A STUDY OF RATIONAL TOEPLITZ OPERATORS

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by

Leonard Charles Gambler

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AT STONY BROOK

Leonard Charles Gambler

We, the dissertation committee for the above candidate for the Ph.D. degree, hereby recommend acceptance of the dissertation.

Chih-han Sah, Professor
Committee Chairman

Ronald Douglas, Professor
Thesis Advisor

Henry Laufer, Professor

Israel Gohberg, Professor

Barry McCoy, Professor

The dissertation is accepted by the Graduate School.

Herbert Weisinger, Dean

August, 1977

Abstract of the Dissertation

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We show that for φ in $L^{\infty}(\mathbb{T})$, φ analytic in a neighborhood of $\mathbb{T}$ (the unit circle), for which neither the Toeplitz operator T_{φ} nor its adjoint $T_{\bar{\varphi}}$ has an eigenspace, T_{φ} is strongly decomposable. As a corollary, we show that for φ in $L^{\infty}(\mathbb{T})$, φ non-constant and analytic in a neighborhood of $\mathbb{T}$, T_{φ} has a non-trivial hyperinvariant subspace. The decomposability result is obtained by first using analytic-coanalytic factoring of invertible rational φ in conjunction with analytic-coanalytic factoring of invertible φ with absolutly convergent Fourier series to show the growth condition: $\| T_{\varphi-\lambda}{}^{-1} \| < K(\text{dist}(\lambda,\sigma(T_{\varphi})\ )\ )^{-2}$ for λ not in $\sigma(T_{\varphi})$, K a positive constant where φ is analytic in a neighborhood of $\mathbb{T}$ with neither T_{φ} nor $T_{\bar{\varphi}}$ having an eigenspace. Then by arguments close to those in (M. Radjabalipour, "Growth conditions and decomposable

operators", Canad. J. Math. 26 (1974), 1372-1379), we establish the desired result.

Included here are a few techniques used in the study of rational Toeplitz operators. Many known results are reproved for the rational case using these techniques. Also included is a progress report on the problem of determining the span of the eigenfunctions of T_φ and $T_{\bar{\varphi}}$ where φ in $L^\infty(\mathbb{T})$ is rational.

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Acknowledgments

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Third, I express my special appreciation to Prof. Ronald Douglas. Without his persistent but patient mathematical guidance this dissertation would surely not have been written.

CHAPTER 0: INTRODUCTION

The original problem for this thesis was the spanning characteristics of the eigengunctions of the Toeplitz operators T_{φ} and $T_{\overline{\varphi}}$, for φ rational. Whether or not these eigengunctions span depends on the characteristics of φ, e.g., if $\varphi(\mathbb{T})$ is a subset of $\mathbb{R}$, or any straight line, then there are no eigengunctions at all. So part of the problem was to find appropriate conditions on φ which would guarantee spanning of the eigenfunctions. Since it is not hard to prove that the eigenfunctions of a non-trivial coanalytic Toeplitz operator do span $H^2(\mathbb{T})$, it had been speculated that the same thing should be true for those φ which satisfy some reasonable hypotheses(e.g., the graph, $\varphi(\mathbb{T})$, has a finite number of self-intersection points), and would not be too hard to prove, at least in the case where φ was rational. This problem has turned out to be very truculant and the speculation is false in general, both for the spanning and the difficulty.

Partial results in this area have been attained by Prof. D. Clark and his student J. Morrel. They were able to show in (4) that for F(z), a rational function such that $t \rightarrow F(e^{it})$ is a simple closed positively oriented curve and F is one-to-one in some annulus $s \leq |z| \leq 1$, the eigenfunctions of $T_{\overline{F}}$ span all of $H^2(\mathbb{T})$. Clark, in personal communications with Prof. Douglas, has given results which indicates this problem's depth: he has found examples of Toeplitz operators whose eigenfunctions do not span. In these examples the symbols are rational with the property that they map the circle, $\mathbb{T}$, into $\mathbb{T}$. Further progress awaits new ideas.

The study of this problem, as in the case of many difficult research problems, although not solved, has led us to other significant discoveries. In this case the discoveries are in the area of the structure of Toeplitz operators with rational symbols. Namely, we found that rational Toeplitz operators have non-trivial hyperinvariant subspaces and, in fact, the ones where neither T_φ nor $T_{\bar{\varphi}}$ have eigenspaces are strongly decomposable. This fact was announced in the Notices (11). Subsequently, we generalized these results to the case where the symbol is analytic in some neighborgood of the unit circle, $\mathbb{T}$.

There has been much interest and activity in the area of invariant and hyperinvariant subspaces for bounded linear operators on separable Hilbert spaces. This thesis answers a question in a recent article of Halmos (14): he asks whether or not Toeplitz operators with polynomial symbols have non-trivial invariant subspaces.

In Chapter I, we give basic definitions, set most of the notation and state the known results which are used in later proofs. The main references are Halmos(13) for basic operator theory on separable Hilbert space and Douglas(6, Chap. 6) for Hardy space theory and (6, Chap. 7) for Toeplitz operator theory.

Chapter II is concerned with the developement of techniques used in the study of how the properties of rational functions affect the properties of the corresponding Toeplitz operators. Several well known results are reproved using a minimum of machinery.

Chapter III contains the main results of this thesis. We show that the resolvents of rational Toeplitz operators, for which neither T_φ nor $T_{\bar{\varphi}}$ possesses eigenspaces, possess certain growth restrictions near their spectrums. As a consequence we get that Toeplitz operators with symbol analytic in a neighborhood of $\mathbb{T}$, for which neither T_φ nor $T_{\bar{\varphi}}$ possess eigenspaces, have the same growth restrictions and thus by arguments close to (16) are strongly decomposable. Then as a corollary, we get that the Toeplitz operators with symbol analytic near $\mathbb{T}$ have non-trivial hyperinvariant subspaces.

Chapter IV consists of a short report on the progress and techniques used in the study of the spanning characteristics of the eigenfunctions of T_φ and $T_{\bar{\varphi}}$, where φ is rational. Using our techniques, the problem reduces to one dealing with functional relations. Also, in the course of study, certain connections with Carléson measures and compact composition operators were observed.

CHAPTER I: BASICS

We begin by letting $\mathbb{T}$ denote the unit circle in the complex plane and μ the Lebesgue measure on $\mathbb{T}$ normalized so that $\mu(\mathbb{T}) = 1$. We can define the Lebesgue spaces $L^p(\mathbb{T})$ with respect to μ. The Hardy spaces $H^p(\mathbb{T})$ are defined as closed subspaces of $L^p(\mathbb{T})$. Using the standard symbol $\mathbb{Z}$ for the integers, let, for n in $\mathbb{Z}$, x_n be the function on $\mathbb{T}$ defined by $x_n(e^{it}) = e^{int}$. For $p = 2$ or ∞, we define the Hardy space

$$H^p = H^p(\mathbb{T}) = \left\{ f \text{ in } L^p(\mathbb{T}) : \int_0^{2\pi} f(e^{it}) x_n(e^{it})\, dt = 0 \text{ for } n > 0 \right\}.$$

For a little insight into these definitions we first look closer at the definition for $p = 2$. By elementary calculus the functions x_n form an orthonormal set in $L^2(\mathbb{T})$ and it is an easy consequence of standard approximation theorems (e.g., the Weierstrass theorem on approximation by polynomials) that the x_n's form an orthonormal basis for $L^2(\mathbb{T})$. (Finite linear combinations of the x_n's are called trigonometric polynomials.) The space H^2 is then the subspace of $L^2(\mathbb{T})$ spanned by the x_n's with $n \geq 0$; equivalently, (this is the above definition) H^2 is the orthogonal complement in $L^2(\mathbb{T})$ of $\{x_{-1}, x_{-2}, \dots\}$.

Fourier expansions with respect to the orthonormal basis $\{x_n : n = 0, \pm 1, \pm 2, \dots\}$ are formally similar to the Laurent expansions that occur in analytic function theory.

The analogy motivates calling the functions in H^2 the analytic elements of $L^2(\mathbb{T})$; the functions in $L^2(\mathbb{T})$ whose complex conjugates are in H^2 are called coanalytic.

The subset, H^∞, is the linear manifold (not a subspace) of H^2 which consists of the bounded functions in H^2 ; equivalently, H^∞ consists of all those functions in $L^\infty(\mathbb{T})$ for which all the negative Fourier coefficients vanish.

If f is in H^2, with Fourier expansion $f = \sum_{n=0}^{\infty} a_n x_n$, then $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, and therefore the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n^2 z^n$ is greater than or equal to 1. It follows from the usual expansion for the radius of convergence in terms of the coefficients that the power series $\sum_{n=0}^{\infty} a_n z^n$ defines an analytic function $\hat{f}$ in the open unit disc $\mathbb{D}$. In fact, one way to look at H^2 is as analytic functions in the unit disc with square-summable Taylor series. For further properties see(6, Chap. 6).

The main objects of scrutiny here are the so called Toeplitz operators from H^2 to H^2. Let P be the projection of $L^2(\mathbb{T})$ onto H^2. For φ in $L^\infty(\mathbb{T})$ the Toeplitz operator T_φ on H^2 is defined by $T_\varphi f = P(\varphi f)$ for f in H^2.

The original context in which Toeplitz operators were studied was not that of the Hardy spaces but rather as operators on $l^2(\mathbb{Z}_+)$. Consider the orthonormal basis $\{x_n\colon n \text{ in } \mathbb{Z}_+\}$ for H^2, and the matrix for a Toeplitz operator with respect to it. If φ is a function in $L^\infty(\mathbb{T})$ with Fourier coefficients $\hat{\varphi}(n) = \frac{1}{2\pi}\int_0^{2\pi}\varphi x_{-n}dt$, then the matrix $\{a_{m,n}\}_{m,n \text{ in } \mathbb{Z}_+}$ for T_φ with respect to $\{x_n: n \text{ in } \mathbb{Z}_+\}$ is

$$a_{m,n} = (T_\varphi x_n, x_m) = \frac{1}{2\pi}\int_0^{2\pi}\varphi x_{n-m}dt = \hat{\varphi}(m-n).$$

Thus the matrix for T_φ is constant on diagonals. Such a matrix is called a Toeplitz matrix, and it can be shown that if the matrix defines a bounded operator, then its diagonal entries are the Fourier coefficients of a function in $L^\infty(\mathbb{T})$.

Our basic desire in this thesis is to look closely at the Toeplitz operators for which the symbol, i.e., function φ, in $L^\infty(\mathbb{T})$ is taken to be rational. But we first look at some general elementary properties of Toeplitz operators and the mapping ξ from $L^\infty(\mathbb{T})$ to $\mathcal{L}(H^2)$ (the space of bounded linear operators on H^2) defined by $\xi(\varphi) = T_\varphi$.

Proposition 1.1: The mapping ξ is a contractive $*$-linear mapping from $L^\infty(\mathbb{T})$ into $\mathcal{L}(H^2)$.

Proof: (6)///

The mapping ξ is in general not multiplicative, and hence ξ is not a homomorphism. In special cases though, ξ is multiplicative, and this will be important in what follows.

Proposition 1.2: If φ is in $L^\infty(\mathbb{T})$ and ψ and $\bar{\theta}$ are functions in H^∞, then $T_\varphi T_\psi = T_{\varphi\psi}$ and $T_\theta T_\varphi = T_{\theta\varphi}$.

Proof: (6)///

The converse of this proposition is also true (3) but will not be needed in what follows.

The next result is used to show that ξ is an isometry.

Proposition 1.3: If φ is a function in $L^\infty(\mathbb{T})$ such that T_φ is invertible, then φ is invertible in $L^\infty(\mathbb{T})$.

Proof: see (6) ///

As a corollary one can obtain the spectral inclusion theorem, where $\mathcal{R}(f)$ is the essential range of f and $M_\varphi f = \varphi f$.

Corollary(Hartman-Wintner) 1.4: If φ is in $L^\infty(\mathbb{T})$, then $\mathcal{R}(\varphi) = \sigma(M_\varphi) \subseteq \sigma(T_\varphi)$.

Proof: see (6)///

This result enables one to complete the elementary properties of ξ.

Corollary 1.5: The mapping ξ is an isometry from $L^\infty(\mathbb{T})$ into $\mathcal{L}(H^2)$.

Proof: see (6)///

CHAPTER II: RATIONAL SYMBOL TECHNIQUES

Although most of what follows is known, there does not seem to be any one place where it all can be found. (Duren has done work with Toeplitz operators with rational symbol, (7), (8), (9), as have Clark and Morrel, (4).) Also, the discussions yield insight into some techniques and ideas which may be used in the study of Toeplitz operators with rational symbol.

We begin our study with:

Theorem 2.1: If φ in $L^\infty(\mathbb{T})$ is rational and f is in $\mathrm{Ker}(T_\varphi)$, then f is also rational.

Proof: The function f is in $\mathrm{Ker}(T_\varphi)$ iff $P\varphi f = 0$ iff $\varphi f = g$ is in the orthogonal complement of H^2 in $L^2(\mathbb{T})$, i.e., $\bar{g}(z)$ is analytic with $\bar{g}(0) = 0$ or in other words $g(\infty) = 0$.

If we take $\varphi(z) = \frac{p(z)}{q(z)}$ where p(z) and q(z) are polynomials with no common factor, we get $\frac{p(z)}{q(z)} f = g$ or $pf = gq$.

Since p is a polynomial and f is in H^2, pf has a valid power series expansion in the closed unit disc. Since q is a polynomial and g is coanalytic, gq has a valid power series expansion outside the closed unit disc. Putting these together we have that $pf = gq$ has a power series valid in the whole plane, i.e., pf is an entire function.

Further, since $\frac{pf}{q} = g$ is zero at infinity and pf is entire, pf must be a polynomial and have degree less than the degree of q. So if we put $pf = p^{\#}$, we have $f = \frac{p^{\#}}{p}$, a rational function.///

This proof yields quite a lot of information concerning the $Ker(T_\varphi)$ where φ is rational:

1) If $\varphi = \frac{p}{q}$ has no poles(i.e., q has no zeros) in the open disc, then φ is in H^∞ and $Ker(T_\varphi)$ is trivial.

2) With $\varphi = \frac{p}{q}$ and $f = \frac{p^{\#}}{p}$ as in the above proof, we have degree $p^{\#} =$ deg $p^{\#} <$ deg q. In order for f to be in H^2, every zero of p in the closed disc must also be a zero of $p^{\#}$(since they must cancel). Hence if the number of zeros of p in the closed disc is greater than or equal to the number of zeros of q in the open disc, then $Ker(T_\varphi)$ will be trivial. (The Argument Principle translates this into φ has positive winding number and hence $Ker(T_\varphi)$ is trivial.)

3) Since $p^{\#} = gq$ for some g coanalytic with $g(\infty) = 0$, the zeros of q(i.e., poles of φ) outside the closed disc become zeros of $f = \frac{p^{\#}}{p}$ (unless, of course, they are already zeros of p of the same or higher order).

4) Since $f = \frac{p^{\#}}{p}$, the poles of f are among the zeros of p (i.e., zeros of φ) which lie outside the closed disc.

5) Since deg $p^{\#} <$ deg q, the zeros of f can not include all the poles of φ.

6) The dimension of the subspace generated by the functions f corresponding to a given φ, i.e., the dimension of $\mathrm{Ker}(T_\varphi)$, is the number of poles of φ in the closed disc minus the number of zeros of φ which lie in the closed disc(again the Argument Principle could be used to phrase this as: "dim $\mathrm{Ker}(T_\varphi)$ is the negative of the winding number of φ about the origin").

We now use these observations to study the spectrum of T_φ where again φ is rational.

Using the notation from the proof of Theorem 2.1: $\varphi = \frac{p}{q}$, we have $\varphi(z)-\lambda = \frac{p(z)}{q(z)} - \lambda = \frac{p-\lambda q}{q}$ for λ is $\mathbb{C}$. Further let M be the number of zeros of q in $\mathbb{D}$(the open unit disc) and N_λ be the number of zeros of $p-\lambda q$ in $\overline{\mathbb{D}}$(the closed unit disc). It is clear, but we note: M is the number of poles of $\varphi-\lambda$ in $\mathbb{D}$.

From the above observations, we have that the functions in $\mathrm{Ker}(T_{\varphi-\lambda})$ can be expressed in the form:

$$f = \frac{a_M + a_{M-1}z + \cdots + a_1 z^{M-1}}{p-\lambda q}, \text{ where } a_i,\ i = 1, 2, \ldots, M$$

are arbitrary constants.

(*) If $N_\lambda \geq M$, observation 2) says that $\mathrm{Ker}(T_{\varphi-\lambda})$ is trivial. If $M > N_\lambda$, we may write f as

$$f = \frac{A_1(z-A_2)\ldots(z-A_M)}{(z-B_1)(z-B_2)\ldots(z-B_{N_\lambda})(z-B_{N_\lambda+1})\ldots(z-B_n)}$$

$$f = \frac{A_1(z-A_2) \ \ldots \ (z-A_{M-N_\lambda})}{(z-B_{N_\lambda+1}) \ \ldots \ (z-B_n)}$$

where $|B_i| \leq 1$, $i = 1, 2, \ldots, N_\lambda$ and n is the degree of $p-\lambda q$. N_λ of the arbitrary constants in the numerator had to be chosen in such a way that they cancelled the N_λ zeros of $p-\lambda q$ which lay in $\overline{D}$, i.e., these N_λ zeros become removable singularities rather than poles. Furthermore, by expressing the f in this manner we see that $M-N_\lambda$ is the number of generators of $\mathrm{Ker}(T_{\varphi-\lambda})$.

We could have used the Argument Principle to get $N_\lambda-M$ to be the winding number of φ about λ and hence by (6, 7.24 and 7.26, where 7.26 states: If φ is a continuous function on $\mathbb{T}$, then the operator T_φ is a Fredholm operator iff φ does not vanish and in this case the index is equal to minus the winding number of the curve traced out by φ with respect to the origin.) $M-N_\lambda$ is the dimension of $\mathrm{Ker}(T_{\varphi-\lambda})$. But we would not have seen the structure of the functions making up $\mathrm{Ker}(T_{\varphi-\lambda})$.

We continue using these techniques of rational functions to show that if the winding number $N_\lambda-M$ is positive, we get $N_\lambda-M$ as the dimension of $\mathrm{Ker}(T_{\overline{\varphi-\lambda}})$. This is again, of course, (6, 7.24 and 7.26).

If we write $\varphi-\lambda = \dfrac{p-\lambda q}{q} = \dfrac{c(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_n)}{(z-\beta_1)(z-\beta_2)\ldots(z-\beta_m)}$

where n is $\deg(p-\lambda q)$ and m is $\deg q$, then

$$\overline{\varphi-\lambda} = \frac{\bar{c}(\bar{z}-\bar{\alpha}_1)(\bar{z}-\bar{\alpha}_2)\ldots(\bar{z}-\bar{\alpha}_n)}{(\bar{z}-\bar{\beta}_1)(\bar{z}-\bar{\beta}_2)\ldots(\bar{z}-\bar{\beta}_m)}$$

$$= \frac{\bar{z}^n\bar{c}(1-\bar{\alpha}_1 z)(1-\bar{\alpha}_2 z)\ldots(1-\bar{\alpha}_n z)}{\bar{z}^m(1-\bar{\beta}_1 z)(1-\bar{\beta}_2 z)\ldots(1-\bar{\beta}_m z)}$$

$$= \frac{z^m\bar{c}(1-\bar{\alpha}_1 z)(1-\bar{\alpha}_2 z)\ldots(1-\bar{\alpha}_n z)}{z^n(1-\bar{\beta}_1 z)(1-\bar{\beta}_2 z)\ldots(1-\bar{\beta}_m z)}$$

since we are working on $|z|=1$, where $\bar{z}=\frac{1}{z}$.

From this we observe that if $\varphi-\lambda$ had a zero at α, $\overline{\varphi-\lambda}$ has a zero at $\frac{1}{\bar{\alpha}}$. If $\varphi-\lambda$ had a pole at β, $\overline{\varphi-\lambda}$ has a pole at $\frac{1}{\bar{\beta}}$. In other words the zeros and poles that were in $\mathbb{D}$ for $\varphi-\lambda$ correspond to zeros and poles, respectively, that are outside $\bar{\mathbb{D}}$ for $\overline{\varphi-\lambda}$ and the zeros and poles of $\varphi-\lambda$ that were outside $\bar{\mathbb{D}}$ corrrspond to ones inside $\mathbb{D}$ for $\overline{\varphi-\lambda}$. The zeros and poles for $\varphi-\lambda$ which lay on $\mathbb{T}$ correspond to zeros and poles, respectively, of $\overline{\varphi-\lambda}$ which lie on $\mathbb{T}$. Also a zero or a pole at the origin is introduced for $\overline{\varphi-\lambda}$ according to the relative sizes of m and n.

Now let $\bar{M}$ be the number of poles of $\overline{\varphi-\lambda}$ in $\mathbb{D}$ and $\bar{N}_\lambda$ be the number of zeros of $\overline{\varphi-\lambda}$ in $\bar{\mathbb{D}}$. As in the case of $\varphi-\lambda$, we have (**): If $\bar{N}_\lambda \geq \bar{M}$, then $\mathrm{Ker}(T_{\overline{\varphi-\lambda}})$ is trivial. Also if $\bar{M} > \bar{N}_\lambda$, then $\bar{M}-\bar{N}_\lambda$ is the number of generators of $\mathrm{Ker}(T_{\overline{\varphi-\lambda}})$.

We use our observations on the shifting of zeros and

and poles under complex conjugation to get: if $\deg(p-\lambda q) = n$ $n < \deg q = m$, then $\overline{M} = m-M$ and $\overline{N}_\lambda = m-N_\lambda$ and if $n \geq m$, then $\overline{M} = n-M$ and $\overline{N}_\lambda = n-N_\lambda$. In either case we have $\overline{M}-\overline{N}_\lambda = N_\lambda - M$ (clearly we are assuming that λ is not in the image of $\mathbb{T}$ under φ). So if $N_\lambda - M$ is positive, it is the dimension of $\mathrm{Ker}(T_{\overline{\varphi-\lambda}})$. We note here that we have $N_\lambda = M$ iff both $\mathrm{Ker}(T_{\overline{\varphi-\lambda}})$ and $\mathrm{Ker}(T_{\varphi-\lambda})$ are trivial.

Although φ has no winding number about any λ in the image of $\mathbb{T}$ under φ, the expression $N_\lambda - M$ is still defined. $N_\lambda - M$ can be viewed as a generalization of the winding number, since it is the winding number of φ about any λ which is not in $\varphi(\mathbb{T})$.

Definition 2.2: For every λ, we define the generalized winding number of φ about λ to be $N_\lambda - M$.

Using the generalized winding number, we can generalize the usual statements concerning winding numbers to include the points on the graph of a rational function.

Theorem 2.3: Let φ in $L^\infty(\mathbb{T})$ be rational and w_λ be the generalized winding number of φ about λ. If $w_\lambda > 0$, then $\overline{\lambda}$ is an eigenvalue of $T_{\overline{\varphi}}$ and $\mathrm{Ker}(T_{\overline{\varphi-\lambda}})$ has dimension w_λ. If $w_\lambda < 0$, then λ is an eigenvalue of T_φ and $\mathrm{Ker}(T_{\varphi-\lambda})$ has dimension $-w_\lambda$. If $w_\lambda = 0$, then λ is not an eigenvalue for T_φ nor is $\overline{\lambda}$ an eigenvalue of $T_{\overline{\varphi}}$.

Proof: The result for λ not on the graph of $\varphi(\mathbb{T})$ was shown above, so we assume λ is on the graph of $\varphi(\mathbb{T})$. The cases of $w_\lambda > 0$ and $w_\lambda < 0$ are also contained in the discussions of $\varphi-\lambda$ and $\overline{\varphi-\lambda}$ above. We are left only with $w_\lambda = 0$.

Put $K_\lambda = \text{card}(\varphi^{-1}(\lambda) \cap \mathbb{T})$. So λ is on the graph of $\varphi(\mathbb{T})$ iff $K_\lambda > 0$. To say $w_\lambda = 0$ means $N_\lambda - M = 0$, i.e., $N_\lambda = M$, and so from (*), $\text{Ker}(T_{\varphi-\lambda})$ is trivial. If $\deg(p-\lambda q) = n < \deg q = m$, we get $\overline{M} = m-M$ and $\overline{N}_\lambda = m - N_\lambda + K_\lambda$. This last equality follows since the K_λ zeros of $\varphi-\lambda$ on $\mathbb{T}$ remain K_λ zeros of $\overline{\varphi-\lambda}$ on $\mathbb{T}$. So we have $\overline{N}_\lambda = \overline{M} + K_\lambda > \overline{M}$ and hence, from (**), $\text{Ker}(T_{\overline{\varphi-\lambda}})$ is trivial. Now if $n \geq m$ we have $\overline{M} = n-M$ and $\overline{N}_\lambda = n - N_\lambda + K_\lambda$. So again $\overline{N}_\lambda > \overline{M}$ and $\text{Ker}(T_{\overline{\varphi-\lambda}})$ is trivial.///

This theorem is known, e.g., (4), but, as before, we are not aware of a proof which relies solely on rational functional argrments. This concept of generalized winding number can, of course, only be used with rational functions.

We continue and see what other results can be proven with rational argrments. This first lemma is well known and we do not prove it here.

Lemma 2.4: Let $\Omega \subseteq \mathbb{D}$ be any set with an accumulation point in $\mathbb{D}$. Then H^2 is generated by functions in the form $\frac{1}{1-\lambda z}$ where λ is taken from Ω.

<u>Lemma 2.5</u>: Let φ in $L^\infty(\mathbb{T})$ be rational. Then $\varphi = \frac{p}{q}$ is coanalytic iff $\deg q = d_q \geq d_p = \deg p$ and all the zeros of q are in $\mathbb{D}$.

Proof: The function φ is coanalytic iff $\bar{\varphi}$ is analytic iff $\bar{\varphi}$ has no poles in $\bar{\mathbb{D}}$.

If $d_q < d_p$, $\bar{\varphi}$ would have a pole at zero. If one zero of q was outside $\mathbb{D}$, then $\bar{\varphi}$ would have a pole in $\bar{\mathbb{D}}$. In either case φ would not be analytic.

If $d_q \geq d_p$ and all the zeros of q are in $\mathbb{D}$, then $\bar{\varphi}$ has no poles in $\bar{\mathbb{D}}$ and hence φ is coanalytic.///

The functions $\frac{1}{1-\lambda z}$ can easily be shown to be eigenfunctions of T_φ where φ is coanalytic and so we would have the next theorem by Lemma 2.4. But we prove a little more. This exercise demonstrates rational techniques which may find other uses.

<u>Theorem 2.6</u>: Let φ in $L^\infty(\mathbb{T})$ be rational and coanalytic. The span of $\{f \text{ in } \operatorname{Ker}(T_{\varphi-\lambda}) : \lambda \text{ in } \mathbb{C}\}$ is all H^2.

Proof: We may take $\varphi = \frac{p}{q}$ where p and q are monic polynomials with no common factor. Put $d_\lambda = \deg(p-\lambda q)$. Using the notation of Lemma 2.5, it follows from Lemma 2.5 that $d_q = d_\lambda$ except for $\lambda = 1$. But we may safely ignore this one point and take $d_q = d_\lambda$, i.e., d_λ is conatant.

From the discussions above, $\mathrm{Ker}(T_{\varphi-\lambda})$ is generated by functions of the form

$$\frac{A_1(z-A_2)\ldots(z-A_{d_q})}{(z-B_1)\ldots(z-B_{d_q})}$$

where the A_i are arbitrary constants and the B_i are the zeros of $p-\lambda q$.

If we could guarantee $|B_1| > 1$, we could choose $A_2 = B_2$, $A_3 = B_3$, ..., $A_{d_q} = B_{d_q}$ to get a generator in the form $\dfrac{A_1}{z-B_1}$. Then we could use a result of Ostrowski(15, Appendix A) which states that the zeros of a polynomial are continuous functions of the coefficients. The coefficients of our polynomial $p-\lambda q$ are continuous functions of λ. So putting these together we have that B_1 is continuous in λ. So if B_1 is not constant we could use Lemma 2.4 to get the result. In fact B_1 is not constant, for suppose B_1 is constant for λ_1 and λ_2, $\lambda_1 \neq \lambda_2$, then $(z-B_1)$ divides $(p-\lambda_1 q)$ and $(z-B_1)$ divides $(p-\lambda_2 q)$ which implies $(z-B_1)$ divides $(\lambda_2-\lambda_1)q$ which implies $(z-B_1)$ divides q which implies $(z-B_1)$ divides $(p-\lambda_1 q + \lambda_1 q)$ which implies $(z-B_1)$ divides p(a contradiction to the fact that p and q have no common divisors).

We now show that $|B_1|$ may be taken greater then 1.

Take $p(z) = z^{d_p} + a_1 z^{d_p - 1} + \dots + a_{d_p}$ and

$q(z) = z^{d_q} + b_1 z^{d_q - 1} + \dots + b_{d_q}$.

Case I) $d_q = d_p = d$.

Since $p \neq q$, there exists an i such that $a_i \neq b_i$.

Make $p - \lambda q$ monic by dividing by $1 - \lambda$. (Recall we are taking $\lambda \neq 1$.)

The coefficient of z^{d-i} is then $\frac{a_i - \lambda b_i}{1 - \lambda}$. By proper choice of λ, $\left| \frac{a_i - \lambda b_i}{1 - \lambda} \right|$ can be made arbitrarily large. But this coefficient is an elementary symmetric polynomial in the roots of $\frac{p - \lambda q}{1 - \lambda}$, i.e., it is the sum of products of the roots taken i at a time. So one such product can be made to have modulus greater than one. Hence one zero can be taken to have modulus greater than one. Since A_1 is arbitrary we lose nothing by dividing the top and bottom of $\frac{A_1}{p - \lambda q}$ by a constant $\frac{1}{1 - \lambda_0}$ and hence we have what we want.

Case II) $d_q > d_p$.

Choose $a_i \neq 0$. Then $\left| \frac{a_i - \lambda b_i}{-\lambda} \right|$ may be made arbitrarily large and the argument of Case I) applies.///

CHAPTER III: DECOMPOSABILITY AND HYPERINVARIANT SUBSPACES

In this chapter, we show the main results of this thesis. For φ in $L^\infty(\mathbb{T})$, φ analytic in some neighborhood of $\mathbb{T}$, we show that if neither T_φ nor $T_{\bar{\varphi}}$ has an eigenspace then T_φ is strongly decomposable. As a result of this, for φ analytic in some neighborhood of $\mathbb{T}$, T_φ has non-trivial hyperinvariant subspaces. Recall: The subspace $\mathcal{M}$ is invariant for the operator T iff $T|_{\mathcal{M}} \subseteq \mathcal{M}$. The subspace $\mathcal{M}$ is a hyperinvariant subspace for T if $\mathcal{M}$ is an invariant subspace for each operator which commutes with T. For more information on invariant and hyperinvariant subspaces, see (17).

The fact that functions can be factored into analytic and coanalytic parts is known for a broader class of functions than we are concerned with here, e.g., see (12). Also, the relation between such factorizations and invertibility explored in Propositions 3.1 and 3.2 are known, see, e.g., (19) and (20), but as in the previous chapter, the following proposition is written and proved entirely in terms of rational functions. Also, the proof yields insight into the desired results for φ analytic in a neighborhood of $\mathbb{T}$.

Proposition 3.1: Let φ in $L^\infty(\mathbb{T})$ be a rational function. T_φ is invertible iff there exists a factorization $\varphi = \varphi^- \varphi^+$ with φ^+, $\frac{1}{\varphi^+}$ analytic while φ^-, $\frac{1}{\varphi^-}$ are coanalytic.

Proof: Suppose $\varphi^-\varphi^+ = \varphi$ is such a factorization. Using

Proposition 1.2, it is an easy calculation to see

$T_{\varphi}^{-1} = T_{\frac{1}{\varphi^+}} T_{\frac{1}{\varphi^-}}$:

$$(T_{\frac{1}{\varphi^+}} T_{\frac{1}{\varphi^-}})(T_{\varphi}) = (T_{\frac{1}{\varphi^+}} T_{\frac{1}{\varphi^-}})(T_{\varphi^- \varphi^+}) = (T_{\frac{1}{\varphi^+}} T_{\frac{1}{\varphi^-}})(T_{\varphi^-} T_{\varphi^+})$$

$$T_{\frac{1}{\varphi^+}}(T_{\frac{1}{\varphi^-}} T_{\varphi^-})T_{\varphi^+} = T_{\frac{1}{\varphi^+}} T_{\varphi^+} = T_1 = I.$$

$$T_{\varphi}(T_{\frac{1}{\varphi^+}} T_{\frac{1}{\varphi^-}}) = T_{\varphi^- \varphi^+}(T_{\frac{1}{\varphi^+}} T_{\frac{1}{\varphi^-}}) = (T_{\varphi^-} T_{\varphi^+})(T_{\frac{1}{\varphi^+}} T_{\frac{1}{\varphi^-}})$$

$$= T_{\varphi^-}(T_{\varphi^+} T_{\frac{1}{\varphi^+}})T_{\frac{1}{\varphi^-}} = T_{\varphi^-} T_{\frac{1}{\varphi^-}} = T_1 = I.$$

Conversely, let T_{φ} be invertible. Put $\varphi = \frac{p}{q}$, where p and q are polynomials with no common factors. Then by Proposition 1.3, we have that φ has no zeros or poles on $\mathbb{T}$ and from observation 6), page 10, we have that the number of zeros of φ in $\mathbb{D}$ is equal to the number of poles of φ in $\mathbb{D}$. So φ may be written:

$$\varphi = \frac{p}{q} = \frac{a(z-a_1)\ldots(z-a_n)(z-b_1)\ldots(z-b_m)}{(z-c_1)\ldots(z-c_n)(z-d_1)\ldots(z-d_k)}$$

where a is a constant; $|a_i| < 1$, $|c_i| < 1$ for $i = 1, \ldots, n$; $|b_i| > 1$, $i = 1, \ldots, m$; $|d_i| > 1$, $i = 1, \ldots, k$.

Put $\varphi^+(z) = \dfrac{a(z-b_1)\ldots(z-b_m)}{(z-d_1)\ldots(z-d_k)}$ and

$$\varphi^-(z) = \frac{(z-a_1)\ldots(z-a_n)}{(z-c_1)\ldots(z-c_n)} .$$

Clearly φ^+ and $\frac{1}{\varphi^+}$ are analytic. That φ^- and $\frac{1}{\varphi^-}$ are coanalytic follows from Lemma 2.5.///

For notational convenience, we put $\varphi(z)-\lambda = \varphi_\lambda(z)$, λ in $\mathbb{C}$.

For φ, $\varphi_\lambda^+(z)$, and $\varphi_\lambda^-(z)$ defined according to the constructions in the above proof, we have the following observations. By the Ostrowski result, (15), we know the zeros of a polynomial are continuous functions of the coefficients and so the zeros of our rational $\varphi_\lambda(z)$ are continuous in λ. Since the zeros of $\varphi_\lambda^+(z)$ and $\varphi_\lambda^-(z)$ are the zeros of $\varphi_\lambda(z)$, they are continuous in λ. In particular, if L is a line segment which meets $\varphi(\mathbb{T})$ only in $\{\lambda_0\}$, and $\varphi_\lambda(z)$ is invertible for every λ on L, $\lambda \neq \lambda_0$, then both φ_λ^+ and φ_λ^- have one sided limits at λ_0 along L. We can speak only of one sided limits even if L passes through λ_0, for the internal make-up of φ_λ^+ and φ_λ^- may be entirely different on the two sides of λ_0. But we can see that both $|\varphi_\lambda^+(z)|$ and $|\varphi_\lambda^-(z)|$ are bounded above as $\lambda \to \lambda_0$, along L, for every z in $\mathbb{T}$, and these bounds may be taken independent of λ on L-$\{\lambda_0\}$.

The following proposition is again well known but we include a proof both for completeness and our later convenience.

Proposition 3.2: Let φ in $L^\infty(\mathbb{T})$ have an absolutely convergent Fourier series. T_φ is invertible iff there exists a factorization $\varphi = \varphi^-\varphi^+$ with φ^+, $\frac{1}{\varphi^+}$ analytic while φ^-, $\frac{1}{\varphi^-}$ are coanalytic.

Proof: If $\varphi^-\varphi^+ = \varphi$ is such a factorization then as in the proof of Proposition 3.1, $T_\varphi^{-1} = T_{\frac{1}{\varphi^+}} T_{\frac{1}{\varphi^-}}$.

So suppose T_φ is invertible. Then (6, Chap. 7) φ does not vanish and the winding number of φ is zero. Hence $\varphi(\mathbb{T})$ can be continuously deformed to a point without crossing the origin.

Let W be the commutative Banach Algebra of $\mathbb{C}$-valued functions on $\mathbb{T}$ which have absolutely convergent Fourier series. Then (6, 2.15) $\exp(W) = G_o$, where G is the group of invertible elements in W and G_o is the connected component in G which contains the identity. The above paragraph says that φ is in G_o; and so can be expressed as $\varphi = e^\psi$ where ψ has an absolutely convergent Fourier series, i.e., $\psi = \sum_{n=-\infty}^{\infty} a_n z^n$, where $z = e^{it}$, and $\sum_{n=-\infty}^{\infty} |a_n| < \infty$.

We take $\psi_+ = \sum_{n=o}^{\infty} a_n z^n$ and $\psi_- = \sum_{n=-\infty}^{-1} a_n z^n$ and define

$\varphi^+ = e^{\psi_+}$ and $\varphi^- = e^{\psi_-}$.

Since e^z is entire, it is clear that φ^+ and $\frac{1}{\varphi^+}$ are analytic while φ^- and $\frac{1}{\varphi^-}$ are coanalytic.///

For φ with absolutely convergent Fourier series, $\varphi_\lambda(z) = \varphi(z) - \lambda = e^{\psi(z,\lambda)}$ is continuous in λ at each λ_0. We would like both φ_λ^+ and φ_λ^- also to be continuous in λ at each λ_0. This is the case and is not too difficult to see since we are assured that φ_λ is non-zero for all λ involved: Since $\varphi_\lambda = e^{\psi(\lambda)}$ says that $\psi(\lambda) = \log(\varphi_\lambda)$ and log can be taken to be a well-defined continuous function in a neighborhood of φ_{λ_0}, we have that $\psi(\lambda)$ is continuous in λ at λ_0. We then note that $\psi_+(\lambda)$ and $\psi_-(\lambda)$ are projections of $\psi(\lambda)$ into the analytic and coanalytic subalgebras of the Wiener algebra and so are also continuous in λ. Putting these together, we have it. (This could also have been argued using (12, Lemma 5.1.)

We note here that in these factorizations φ_λ^- is one at infinity. From this we get uniqueness of the factorization: if $\varphi = \varphi_1^+ \varphi_1^- = \varphi_2^+ \varphi_2^-$ then $\frac{\varphi_1^+}{\varphi_2^+} = \frac{\varphi_2^-}{\varphi_1^-}$ which must be constantly one since $\frac{\varphi_1^+}{\varphi_2^+}$ is an entire function which is one at infinity. That is $\varphi_1^+ = \varphi_2^+$ and $\varphi_1^- = \varphi_2^-$.

To get our main results, we use certain growth conditions (16), (17), on the resolvents near the spectra of our operators. In order to get these conditions we need to know certain geometric properties of the spectrum. One involves the next definition.

<u>Definition 3.3</u>: A smooth Jordan arc is a one-to-one function $z(t) = x(t) + iy(t)$ from $(0,1)$ into $\mathbb{C}$ such that $\frac{d^2z}{dt^2}$

exists everywhere in (0,1). If T is a bounded linear operator, then $\sigma(T)$ contains an exposed arc J if there exists an open disc M with $M \cap \sigma(T) = J$ is a smooth Jordan arc.

For φ analytic in some neighborhood of $\mathbb{T}$, for which neither T_φ nor $T_{\overline{\varphi}}$ have eigenspaces, we need not only that the graph of φ, i.e., $\varphi(\mathbb{T})$, contains such an exposed arc but also that it have only a finite number of isolated points of self-intersection (i.e., points where the graph actually crosses itself or crossing points). Non-trivial arcs of the graph which are traced several times are not included in this set, of course.

We show that the graph, $\varphi(\mathbb{T})$, has only a finite number of loops, twists and turns by looking at the points where the graph changes concavity. Changes in concavity occur only when $\frac{v(t)}{u(t)}$ attains a local maximum or minimum (infinity and minus infinity are included in this set of maxima and minima, i.e., the zero-points of u(t)), where $\varphi'(e^{it}) = \frac{d\varphi}{dt} = u(t) + iv(t)$. Since the functions u and v are real analytic, neither can be zero more than a finite number of times unless the graph, $\varphi(\mathbb{T})$, consists of a horizontal or vertical line segment or a point. Avoiding these trivial φ, we may assume that the number of zeros of u(t) is finite. Similarly, for non-trivial φ, the zero sets of $v'(t)$ and $u'(t)$ are also finite. Hence, by elementary calculus, we see that the local maxima and minima of $\frac{v(t)}{u(t)}$ occur only when $uv' = u'v$, i.e., by avoiding the finite sets of zeros of these functions, $\frac{v(t)}{u(t)} = \frac{v'(t)}{u'(t)}$. This last equality means that the functions $\frac{\varphi'}{\varphi''}$ and $\frac{\varphi''}{\varphi'}$ are real at the points corresponding to the local maxima and minima of $\frac{v}{u}$ ($\varphi''(t) = u' + iv' = u'(1 + i\frac{v'}{u'}) = u'(1 + i\,\frac{v}{u}) = \frac{u'}{u}(u(1 + i\frac{v}{u}))$ $= \frac{u'(t)}{u(t)}\,\varphi'(t)$). Since one of the functions $\frac{\varphi'}{\varphi''}$ or $\frac{\varphi''}{\varphi'}$ can be defined as an analytic function in a closed neighborhood of each point, neither function can be real for an infinity of points without being real for all t. Further, if $\frac{\varphi'}{\varphi''}$ or $\frac{\varphi''}{\varphi'}$ is real for a segment, then the graph of is a straight line on this segment. From all this, we conclude that there can be only a finite number of changes in the concavity of the graph, $\varphi(\mathbb{T})$.

For φ analytic in a neighborhood of $\mathbb{T}$, the graph, $\varphi(\mathbb{T})$, is well behaved: it has only a finite number of crossing points and, of course, has exposed arcs.

We now use the factors obtained in the proof of Proposition 3.2 to get growth conditions on the resolvent of T_φ in the case where neither T_φ nor $T_{\bar\varphi}$ has eigenspaces, i.e., where $\sigma(T_\varphi)$ consists precisely of the set $\varphi(\mathbb{T})$.

Lemma 3.4: Let T_φ be such that neither T_φ nor $T_{\bar{\varphi}}$ has an eigenspace, φ in $L^\infty(\mathbb{T})$, analytic in a neighborhood of $\mathbb{T}$. For each point λ_0 of $\sigma(T_\varphi)$ which is not a cusp point (a point where the derivative of φ vanishes) or an isolated point of self-intersection of the graph (i.e., a point that has a neighborhood which intersects $\sigma(T_\varphi)$ in an exposed arc C) and each closed line segment L which meets $\sigma(T_\varphi)$ only in $\{\lambda_0\}$ and which is not tangent to $\sigma(T_\varphi)$ at λ_0, there exists a constant K such that

$$\|T^{-1}_{\varphi-\lambda}\| < K|\lambda-\lambda_0|^{-2}$$

for all λ on L other than λ_0.

Proof: Take λ in L with $\lambda \neq \lambda_0$, so that T_{φ_λ} is invertible (here as before $\varphi_\lambda(z) = \varphi(z) - \lambda$). Since φ is analytic in a neighborhood of $\mathbb{T}$, φ has an absolutely convergent Fourier series and so from Proposition 3.2, φ_λ can be factored into analytic and coanalytic parts:

$$\varphi_\lambda = \varphi_\lambda^+ \varphi_\lambda^- \quad \text{with } T^{-1}_{\varphi_\lambda} = T_{\frac{1}{\varphi_\lambda^+}} T_{\frac{1}{\varphi_\lambda^-}}.$$

Now by Proposition 1.5, we have

$$||T^{-1}_{\varphi_\lambda}|| = ||T_{\frac{1}{\varphi^+_\lambda}} T_{\frac{1}{\varphi^-_\lambda}}|| \le ||T_{\frac{1}{\varphi^+_\lambda}}|| \, ||T_{\frac{1}{\varphi^-_\lambda}}|| = \left\|\frac{1}{\varphi^+_\lambda}\right\|_\infty \left\|\frac{1}{\varphi^-_\lambda}\right\|_\infty .$$

The next step is to show that both $\left\|\frac{1}{\varphi^+_\lambda}\right\|_\infty$ and $\left\|\frac{1}{\varphi^-_\lambda}\right\|_\infty$ are bounded on $L-\{\lambda_0\}$ by some constant multiple of $\left\|\frac{1}{\varphi_\lambda}\right\|_\infty$, where the constant is independent of λ. We begin by noting for each fixed λ, each of the functions $\left|\frac{1}{\varphi_\lambda(z)}\right|$, $\left|\frac{1}{\varphi^+_\lambda(z)}\right|$, and $\left|\frac{1}{\varphi^-_\lambda(z)}\right|$ is continuous in z on the compact set $\mathbb{T}$. So each takes on its supremum at some point of $\mathbb{T}$, i.e., there are points z_1, z_2, z_3 in $\mathbb{T}$ with

$$\left|\frac{1}{\varphi_\lambda(z_1)}\right| = \left\|\frac{1}{\varphi_\lambda}\right\|_\infty, \quad \left|\frac{1}{\varphi^+_\lambda(z_2)}\right| = \left\|\frac{1}{\varphi^+_\lambda}\right\| \quad \text{and}$$

$$\left|\frac{1}{\varphi^-_\lambda(z_3)}\right| = \left\|\frac{1}{\varphi^-_\lambda}\right\|_\infty .$$

We assume, for the time being, that there are positive constants $\frac{1}{m^+}$ and $\frac{1}{m^-}$ such that $|\varphi^+_\lambda(z)| < \frac{1}{m^+}$ and $|\varphi^-_\lambda(z)| < \frac{1}{m^-}$ for all λ in $L-\{\lambda_0\}$ and z in $\mathbb{T}$. In other words,

$$\left|\frac{1}{\varphi_\lambda^+(z)}\right| \geq m^+ > 0 \text{ and } \left|\frac{1}{\varphi_\lambda^-(z)}\right| \geq m^- > 0 \text{ for all } \lambda \text{ in } L-\{\lambda_0\},$$

z in $\mathbb{T}$.

Given these constants, we continue

$$m^- \left\|\frac{1}{\varphi_\lambda^+}\right\|_\infty = m^- \left|\frac{1}{\varphi_\lambda^+(z_2)}\right| \leq \left|\frac{1}{\varphi_\lambda^-(z_2)}\right| \left|\frac{1}{\varphi_\lambda^+(z_2)}\right| = \left|\frac{1}{\varphi_\lambda(z_2)}\right| \leq$$

$\left\|\frac{1}{\varphi_\lambda}\right\|_\infty$, i.e., $m^- \left\|\frac{1}{\varphi_\lambda^+}\right\|_\infty \leq \left\|\frac{1}{\varphi_\lambda}\right\|_\infty$ for $\lambda \neq \lambda_0$ in L.

In a similar manner we get

$$m^+ \left\|\frac{1}{\varphi_\lambda^-}\right\|_\infty \leq \left\|\frac{1}{\varphi_\lambda}\right\|_\infty \quad \lambda \neq \lambda_0 \text{ on } L.$$

Putting these last two inequalities together

$$\left\|\frac{1}{\varphi_\lambda^+}\right\|_\infty \left\|\frac{1}{\varphi_\lambda^-}\right\|_\infty \leq \frac{1}{m^- m^+}\left\|\frac{1}{\varphi_\lambda}\right\|_\infty^2 \text{ for all } \lambda \neq \lambda_0 \text{ in } L.$$

From here the proof follows easily:

Let the line segment L make the non-zero angle θ with the tangent at λ_0.

If $\theta = \frac{1}{2}\pi$, put $\delta(\lambda) = \left\|\frac{1}{\varphi_\lambda(z)}\right\|_\infty - \left|\frac{1}{\lambda-\lambda_0}\right|$. $\delta(\lambda)$ is continuous in λ on $L-\{\lambda_0\}$ and from elementary calculus $\delta(\lambda) \to 0$ as $\lambda \to \lambda_0$ along L. So δ is bounded on $L-\{\lambda_0\}$ while

$\left|\frac{1}{\lambda-\lambda_0}\right|$ has a minimum at one end of L. We can surely find some constant K_1 with $K_1\left|\frac{1}{\lambda-\lambda_0}\right| \geq \left|\frac{1}{\lambda-\lambda_0}\right| + \partial(\lambda) = \left\|\frac{1}{\phi_\lambda(z)}\right\|_\infty$ for all λ on $L-\{\lambda_0\}$. Hence

$$\frac{1}{m^- m^+}\left\|\frac{1}{\phi_\lambda}\right\|_\infty^2 \leq \frac{K_1^2}{m^- m^+}\left|\frac{1}{\lambda-\lambda_0}\right|^2 = K|\lambda-\lambda_0|^{-2} \text{ with } K = \frac{K_1^2}{m^- m^+}.$$

If $\theta \neq \frac{1}{2}\pi$, we may take θ acute and

$$\frac{\dfrac{1}{\|\phi_\lambda\|^2}}{\dfrac{1}{|\lambda-\lambda_0|^2}} = \left|\csc^2\theta + \delta_1(\lambda)\right| \qquad \text{or}$$

$$\left\|\frac{1}{\phi_\lambda}\right\|_\infty^2 = \left|\csc^2\theta + \delta_1(\lambda)\right| \frac{1}{|\lambda-\lambda_0|^2} \quad \text{where } \delta_1(\lambda)\to 0 \text{ as } \lambda\to\lambda_0$$

along L. We may take $\delta_1(\lambda)$ continuous on $L-\{\lambda_0\}$, so we have it bounded on L, and hance we can find a constant K_1 with

$$K_1 \geq \left|\csc^2\theta + \delta_1(\lambda)\right| \qquad \lambda \text{ in L}.$$

So

$$\frac{1}{m^+ m^-} \left\| \frac{1}{\phi_\lambda} \right\|_\infty^2 \le \frac{K_1}{m^+ m^-} |\lambda - \lambda_0|^{-2} = K|\lambda - \lambda_0|^{-2}$$

where $K = \dfrac{K_1}{m^+ m^-}$.

We have only to show that the upper bounds $\frac{1}{m^+}$ and $\frac{1}{m^-}$ of $|\phi_\lambda^+(z)|$ and $|\phi_\lambda^-(z)|$, respectively, exist.

Note that if φ were rational, the discussion following Proposition 3.1 would insure the existence of such bounds. We use the bounds for rational φ to guarantee such bounds for φ analytic in a neighborhood N of $\mathbb{T}$.

Without loss of generality, we may take $\varphi^{-1}(\lambda_0) \cap N = \varphi^{-1}(\lambda_0) \cap \mathbb{T}$. We note, this set must necessarily be finite since the λ_0-points of a non-constant analytic function may not cluster and any infinite subset of a compact set always clusters. Hence, we may put $\varphi^{-1}(\lambda_0) \cap \mathbb{T} = \{a_1, a_2, \ldots, a_m\}$.

From well known properties of analytic functions, e.g., (1), φ can be expressed in the form $\varphi(z) = (z-a_1)^{n_1} \ldots (z-a_m)^{n_m} \phi_1(z) + \lambda_0$ for some $n_1, n_2, \ldots, n_m$ where $\phi_1(z)$ is analytic and non-zero in N.

Choose R_1, a positive constant, small enough to assure $|(z-a_1)^{n_1}\ldots(z-a_m)^{n_m}|R_1 < \frac{|\lambda_o|}{2}$ for each z in T. This can be done since $(z-a_1)^{n_1}\ldots(z-a_m)^{n_m}$ is continuous and its image of T is aubsequently compact (in particular, bounded).

Also choose n, an integer, such that $\frac{z^n}{\varphi_1(z)}$ has zero winding number about the origin. That such an integer exists follows from the fact that the winding number of a product is the sum of the winding numbers of the factors, i.e., we put n equal to the negative of the winding number of $\frac{1}{\varphi_1}$.

Put $R(z) = (z-a_1)^{n_1}\ldots(z-a_m)^{n_m}R_1 z^n + \lambda_o$.

Then $\frac{R(z)-\lambda_o}{\varphi(z)-\lambda_o} = \frac{(z-a_1)^{n_1}\ldots(z-a_m)^{n_m}R_1 z^n}{(z-a_1)^{n_1}\ldots(z-a_m)^{n_m}\varphi_1(z)}$ and so has removable singularities at the points $a_1,\ldots, a_m$. Hence, we may take $\frac{R_{\lambda_o}(z)}{\varphi_{\lambda_o}(z)} = \frac{R_1 z^n}{\varphi_1(z)}$. Hence, $\frac{R_{\lambda_o}}{\varphi_{\lambda_o}}$ is non-zero with zero winding number about the origin, so it can be factored into analytic and coanalytic parts.

The quotient $\frac{R_\lambda(z)}{\varphi_\lambda(z)}$ is continuous in λ at λ_o, $z \neq a_1$, a_2, ..., a_m. From the uniqueness of the factorization, we have $\left(\frac{R_\lambda}{\Phi_\lambda}\right)^+ = \frac{R_\lambda^+}{\Phi_\lambda^+}$ and $\left(\frac{R_\lambda}{\Phi_\lambda}\right)^- = \frac{R_\lambda^-}{\Phi_\lambda^-}$. From the discussion on the continuity of factors, we have $\frac{R_\lambda^+}{\Phi_\lambda^+}$ and $\frac{R_\lambda^-}{\Phi_\lambda^-}$ are continuous in λ at λ_o, i.e.,

$$\frac{R_\lambda^+(z)}{\Phi_\lambda^+(z)} \to \frac{R_{\lambda_o}^+(z)}{\Phi_{\lambda_o}^+(z)} \quad \text{(non-zero, well defined for } z \neq a_1, \ldots, a_m\text{)}$$

and $$\frac{R_\lambda^-(z)}{\Phi_\lambda^-(z)} \to \frac{R_{\lambda_o}^-(z)}{\Phi_{\lambda_o}^-(z)} \quad \text{(non-zero, well defined for } z \neq a_1,$$

$a_2, \ldots, a_m$). Note, these factors are continuous as quotients even though the individual functions $R_\lambda^+(z)$, $R_\lambda^-(z)$, $\Phi_\lambda^+(z)$, and $\Phi_\lambda^-(z)$ may not even be defined at λ_o. Hence, $\left|\frac{R_\lambda^+}{\Phi_\lambda^+}\right|$ and $\left|\frac{R_\lambda^-}{\Phi_\lambda^-}\right|$ are both uniformly bounded away from zero and infinity for λ in L, i.e., there are, in particular, constants k_+ and k_- with $\frac{|R_\lambda^+|}{|\Phi_\lambda^+|} \geq k_+ > 0$ and $\frac{|R_\lambda^-|}{|\Phi_\lambda^-|} \geq k_- > 0$ for λ in L. Hence, $\frac{1}{k_+}|R_\lambda^+| \geq |\Phi_\lambda^+|$ and $\frac{1}{k_-}|R_\lambda^-| \geq |\Phi_\lambda^-|$ for λ in L. Since $|R_\lambda^+|$ and $|R_\lambda^-|$ are uniformly bounded on L, it is now clear that $|\Phi_\lambda^+|$ and $|\Phi_\lambda^-|$ are also uniformly bounded on L.///

We now follow a series of arguments close to those in (16). We begin by defining, for a closed subset F of the plane and an operator S in some Banach space Y, $X_S(F) = \{x \text{ in } Y : (z-S)^{-1}x$ has an analytic continuation to the complement of F in the complex plane$\}$.

The operator S is said to have the single valued extension property if $x(z)$ is an analytic function from an open subset of the plane into Y with $(z-S)x(z) \equiv 0$, then $x(z) \equiv 0$. It is shown in (5) that if S has the single valued extension property and $X_S(F)$ is closed, then $X_S(F)$ is a maximal spectral subspace of S, i.e., $X_S(F)$ is an invariant subspace of S and if M is another invariant subspace of S with the property that $\sigma(S|M) \subseteq \sigma(S|X_S(F))$ then $M \subseteq X_S(F)$. Moreover, $X_S(F)$ is a hyperinvariant subspace of S and $\sigma(S|X_S(F)) \subseteq \sigma(S) \wedge F$.

In the following, J is a piecewise smooth closed rectifiable curve with at most a finite number of isolated self-intersections, T is a bounded linear operator on the Banach space $\mathfrak{X}$, and $\sigma(T) \subseteq J$.

<u>Lemma 3.5</u>: Let $X_T(F)$ be closed for any closed connected subset F of J. Let F_1 and F_2 be two disjoint closed subsets of the plane. Then $X_T(F_1)$, $X_T(F_2)$ are closed and $X_T(F_1 \cup F_2) = X_T(F_1) \oplus X_T(F_2)$.

Proof: Since every closed subset of J is the intersection of a countable set of closed connected subsets of J, it follows that $X_T(F) = X_T(F \cap J)$ is closed for all closed subsets F of the plane. Therefore $X_T(F_1 \cup F_2)$ is closed and thus by (2, Prop. 1.2.3) we have

$$X_T(F_1 \cup F_2) = X_T(F_1) \oplus X_T(F_2).///$$

Lemma 3.6: Let T be a bounded linear operator defined on some Banach space $\mathcal{X}$. Let F be a closed subset of $\mathbb{C}$. Assume T has the single valued extension property and $X_T(F)$ is closed. Then $\sigma(T) = \sigma(T|X_T(F)) \cup \sigma(T^F)$ where T^F denotes the operator induced on the quotient $\mathcal{X}/X_T(F)$ by T. Moreover, $\sigma(T^F)$ cannot be the disjoint union of two non-empty sets E_1 and E_2 with $E_1 \subseteq F$.

Proof: This is (16, Lemma 2).///

Proposition 3.7: Assume that for any closed connected subset F of J

1.) $X_T(F)$ is closed and

2.) $\sigma(T^F) \subseteq J-F$ where T^F denotes the operator induced on $X/X_T(F)$ by T.

Let F_1 and F_2 be two closed connected subsets of J with the property that $F_1 \cap F_2$ contains no isolated point. Then $X_T(F_1 \cup F_2) = X_T(F_1) + X_T(F_2)$.

Proof: This is (16, Prop. 1) with closed subarcs replaced by closed connected subsets and J a smooth Jordan curve replaced with our definition of J. The proof of (16, Prop. 1) still holds.///

Now by induction we have:

Corollary 3.8: Let T be an in Proposition 3.7. Let F_j, $j = 1, 2, \ldots, n$, be n closed connected subsets of J with the property that $F_i \cap F_j$ contains no isolated point for all i, j. Then $X_T(\cup F_j) = \sum X_T(F_j)$.

Definition 3.9: An operator T is called decomposable if, for every finite open covering G_i($i = 1, 2, \ldots, n$) of $\sigma(T)$, there exists a set of maximal spectral subspaces Y_i of T such that

(1) $\sigma(T|Y_i) \subseteq \overline{G}_i$, $i = 1, 2, \ldots, n$, and

(2) $X = Y_1 + Y_2 + \ldots + Y_n$.

Moreover, T is called strongly decomposable if its restriction to an arbitrary maximal spectral subspace is again decomposable.

Theorem 3.10: Let T be as in Proposition 3.7. Then T is decomposable.

Proof: The proof is the same as the proof of (16, Thm. 1) with the obvious changes that take place when arc is replaced with connected subset.///

The proof of the following lemma is a reworking of the proof of (17, Thm. 6.3).

<u>Lemma 3.11</u>: Let T_φ be the Toeplitz operator with symbol φ in $L^\infty(\mathbb{T})$, φ analytic in a neighborhood of $\mathbb{T}$ where neither T_φ nor $T_{\bar{\varphi}}$ has an eigenspace, i.e., $\sigma(T_\varphi) = \varphi(\mathbb{T})$. For any closed connected subset F of J, we have

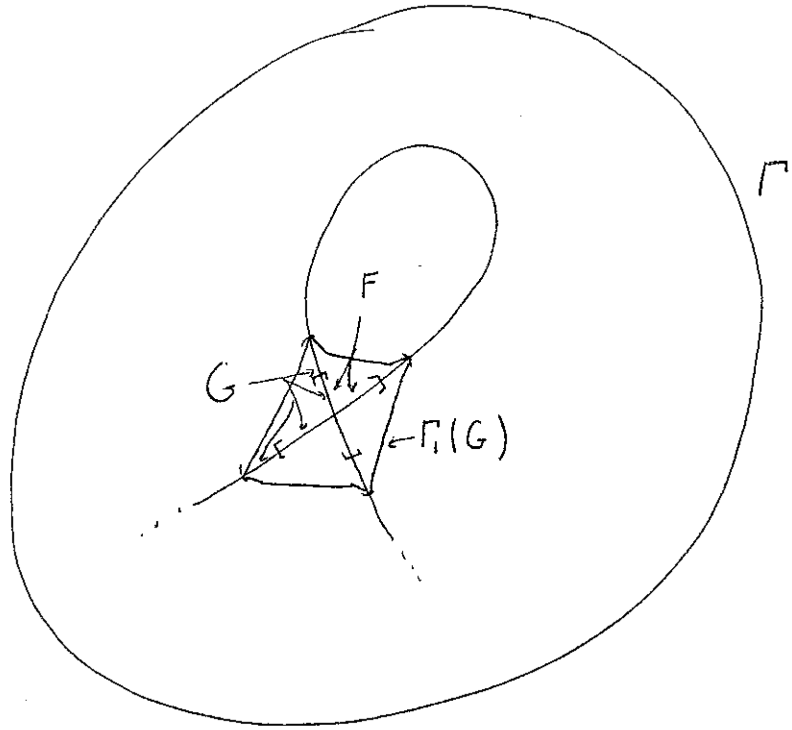
1.) $X_{T_\varphi}(F)$ is closed, and

2.) $X_{T_\varphi}(F) \neq \{0\}$, where J is such that $J \supseteq \sigma(T_\varphi)$.

(F contains more than one point.)

Proof: We first show that $X_{T_\varphi}(F)$ is closed.

Let $\{x_n\}$ in $X_{T_\varphi}(F)$ be such that $x_n \to x$. Let, for each n, $R_n(\lambda)$ denote the analytic continuation of $(T^{-1}_{\varphi-\lambda})x_n$ to the complement of F. To prove that $R_x(\lambda) = (T^{-1}_{\varphi-\lambda})x$ has an analytic continuation to the complement of F, it suffices to show that R_x has an analytic continuation to the complement of $\bar{G}$ for each open connected subset G of J such that $G \supseteq F$. We may take without loss of generality that none of the endpoints of G are cusps or isolated points of self-intersection (there are only a finite number of such points).



The number of lines intersecting at any one point is finite and the number of isolated self-intersections is finite, so G will have a finite number of endpoints. Suppose G has endpoints $z_1, z_2, \ldots, z_n$, where $Re(z_1) \leq Re(z_2) \leq \ldots \leq Re(z_n)$, we construct a simple closed Jordan polygon $\Gamma_1(G)$ intersecting J only at $z_1, z_2, \ldots, z_n$ by beginning with the angles at z_j determined by the rays through z_j making angles of $\pm \frac{\pi}{10}$ with the tangent at z_j in the direction of F, $j = 1, \ldots, n$. Then connect these angles by line segments not intersecting G, getting a polygon as was done in the figure.

Let $\Gamma_2(G)$ be the union of $\Gamma_1(G)$ and any fixed circle Γ

containing J and $\Gamma_1(G)$ in its interior. Let K be the open annulus-like region whose boundary is $\Gamma_2(G)$.

Define the function m by

$$m(z) = \begin{cases} \exp\{-e^{4b_1 i}(z-z_1)^{-4} - e^{4b_2 i}(z-z_2)^{-4} - \dots - e^{4b_n i}(z-z_n)^{-4}\} \\ 0 \end{cases}$$

for $z \neq z_1, z_2, \dots, z_n$ and for $z = z_1, z_2, \dots, z_n$, respectively, where b_i is the angle the tangent at z_i, in the direction of F, makes with the positinve real axes.

It follows from the choice of angles of $\Gamma_1(G)$ at z_1, z_2, ..., z_n that m(z) is continuous on $\overline{K}$. Thus $m(z)R_n(z)$ is analytic on K and continuous on $\overline{K}$. We show that the sequence $\{m(z)R_n(z)\}$ converges to an analytic function on K. For z in K,

$$\|m(z)R_n(z)-m(z)R_m(z)\| \leq \sup_{w \in \Gamma_2(G)} \|m(w)(R_n(w)-R_m(w))\|$$

by the maximum modulus principle. Let L denote one of the line segments of $\Gamma_2(G)$ with z_1 as an endpoint. Then, by the growth conditions on the resolvent of T_φ, for w in L, $w \neq z_1$, we have

$$\|m(w)(R_n(w)-R_m(w))\| = \| m(w)T^{-1}_{\varphi-\lambda}(x_n-x_m)\|$$

$$\leq \|x_n-x_m\| \,\big|\exp\{-e^{-4b_1 i}(w-z_1)^{-4} - \dots - e^{-4b_n i}(w-z_n)^{-4} + k|w-z_1|^{-2}\}\big|$$

$\leq \|x_n - x_m\| \, N \exp\{\mathrm{Re}(-e^{-4b_1 i}(w-z_1)^{-4} + k|w-z_1|^{-2})\}$,

where $N = \sup_{w \text{ in } L} |\exp\{-e^{-4b_2 i}(w-z_2)^{-4} - \ldots - e^{-4b_n i}(w-z_n)^{-4}\}|$.

Since w and z_1 are on L, $e^{-4b_1 i}(w-z_1) = |w-z_1| e^{it}$, where t is either $\frac{\pi}{10}$ or $\frac{\pi}{-10}$. It follows that $\exp\{\mathrm{Re}(-e^{-4b_1 i}(w-z_1)^{-4} + k|w-z_1|^{-2})\}$ is bounded on L. Thus for w in L

$\|m(w)(R_n(w)-R_m(W))\| \leq M_1 \|x_n - x_m\|$

for some constant M_1.

In exactly the same manner it can be shown that there exist constants M_j such that $\|m(w)(R_n(w)-R_m(w))\| \leq M_j \|x_n - x_m\|$ for all w on the line segments of $\Gamma_2(G)$ through z_j, $j = 2, 3, \ldots, n$, respectively. Since a similar assertion is obviously true for the line segments of $\Gamma_2(G)$ which do not meet any of $z_1, z_2, \ldots, z_n$, it follows that the sequence $\{m(z)R_n(z)\}$ is a uniform Cauchy sequence on $\overline{K}$. Hence, this sequence converges uniformly to some function s(z) analytic on K and continuous on $\overline{K}$.

For each z in $\rho(T_\varphi)$, $(m(z))^{-1} m(z) R_n(z) = T_{\varphi-z}^{-1} x_n$ and thus $(m(z))^{-1} s(z) = T_{\varphi-z}^{-1} x$. Hence, the function $\frac{s(z)}{m(z)}$ is an analytic continuation of $T_{\varphi-z}^{-1} x$ to the complement of $\overline{G}$. For each G, then, $T_{\varphi-z}^{-1} x$ can be analytically continued to the complement of $\overline{G}$, and it follows that $T_{\varphi-z}^{-1} x$ has an analytic continuation to the complement of F. Thus x is in $X_{T_\varphi}(F)$.

It remains to show that $X_{T_\varphi}(F) \neq \{0\}$.

It is an easy result (5, 1-1.2) that $F_1 \subseteq F_2 \Rightarrow X_T(F_1) \subseteq X_T(F_2)$. We may take a closed connected subset F_1 of F which is a C^2-Jordan arc, then use the proof of (17, Thm. 6.3) to get $X_{T_\varphi}(F_1) \neq \{0\}$ and hence $X_{T_\varphi}(F) \neq \{0\}$. Or we could follow the last part of the proof of (16, Lemma 3).///

Now the main theorem.

<u>Theorem 3.12</u>: Let φ in $L^\infty(\mathbb{T})$ be analytic in a neighborhood of $\mathbb{T}$, where neither T_φ nor $T_{\bar\varphi}$ has an eigenspace. Then T_φ is strongly decomposable.

Proof: In view of Lemmas 3.5 and 3.10, $X_{T_\varphi}(F)$ is a closed invariant subspace of T_φ for all closed subsets F of $\mathbb{C}$. Therefore $\sigma(T|X_{T_\varphi}(F)) \subseteq J$ and thus $T|X_{T_\varphi}(F)$ also satisfies the growth conditions of Lemma 3.4. Hence, it suffices to show that such an operator is decomposable.

We observe that in the previous lemma the spectrum did not have to fill all of J, it had only to be contained in it. In light of this, Theorem 3.11 and Lemma 3.12, we need only show that $\sigma(T_\varphi{}^F) \subseteq \overline{J-F}$ for all closed connected subsets F of J, where $T_\varphi{}^F$ denotes the operator induced on $H^2/X_{T_\varphi}(F)$ by T_φ. To do this we follow the proof of (16, Thm. 2).///

Now we have another main result.

Corollary 3.13: For φ in $L^{\infty}(\mathbb{T})$, φ analytic in a neighborhood of $\mathbb{T}$, φ non-constant, T_{φ} has a non-trivial hyperinvariant subspace.

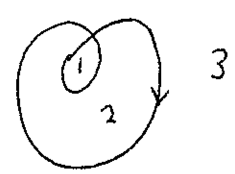
Proof: If either T_{φ} or $T_{\bar{\varphi}}$ has an eigenvalue, we are finished since an eigenspace for either of these operators is such a hyperinvariant subspace.

If neither T_{φ} nor $T_{\bar{\varphi}}$ has an eigenvalue, then T_{φ} is strongly decomposable. Hence, by (5, 1-3.2), T_{φ} has many hyperinvariant subspaces.///

CHAPTER IV: PROGRESS REPORT

To demonstrate the ideas and techniques involved we discuss examples. The original idea was to solve the problem of spanning of eigenfunctions for a specific example, then expand this solution into one which would be valid for all rational functions or at least the quadratic ones.

We start by looking at $\phi(z) = \frac{z-\alpha}{z^2}$. This function is coanalytic by Lemma 2.5, so its eigenfunctions span all H^2. In particular, $\mathrm{Ker}(T_{\phi-\lambda})$ is generated by functions in the form $\frac{Az+B}{-\lambda z^2 + z - \alpha}$, where A and B are arbitrary constants. The graph, $\phi(\mathbb{T})$, looks like



where the arrow indicates the natural ordering induced by that of $\mathbb{T}$. We look closely at the geometry, the position of the roots of $-\lambda z^2 + z - \alpha$, to see its relation to the $\mathrm{Ker}(T_{\phi-\lambda})$ and its generators. Area 1 consists of those λ for which both roots of $-\lambda z^2 + z - \alpha$ lie outside $\overline{\mathbb{D}}$. Hence, $\mathrm{Ker}(T_{\phi-\lambda})$ has two generators here. Area 2 consists of those λ for which one root of $-\lambda z^2 + z - \alpha$ lies in $\mathbb{D}$ and the other outside $\overline{\mathbb{D}}$. Here one arbitrary constant is used to compensate for the inside root and there is only one generator of $\mathrm{Ker}(T_{\varphi-\lambda})$. Area 3 consists of those λ for which both roots of $-\lambda z^2 + z - \alpha$ lie in $\mathbb{D}$. Here, of course, $\mathrm{Ker}(T_{\varphi-\lambda})$ is trivial. If λ is on the graph, then at least one root is on $\mathbb{T}$.

In general, if $\varphi = \dfrac{p(z)}{q(z)}$, the position of the roots of $p-\lambda q$ is related to λ and the number of generators of $\mathrm{Ker}(T_{\varphi-\lambda})$ in a similar way. We would like to somehow take advantage of this knowledge to find the spanning characteristics of eigenfunctions for those φ which are not analytic nor coanalytic. Techniques derived from these type of observations, helped Clark and Morrell (4) show the spanning of eigenfunctions for certain rational φ which have for graphs, simple closed curves.

We now look at $\varphi = \dfrac{z^2-\alpha}{z-\beta}$, and note that if we could solve the problem for this function, we would be able to solve it for all quadratic rational functions. We have $\varphi(z)-\lambda = \dfrac{z^2-\lambda z-\alpha+\lambda\beta}{z-\beta}$. $\mathrm{Ker}(T_{\varphi-\lambda})$ is generated by functions of the form

$$\frac{C_1}{z^2-\lambda z-\alpha+\lambda\beta} = \frac{C_1}{(z-A)(z-B)} \quad \text{with } |A|, |B| > 1,$$

C_1 an arbitrary constant.

It turns out that functions which are perpendicular to

$\mathrm{Ker}(T_{\varphi-\lambda})$ for every λ are of the form

$\frac{1}{\bar{A}} f(\frac{1}{\bar{A}}) = \frac{1}{\bar{B}} f(\frac{1}{\bar{B}})$ with $\frac{AB+\alpha}{A+B} = \beta$, or by renaming

$Af(A) = Bf(B)$ with $\frac{1+\bar{\alpha}AB}{A+B} = \bar{\beta}$ or $B = \frac{1-\bar{\beta}A}{\bar{\beta}-\bar{\alpha}A}$, i.e.,

(1.) $\quad zf(z) = (\frac{1-\bar{\beta}z}{\bar{\beta}-\bar{\alpha}z})f(\frac{1-\bar{\beta}z}{\bar{\beta}-\bar{\alpha}z})$ on the set of z with

both $|z|$ and $\left|\frac{1-\bar{\beta}z}{\bar{\beta}-\bar{\alpha}z}\right|$ less then one. This set consists of

the intersection of $\mathbb{D}$ with the interior of the circle centered at $\frac{\alpha\bar{\beta}-\beta}{|\alpha|^2-|\beta|^2}$ with radius $\frac{|\alpha-\beta^2|}{\left||\alpha|^2-|\beta|^2\right|}$ if $|\alpha|<|\beta|$

and it consists of the intersection of $\mathbb{D}$ with the exterior of the above circle if $|\alpha|>|\beta|$.

We have $\mathrm{Ker}(T_{\overline{\varphi-\lambda}})$ generated by $\frac{C_2(1-\bar{\beta}z)}{1-\bar{\lambda}z+(\overline{\lambda\beta-\alpha})z^2} =$

$\frac{C_2(1-\bar{\beta}z)}{(1-Cz)(1-Dz)}$ with $|C|, |D| < 1$ and C_2 an arbitrary constant.

Functions which are perpendicular to $\mathrm{Ker}(T_{\overline{\varphi-\lambda}})$ for every λ are of the form $(\bar{C}^2-\alpha)f(\bar{C}) = (\bar{D}^2-\alpha)f(\bar{D})$ or by renaming $(C^2-\alpha)f(C) = (D^2-\alpha)f(D)$ with $\frac{CD-\alpha}{C+D} = \beta$ or $D = \frac{BC-\alpha}{C-\beta}$, i.e.,

(2.) $\quad (z^2-\alpha)f(z) = ((\frac{\beta z-\alpha}{z-\beta})^2-\alpha)f(\frac{\beta z-\alpha}{z-\beta})$

on the set of z with both $|z|$ and $\left|\frac{\beta z-\alpha}{z-\beta}\right|$ less than one.

This set consists of the intersection of $\mathbb{D}$ with the exterior of the circle centered at $\frac{\bar{\beta} - \bar{\alpha}\beta}{1-|\beta|^2}$ with radius $\frac{|\alpha - \beta^2|}{1-|\beta|^2}$.

If the above circles intersect $\mathbb{T}$, they intersect it at the same points. They seem to intersect $\mathbb{T}$ iff the graph of ϕ is a figure eight. If $|\alpha| = 1$ and $\mathrm{Re}(\beta) = \mathrm{Re}(\alpha\bar{\beta})$, then both circles coincide with $\mathbb{T}$, indicating that no λ has a non-zero winding number. Further, it seems that the ϕ which have straight line segments as graphs are of the form $\phi(z) = z - \alpha\bar{z}$ where $|\alpha| = 1$.

The most promising fact is that all the functions in the orthogonal complement of the span of the eigenfunctions of T_ϕ and $T_{\bar{\phi}}$ must satisfy both functional relations (1.) and (2.). If we could show that $\phi \equiv 0$ was the only function in H^2 which could satisfy both (1.) and (2.), we, of course, would be finished. Although we can not yet do this, we can show that no rational function can satisfy both and , in fact, no non-zero function which is in H^2 and continuous on $\mathbb{T}$ can satisfy both conditions simultaneously.

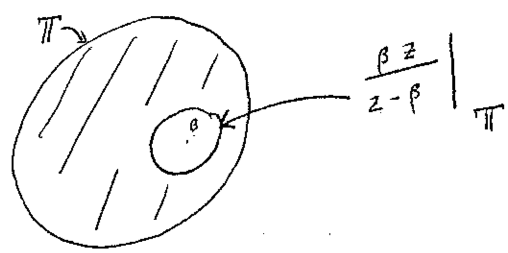
We look in detail at the slightly less general $\phi = \frac{z^2}{z-\beta}$, i.e., we are setting $\alpha = 0$. For this case, if $|\beta| < \frac{1}{2}$, the graph is a simple closed curve, while if $\frac{1}{2} < |\beta| < 1$, the graph is a figure eight. Also, the functional relations

(1.) and (2.) become, respectively,

$$(1') \quad zf(z) = \frac{1-\bar{\beta}z}{\bar{\beta}} f\left(\frac{1-\bar{\beta}z}{\bar{\beta}}\right) \quad \text{on } \mathbb{D} \cap \left\{z : \left|\frac{1-\bar{\beta}z}{\bar{\beta}}\right| < 1\right\}$$

$$(2') \quad z^2 f(z) = \left(\frac{\beta z}{z-\beta}\right)^2 f\left(\frac{\beta z}{z-\beta}\right) \quad \text{on } \mathbb{D} \cap \left\{z : \left|\frac{\beta z}{z-\beta}\right| < 1\right\}.$$

If $|\beta| < \frac{1}{2}$, then the set on which (1') is defined is empty and (2') is defined in the exterior of $\left.\frac{\beta z}{z-\beta}\right|_{\mathbb{T}}$ intersected with $\mathbb{D}$(see figure)



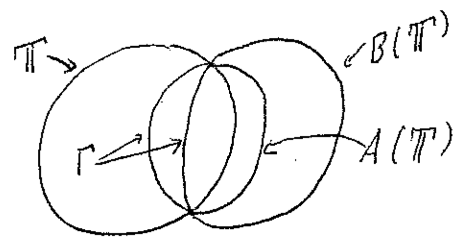
We define

$$w(z) = \begin{cases} z^2 f(z) & |z| < 1 \\ \left(\frac{\beta z}{z-\beta}\right)^2 f\left(\frac{\beta z}{z-\beta}\right) & |z| \geq 1. \end{cases}$$

Since $\frac{\beta z}{z-\beta}$ is analytic outside $\mathbb{D}$ and f is analytic in $\mathbb{D}$, w(z) is a well defined entire function. Further w(z) is bounded at infinity, and so by Liouville's Theorem, w(z) is constant. But $w(0) = 0$, hence, $w(z) \equiv 0$, hence, $f(z) \equiv 0$. From here we conclude that for $|\beta| < \frac{1}{2}$, the eigenfunctions of T_ϕ and $T_{\bar{\phi}}$ span all of H^2, since the orthogonal complement of the span consists of the zero function alone.

Although at present we do not know what happens when $|\beta| > \frac{1}{2}$, several interesting questions arise from knowing what occurs when $|\beta| < \frac{1}{2}$. One of the more interesting ones is: if H^2 is generated by functions of the form $\frac{A(1-\bar{\beta}z)}{1-\lambda z+\bar{\beta}\lambda z^2}$ for λ in some domain and $|\beta| < \frac{1}{2}$, why is H^2 not generated by the same type of functions for $|\beta| > \frac{1}{2}$? The case where $|\beta| = \frac{1}{2}$ leads to even more questions.

Let's look still closer at (1') and (2'). If we define $A(z) = \frac{\beta z}{z-\beta}$ and $B(z) = \frac{1-\bar{\beta}z}{\bar{\beta}}$, then A and B are Möbius transformations which are both inverses for themselves, i.e., $A \circ A(z) = z$ and $B \circ B(z) = z$. For $\frac{1}{2} < |\beta| < 1$, we look at the image of $\mathbb{T}$ under A and B.



Put $\Gamma = \{z \text{ in } \mathbb{D}: z \text{ in } \{B(\mathbb{T}) \cap \mathbb{D}\} \cup \{A(\mathbb{T}) \cap \mathbb{D}\}\}$. Let η be the Carleson measure(for definition and properties see (10)) induced on $\mathbb{D}$ by arclength measure on Γ. Further, let I_η be the mapping of $H^2(\mathbb{T})$ into $L^2(\mathbb{D},\eta)$ defined by $I_\eta f = f$, i.e., simply identifying the elements of H^2 as elements of $L^2(\mathbb{D},\eta)$. A very interesting property of this I_η is that it is isometric on the subspace M, the orthogonal complement of the span of the eigenfunctions of T_φ and $T_{\bar{\varphi}}$ in H^2. If we could show this map compact, we would have M at most finite dimensional. Unfortunately, due to observations of Prof. Douglas, which led to a characterization of compact composition operators (for definition and properties, see, e.g., (18)) in terms of induced Carleson measures, the map I_η can not be compact.

The problem of the spanning characteristics of eigenfunctions has been changed or exchanged for a problem of studying a certain operator. We still have hope that these techniques will eventually show M to be either empty or at most finite dimensional for $\varphi = \dfrac{z^2}{z-\beta}$, and then at least the same result should be easy to show for quadradic rational functions.

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