2A$.

4) For each $i = s + 1, n$ $\exists \sigma \in Z_2^k \ni W\sigma,i \neq 0$.

5) Each v_i is of the form $p_i/m_i X_i + 1/_{mi} W_i$ where $W_i \in A$ and where $\forall \sigma \in Z_2^k$, $\sigma \cdot x_i = \pm x_i$, and p_i, $m_i \in Z$.

Lemma 5. Let Z^n be a Z_2^k-module. Then Z^n contains a special set.

Proof. Consider the Z_2^k-module, $Z^n \otimes_Z Q$. By Fact 1d, $Z^n \otimes_Z Q$ is Z_2^k-isomorphic to a direct sum of one dimensional submodules. Let D be the collection of all such direct sum decompositions of $Z^n \otimes_Z Q$. To each element, $c = \ell_1 \oplus \cdots \oplus \ell_n$ of D we associate a submodule, Ac, of Z^n as follows. Start with ℓ_1 and let $\ell_1, \ell_2, \cdots, \ell_p$ be the longest sequence of direct factors in c such that $(\ell_1 \oplus \cdots \oplus \ell_p) \cap Z^n = (\ell_1 \cap Z^n) \oplus \cdots \oplus (\ell_p \cap Z^n)$. Define $Ac = (\ell_1 \oplus \cdots \oplus \ell_p) \cap Z^n$. Now let $\overline{c} \in D$ be such that the number of direct factors in $A\overline{c}$ is a maximum. Let $s = \dim_Z(A\overline{c})$ and set $A\overline{c} = A$. We have

$$A = (\ell_1 \cap Z^n) \oplus \cdots \oplus (\ell_s \cap Z^n) \text{ and}$$

$$\overline{c} = \ell_1 \oplus \cdots \oplus \ell_s \oplus \ell_{s+1} \oplus \cdots \oplus \ell_n = Z^n \otimes Q$$

Let e_i be a Z-generator of $\ell_i \cap Z^n$. Since ℓ_i is one dimensional if σ is an element of Z_2^k, then $\sigma \cdot e_i = \pm e_i$. Thus,

$e_1, \cdots, e_s$ satisfy 1) of Definition 5. Note also that A is the subgroup of Z^n generated over Z by $e_1, \cdots, e_s$.

We next extend $e_1, \cdots, e_s$ to a set $v_{s+1}, \ldots, v_n, e_1, \cdots, e_s$ of Z^n satisfying conditions 2), 3), 4), and 5) of Definition 5. If $A = Z^n$, then there is nothing to do since conditions 2), 3), 4), and 5) are satisfied vacuously. If $A \neq Z^n$, then for each $i > s$ choose a Z-generator, X_i, of $\ell_i \cap Z^n$. Now $e_1, \cdots e_s, x_i$ do not generate a pure subgroup of Z^n for then $(\ell_i \oplus (A \otimes_Z Q)) \cap Z^n$ would be a Z_2^k-isomorphic to $(\ell_i \cap Z^n) \oplus A$ contradicting the maximality of A. Therefore, there exists $v_i \notin (\ell_i \cap Z^n) \oplus A$ and an integer $m_i \neq \pm 1, 0$ such that $m_i \cdot v_i \in (\ell_i \cap Z^n) \oplus A$ and such that $e_1, \cdots, e_s$, v_i is a Z-basis of $(\ell_i \oplus (A \otimes_Z Q)) \cap Z^n$. Notice that $m_i v_i \notin A$ because A is a pure subgroup of Z^n. Thus, $m_i v_i = p_i \cdot x_i + W_i$ for some integer, p_i, and for some $W_i \in A$.

Now let $\sigma \in Z_2^k$. If $\sigma \cdot x_i = x_i$, then $\sigma \cdot v_i = p_i/m_i x_i + 1/m_i \sigma \cdot W_i$ $= p_i/m_i x_i + 1/m_i W_i + 1/m_i(\sigma \cdot W_i - W_i) = v_i + 1/m_i(\sigma \cdot W_i - W_i)$. If $\sigma \cdot x_i = -x_i$, then $\sigma \cdot v_i = -p_i/m_i x_i + 1/m_i \sigma \cdot W_i$
$= -p_i/m_i x_i - 1/m W_i + 1/m_i(\sigma \cdot W_i + W_i) = -v_i + 1/m_i(\sigma \cdot W_i + W_i)$.

Therefore if we set $W_{\sigma, i} = \begin{cases} \dfrac{\sigma \cdot W_i - W_i}{m_i} & \sigma \cdot x_i = x_i \\ \dfrac{\sigma \cdot W_i + W_i}{m} & \sigma \cdot x_i = -x_i \end{cases}$

condition 2) of Definition 5 is satisfied since $W_{\sigma, i} \in A$, since $m W_{\sigma, i} = \sigma \cdot W_i \pm W_i \in A$ and A is pure. However, it might happen that for some $\sigma \in Z_2^k$, $1/m_i(\sigma\cdot W_i - W_i)$(respectively $1/m_i(\sigma\cdot W_i + W_i)$) is a non-zero element of 2A, so that condition 3) of Definition may not be automatically satisfied. We show that v_i can be adjusted if necessary so that 3) of Definition 5 is satisfied. Suppose e.g. that $\sigma\cdot x_i = -x_i$ and that $1/m_i(\sigma\cdot W_i + W_i) \in 2A$. Let W_i^+ be the component of W_i with respect to $e_1,\cdots,e_s$ which σ fixes (rather than negates). Set $\overline{W}_i = W_i - W_i^+$. Then $\sigma\cdot\overline{W}_i{}^{+\overline{W}_i} = 0$. Applying this adjustment for each element of Z_2^k, one finally obtains after finitely many steps a vector W_i' such that $\forall\sigma \in Z_2^k$, $1/m_i(\sigma\cdot W_i' + W_i')$ (respectively $1/m_i(\sigma\cdot W_i' - W_i') \in 2A \Leftrightarrow 1/m_i(\sigma\cdot W_i' + W_i') = 0$ (respectively $1/m_i(\sigma\cdot W_i' - W_i' = 0)$. $v_i^1 = p_{i/m_i}x_i + 1/m_i\cdot W_i'$ clearly satisfies condition 3) of Definition 5. For 4) of Definition 5 if all $\sigma \in Z_2^k$, $W_{\sigma,i} = 0$ then $\forall\sigma \in Z_2^k$, $\sigma\cdot v_i = \pm\, v_i$. Thus, if ℓ' denotes the one dimensional sublattice of Z^n spanned over Z by v_i, then the group spanned over Z by $e_1,\cdots, e_s, v_i$ is Z_2^k-isomorphic to $\ell' \oplus A$. If we can show that $\ell' \oplus A$ is a pure submodule of Z^n, this will contradict the maximality of A. For this we must show that if V_i needs to be adjusted to v_i then $e_1,\cdots,e_s$ is still a Z-basis of $(\ell_i \oplus (A\otimes_Z Z^n)) \cap Z^n$. If for some $\sigma \in Z_2^k$, $W_i' = W_i' - W_i^+$, then $v_i^1 = v_i - 1/m_i W_i^+$. Since $1/m_i W_1^+$ can be written as a Z-linear combination of $e_1,\cdots,e_s,e_1,\cdots e_s$, v_i' is a Z-basis of $(\ell i \oplus (A\otimes_Z Q)) \cap Z^n$ if and only if $e_1,\cdots,e_s$, v_i is a Z-basis of $(\ell i \oplus (A\otimes_Z Q)) \cap Z^n$.

Since in general, W_i' is constructed from W_i by subtracting off a finite number of vectors of the form W_i^+, W_i^-, a straightforward induction gives that $e_1, \cdots e_s$, v_i' is a Z-basis of $(\ell i \oplus (A \otimes_Z Q)) \cap Z^n$. Ω of Definition 5 is true by construction of the v_i's. This proves the lemma. ∎

Example 1. Let $k = 1$ and $n = 2$ and let Z_2 act on $Z \times Z$ by $\sigma \cdot (1,0) = (0,1)$, $\sigma \cdot (0,1) = (1,0)$. $Z^2 \otimes_Z Q \approx \ell_1 \oplus \ell_2$ where ℓ_1 is generated by $(1,1)$ and ℓ_2 is generated by $(-1,1)$. Thus, $e_1 = (1,1)$, $x_2 = (-1,1)$ and $\sigma \cdot e_1 = e_1$, $\sigma \cdot x_2 = -x_2$. Note that $A = \ell_1 \cap Z^2$. Set $v_2 = -\frac{1}{2}x_2 + \frac{1}{2}e_1$. Since $v_2 = (1,0)$, v_2, e_1 span Z^2. Set $W_2 = e_1$. Then $v_2 = -\frac{1}{2}x_2 + \frac{1}{2}W_2$, so $p_1 = -1$, $m_2 = 2$. Now

$\sigma \cdot v_2 = -\frac{1}{2}\sigma \cdot x_2 + \frac{1}{2}\sigma \cdot W_2 = -(-\frac{1}{2}x_2) + \frac{1}{2}\sigma \cdot W_2$

$= -(-\frac{1}{2}x_2) - \frac{1}{2}W_2 + \frac{1}{2}(\sigma \cdot W_2 + W_2) = -v_2 + \frac{1}{2}(\sigma \cdot W_2 + W_2)$. Since $W_2 = (1,1)$, $\frac{1}{2}(\sigma \cdot W_2 + W_2) = \frac{1}{2}(\sigma \cdot (1,1) + (1,1)) = \frac{1}{2}(2,2) = (1,1)$. Thus, $\frac{1}{2}(\sigma \cdot W_2 + W_2) \notin 2A$ since $A = \ell_1 \cap Z^2$. The matrix of σ_1 with respect to v_2, r_1 is $\begin{matrix} -1 & 0 \\ 1 & 1 \end{matrix}$. In general, the matrices of $\sigma \in Z_2^k$ with respect to a special basis are lower triangular with entries consisting completely of odd integers.

<u>Example 2</u>. Let $0 \to Z^3 \to \pi \to Z_2 \to 1$ be the torsion free Bieberbach group generated by the standard lattice, Z^3, in

in R^3 together with the rigid motion $\left((0,0,\tfrac{1}{2}), \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$.

Here $e_1 = (0,0,1)$ and $e_2 = (0,1,0)$, $v_3 = (1,0,0)$. A is the intersection of Z^3 with the y,z-plane. Since σ_1 acts trivially on A, the maps $x \to x + \tfrac{1}{2}e_1$, $x \to \tfrac{1}{2}e_2$, $x \to x + \tfrac{1}{2}(e_1+e_2)$ are translational involutions of R^n/π. Now $x \to \tfrac{1}{2}e_2 + x$ is fixed point free while $x \to \tfrac{1}{2}e_1 + x$ fixes (mod Z^3) the hyperplanes $(0,y,z)$, $(\tfrac{1}{2},y,Z)$. Thus, the intersection of the fixed points sets of these involutions is empty. In general, the module, A, will determine a Z_2^{-s}- action on R^n/π for an arbitrary Z_2^k - manifold. We will show that this action never has a stationary point.

<u>Theorem 4</u>. Every Z_2^k-manifold is a boundary.

<u>Proof</u>. Let $0 \to Z^n \to \pi \to Z_2^k \to 1$ be a torsion free Bierberbach group. Let $e_1,\cdots,e_s$, $v_{s+1},\cdots,v_n$ be a special basis of Z^n. Since $\forall\sigma \in Z_2^k \sigma\cdot e_i = \pm e_i$, the projection of A to $Z^n/2Z^n$ is a subgroup of $(Z^n/2Z^n)^{Z_2^k}$ whose dimension over Z_2 is s. Thus the maps from R^n into R^n defined by $x \to x + \tfrac{1}{2}(e_{i_j}+\cdots+e_{i_j})$ induce translational involutions of R^n/π. If R^n/π is not a boundary then by Theorem (30.1) of Conner and Floyd these translational involutions have a common fixed point. If $\bar{A}$ denotes the projection of A to $Z^n/2Z^n$ then by Lemma 1, $J(\bar{A})$ is a subgroup of Z_2^k isomorphic to Z_2^s. Choose co-ordinates in R^n so that this common fixed point is the origin. Then

again by Lemma 1, $\exists$ for each $i = 1, s$, $\sigma_i \in \overline{J}(\overline{A})$ such that $0 + \frac{1}{2}e_i = (T_i,\sigma_i)\cdot 0$ for some lift, (T_i,σ_i), of σ_i to π. Multiplying the right hand side of this equation out gives

$$0 + \tfrac{1}{2}e_i = T_i + \sigma_i 0 \Rightarrow \tfrac{1}{2}e_i = T_i.$$

Using this equation and the fact that A is a Z_2^k-submodule of Z^n we see that the subgroup, π', of π generated by A and the set, $\{(T_i,\sigma_i),\ i = 1,s\}$ satisfies the exact sequence, $0 \to A \to \pi' \to J(\overline{A}) \to 0$. Furthermore since π' is a subgroup of π, this sequence is a torsion free extension of A by $J(\overline{A})$.

Since each element of A acts on R^n by translation, A is contained in the subgroup of pure translations in π'. Note that A may not equal the whole group of pure translations in π' because if there exists $\sigma \in J(\overline{A})$ such that σ acts trivially on A, then any lift, (T,σ), of σ to π' acts by the pure translation $x \to x + T$. Said another way, $J(\overline{A})$ may not equal the holonomy group of π'. We intend to extend π' to a larger torsion free Bieberbach group, π'', whose holonomy _is_ equal to $J(\overline{A})$. This requires some technical work which we now begin. Note that in the process we will naturally get the equations below which will play an important role in the proof of Lemma 6. Suppose that $J(\overline{A})$ does not act faithfully on A. Decompose $J(\overline{A})$ into $H \times F$ where H is the subgroup which acts trivially on A and where F is a maximal subgroup which acts faithfully on A. Since

$J(\bar{A}) \approx Z_2^s$, $H \approx Z_2^p$ and $F \approx Z_2^{s-p}$ for some integer p. From 5) of Definition , we know that each of the vectors, v_i, in the special set $e_1, \cdots e_s, v_{s+1}, \cdots v_n$ is of the form $v_i = p_i/m_i X_i + 1/m_i W_i$ where $W_i \in A$ and where $\forall \sigma \in J(\bar{A})$, $\sigma \cdot x_i = \pm x_i$. If we let $\ell(x_i)$(respectively $\ell(e_j)$) denote the one dimensional subgroup of Z^n generated by x_i (respectively e_j) then the subgroup, L, of Z^n generated by $e_1, \cdots e_s, x_{s+1}, \cdots, x_n$ is $J(\bar{A})$-isomorphic to $\ell(e_1)\oplus\cdots\oplus\ell(e_s) \oplus \ell(x_{s+1})\oplus\cdots\oplus\ell(x_n)$. L clearly has finite index in Z^n. Thus, since $J(\bar{A})$ is faithfully represented on Z^n, $J(\bar{A})$ is also faithfully represented on L. So by Fact 2d there exists vectors $x_{i1}, \cdots x_{ip}, e_{j1}, \cdots, e_{js-p}$ and generators $\tau_1, \cdots, \tau_p$ of H and $\eta_1, \cdots \eta_{s-p}$ of F such that

$$(*) \qquad \tau_h \cdot x_{i_t} = \begin{cases} -x_{i_t} & t = h \\ x_{i_t} & t \neq h \end{cases} \quad t=1,p, \quad \tau_h \cdot e_{j_t} = e_{j_t} \quad h,t=1,s-p$$

and

$$\eta_h \cdot x_{i_t} = x_{i_t} \quad t = 1,p, \quad \eta_h \cdot e_{j_t} = \begin{cases} -e_{j_t} & t = h \\ e_{j_k} & t \neq h \end{cases} \quad h,t = 1,s-p$$

(We will need these equations in the proof of Proposition 1 below). Let π'' be the subgroup of π generated by π' together with the lattice points $v_{i_1}, \cdots, v_{i_p}$. Note that if M denotes the $J(\bar{A})$submodule of Z^n generated by A together with

$v_{i_1}, \cdots v_{i_p}$, then $J(\overline{A})$ is faithfully represented on M. Thus, π'' is the subgroup of π whose holonomy group is $J(\overline{A})$ which we have been seeking. Clearly, π'' satisfies the exact sequence,

$$0 \to M \to \pi'' \to J(\overline{A}) \to 0 .$$

Remember that π'' is a torsion free Bieberbach group because it is a subgroup of π. Below we will employ other arguments to show that π'' is not a torsion free Bieberbach group. The source of this contradiction is the assumption that the translational involutions of R^n/π determined by the projection, $\overline{A}$, of A to $(Z^n/2Z^n)^{Z_2^k}$ have a common fixed point. Thus, we are forced to conclude that these involutions do not have a common fixed point. It then follows from Theorem (30.1) of Conner and Floyd that R^n/π _is_ a boundary, and the theorem is proved.

We begin the final phase of the proof, the arguments which show that π'' is not a torsion free Bieberbach group. These are contained in Lemmas 5 and 6 below. Lemma 5 provides a purely technical fact needed in the proof of Proposition 1.

Lemma 5. Let G be an additive group of homomorphisms from Z_2^m into Z_2^m with following property: There exists elements $h_1, \cdots, h_m$ of G and a set of generators $\sigma_1, \cdots, \sigma_m$ of Z_2^m

such that $\forall_i = 1,m$, Kernel (h_i) is the subgroup of Z_2^m generated by $\sigma_1, \cdots, \sigma_{i-1}, \sigma_{i+1}, \cdots \sigma_m$. Then $\exists g \in G$ and $r \in Z_2^m \ni g(r) = r$.

<u>Proof</u>. For simplicity we co-ordinatize Z_2^m with respect to the set of generators $\sigma_1, \cdots, \sigma_k$ by k-tuples of elements of the set $\{0,1\}$. We define a <u>chain</u> of length s of elements of the set, $\{\sigma_1, \cdots, \sigma_m\}$ to be a sequence, $\sigma_{j1}, \cdots \sigma_{js}$ such that $\forall i < s$ $h_{ji}(\sigma_{ji})$ has a non-zero co-ordinate in the j_{i+1}-st place and zero co-ordinates in each of the $j_1, \cdots, j_i$-places. For instance, each element $\sigma_j \in \{\sigma_1, \cdots \sigma_m\}$ is a chain of length one because if we write $\sigma_{j1} = \sigma_i$ then there are no indices j_i with $i < 1$, so that the defining condition of a chain is satisfied vacuously. Note that in a general chain no restriction has been placed on the values of $h_{js}(\sigma_{js})$ but only upon the values of $h_{ji}(\sigma_{ji})$ for $i < s$.

The set of all chains of all lengths is non-empty since each $\sigma_j \in \{\sigma_1, \cdots, \sigma_m\}$ determines a chain of length one. This set is partially ordered by the relation of left subchain i.e. $\sigma_{j1} \cdots \sigma_{js} \leq \sigma_{j1} \cdots \sigma_{js} \sigma_{js+1} \cdots \sigma_{js+t}$. Let $\sigma_{js}, \cdots, \sigma_{js}$ be a maximal chain with respect to this partial ordering. The condition that $\sigma_{j1} \cdots \sigma_{js}$ is maximal implies that $h_{js}(\sigma_{js})$ has a non-zero co-ordinate in at least one of the $j1 \cdots js$ places. Let ℓ be the greatest integer such that $h_{js}(\sigma_j)$ has a non-zero co-ordinate in the $j\ell$'th place.

the homomorphism $h_{j\ell} \cdots h_{js}$. Decompose Z_2^m as $H \times L$ where H is the subgroup of Z_2^m generated by $\sigma_{j\ell} \cdots \sigma_{js}$ and where L is the subgroup generated by the set $\{\sigma_1, \cdots, \sigma_k\} - \{\sigma_{j\ell} \cdots \sigma_{js}\}$. Define the homomorphisms $\bar{h}_{j\ell}, \cdots \bar{h}_{js}$ from H into H to be $p \circ h'_{j\ell}, \cdots, p \circ h'_{js}$ where $p : H \times L \to H$ is the projection homomorphism and where h'_{ji} is the restriction of h_{ji} to H. We claim that there exists a linear combination of the $\bar{h}_{ji}$'s which sends $\sigma_{j\ell} + \cdots + \sigma_{js}$ to itself. First notice that the vectors $\{\bar{h}_{ji}(\sigma_{ji})\}$ form a Z_2-basis of H. $\{\bar{h}_{ji}(\sigma_{ji})\}$ implies a basis that there is a linear combination of $\{\bar{h}_{ji}(\sigma_{ji}) i = \ell, s\}$ which equals $\sigma_{j\ell} + \cdots + \sigma_{js}$. Denote this linear combination by

$$\bar{h}_{m_1}(\sigma_{m_1}) + \cdots + \bar{h}_{m_p}(\sigma_{m_p}).$$

Then $\bar{h}_{m_1} + \cdots + \bar{h}_{m_p}(\sigma_{j\ell} + \cdots + \sigma_{js}) = \sum_{i=1}^{p} \bar{h}_{m_i}(\sigma_{mi})$ because $\bar{h}_{mi}(\sigma_j) \neq 0 \Leftrightarrow \sigma_j = \sigma_{mi}$. But $\sum_{i=1}^{p} \bar{h}_{m_i}(\sigma_{m_i}) = \sigma_{j\ell} + \cdots + \sigma_{js}$.

Now consider the element γ of Z_2^m defined by $\gamma = (\sum_{i=1}^{p} h_{m_i})(\sigma_{j\ell} + \cdots + \sigma_{js})$. Then clearly $(\sum_{i=1}^{p} h_{m_i})(\gamma) = \gamma$ ∎

Example. Let $m = 2$. Let $h_1(\sigma_1) = \sigma_1 + \sigma_2$, $h_1(\sigma_2) = 0$ and let $h_2(\sigma_2) = \sigma_1 + \sigma_2$, $h_2(\sigma_1) = 0$. σ_1 is a maximal chain of length one because $h_1(\sigma_1)$ has a non-zero co-ordinate in the σ_1 direction. Thus, $\sigma_{j\ell} = \sigma_1$ and H is just the subgroup $\langle \sigma_1 \rangle$ of $Z_2 \times Z_2$ generated by σ_1. $\bar{h}_1$ is the map which sends σ_1 to itself. Obviously $\bar{h}_1(\sigma_1)$ is a Z_2-basis of $\langle \sigma_1 \rangle$

and $h_1(\sigma_1) = \sigma_1$. Now $\gamma = h_1(\sigma_1) = \sigma_1 + \sigma_2$. $h_1(\gamma)$ $= h_1(\sigma_1+\sigma_2) = h_1(\sigma_1) + h_1(\sigma_2) = \sigma_1 + \sigma_2 + 0 = \sigma_1 + \sigma_2$, so h_1 sends $\sigma_1 + \sigma_2$ to itself.

Lemma 6. π'' is not a torsion free Bieberbach group.

Proof. We first look at π'' from the cohomological standpoint. In particular, we construct a section, s, of $\overline{U}(\overline{A})$ in π'' and derive from s a cocycle $\alpha \in Z^2(J(\overline{A});M)$ representing the cohomology class of the extension
$0 \to M \to \pi'' \to J(\overline{A}) \to 0$. The image of α will actually lie in A. Set $s(id) = id_{R^n}$. To each of the generators $\sigma_1, \cdots \sigma_s$ of $J(\overline{A})$ associate the element $s(\sigma_1) = (\frac{1}{2}e_i, \sigma_i)$. To each element $\sigma_{i_1} \cdot \cdots \cdot \sigma_{i_p}$ of $J(\overline{A})$ with $i_1 < i_2 < \cdots < i_p$ associate the element
$s(\sigma_{i_1} \cdot \cdots \cdot \sigma_{i_p}) = (\frac{1}{2}e_{i_1} + \cdots \frac{1}{2}e_{i_p}, \sigma_{i_1} \cdot \cdots \cdot \sigma_{i_p})$. Note that $s(\sigma_{i_1} \cdots \sigma_{i_p})$ is not the natural choice of a lift of $\sigma_{i_1} \cdot \cdots \cdot \sigma_{i_p}$ to π'' since in general it is not generated by products of the $s(\sigma_i)$'s.

Since $s(id) = id_{R^n}$, we may compute $\alpha \in Z^2(J(\overline{A});M)$ by $\alpha(\sigma,\sigma) = s(\sigma)^2$ for each $\sigma \in J(\overline{A})$. If we write σ as $\sigma_{i_1} \cdot \cdots \cdot \sigma_{i_p}$ this equation becomes

$$\alpha(\sigma_{i_1} \cdot \cdots \cdot \sigma_{i_m}, \sigma_{i_1} \cdot \cdots \cdot \sigma_{i_m}) = (\tfrac{1}{2}e_{i_1} + \cdots + \tfrac{1}{2}e_{i_m}, \sigma_{i_1} \cdot \cdots \cdot \sigma_{i_m})^2$$

$= \frac{1}{2}(e_{i_1} + \cdots + e_{i_m}) + \sigma_{i_1} \cdot \cdots \cdot \sigma_{i_m} \cdot (\frac{1}{2}e_{i_1} + \cdots + \frac{1}{2}e_m)$

$= e_{i_1} + \cdots + e_{i_m} - N_{i_1} \cdots {i_m}$ where $N_{i_1} \cdots {i_m}$ is the sum of those special basis vectors in the set $\{e_{i_1}, \cdots, e_{i_m}\}$ upon which $\sigma_{i_1} \cdot \cdots \cdot \sigma_{i_m}$ acts by negation. This fact will be needed later so we state it as

Fact 3. Let $\sigma \in J(\bar{A})$, and let $s(\sigma)$ be written as (T,σ). Then $\alpha(\sigma,\sigma) = 2T^+$ where T^+ is the component of 2T with respect to the special basis vectors $e_1, \cdots, e_s$ upon which σ acts trivially.

Recall the equations above. We use these equations to compute the actions of $\tau_1, \cdots, \tau_p$, $\eta_1, \cdots, \eta_{s-p}$ upon the vectors $v_{i_1}, \cdots, v_{i_p}$, $e_{j1}, \cdots, e_{js-p}$ chosen above. We first list these actions and then provide the necessary discussion immediately afterwards.

$$\tau_n \cdot vi_t = \begin{cases} -vi_t + W_{\tau_n, i_t} & \text{and } W_{\tau_n, i_t} \neq 0, t = h \ W_{\tau_n, i_t} \in A \\ vi_t & t \neq h \end{cases}$$

$\tau_n \cdot e_{jt} = e_{jt}$ and

$$\eta_h \cdot vi_t = vi_t + Wn_k, i_t, \quad \eta_h \cdot e_{jt} = \begin{cases} -e_{jt} & t = h \\ e_{jt} & t \neq h \end{cases} \quad Wn_k, i_t \in A$$

The equations for the η_h's follow directly from Definition 5. $\tau_h \cdot e_{jt} = e_{jt}$ is true because it is true in * . $\tau_h \cdot vi_h = -vi_h + W\tau_h, i_h$ also follows directly from Definition 5, but the claim, $W\tau_h, i_h \neq 0$, requires some proof. By 4) of Definition 5, $\exists g \in Z_2^k$ such that $g \cdot v_{i_h} + W_{g, i_h}$ where $W_{g, i_h} \neq 0$. Suppose $W_{\tau_h, i_h} = 0$. Since

τ_h and g commute, we get $\tau_h \cdot g(v_{i_h}) = g \cdot \tau_h(v_{i_h})$
$\Rightarrow \tau_h(\pm v_{i_h} + W_{g,i_h}) = g(-v_{i_h}) \Rightarrow \mp v_{i_h} + \tau_h \cdot W_{g,i_h} = -g(v_{i_h})$
$\Rightarrow \mp v_{i_h} + W_{g,i_h} = \mp v_{i_h} - W_{g,i_h} \Rightarrow w_{g,i_h} = - W_{i_h,g}$ contradicting $W_{g,i_h} \neq 0$.

$\tau_h \cdot v_{i_t} = v_{i_t}$ $t \neq h$ also requires some proof. From Definition , we know that $\tau_h \cdot v_{i_t} = v_{i_t} + W_{\tau_h, i_t}$. We must show that $W_{\tau_h, i_t} = 0$. Since $\tau_h^2 = 1$, $v_{i_t} = \tau_k^2 \cdot v_{i_t}$ $= \tau_h(v_{i_t} + W_{\tau_n, i_t}) = v_{i_t} + W_{\tau_n, i_t} + \tau_h \cdot W_{\tau_n} = v_{i_t} + 2W_{\tau_n}, it$ because τ_n acts trivially on A. Thus, $2W_{\tau_n, i_t} = 0$ $\Rightarrow W_{\tau_n, i_t} = 0$.

We next use the equations (*) to construct an additive group, G, of homomorphisms of $J(\overline{A})$ into $J(\overline{A})$. Now $A/2A \approx Z_2^s \approx J(\overline{A})$. We choose a particular isomorphism, p, from A/2A to $J(\overline{A})$ by sending each coset $e_i + 2A$ to σ_i. Let $\overline{v}_1, \cdots, \overline{v}_p$ be the projections of $vi_1, \cdots, vi_p$ to $Z^n/2Z^n$. For each $j = 1,p$ define the homomorphism h; from $J(\overline{A})$ to $J(\overline{A})$ by

$$h_j(\tau_i) = p(\tau_i \cdot \overline{v}_j + v_j) \text{ and } h_j(\eta_i) = \rho(\eta_i \cdot \overline{v}_j + \overline{v}_j).$$

From (*) we compute h_j explicitly as

$h_j(\tau_i) = 0 \quad i \neq j$

$= \rho(\overline{W}_j, \tau_i)$ where $\overline{W}_j, \tau_i$ is the projection of W_j, τ_i to $\overline{A}$, and $i = j$.

$h_j(\eta_i) = \rho(\overline{W}_j, \eta_i)$ where $\overline{W}_j, \eta_i$ is the projection of W_j, η_i to $\overline{A}$

Define $\forall_j = p + 1, s$ the homomorphism $g_j : J(\bar{A}) \to J(\bar{A})$ by

$$g_j(\tau_i) = \sigma \quad \forall i = 1, p$$

$$g_j(\eta_u) = 0 \text{ if } \eta_u \cdot ei_j = e_{i_j}$$

$$= p(\bar{e}_{i_j}) \text{ if } \eta_u \cdot e_{i_j} = - e_{i_j} \text{ and where } \bar{e}_{i_j}$$

is the projection of e_{i_j} to $\bar{A}$.

Next define $\bar{h}_j : J(\bar{A}) \to J(\bar{A})$ by $\bar{h}_j(\tau_i) = h_j(\tau_i)$ and $\bar{h}_j(\eta_i) = 0$. Let G be the group of homomorphisms generated additively by $g_{p+1}, \cdots, g_s, \bar{h}_1, \cdots, \bar{h}_p$. G satisfies the conditions of Lemma 5 from which it follows that there exists $f \in G$ and $\sigma \in J(\bar{A}) \ni f(\sigma) = \sigma$. We will show that σ cancels $\alpha(\sigma, \sigma)$. By Lemma 2 this is sufficient to prove Proposition 1.

Since the homomorphisms $\bar{h}_1, \cdots, \bar{h}p$, $g_{p+1}, \cdots, g_{n-s}$ are a free Z_2-basis of G, f can be written uniquely as a sum

(1) $$f = h_{i_1} + \cdots + h_{i_t} + g_{j_1} + \cdots + g_{j_r}$$

We may assume that $h_{i_k}(\sigma)$ (respectively $g_{i_k}(\sigma) \neq 0) \forall k = 1, t (k = 1, r)$. Since any of the factors of f which send σ to zero may be deleted without losing the condition $f(\sigma) = \sigma$. From this equation and from the definition of $\bar{h}_1, \cdots, \bar{h}_p$, $g_{p+1}, \cdots, g_{n-s}$ we see that the condition $f(\sigma) = \sigma$ implies that σ contains the factor $\tau_{i_1} \cdot \cdots \cdot \tau_{i_k} \cdot \eta_{j_1} \cdot \cdots \cdot \eta_{j_r}$.
Furthermore, consider the translational component, T, of

$s(\sigma)$. In particular, note that the projection of 2T to $\bar{A}$ is sent to σ by the homomorphism, ρ, i.e. $\rho(\overline{2T}) = \sigma$. From this fact and from the equation, $f(\sigma) = \sigma$, we get

(3) $$2T = (\tau_{i_1} \cdot v_{i_1} + v_{i_1}) + \cdots + (\tau_{i_t} \cdot v_{i_t}) + e_{j_1} + \cdots + e_{j_r} .$$

Set $v = \sum_{j=1}^{t} v_{i_j}$; $\tau = \tau_{i_1} \cdots \tau_{i_t}$; $\eta = \eta_{j_1} \cdot \cdots \cdot \eta_{j_r}$, and $W_\tau = \sum_{j=1}^{t} (\tau_{i_j} v_{i_j} + v_{i_j})$. In this new notation we have $\sigma = \tau\eta$ and $\tau \cdot v = - v + W_\tau$.

Since by construction $\forall u = 1, r$

$$\eta_{j_u} \cdot v_{i_1} = v_{i_1} + W_{i_1}, \ldots, \eta_{j_u} \cdot v_{i_t} = v_{i_t} + W_{i_t}, \eta_{j_u}$$

$\eta \cdot v$ is of the form $\eta \cdot v = v + W_\eta$. From this we get

$$\sigma \cdot v = \tau_\eta \cdot v = \tau(v + W_\eta) = - v + W_\tau + \tau \cdot W_\eta = - v + W_\tau + W_\eta$$

(because τ acts trivially on A).

Now σ acts trivially on $W_\tau + W_\eta$ because

$$v = \sigma^2 \cdot v = \sigma \cdot (-v + W_\tau + W_\eta) = - \sigma \cdot v + \sigma \cdot (W_\tau + W_\eta) = -v + (W_\tau + W_\eta)$$

$$+ \sigma \cdot (W_\tau + W_\eta) \Rightarrow 0 = - (W_\tau + W_\eta) + \sigma \cdot (W_\tau + W_\eta) \text{ or } \sigma \cdot (W_\tau + W_\eta) = W_\tau + W_\eta$$

Since τ acts trivially on A, $\sigma \cdot W_\eta = \eta \cdot W\eta$ from which it follows that $\sigma \cdot W_\eta = - W_\eta$ because $v = \eta^2(v) = \eta(v + W_\eta) = v + W_\eta + \eta \cdot W_\eta$. Now if W_η has a non-zero component in one of the e_i directions in which W_τ has a zero component, then $W_\tau + W_{\eta'}$ has a non-zero component in this direction as well. Since $\sigma \cdot W_{\eta'} = - W_{\eta'}$ it follows that $\sigma \cdot (W_\tau + W_{\eta'}) \neq W_\tau + W_{\eta'}$ which contradicts the fact that σ acts trivially on $W_\tau + W_{\eta'}$. Thus, $W_\tau + W_{\eta'}$ is the component of W_τ with respect to the special basis vectors $e_1, \cdots, e_s$ upon which σ acts trivially. Now from equations 3) we have $W_\tau = 2T - (e_{j_1} + \cdots + e_{j_r})$ mod $2A$. Thus, mod $2A$ $W_\tau + W_{\eta'}$ is the component of $2T - (e_{j_1} + \cdots + e_{j_r})$ upon which σ acts trivially. But $\forall_i = 1, r, \sigma \cdot e_{j_i} = -e_{j_i}$ because from equation 1), $g_{j_i}(\sigma) \neq 0$ $\forall_i = 1, r$. From equation 2) we know that $e_{j_1}, \cdots, e_{j_r}$ is a complete list of the e_i's upon which σ both acts by negation, and in which σ has a non-trivial translation component. Thus by the same type reasoning it follows that the component of $2T - (e_{j_1} + \cdots + e_{j_r})$ upon which σ acts trivially is equal to the component of $2T$ upon which σ acts trivially. But this component is $2T^+$ (see Fact 3). Thus, we have

$$\sigma \cdot v + v = 2T^+$$

By Fact 3, $2T^+ = \alpha(\sigma, \sigma)$. Thus, σ cancels $\alpha(\sigma, \sigma) \Rightarrow$ by Lemma 2 that π'' is not a torsion free Bieberbach group. ∎

Since π'' is a subgroup of π Proposition 1 implies that π is not a torsion free Bieberbach group. This is a contradiction so we are forced to conclude that π'' does not exist. But π'' arose from π solely on the assumption that the translational involutions determined by $\bar{A}$ have a common fixed point. Thus, we see that these involutions do not have a common fixed point. Theorem 4 now follows from Theorem (30.1) of Conner and Floyd. ■

We give an example of the situation of Proposition 1.

<u>Example</u>. Let $\tau = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $\eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$,

$2T = (0,1,1)$, $v_3 = (1,0,0)$, $e_1 = (0,0,1)$, and $e_2 = (0,1,0)$. Here $\alpha(\sigma,\sigma) = (0,1,0)$. We have $\bar{h}_1(\tau) = \bar{h}_1(\sigma) = (0,1,1)$. So $f = \bar{h}$. Thus, $W_\tau = (0,1,1)$. Now

$$\sigma = \tau_\eta = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow \sigma\cdot(1,0,0) + (1,0,0) = (0,1,0) = 2T^+$$

= the component of W_τ upon which σ acts trivially.

<u>Remark</u>. The author feels that these same techniques may be applicable to the case where ϕ is an arbitrary abelian 2-group. Proving this would involve the construction of the group, π'', in this new setting which in turn amounts to proving the analogue of Lemma 4.

CHAPTER IV

COHOMOLOGICAL TECHNIQUES

In this section we describe a different approach to the problem. Since the classifying map of the tangent bundle can be factored through $B(\phi)$, every Stiefel-Whitney class of R^n/π is of the form $\rho^*(W_i)$ where $\rho : R^n/\pi \to B(\phi)$ is the projection map; $W_i = r^*(W_i)$ where $r : B(\phi) \to BO(n)$ is induced by the action of ϕ on R^n; and where W_i is the i'th universal Stiefel-Whitney class. One hopes to solve cobordism problems by showing that if $W_{i_1} \cup \cdots \cup W_{i_k} \in H^n(\phi;Z_2)$ is a product of Stiefel-Whitney classes then $\rho^*(W_{i_1} \cup \cdots \cup W_{i_k}) = 0$.

<u>Example 1. The classifying map of the tangent bundle can be factored through BU(n).</u>

Here since r factors through $BU(n)$ any class $r^*(W_i)$ is the mod 2 reduction of an element $X_i \in H^i(\phi;Z)$. Thus if $\rho^*(W_{i_1} \cup \cdots \cup W_{i_k})$ is a Stiefel-Whitney number of R^n/π, $\rho^*(W_{i_1} \cup \cdots \cup W_{i_k})$ is the mod 2 reduction of $\rho^*(X_{i_1} \cup \cdots \cup X_{i_k})$. Since R^n/π is orientable, $H^n(\pi;Z) \approx Z$. But $H^n(\phi;Z)$ is a finite abelian group so that $\rho^* : H^n(\phi;Z) \to H^n(\pi;Z)$ must be the zero map. Thus, all of the Stiefel-Whitney numbers of R^n/π are zero.

<u>Example 2. Abelian 2-groups with no direct factors isomorphic to Z_2.</u>

We begin with some facts about the Steenrod algebra of ϕ.

Lemma 6. Let ϕ be a finite abelian 2-group with no direct factors isomorphic to Z_2. Then the maps, Sq^{odd}, are all zero.

Proof. First assume that ϕ is cyclic of order 2^{2+p}. We prove Lemma 6 in this case by induction.

$Sq^1 = 0$: Sq^1 is the mod 2 reduction of δ^*, the connecting homomorphism associated to the Bockstein co-efficient sequence determined by $0 \to Z \xrightarrow{x2} Z \xrightarrow{/2} Z_2 \to 0$. Now $H^{odd}(\phi,Z) = 0$; $H^{even}(\phi;Z) = Z/2^{2+p}Z$; and $H^*(\phi;Z_2) = Z_2$. From even to odd dimensions, $Sq^1 = 0$ because $H^{odd}(\phi;Z) = 0$. From odd to even dimensions we see that since $2\ Sq^1(x) = 0$ and since $H^{even}(\phi;Z) = Z/2^{2+p}Z$, $/2(\delta^*(x)) = 0$.

$Sq^{2j+1} = 0$: Assume inductively for all odd integers, $k < 2_j + 1$, that $Sq^k = 0$. Consider Sq^{2j+1}. By the Adem relations (see Steenrod and Epstein [11])

$$Sq^{2j+1} = Sq^j Sq^{j+1} + \sum_{s>0} \begin{pmatrix} j - s \\ j - 2s \end{pmatrix} Sq^{2j+1-s} Sq^3 .$$

Since all terms on the right contain an odd square of dimension strictly less than $2j + 1$, $sq^{2j+1} = 0$. Now let ϕ be the Cartesian product of cyclic 2-groups of order greater than 2, $\phi = G_1 x \cdots x G_m$. One knows from the Kunneth formula that $H^*(\phi;Z_2) \approx \otimes_{j=1}^{m} H^*(G_j;Z_2)$. Let $i_j : G_j \to \phi$ be the inclusion homomorphism. Then

$x \in H^{o}(G_1;Z_2)\otimes\cdots\otimes H^{k}(G_j;Z_2)\otimes\cdots\otimes H^{o}(G_m;Z_2)$ is non-zero $\Leftrightarrow i_j^*(x) \neq 0$. Thus $Sq^{odd}(x) = 0$ by the above. Now consider an arbitrary element $x_1 \cup\cdots\cup x_m \in H^{j_1}(G_1;Z_2)\otimes\cdots\otimes H^{j_m}(G_m;Z_2)$. By the Cartan formula,

$$Sq^{2j+1}(x_1\cup\cdots\cup x_m) = \sum_{s=0}^{2j+1} Sq^s(x_1)\cup Sq^{2j+1-s}(x_2\cup\cdots\cup x_m).$$

If s is odd then $Sq^s(x_1) = 0$. If s is even, then $2j+1-s$ is odd so that Lemma 6 now follows from a straightforward induction. ∎

The odd dimensional case

Let n be odd. We need two standard facts. 1) For any vector bundle over any space, $W_{2j+1} = Sq^1(W_{2j}) + W_{2j}\cup W_1$. 2) If M^n is a manifold and if W_1 is the first Stiefel-Whitney class of the tangent bundle of M^n then for each $x \in H^{n-1}(M^n;Z_2)$, $Sq^1(x) = x \cup W_1$. (See Spanier [9] pg. 350). Since n is odd, any Stiefel-Whitney number must contain an odd Stiefel-Whitney class as a factor. Thus, $\rho^*(W) = \rho^*(\overline{W})\cup \rho^*(W_{2j+1})$. Since $Sq^{odd} = 0$ in $H^*(\phi;Z_2)$, $W_{2j+1} = W_{2j}\cup W_1 \Rightarrow \rho^*(W)$ $= \rho^*(\overline{W})\cup \rho^*(W_{2j})\cup \rho^*(W_1) = Sq^1(\rho^*(\overline{W}\cup W_{2j}) = \rho^* Sq^1(\overline{W}\cup W_{2j}) = 0$ by Lemma 6. Thus R^n/π is a boundary.

The even dimensional orientable case

Let n be even and suppose that R^n/π is orientable. $W_{2i+1} = W_{2i}\cup W_1 + Sq^1(W_{2i}) \Rightarrow W_{2i+1} = 0$ since $Sq^1 = 0$ in ϕ and since $W_1 = 0$. Thus in order to show that R^n/π is a boundary it is sufficient to show that $\rho^*(x_1 \cup \cdots \cup x_u) = 0$ where $\rho^*(x_i)$ is an even Stiefel-Whitney class. For this we need to discuss the Wu classes of a manifold (See Spanier [9] pg. 350). To every n-manifold, M^n, there exist classes $v_i \in H^i(M^n;Z_2)$, $i = 0, n$ called Wu classes which satisfy the following two properties; 1) For each $x \in H^{n-i}(M^n;Z_2)$, $Sq^i(x) = x \cup v_i$; 2) the total Steenrod operation applied to the total Wu class is equal to the total Stiefel-Whitney class of M^n i.e.

$$(\sum_{j=0}^{\infty} Sq^j)(\sum_{i=0}^{n} v_i) = \sum_{i=0}^{n} W_i \quad .$$

<u>Lemma 7</u>: $v_{2j+1} = 0 \quad \forall j$.

<u>Proof</u>. (By induction on j). $v_1 = W_1 = 0$ because R^n/π is orientable. Now $W_{2j+1} = v_{2j+1} + Sq^1(v_{2j}) + \cdots + Sq^{2j+1}(v_o)$. All of the terms on the right are zero except possibly v_{2j+1} because $Sq^{odd} \equiv 0$ in $H^*(\phi;Z_2)$ and by the inductive hypothesis. Thus, $W_{2j+1} = W_{2j+1}$. But $W_{2j+1} = Sq^1(W_{2j}) + W_{2j} \cup W_1 = 0$. ∎

<u>Lemma 8</u>. Let $x \in H^{2i}(\phi;Z_2)$. Then $Sq^{2j}(x) = x \cup y$ where y is the mod 2 reduction of an element, $\overline{y} \in H^*(\phi;Z)$.

<u>Proof</u>. If G_i is any one of the cyclic factors of ϕ, then $H^*(G_i;Z_2)$ is a polynomial algebra in even dimensions on one generator, $\beta_i \in H^2(G_i;Z_2)$ tensored with a truncated polynomial algebra generated by the non-zero element $x_i \in H^1(G_i;Z_2)$

where $x_i \cup x_i = 0$. Therefore by the Kunneth formula, an element of $H^*(\phi;Z_2)$ can be factored as a product of the form $\beta_{i_1}\cup\cdots\cup\beta_{i_k}\cup x_{j_1}\cup\cdots\cup x_{j_s}$. The lemma now follows from $Sq^2(\beta_j) = \beta_j^2$, $Sq^2(x_j) = 0$, the Cartan formula, and a straightforward induction. ∎

Now let $\rho^*(W_{i_1},\cup\cdots\cup W_{i_k})$ be a Stiefel-Whitney number of R^n/π. Since all of the odd Whitney classes are zero, we may assume that the W_{i_j}'s are even with $i_j \leq i_{j+1}$. Consider first $\rho^*(W_2)$. $\rho^*(W_2) = v_2$ so that if $\rho^*(W_{i_1}\cup\cdots\cup W_{i_k})$ is of the form $\rho^*(W_2)\cup\rho^*(W)$ then

$\rho^*(W_{i_1}\cup\cdots\cup W_{i_k}) = \rho^*(W_2)\cup\rho^*(W) = Sq^2(\rho^*(W)) = \rho^*Sq^2(W)$

$= \rho^*(W)\cup\rho^*(y)$ where $\rho^*(y)$ is the mod 2 reduction of $\rho^*(\bar{y}) \in H^*(\pi;Z)$. Assume inductively that if $i_1<\cdots<i_j < 2m$ then $\rho^*(W_{i_1}\cup\cdots\cup W_{i_j}\cup W_{i_{j+1}}\cup\cdots\cup W_{i_k}) = \rho^*(W_{i_{j+1}}\cup\cdots\cup W_{i_k})\cup\rho^*(y')$ where $\rho^*(y')$ is the mod 2 reduction of $\rho^*(\bar{y}')$. Let $i_{j+1}=2m$. Then $\rho^*(W_{i_{j+1}}\cup\cdots\cup W_{i_k})\cup\rho^*(y') = \rho^*(W_{2m})\cup\rho^*(W_{i_{j+2}}\cup\cdots\cup W_{i_k})\cup\rho^*(y')$.

Now $\rho^*(W_{2m}) = v_{2m} + Sq^2(v_{2m-2})+\cdots+Sq^{2m-2}(v_2)$

$=$ (by Lemma 8) $v_{2m} + v_{2m-2}\cup\rho^*(y_{2m-2})+\cdots+v_2\cup\rho^*(y_2)$ where y_{2m-2i} have lifts, $\bar{y}_{2m-2i} \in H^*(\phi;Z)$. Thus,

$\rho^*(W_{2m})\cup\rho^*(W_{i_{j+2}}\cup\cdots\cup W_{i_k})\cup\rho^*(y')$

$= \sum_{s=1}^{m} Sq_{2s}(\rho^*(W_{i_{j+2}}\cup\cdots\cup W_{i_k})\cup\rho^*(y')\cup\rho^*(y_{2s}))$

$= \sum_{s=1}^{m} (\rho^*(W_{i_{j+2}}\cup\cdots\cup W_{i_k})\cup\rho^*(y')\cup\rho^*(y_{2s}))\cup\rho^*(y''_{2s})$ by

Lemma 8.

Rewriting this last term as $\rho^*(W_{i_{j+2}} \cup \cdots \cup W_{i_k}) \cup \rho^*(y' \cup \sum_{s=1}^{m} y'_{2s} \cup y''_{2s})$ we see that the induction is complete. We conclude from this that $\rho^*(W_{i_1} \cup \cdots \cup W_{i_k}) = \rho^*(y)$ where y is the mod 2 reduction of $\overline{y}$. Thus, $\rho^*(W_{i_1} \cup \cdots \cup W_{i_k}) = /2\rho^*(\overline{y})$. We now use the fact that $H^n(\phi;Z) \approx Z$ while $H^n(\phi;Z)$ is a finite abelian group to conclude that $\rho^*(W_{i_1} \cup \cdots \cup W_{i_k}) = 0$. We summarize these results as

Theorem 5. If ϕ is a finite abelian 2-group with no direct factors isomorphic to Z_2 and if n is odd, or if n is even and R^n/π isorientable, then R^n/π is a boundary.

Remark 1. It is typical that these techniques provide little or no information about the unorientable even dimensional case. More generally, these techniques fail because the Steenrod algebras of arbitrary 2-groups may not satisfy relations which permit conclusions about cobordism. For instance, the Steenrod Algebra of Z_2^k satisfies no relations other than those contained in the axioms for the Steenrod Algebra itself. So far the author knows of no other theorems derivable from this approach.

Remark 2. Theorem 5 can be extended slightly to give partial information in the unorientable even dimensional case e.g. since n is even any Whitney number involving one odd Whitney class must involve at least two odd Whitney classes, W_{2i+1}, W_{2j+1}. Since $W_{2i+1} = W_{2i} \cup W_1$ and $W_{2j+1} = W_{2j} \cup W_1$, $W_{2i+1} \cup W_{2j+1} = W_{2i} \cup W_{2j} \cup W_1^2 = 0$ because $W_1^2 = Sq^1(W_1) = 0$. Other facts derive from relations in the Steenrod Algebra e.g. suppose $n = 8$ and consider $W_6 \cup W_2 = Sq^2(W_6)$. Now $W_6 = v_6 + Sq^1(v_5) + Sq^2(v_4) + Sq^3(v_3) = Sq^2(v_4)$. Thus $W_6 \cup W_2 = Sq^2 Sq^2(v_4) = Sq^1 Sq^2 Sq^1(v_4) = 0$. $Sq^2 Sq^2 = Sq^1 Sq^2 Sq^1$ is one of many universal relations between iterated even squares and mixed even and odd squares. The interested reader should consult Steenrod and Epstein [11].

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