AUTOMORPHISMS OF THE DEFORMATION SPACE OF A KLEINIAN GROUP

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Abstract of the Dissertation

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One of the main objectives of this dissertation is to determine the biholomorphic automorphisms of the Deformation space of a (finitely generated) Kleinian group G, denoted $T(G,\Omega)$. It has been proven that the universal holomorphic covering space of $T(G,\Omega)$ is a cross-product of Teichmüller spaces of Fuchsian groups, that is, of the form $T(\Gamma_1) \times \ldots \times T(\Gamma_n)$, where the Γ_i, $1 \le i \le n$, are Fuchsian. Hence we approach the problem by first determining the biholomorphic automorphisms of cross-products of Teichmüller spaces and then to study the covering mapping. It should be pointed out here, that under certain conditions on G, namely the components of G being simply connected, the covering mapping is actually biholomorphic so that the Deformation space $T(G,\Omega)$ is (biholomorphically equivalent to) a cross-product of Teichmüller spaces.

We approach the problem of determining the biholomorphic automorphisms of a cross-product of Teichmüller spaces by characterizing the mapping induced on the cotangent space by an arbitrary biholomorphic automorphism of the cross-product and apply results of Royden. In the process we discover properties of $T(G,\Omega)$ as well as prove theorems interesting in their own right. In particular, we prove that $T(G,\Omega)$ is not a homogeneous space in general.

The other objective is to describe biholomorphic mappings between Deformation spaces. To this end, we study biholomorphic mappings between cross-products of Teichmüller spaces where the number of factors in each product is possibly different. Via the cotangent space approach, we prove that in general there do not exist biholomorphic mappings between cross-products of Teichmüller spaces with a different number of factors. As a special case, it follows that a cross-product of non-trivial Teichmüller spaces is never a Teichmüller space. We conclude from the above that in general a biholomorphic mapping between Deformation spaces implies that the number of inequivalent components of each Kleinian group is in one-to-one correspondence. We study properties of these biholomorphic mappings between Deformation spaces.

to those I love --

to my parents, to my grandparents, to my dearest Barbara

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I want to express my most sincere and deepest gratitude to my teachers - Irwin Kra, Bernard Maskit, and Lipman Bers for the knowledge and inspiration into a fascinating, creative field, which only they can provide.

"I am honored to have had the privilege to contribute one small stone in a huge pyramid, designed and built by great mathematicians during the last one-hundred years."

CHAPTER 0

INTRODUCTION

Assuming the knowledge of the concept of a Riemann surface, originally conceived by Riemann as an object on which multi-valued complex functions in the complex plane could be represented as single-valued, a natural question to ask is the following: When are two Riemann surfaces of the same topological type conformally equivalent? The answer to the special case where the surface is compact and of genus one, that is, a torus, is classically known. The answer to the general case is, however, evasive. To study this question and, in general, a wide variety of questions relating to the collection of all Riemann surfaces of a given topological type, the notion of Teichmüller space arose. A point in Teichmüller space in an equivalence class of Riemann surfaces, and a remarkable and attractive fact, among other things, is that this object is naturally a complex manifold. Since its conception by O. Teichmüller, many prominent mathematicians - including those appearing in the acknowledgements and bibliography, have spent years of their lives developing this and related concepts to its present beautiful existence.

In recent years, a more general object called the Deformation space of a Kleinian group was created, in which the Teichmüller space is a very special case. Loosely speaking, one may think of the Deformation space as the analogue of the Teichmüller space of a union of Riemann surfaces, once one is familiar with the fact that certain collections of Riemann surfaces are obtainable via the action of certain subgroups of the group of conformal automorphisms, called Kleinian groups, operating on the extended complex plane. The Deformation space also has the attractive property of being naturally a complex manifold.

My objective in this paper is to classify the biholomorphic automorphisms of the Deformation space of a Kleinian group and more generally the biholomorphic mappings between Deformation spaces. In the process, I have proven some lemmas interesting in their own right, as well as discovering properties of these spaces. I have structured and included those items in this paper which I felt were needed to unify the subject as a whole - a need, which I feel, is strongly lacking at present.

I sincerely hope that this paper stirs up the curiosity and interest of the reader, as has the entire subject to the author.

CHAPTER I

THE TEICHMÜLLER SPACE OF A RIEMANN SURFACE AND ITS MODULAR GROUP

§1. Riemann surfaces

A Riemann surface S is one complex dimensional complex analytic manifold; that is, a connected topological space which is Hausdorff, together with a collection of objects $\mathcal{C} = \{(U_\alpha, z_\alpha)\}_{\alpha \in I}$, where the U_α are open subsets of S whose union is S and to each U_α is assigned a homeomorphism $z_\alpha : U_\alpha \to C$ onto an open subset of the complex plane with the following property: if $U_\alpha \cap U_\beta \neq \emptyset$ then the function $z_\beta \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap U_\beta) \to z_\beta(U_\alpha \cap U_\beta)$ is holomorphic. More correctly, the collection $\mathcal{C} = \{(U_\alpha, z_\alpha)\}_{\alpha \in I}$ is required to be maximal, that is, if an arbitrary pair (U, z), U an open subset of S, $z : U \to C$ a homeomorphism, has the property that $z_\alpha \circ z^{-1}$ and $z \circ z_\alpha^{-1}$ are holomorphic whenever defined, $\alpha \in I$, then one requires that $(U, z) \in \mathcal{C}$. However, the existence of a maximal collection, given any collection with the above properties, follows from Zorn's lemma [27].

A holomorphic function f on S is a continuous mapping $f : S \to C$ such that for each (U_α, z_α) from S, $f|_{U_\alpha} \circ z_\alpha^{-1} : z_\alpha(U_\alpha) \to C$ is holomorphic in the usual sense. Let S' be another Riemann surface with collection $\mathcal{C}' = \{(V_\beta, w_\beta)\}_{\beta \in J}$. More generally, a continuous mapping $f : S \to S'$ between two Riemann surfaces is said to be

<u>holomorphic</u> if for (U_α, z_α) from S, (V_β, w_β) from S', the function $w_\beta \circ f|_{U_\alpha} \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap f^{-1}(V_\beta)) \to w_\beta(V_\beta)$ is holomorphic in the usual sense. A holomorphic function is said to be <u>conformal</u> if it is bijective [27].

§ 2. Covering space theory

Let S be a Riemann surface. The Uniformization theorem for Riemann surfaces states that the holomorphic universal covering space of S is either the Riemann sphere $\hat{C}$, the complex plane C, or the upper half plane U [27]. Let X denote any of these possibilities and $X \overset{\pi}{\to} S$ the holomorphic covering mapping. Let G be the group (perhaps trivial) of deck transformations, that is, the group of conformal homeomorphisms $h : X \to X$ with the property that $\pi \circ h = \pi$.* The general theory of covering spaces then implies that $S \cong X/G$ ($\cong$ conformal equivalence) and the fundamental group $\pi_1(S)$ is isomorphic to the fixed point free group G [27]. The Riemann surface corresponding to $X = \hat{C}$ is $\hat{C}$, those corresponding to X = C are C, C−{0}, and the tori. All other Riemann surfaces have X = U. If X = U, then G is classically called a <u>Fuchsian group</u> and will be denoted by Γ.

More generally one can define a Riemann surface S with distinguished points. This is a Riemann surface S in the usual sense in which a discrete set $\{P_i\}$ of points on S has been distinguished and to each P_i a number v_i, $2 \le v_i \le \infty$ has been assigned. Analogously, one has a holomorphic branched universal** covering space $\pi : X \to S$ of S, that is, X is simply connected, $\pi|_{\text{rest.}} : X-\pi^{-1}(\bigcup_i' \{P_i\}) \to S-\bigcup_i' \{P_i\}$ a holomorphic

* We remark that G is a subgroup of $M = \{z \to \frac{az+b}{cz+d} / a,b,c,d \in C, ad-bc = 1\}$, the group of fractional linear transformations.

** There exist some exceptions; see [21].

covering space of $S-\bigcup_i'\{P_i\}$, and over each $P_i \in S$ the covering mapping π is locally v_i to one. Points $P_i \in S$ with $v_i = \infty$ correspond to punctures on S. The group G in general is no longer fixed point free and in fact the fixed points occur precisely at those P_i for which the corresponding v_i is finite. As before, we call G a Fuchsian group and denote it by Γ if X = U [21].

Note: $\bigcup_i'\{P_i\} = \{P_i \in S/v_i < \infty\}$.

§3. Quasiconformal mappings

A concept which is weaker than conformal mapping and hence more flexible is that of a quasiconformal mapping $f : S \to S'$ between two Riemann surfaces. A homeomorphism $g : D \to C$, D a domain in the complex plane C, is said to be quasiconformal if g has locally square integrable generalized derivatives and has the property that $\frac{|g_{\bar{z}}|}{|g_z|} \leq k < 1$ a.e. for some k. A homeomorphism $f : S \to S'$ between two Riemann surfaces is called quasiconformal if locally, that is, for $g_{\alpha,\beta} = w_\beta \circ f|_{U_\alpha} \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap f^{-1}(V_\beta)) \to w_\beta(V_\beta)$, it is quasiconformal in the complex plane with globally bounded k. We remark that every conformal mapping is quasiconformal [2]. In contrast to conformal mappings which preserve the complex structure (that is, the collection $C = \{(U_\alpha, z_\alpha)\}_{\alpha \in I}$), quasiconformal mappings in general do not. Hence, one is led to the study of the various complex structures one can put on a Riemann surface S of a given topological type via quasiconformal mappings.

The Teichmüller space, denoted by $T(S)$, of a Riemann surface S was specifically developed for this study. With the concepts already defined, we proceed to define the Teichmüller space of a Riemann surface.

§4. Teichmüller spaces

Let S be a fixed Riemann surface. From now one we will assume that S is of <u>finite type</u>, that is, a compact surface from which possibly a finite number of points have been removed and that its universal covering space $X = U$. We look at the collection of all quasiconformal mappings (also called quasiconformal deformations) $f : S \to S'$ onto another Riemann surface S'. We put an equivalence relation on this collection as follows: $f : S \to S'$ and $g : S \to S''$ will be called equivalent if there exists a conformal mapping $h : S' \to S''$ such that the composite mapping $g^{-1} \circ h \circ f : S \to S$ is homotopic to the identity self-mapping of S (i.e. $g^{-1} \circ h \circ f \sim id$). Each equivalence class, denoted by (S, f, S'), will be a point in the <u>Teichmüller space</u> of S denoted by $T(S)$* [6].

One has a rather simple interpretation of the equivalence relation in terms of the fundamental group of the deformed surfaces as follows: One selects and fixes a set of generators for $\pi_1(S)$, the fundamental group of the given Riemann surface S. Each quasiconformal deformation $f : S \to S'$ induces an isomorphism $f_* : \pi_1(S) \to \pi_1(S')$ thus determining a set of generators for the fundamental group of S', that is, the images under f_* of the chosen set of generators for $\pi_1(S)$. Suppose $g : S \to S''$ is another deformation with induced map

*One may analogously define $T(S)$ for an arbitrary Riemann surface S, although we do not in this paper.

$g_* : \pi_1(S) \to \pi_1(S")$ and thus determining a set of generators for the fundamental group of S" in the same manner. Then $(S,f,S') \sim (S,g,S")$ if there exists a conformal mapping $h : S' \to S"$ such that the induced mapping $h_* : \pi_1(S') \to \pi_1(S")$ maps the generators of $\pi_1(S')$ determined by f_* onto the generators of $\pi_1(S")$ determined by g_*.

It is well-known that the Teichmüller space T(S) is a complex analytic manifold and of finite complex dimension $3g-3+n$, whenever S is of finite type, where g is the genus of $\bar{S}$, the closure of S, and n is the number of deleted points. T(S) is also contractible, hence simply connected [6].

§5. The Modular group

The Modular group, denoted by Mod(S), acts as a group of self-mappings of T(S). Each element θ of Mod(S) is induced by a quasiconformal self-mapping $\omega : S \to S$ as follows:

$$(S,f,S') \overset{\theta}{\mapsto} (S,f\circ\omega^{-1},S').$$

It is quite clear from the definition of T(S) that if $\omega_1,\omega_2 : S \to S$ are quasiconformal and ω_1 is homotopic to ω_2, then they induce the same action on T(S). Hence it is suitable to define Mod(S) as equivalence classes of quasiconformal self-mappings of S, the equivalence being homotopy.

A well-known fact is that Mod(S) acts as a group of biholomorphic automorphism of T(S) [11]. Another important fact which we will now prove is that Mod(S) acts effectively on T(S) if $2g+n > 4$. Hence Mod(S) is not effective in only a few cases.

<u>Theorem</u>. Mod(S) acts effectively on T(S) if $2g+n > 4$.

<u>Proof</u>. [11] Suppose Mod(S) is not effective, that is, there exists $\theta_1 \neq \theta_2 \in$ Mod(S) such that $\theta_1(x) = \theta_2(x)$ for all $x \in T(S)$. Let $\theta = \theta_2^{-1}\circ\theta_1$, hence $\theta(x) = x$ the identity self-mapping of T(S), yet θ is not the identity element of Mod(S). Hence if $\omega : S \to S$ quasiconformal induces θ, we can assume $\omega \not\simeq$ id.

$\theta(x) = x$ implies $(S,f\circ\omega^{-1},S') = (S,f,S')$ for all points $(S,f,S') \in T(S)$. In particular, for the point (S,id,S) one has

$$(S, \omega^{-1}, S) = (S, id, S)$$

that is, there exists a conformal mapping $h : S \to S$ such that $id^{-1} \circ h \circ \omega^{-1} \sim id$ or $h \sim \omega$. Since homotopic mappings induce the same action on $T(S)$, one can assume $\omega : S \to S$ is conformal. Let $(S, g, S'') \in T(S)$ be arbitrary. Then as before

$$(S, g \circ \omega^{-1}, S'') = (S, g, S'')$$

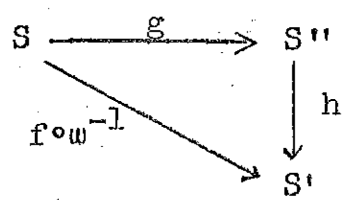
Hence there exists a conformal mapping $k : S'' \to S''$ such that $(g \circ \omega^{-1})^{-1} \circ k \circ g = \omega \circ g^{-1} \circ k \circ g \sim id$. Now $k \neq id$, for if $k = id$ then $w \circ g^{-1} \circ k \circ g = \omega \sim id$, a contradiction. Hence, this implies that every Riemann surface of type S has non-trivial conformal automorphisms, but if $2g+n > 4$, it is known that in each type (g,n) there exists a Riemann surface with trivial conformal automorphism group, a contradiction, hence Mod(S) is effective if $2g+n > 4$.

In the next chapter it will become apparent that each element $\theta \in Mod(S)$ is an isometry in the so-called Teichmüller metric. Mod(S) acts <u>properly discontinuously</u> on T(S) [11]; this means that for each point $x \in T(S)$, $Mod_x(S)$ the stability subgroup of x, is finite and for each $x \in T(S)$ there exists a neighborhood U of x such that $\theta(U) = U$ for $\theta \in Mod_x(S)$ and $\theta(U) \cap U = \emptyset$ for all $\theta \in Mod(S) - Mod_x(S)$. The <u>Riemann space</u> $\mathcal{R}(S)$ by definition is $T(S)/Mod(S)$. It is a complex normal space, but in general, is not a complex analytic manifold.

<u>Proposition.</u> [6] Let (S,f,S') and (S,g,S'') be two points in $T(S)$. Then there exists an element $\theta \in \mathrm{Mod}(S)$ such that $\theta(S,f,S') = (S,g,S'')$ if and only if S' is conformally equivalent to S''.

<u>Proof.</u> $\Rightarrow$ Let $\theta \in \mathrm{Mod}(S)$ be such that $\theta(S,f,S') = (S,g,S'')$. Let $\omega : S \to S$ be a quasiconformal mapping which induces θ. Then $(S, f\circ\omega^{-1}, S') = (S,g,S'')$ which by definition implies the existence of a conformal mapping $h : S' \to S''$.

$\Leftarrow$ Let $h : S'' \to S'$ be conformal. We must find $\omega : S \to S$ quasiconformal such that $(S, f\circ\omega^{-1}, S') = (S,g,S'')$. In other words, we want ω such that the diagram



commutes up to homotopy, that is, $(f\circ\omega^{-1})^{-1}\circ h\circ g \sim id$. Let $\omega^{-1} = f^{-1}\circ h\circ g$, then $\omega^{-1} : S \to S$ and is quasiconformal since the composition of quasiconformal mappings and their inverses are quasiconformal, then clearly

$(f\circ\omega^{-1})^{-1}\circ h\circ g = (f\circ f^{-1}\circ h\circ g)^{-1}\circ h\circ g = g^{-1}\circ h^{-1}\circ h\circ g = id$ so that in particular $(f\circ\omega^{-1})^{-1}\circ h\circ g \sim id$.

$R(S)$ is therefore also called the space of conformal equivalence classes of Riemann surfaces of the given topological type S.

CHAPTER II

THE TEICHMÜLLER AND KOBAYASHI METRICS

§1. The Teichmüller metric

Along with each Teichmüller space $T(S)$ there is a rather natural metric τ called Teichmüller's metric. Let $g : D \to C$ be a quasiconformal mapping of a domain D in the complex plane. Let $\mu(z) = \frac{g_{\bar{z}}(z)}{g_z(z)}$ for $z \in D$. The function μ is called the <u>Beltrami coefficient</u> of g. We define a function K, called the <u>dilatation</u>, on the set of all such quasiconformal mappings $\{g : D \to C / D \text{ any domain in } C\}$ into the real numbers by:

$$K(g) = \frac{1+\|\mu\|_\infty}{1-\|\mu\|_\infty}, \text{ where } \|\mu\|_\infty = \operatorname*{ess\,sup}_{z\in D} |\mu(z)|.$$

We remark that $0 \leq \|\mu\|_\infty < 1$. It is not a difficult computation to show that if g is composed with conformal mappings on either side, the dilatation K is invariant [2].

For quasiconformal mappings $f : S \to S'$ between Riemann surfaces one then has a well-defined meaning of the dilatation K of f locally. Since the Beltrami coefficients defined locally are required to be globally bounded by a constant $k < 1$ a.e., one defines $K(f) = \frac{1+k}{1-k}$, by choosing k as small as possible.

Let $x = (S,f,S')$, $y = (S,g,S'')$ be two points in $T(S)$. Define τ by:

$$\tau(x,y) = \inf_{\substack{f\in x\\ g\in y}} \log K(f\circ g^{-1}).$$

It can be shown that the above is well-defined and a metric on $T(S)$ [6]. An important and yet very simple result is the following:

<u>Theorem.</u> [6] Let $\theta \in \mathrm{Mod}(S)$. Then θ is an isometry in the Teichmüller metric τ.

<u>Proof.</u> Let $\omega : S \to S$ be quasiconformal and induce θ. Let $x = (S,f,S')$ and $y = (S,g,S'')$ be two points in $T(S)$. We want to show that $\tau(\theta(x),\theta(y)) = \tau(x,y)$. Now $\theta(x) = (S, f\circ\omega^{-1}, S')$ and $\theta(y) = (S, g\circ\omega^{-1}, S'')$ hence:

$$\tau(\theta(x),\theta(y)) = \inf_{\substack{f\circ\omega^{-1}\in\theta(x)\\ g\circ\omega^{-1}\in\theta(y)}} \log K((f\circ\omega^{-1})\circ(g\circ\omega^{-1})^{-1})$$

$$= \inf_{\substack{f\in x\\ g\in y}} \log K((f\circ\omega^{-1})\circ(g\circ\omega^{-1})^{-1})$$

$$= \inf_{\substack{f\in x\\ g\in y}} \log K(f\circ g^{-1}) = \tau(x,y).$$

§2. The Kobayashi metric.

Let M be any complex analytic manifold. Let Δ be the unit disk in the complex plane and ρ the Poincaré metric on Δ. Let $p,q \in M$. We choose points $p = p_0,\dots,p_k = q$ on M, points $a_1,\dots,a_k$, $b_1,\dots,b_k$ on Δ and holomorphic mappings $f_1,\dots,f_k$ of Δ into M such that $f_i(a_i) = p_{i-1}$ and $f_i(b_i) = p_i$ for $i = 1,\dots,k$. We define the Kobayashi metric k as:

$$k(p,q) = \inf \rho(a_1,b_1) + \dots + \rho(a_k,b_k)$$

where the infimum is taken over all possible choices of points and mappings thus made. It can be shown that k is in general a pseudo-metric on M, that is, has all the properties of a metric except that $k(p,q) = 0$ need not imply that $p = q$.

One of the most important intrinsic properties of the Kobayashi metric is the following: Let $f : M \to N$ be holomorphic, where M and N are complex analytic manifolds. Let k_M and k_N be the associated Kobayashi metrics. Then $k_M(p,q) \geq k_N(f(p),f(q))$ for $p,q \in M$, that is, f is distance decreasing. Hence if $f : M \to N$ is biholomorphic, f is an isometry in the respective Kobayashi metrics [17].

§3. Teichmüller metric = Kobayashi metric

An important result of Royden and crucial result to this paper is that the Teichmüller and Kobayashi metrics are the same for every Teichmüller space T(S). Royden proves that if $x,y \in T(S)$, then $\tau(x,y) = \inf_f \rho(f^{-1}(x),f^{-1}(y))$, where the infimum is taken over all holomorphic mappings $f : \Delta \to T(S)$ with the property that $x,y \in f(\Delta)$. It was shown that the Kobayashi metric k as defined above for an arbitrary complex analytic manifold M has this more concise form for $M = T(S)$. Royden uses the existence of a holomorphic mapping $f : \Delta \to T(S)$ such that $x,y \in f(\Delta)$ and isometric with respect to the Poincaré metric ρ on Δ and the Teichmüller metric τ on T(S) to show that $\tau \geq \inf \rho$. In order to show $\tau \leq \inf \rho$, he shows that the Riemannian metric on Δ induced by the Teichmüller metric is less than or equal to the differential form of the Poincaré metric on Δ [25].

It is evident from Royden's result, that every biholomorphic mapping $h : T(S) \to T(S')$ is an isometry in the Teichmüller metric τ, and if $S = S'$, then since every element of Mod(S) is a biholomorphic self-mapping of T(S), we have reproven the fact that Mod(S) acts as a group of isometries of T(S).

§4. The Infinitesimal metric

More precise information about the Teichmüller metric τ is known. It has been shown [22] that there exists a Finsler structure $F_{T(\Gamma)}$ on the tangent bundle of the Teichmüller space $T(\Gamma)$ such that the Teichmüller metric is the integrated form of $F_{T(\Gamma)}$.

A Finsler structure F_M on a complex manifold M modeled on a Banach space E with norm $\|\cdot\|$ is a mapping $F_M : M \times E \to R$ such that for each $x \in M$,

i) $F_M(x,\cdot)$ is a norm on E equivalent to the norm $\|\cdot\|$, and

ii) there exists a neighborhood O of x and a constant C such that

$$|F_M(x_1,e) - F_M(x_2,e)| < C\|e\| \, \|x_1 - x_2\|$$

if $x_1, x_2 \in O$ and $e \in E$.

Proposition. $\tau(x,y) = \inf \int_\gamma F_{T(\Gamma)}(\gamma(t),\gamma'(t))dt$ where the infimum is taken over all differentiable curves $\gamma : [0,1] \to T(\Gamma)$ joining x and y.

Proof. See the literature [22].

It follows from the above proposition and the fact that τ is the Kobayashi metric that if $h : T(\Gamma_1) \to T(\Gamma_2)$ is a holomorphic mapping between Teichmüller spaces $T(\Gamma_1)$, $T(\Gamma_2)$, then $F_{T(\Gamma_2)}(h(x),h_*(\xi)) \leq F_{T(\Gamma_1)}(x,\xi)$ for all (x,ξ) belonging to the tangent bundle of $T(\Gamma_1)$. If $h : T(\Gamma_1) \to T(\Gamma_2)$ is a biholomorphic mapping, then $F_{T(\Gamma_2)}(h(x), h_*(\xi)) = F_{T(\Gamma_1)}(x, \xi)$. Hence not only is every biholomorphic mapping $h : T(\Gamma_1) \to T(\Gamma_2)$ an isometry in the metric sense, but more importantly, an isometry in the sense of manifolds. The latter fact is fundamental to the results in this paper. The Finsler structure $F_{T(\Gamma)}$ will be discussed more explicitly in Chapter IV.

CHAPTER III

AUTOMORPHISMS OF THE TEICHMÜLLER SPACE

§1. The Teichmüller space of a Fuchsian group

A <u>Fuchsian group</u> Γ is a properly discontinuous (in U) subgroup of the group of conformal automorphisms $M = \{z \to \frac{az+b}{cz+d}/a,b,c,d \in C,\ ad-bc = 1\}$ of the Riemann sphere $\hat{C}$ with the property that for each $\gamma \in \Gamma$, $\gamma(U) = U$. By <u>proper discontinuity</u> (in U) one means that for each $x \in U$ there exists a neighborhood N of x such that $\gamma(N) = N$ for $\gamma \in \Gamma_x = \{\gamma \in \Gamma/\gamma(x) = x\}$, Γ_x finite, and $\gamma(N) \cap N = \emptyset$ for $\gamma \in \Gamma - \Gamma_x$. It is well-known that such a group Γ also acts properly discontinuously in L (the lower half plane). A Fuchsian group Γ is said to be of the <u>first kind</u> if it is not discontinuous at each point of $\hat{R}$, otherwise of the <u>second kind</u>. An element $\gamma \in \Gamma$, $\gamma \neq id$ is called <u>elliptic</u> if γ has two fixed points, one of which is in U, and called <u>parabolic</u> if γ has precisely one fixed point in $\hat{C}$. All other elements of $\Gamma - \{id\}$ are called <u>hyperbolic</u>. U/Γ can be made into a Riemann surface with distinguished points by requiring that the projection $\pi : U \to U/\Gamma$ to be holomorphic. The distinguished points $\{P_i\} \in U/\Gamma$ are precisely the projections of those points $x \in U \cup \{$parabolic fixed pts. of $\Gamma\}$ such that $\Gamma_x \neq id$ and the corresponding v_i are the orders of the non-conjugate maximal cyclic subgroups of Γ generated by the elliptic (if $v_i < \infty$) or parabolic (if $v_i = \infty$) elements.

The Teichmüller space $T(\Gamma)$ of a Fuchsian group Γ is defined as follows: One looks at quasiconformal deformations of Γ onto other Fuchsian groups, that is, quasiconformal self-mappings w of the upper half plane U such that $w\Gamma w^{-1}$ is again Fuchsian. We require that w fix $0,1$, and ∞ and call w_1 and w_2 equivalent if $w_1(x) = w_2(x)$ for all $x \in R$. Each equivalence class is a point in the Teichmüller space $T(\Gamma)$ [6]. An important result of Bers and Greenberg is that $T(\Gamma) \cong T(U_\Gamma/\Gamma)$ (biholomorphically), where $T(U_\Gamma/\Gamma)$ is the Teichmüller space of the Riemann surface U_Γ/Γ, U_Γ obtained by deleting those points $x \in U$ such that $\Gamma_x \neq id$ [12]. If Γ is finitely generated and of the first kind, U_Γ/Γ is a compact Riemann surface punctured at finitely many points. Hence to classify the biholomorphic automorphisms of $T(\Gamma)$, one need only consider the case where the Riemann surface U/Γ is of finite type (g,n). Furthermore, since it is known that Riemann surfaces of the same type (g,n) yield biholomorphically equivalent Teichmüller spaces, one needs to classify such mappings of $T(\Gamma)$ for only one surface of a given type [12].

An automorphism $\theta : \Gamma \to \Gamma$ is called geometric if $\theta(\gamma) = w \circ \gamma \circ w^{-1}$, $\gamma \in \Gamma$ for some quasiconformal self-mapping w of U.

The Modular group $Mod(\Gamma)$ associated to the Teichmüller space $T(\Gamma)$ is defined to be the quotient of the group of geometric automorphisms by the (normal) subgroup of inner automorphisms, that is, $\theta_1 \sim \theta_2$ if there exists a $\beta \in \Gamma$ such that

$$\theta_2(\gamma) = \beta \circ \theta_1(\gamma) \circ \beta^{-1}$$

for all $\gamma \in \Gamma$ [15].

§2. Tangent and Cotangent space

Let $S = U/\Gamma$, Γ a finitely generated, fixed point free Fuchsian group of the first kind. Let $L^{\infty}(\Gamma)$ denote the closed subspace of $L^{\infty}(U,C)$ consisting of all $\mu \in L^{\infty}(U,C)$ such that $(\mu\circ\gamma(z))\frac{\overline{\gamma'(z)}}{\gamma'(z)} = \mu(z)$, for all $\gamma \in \Gamma$ and almost all $z \in U$. Also denote by $L^1(\Gamma)$ the Banach space of (equivalence classes) of measurable functions f on U satisfying $(f\circ\gamma)(z)(\gamma'(z))^2 = f(z)$, for all $\gamma \in \Gamma$ and almost all $z \in U$ with norm

$$\|f\| = \tfrac{1}{2} \int_{U/\Gamma} |f(z)| \; |dz\wedge d\bar{z}| < \infty.$$

There is a natural pairing

$$(f,\mu)_{\Gamma} = \tfrac{1}{2} \iint_{U/\Gamma} f(z)\mu(z) \; |dz\wedge d\bar{z}|,$$

$$f \in L^1(\Gamma), \; \mu \in L^{\infty}(\Gamma)$$

which establishes a norm preserving isomorphism of $L^{\infty}(\Gamma)$ onto the conjugate space $L^1(\Gamma)^*$. Let $Q(\Gamma)$ be the space of integrable holomorphic quadratic differentials, i.e. the elements of $L^1(\Gamma)$ which are holomorphic [15].

It has been shown [15] using the above pairing, that the tangent space at the point $(S,f,S') \in T(S)$ where $S' = U/\Gamma'$ is (isomorphic to) $L^{\infty}(\Gamma')/Q(\Gamma')^{\perp}$, where $Q(\Gamma')^{\perp} = \{\mu \in L^{\infty}(\Gamma')/(f,\mu)_{\Gamma'} = 0, \text{ all } f \in Q(\Gamma')\}$ and that the cotangent space at the same point is $Q(\Gamma')$ [15]. $Q(\Gamma')$ is precisely the integrable holomorphic quadratic differentials

$Q(S')$ on the Riemann surface $S' = U/\Gamma$. The tangent and cotangent spaces are dual with dual norms of the L^∞ and L^1 spaces respectively. The cotangent space will be of special importance in what follows.

§3. Automorphisms of the Teichmüller space

Let S be a compact Riemann surface of genus $g \geq 2$. Royden has shown that the biholomorphic self-mappings of T(S), which we denote simply by Aut T(S), is the action of Mod(S) on T(S) whenever S has genus $g \geq 3$, i.e.

<u>Theorem.</u> (Royden) If $g \geq 3$, then Aut T(S) = Mod(S) [25].

We give only a rough outline of the proof to point out the role played by the cotangent space which will be of use in later results.

First of all, if $g \geq 3$ Mod(S) $\subseteq$ Aut T(S). This follows from the fact that Mod(S) acts as a group of biholomorphic automorphisms of T(S) and acts effectively (see Chapter I). We remark that for $g = 2$, Mod(S) does not act effectively on T(S) for in this case S is hyperelliptic and it is not difficult to show that the identity mapping I is not homotopic to J, the hyperelliptic involution, and yet they induce the same mapping of T(S).

To show that Aut T(S) $\subseteq$ Mod(S) and hence the theorem, one first realizes that every $h \in$ Aut T(S) is an isometry in the sense of manifolds in the Teichmüller metric (see Chapter II). Hence the induced map h^* between the cotangent spaces at x and h(x) on T(S) is a complex linear isometry in the L^1 metric. Royden [25] then proves that every such isometry implies that the underlying Riemann surfaces S' and S" are conformally equivalent, where $x = (S, f, S')$ and $h(x) = (S, g, S'')$. This in turn implies the existence of an element $\theta_x \in \mathrm{Mod}(S)$, depending in general on the point x, such that $\theta_x(x) = h(x)$ (see Chapter I). Since this is true for all $x \in T(S)$ and Mod(S) is a discontinuous group of biholomorphic isometries of T(S), there exists a $\theta \in \mathrm{Mod}(S)$ such that $\theta(x) = h(x)$ for all $x \in T(S)$, hence $\mathrm{Aut}\, T(S) \subseteq \mathrm{Mod}(S)$.

Earle and Kra [15] have generalized this theorem to Riemann surfaces S of finite type (g,n). As before, we assume that the universal covering space is U, that is, $2g-2+n > 0$, then

Theorem. $\mathrm{Aut}\, T(S) = \mathrm{Mod}(S)$ unless

$$(g,n) = (0,3),(0,4),(1,1),(1,2) \text{ or } (2,0).$$

Proof. See the literature [15].

It will be useful later to use another result of Earle and Kra:

Theorem. Let $f : Q(S) \to Q(S')$ be a complex linear isometry in the L^1 norm, then S and S' are of the same type (g,n) (and $S \cong S'$) unless

$$(g,n) = (0,3),(0,4),(1,1),(1,2), \text{ or } (2,0).$$

Proof. See the literature [15].

Since every biholomorphic mapping $h : T(S) \to T(S')$ between Teichmüller spaces induces a complex linear isometry between cotangent spaces over corresponding points, then

<u>Corollary.</u> [23] If $h : T(S) \to T(S')$ is biholomorphic and type $(S) = (g,n)$ such that $2g+n > 4$ and type $(S') = (g',n')$ such that $2g'+n' > 4$, then $g = g'$ and $n = n'$, i.e. S and S' have the same type.

CHAPTER IV

CROSS-PRODUCTS OF TEICHMÜLLER SPACES

§1. The Finsler structure

Let Γ be a Fuchsian group. Let $M(\Gamma)$ denote the open unit ball in $L^{\infty}(\Gamma)$. It is known [6] that for each $u \in M(\Gamma)$ there exists a unique quasiconformal mapping $w : U \to U$ which fixes 0, 1 and ∞ and has the property that $\frac{w_{\bar{z}}(z)}{w_z(z)} = u(z)$ for almost all $z \in U$. One defines an equivalence relation $\sim$ on $M(\Gamma)$ as follows: $u_1, u_2 \in M(\Gamma)$ are called equivalent if the corresponding quasiconformal mappings w_1, w_2 have the property that $w_1(z) = w_2(z)$ for all $z \in R$. The quotient space $\frac{M(\Gamma)}{\sim}$ is therefore the Teichmüller space of the Fuchsian group Γ, denoted by $T(\Gamma)$. $T(\Gamma)$ is given a complex structure so that the projection mapping $\Phi : M(\Gamma) \to T(\Gamma)$ is holomorphic.

One obtains a Finsler structure $F_{M(\Gamma)}$ on the tangent bundle of $M(\Gamma)$ by defining the length of the tangent vector $v \in L^{\infty}(\Gamma)$ to $u \in M(\Gamma)$ as $F_{M(\Gamma)}(u,v) = \|\frac{v}{1-|u|^2}\|_{\infty}$. The Finsler structure $F_{T(\Gamma)}$ on the tangent bundle of $T(\Gamma)$ is the quotient Finsler structure defined by

$F_{T(\Gamma)}(\Phi(u), \Phi'(u)v) = \inf_{\lambda}\{F_{M(\Gamma)}(u, v+\lambda)/\Phi'(u)\lambda = 0\}$. It is well-defined [16]. It is apparent from the fact that the Teichmüller metric τ is the integrated form of $F_{T(\Gamma)}$ [16], and that τ is the Kobayashi metric on $T(\Gamma)$ [25], that biholomorphic mappings between Teichmüller spaces are isometries with respect to the corresponding Finsler structures.

In a more general context about Finsler structures, O'Bryne [22] has shown that

$$\tau(\Phi(u_1), \Phi(u_2)) = \inf\{ d_{F(M(\Gamma))}(\tilde{u}_1, \tilde{u}_2) / \Phi(\tilde{u}_1) = \Phi(u_1), \Phi(\tilde{u}_2) = \Phi(u_2)\},$$

where $d_{F(M(\Gamma))}$ is the integrated form of $F_{M(\Gamma)}$, the induced Finsler metric on $M(\Gamma)$. In the next section, we will use the fact that the infimum is taken on by some $\tilde{u}_1, \tilde{u}_2 \in M(\Gamma)$ [15]. Such elements are called <u>extremal</u>.

§2. The Cross-product Kobayashi metric

Let $\Gamma_1,\ldots,\Gamma_n$ be Fuchsian groups. One forms the product $T(\Gamma_1) \times\ldots\times T(\Gamma_n)$ of Teichmüller spaces. The product is a complex manifold with the product complex structure. Hence we may introduce the Kobayashi metric k on the product $T(\Gamma_1) \times\ldots\times T(\Gamma_n)$. If $n = 1$, we recall that the Kobayashi and Teichmüller metrics coincide and thus the Kobayashi metric is the natural metric to consider on the cross-product. The following result shows the relationship between k and the corresponding Teichmüller metrics τ_i on $T(\Gamma_i)$.

<u>Theorem</u>. Let k be the Kobayashi metric on $T(\Gamma_1) \times\ldots\times T(\Gamma_n)$ and τ_i the Teichmüller metric on $T(\Gamma_i)$, $1 \leq i \leq n$. Then for every $x,y \in T(\Gamma_1) \times\ldots\times T(\Gamma_n)$, where $x = (x_1,\ldots,x_n)$, $y = (y_1,\ldots,y_n)$ one has

$$k(x,y) = \max[\tau_1(x_1,y_1),\ldots,\tau_n(x_n,y_n)].$$

<u>Proof</u>. We may assume $x = \Phi(0) = (\Phi_1(0),\ldots,\Phi_n(0))$ because we can replace any $T(\Gamma_i)$ by some $T(\tilde{\Gamma}_i)$ by the existence of certain biholomorphic mappings $R_i : T(\Gamma) \to T(\tilde{\Gamma}_i)$, called <u>right translations</u>, which are isometries in the Teichmüller metric with the property that $R_i(x_i) = \tilde{\Phi}_i(0)$, where $\tilde{\Phi}_i : M(\tilde{\Gamma}_i) \to T(\tilde{\Gamma}_i)$ is the natural projection mapping [15].

Let k be the Kobayashi metric on $T(\Gamma_1) \times\ldots\times T(\Gamma_n)$. It is clear that $\max[\tau_1(\Phi_1(0),y_1),\ldots,\tau_n(\Phi_n(0),y_n)] \leq k(\Phi(0),y)$ because of the holomorphic projections $P_i : T(\Gamma_1) \times\ldots\times T(\Gamma_n) \to T(\Gamma_i)$,

$i = 1,\ldots,n$, which are distance decreasing with respect to the Kobayashi metrics k on $T(\Gamma_1) \times\ldots\times T(\Gamma_n)$ and τ_i on $T(\Gamma_i)$. To prove the opposite inequality and hence the result, one needs to know the following extremal Teichmüller theory: There exist elements $u_i = c_i \frac{\bar{\varphi}_i}{|\varphi_i|} \in M(\Gamma_i)$ such that $y_i = \Phi_i(u_i)$, where $c_i \in C$, $|c_i| < 1$ and $\varphi_i \in Q(\Gamma_i)$ and $\tau_i(\Phi_i(0),y_i) = d_{F_{M(\Gamma_i)}}(0,u_i)$ [15]. (Such elements u_i are called extremal. See §1.)

The mappings $\psi_i : t \mapsto t\frac{\bar{\varphi}_i}{|\varphi_i|}$ of the unit disk Δ into $T(\Gamma_i)$ are holomorphic and isometric in the respective Kobayashi metrics on Δ and $T(\Gamma_i)$ [25]. We may assume that not all c_i are zero, otherwise there is nothing to prove. Let $c_j = \max(c_1,\ldots,c_n)$, hence $0 < |c_j| < 1$ and $|c_i| \leq |c_j|$, $1 \leq i \leq n$. Define $\psi : \Delta \to T(\Gamma_1) \times\ldots\times T(\Gamma_n)$ by $\psi : t \mapsto (\frac{c_1}{c_j}\, t \,\frac{\bar{\varphi}_1}{|\varphi_j|},\ldots,\frac{c_i}{c_j}\, t\, \frac{\bar{\varphi}_i}{|\varphi_i|},\ldots,\frac{c_n}{c_j}\, t\, \frac{\bar{\varphi}_n}{|\varphi_n|})$. Then ψ is holomorphic, $\psi(0) = (\Phi_1(0),\ldots,\Phi_n(0))$ and $\psi(c_j) = (y_1,\ldots,y_n) = y$. Hence

$$k(\Phi(0),y) \leq \rho(0,c_j) = \tau_j(\Phi_j(0),y_j)$$

$$\leq \max[\tau_1(\Phi_1(0),y_1),\ldots,\tau_n(\Phi_n(0),y_n)]$$

and thus the result.

§3. The Cross-product Finsler structure

Our next goal is to introduce a Finsler structure F_T on the tangent bundle of $T = T(\Gamma_1) \times \dots \times T(\Gamma_n)$ so that the Kobayashi metric k on the cross-product is the integrated form of F_T. The approach is analogous to the case n = 1 (see §1). One has the holomorphic projection mapping

$$\Phi : M(\Gamma_1) \times \dots \times M(\Gamma_n) \to T(\Gamma_1) \times \dots \times T(\Gamma_n),$$

where $\Phi = \Phi_1 \times \dots \times \Phi_n$, $\Phi_i : M(\Gamma_i) \to T(\Gamma_i)$. Our objective is then to introduce a Finsler structure F_M on the tangent bundle of $M = M(\Gamma_1) \times \dots \times M(\Gamma_n)$ and thus inducing a Finsler structure F_T on the tangent bundle of $T = T(\Gamma_1) \times \dots \times T(\Gamma_n)$ via the mapping Φ, so that k is the integrated form of F_T.

Define: $F_M(u,v) = \max[F_{M(\Gamma_1)}(u_1,v_1),\dots,F_{M(\Gamma_n)}(u_n,v_n)]$, where $u = (u_1,\dots,u_n)$, $v = (v_1,\dots,v_n)$ and $u_i \in M(\Gamma_i)$, $v_i \in L^\infty(\Gamma_i)$.

Let $F_T(\Phi(u),\Phi'(u)v) = \inf_\lambda\{F_M(u,v+\lambda)/\lambda = (\lambda_1,\dots,\lambda_n) \ni : \Phi_i'(u_i)\lambda_i = 0, \text{ all } i\}$

be the induced Finsler structure on the tangent bundle of $T = T(\Gamma_1) \times \dots \times T(\Gamma_n)$.

<u>Theorem.</u> $F_T(\Phi(u),\Phi'(u)v) = \max[F_{T(\Gamma_1)}(\Phi_1(u_1),\Phi_1'(u_1)v_1),\dots$

$$\dots,F_{T(\Gamma_n)}(\Phi_n(u_n),\Phi_n'(u_n)v_n)].$$

<u>Proof.</u> By definition

$F_T(\Phi(u), \Phi'(u)v)$

$$= \inf_\lambda\{\max[F_{M(\Gamma_1)}(u_1, v_1+\lambda_1), \ldots, F_{M(\Gamma_n)}(u_n, v_n+\lambda_n)]/ \lambda = (\lambda_1, \ldots, \lambda_n) \ni : \Phi_i'(u_i)\lambda_i = 0, \text{ all } i\}$$

and for each i,

$$F_{T(\Gamma_i)}(\Phi_i(u_i), \Phi_i'(u_i)v_i) = \inf_{\lambda_i}\{F_{M(\Gamma_i)}(u_i, v_i+\lambda_i)/\Phi_i'(u_i)\lambda_i = 0\}.$$

Clearly, $F_{T(\Gamma_i)}(\Phi_i(u_i), \Phi_i'(u_i)v_i) \leq F_T(\Phi(u), \Phi'(u)v)$ for all i.

Hence

$$\max[F_{T(\Gamma_1)}(\Phi_1(u_1), \Phi_1'(u_1)v_1), \ldots, F_{T(\Gamma_n)}(\Phi_n(u_n), \Phi_n'(u_n)v_n)] \leq F_T(\Phi(u), \Phi'(u)v).$$

Now strict inequality is not possible since by extremal Teichmüller theory there exists $(\tilde{u}_i, \tilde{v}_i) \in M(\Gamma_i) \times L^\infty(\Gamma_i)$ such that $\Phi_i(\tilde{u}_i) = \Phi_i(u_i)$, $\Phi'(u_i)\tilde{v}_i = \Phi_i'(u_i)v_i$, and

$F_{M(\Gamma_i)}(\tilde{u}_i, \tilde{v}_i) = F_{T(\Gamma_i)}(\Phi_i(u_i), \Phi_i'(u_i)v_i)$, for all i [22].

Thus

$$\max[F_{M(\Gamma_1)}(\tilde{u}_1, \tilde{v}_1), \ldots, F_{M(\Gamma_n)}(\tilde{u}_n, \tilde{v}_n)] = \max[F_{T(\Gamma_1)}(\Phi_1(u_1), \Phi_1'(u_1)v_1), \ldots, F_{T(\Gamma_n)}(\Phi_n(u_n), \Phi_n'(u_n)v_n)].$$

If strict inequality held,

$$\max[F_{M(\Gamma_1)}(\tilde{u}_1, \tilde{v}_1), \ldots, F_{M(\Gamma_n)}(\tilde{u}_n, \tilde{v}_n)] < F_T(\Phi(u), \Phi'(u)v)$$
$$= \inf_\lambda \{\max[F_{M(\Gamma_1)}(u_1, v_1+\lambda_1), \ldots, F_{M(\Gamma_n)}(u_n, v_n+\lambda_n)/ \Phi'(u_i)\lambda_i = 0, \text{ all } i\},$$

a contradiction.

<u>Remark</u>. The theorem can be restated as follows:

$$\inf_{\lambda}\{\max[F_{M(\Gamma_1)}(u_1,v_1+\lambda_1),\ldots,F_{M(\Gamma_n)}(u_n,v_n+\lambda_n)]$$

$$= \max[\inf_{\lambda_1}\{F_{M(\Gamma_1)}(u_1,v_1+\lambda_1)/\Phi_1'(u_1)\lambda_1 = 0\},\ldots$$

$$\ldots,\inf_{\lambda_n}\{F_{M(\Gamma_n)}(u_n,v_n+\lambda_n)/\Phi_n'(u_n)\lambda_n = 0\}].$$

<u>Theorem</u>. $k(x,y) = \inf_{\gamma} \int F_T(\gamma(t),\gamma'(t))dt$, where the infimum is taken over all differentiable curves $\gamma : [0,1] \to T = T(\Gamma_1) \times\ldots \ldots\times T(\Gamma_n)$ which join x and y in T.

<u>Proof</u>. We may assume that $x = (\Phi_1(0),\ldots,\Phi_n(0))$ because of the existence of right translations which are isometries in the Kobayashi metric and Finsler structure [16]. Since $k = \max[\tau_1,\ldots,\tau_n]$ and $F_T = \max[F_{T(\Gamma_1)},\ldots,F_{T(\Gamma_n)}]$, we need to prove that

$$\max[\tau_1(\Phi_1(0),y_1),\ldots,\tau_n(\Phi_n(0),y_n)]$$

$$= \inf_{\gamma_1,\ldots,\gamma_n} \int \max[F_{T(\Gamma_1)}(\gamma_1(t),\gamma_1'(t)),\ldots$$

$$\ldots,F_{T(\Gamma_n)}(\gamma_n(t),\gamma_n'(t))]dt$$

over all differentiable curves $\gamma_i : [0,1] \to T(\Gamma_i)$ which join $\Phi_i(0)$ to y_i in $T(\Gamma_i)$, $i = 1,\ldots,n$. Clearly, $\max[\tau_1,\ldots,\tau_n] \leq \inf \int \max[F_{T(\Gamma_1)},\ldots,F_{T(\Gamma_n)}]dt$ since $F_{T(\Gamma_i)} \leq \max[F_{T(\Gamma_1)},\ldots,F_{T(\Gamma_n)}]$ and $\tau_i = \inf \int F_{T(\Gamma_i)}$ and thus $\tau_i = \inf \int F_{T(\Gamma_i)} \leq \inf \int \max[F_{T(\Gamma_1)},\ldots,F_{T(\Gamma_n)}]dt$, all i.

Now suppose $\max[\tau_1,\ldots,\tau_n] < \inf \int \max[F_{T(\Gamma_1)},\ldots,F_{T(\Gamma_n)}]dt$. Let $u_i \in M(\Gamma_i)$ be such that $u_i = c_i \frac{\varphi_i}{|\varphi_i|}$ where $c_i \in C$,

$|c_i| < 1$ and $\varphi_i \in Q(\Gamma_i)$ and $\Phi_i(u_i) = y_i$, that is, the u_i are extremal elements [15]. If all c_i are zero, $1 \leq i \leq n$,

we are done since the above strict inequality cannot hold.

Let $c_j = \max(c_1,\ldots,c_n)$, hence $0 < |c_j| < 1$ and $|c_i| \leq |c_j|$, $1 \leq i \leq n$. Define

$$\gamma = \gamma_1 \times\ldots\times \gamma_n : \Delta \to T(\Gamma_1) \times\ldots\times T(\Gamma_n)$$

by

$$t \mapsto \left(\frac{c_1}{c_j}\, t\, \frac{\bar{\varphi}_1}{|\varphi_1|},\ldots,\frac{c_i}{c_j}\, t\, \frac{\bar{\varphi}_i}{|\varphi_i|},\ldots,\frac{c_n}{c_j}\, t\, \frac{\bar{\varphi}_n}{|\varphi_n|}\right)$$
$$= (\gamma_1(t),\ldots,\gamma_n(t)).$$

We claim $F_{T(\Gamma_i)}(\gamma_i(t),\gamma_i'(t)) \leq F_{T(\Gamma_j)}(\gamma_j(t),\gamma_j'(t))$, $1 \leq i \leq n$.

$$F_{T(\Gamma_i)}(\gamma_i(t),\gamma_i'(t)) = F_{T(\Gamma_i)}\left(\frac{c_i}{c_j}\, t\, \frac{\bar{\varphi}_i}{|\varphi_i|}, \frac{c_i}{c_j}\, \frac{\bar{\varphi}_i}{|\varphi_i|}\right)$$

$$= \left\| \frac{\frac{c_i}{c_j}\, \frac{\bar{\varphi}_i}{|\varphi_i|}}{1-\left|\frac{c_i}{c_j}\, t\, \frac{\bar{\varphi}_i}{|\varphi_i|}\right|^2} \right\|_\infty = \text{ess sup}\, \frac{\left|\frac{c_i}{c_j}\, \frac{\bar{\varphi}_i}{|\varphi_i|}\right|}{1-\left|\frac{c_i}{c_j}\, t\, \frac{\bar{\varphi}_i}{|\varphi_i|}\right|^2}$$

$$= \text{ess sup}\, \frac{\left|\frac{c_i}{c_j}\right|}{1-\left|\frac{c_i}{c_j}\right|^2 t^2} \leq \text{ess sup}\, \frac{1}{1-t^2}$$

$$= \left\| \frac{\frac{c_j}{c_j}\, \frac{\bar{\varphi}_j}{|\varphi_j|}}{1-\left|\frac{c_j}{c_j} t \frac{\bar{\varphi}_j}{|\varphi_j|}\right|^2} \right\|_\infty = F_{T(\Gamma_j)}(\gamma_j(t),\gamma_j'(t)).$$

Hence $\max[F_{T(\Gamma_1)},\dots,F_{T(\Gamma_n)}] = F_{T(\Gamma_j)}$, and since these curves are extremal [25], we would have by the above strict inequality that

$$\tau_j = \max[\tau_1,\dots,\tau_n] < \inf \int \max[F_{T(\Gamma_1)},\dots,F_{T(\Gamma_n)}]$$

$$\leq \inf \int F_{T(\Gamma_j)} = \tau_j,$$

and hence a contradiction.

<u>Corollary</u>. If $h : T(\Gamma_1) \times\dots\times T(\Gamma_n) \to T(\tilde{\Gamma}_1) \times\dots\times T(\tilde{\Gamma}_n)$ is a holomorphic mapping between cross-products of Teichmüller spaces, then $F_{T(\tilde{\Gamma}_1)\times\dots\times T(\tilde{\Gamma}_n)}(h(x),h_*(\xi)) \leq F_{T(\Gamma_1)\times\dots\times T(\Gamma_n)}(x,\xi)$ for all (x,ξ) belonging to the tangent bundle of $T(\Gamma_1) \times\dots\times T(\Gamma_n)$. In particular, if h is biholomorphic, then

$$F_{T(\tilde{\Gamma}_1)\times\dots\times T(\tilde{\Gamma}_n)}(h(x),h_*(\xi)) = F_{T(\Gamma_1)\times\dots\times T(\Gamma_n)}(x,\xi).$$

<u>Proof</u>. By the existence of right translations which are isometric in the Finsler structure [16], one may assume $x = \Phi(0)$ and $h(x) = \tilde{\Phi}(0)$, where

$$\Phi : M(\Gamma_1) \times\dots\times M(\Gamma_n) \to T(\Gamma_1) \times\dots\times T(\Gamma_n)$$

and

$$\tilde{\Phi} : M(\tilde{\Gamma}_1) \times\dots\times M(\tilde{\Gamma}_n) \to T(\tilde{\Gamma}_1) \times\dots\times T(\tilde{\Gamma}_n)$$

are the natural projections. There exist holomorphic mappings $\gamma : \Delta \to T(\Gamma_1) \times\dots\times T(\Gamma_n)$ and $\tilde{\gamma} : \Delta \to T(\tilde{\Gamma}_1) \times\dots\times T(\tilde{\Gamma}_n)$ defined analogously as in the proof of the above theorem, which are isometries in the corresponding Finsler structures and $\gamma(0) = \Phi(0)$, $\gamma'(0)c = \xi$, $\tilde{\gamma}(0) = \tilde{\Phi}(0)$, $\tilde{\gamma}'(0)\tilde{c} = h_*(\xi)$. Hence $F_\Delta(0,c) = F_{T(\Gamma_1)\times\ldots\times T(\Gamma_n)}(\Phi(0),\xi)$ and $F_\Delta(0,\tilde{c}) = F_{T(\tilde{\Gamma}_1)\times\ldots\times T(\tilde{\Gamma}_n)}(\tilde{\Phi}(0),h_*(\xi))$. Since the Kobayashi metric on a cross-product T of Teichmüller spaces is the integrated form of F_T and h is holomorphic, $h\circ\gamma(\Delta) \subset \tilde{\gamma}(\Delta)$. Thus $\tilde{\gamma}^{-1}\circ h\circ\gamma : \Delta \to \Delta$, is holomorphic, and $\tilde{\gamma}^{-1}\circ h\circ\gamma(0) = 0$. Since F_Δ is known to be the differential form of the Poincaré metric on the unit disk [16], $F_\Delta(0,\tilde{c}) \leq F_\Delta(0,c)$ by Schwarz's Lemma [1], and hence the result.

§4. Cross-product tangent and cotangent spaces

From the previous theorem, one has that the max norm on the direct sum of tangent spaces over each point in the cross-product of Teichmüller spaces induces the Kobayashi metric on the cross-product. The following theorem describes the dual norm on the direct sum of cotangent spaces over each point in the cross-product.

Theorem. Let $B_1,\dots,B_n$ be finite dimensional complex Banach spaces each with norm $\|\cdot\|$. Let $B_1^*,\dots,B_n^*$ be their duals respectively. The dual of $\overset{n}{\underset{j=1}{\oplus}} B_j$ is then $\overset{n}{\underset{j=1}{\oplus}} B_j^*$. Let $\|\cdot\|^*$ be the induced norm on each B_j^*. Then the norm $\max(\|\cdot\|,\dots,\|\cdot\|)$ on $\overset{n}{\underset{j=1}{\oplus}} B_j$ induces the norm $\|\cdot\|^* + \dots + \|\cdot\|^*$ on $\overset{n}{\underset{j=1}{\oplus}} B_j^*$.

Proof. If B is a finite dimensional complex Banach space with norm $\|\cdot\|$, then by the induced norm on B^* we mean $\|X^*\|^* = \sup\limits_{\|x\|=1} |X^*(x)|$ where $X^* \in B^*$, $x \in B$.

Let $(X_1^*,\dots,X_n^*) \in B_1^* \oplus\dots\oplus B_n^*$ and $(x_1,\dots,x_n) \in B_1 \oplus\dots\oplus B_n$. Then $(X_1^*,\dots,X_n^*)(x_1,\dots,x_n) = X_1^*(x_1) + \dots + X_n^*(x_n)$. Hence $\|(X_1^*,\dots,X_n^*)\|^* = \sup\limits_{\max(\|x_1\|,\dots,\|x_n\|)=1} |X_1^*(x_1) + \dots + X_n^*(x_n)|$.

We now remark that the sup is unchanged if we restrict ourselves to those $(x_1,\dots,x_n)$ such that $\max(\|x_1\|,\dots,\|x_n\|) = 1$ and $X_j^*(x_j) \geq 0$ for $1 \leq j \leq n$. This is true because given an arbitrary $(x_1,\dots,x_n) \in B_1 \oplus\dots\oplus B_n$ such that $\max(\|x_1\|,\dots,\|x_n\|) = 1$, we can find θ_j such that $e^{i\theta_j} X_j^*(x_j) \geq 0$, $1 \leq j \leq n$. Now $X_j^*(e^{i\theta_j} x_j) = e^{i\theta_j} X_j^*(x_j)$ and $\|e^{i\theta_j} x_j\| = \|x_j\|$, then

$$|X_1^*(x_1) + \cdots + X_n^*(x_n)| \le |X_1^*(x_1)| + \cdots + |X_n^*(x_n)|$$
$$= |e^{i\theta_1} X_1^*(x_1)| + \cdots + |e^{i\theta_n} X_n^*(x_n)|$$
$$= |X_1^*(e^{i\theta_1} x_1)| + \cdots + |X_n^*(e^{i\theta_n} x_n)|$$
$$= X_1^*(e^{i\theta_1} x_1) + \cdots + X_n^*(e^{i\theta_n} x_n).$$

So

$$\|(X_1^*, \ldots, X_n^*)\|^* = \sup_{\substack{\max(\|x_1\|, \ldots, \|x_n\|)=1 \\ X_j^*(x_j) \ge 0, \text{ all } j}} |X_1^*(x_1) + \cdots + X_n^*(x_n)|$$

$$= \sup_{\substack{\max(\|x_1\|, \ldots, \|x_n\|)=1 \\ X_j^*(x_j) \ge 0, \text{ all } j}} X_1^*(x_1) + \cdots + X_n^*(x_n).$$

If we fix $(x_1, \ldots, x_n)$ such that $\max(\|x_1\|, \ldots, \|x_n\|) = 1$, $X_i^*(x_i) \ge 0$, all j, it is clear that

$$X_1^*(x_1) + \cdots + X_n^*(x_n) \le \sup_{\substack{\|x_1\|=1 \\ X_1^*(x_1) \ge 0}} X_1^*(x_1) + \cdots + \sup_{\substack{\|x_n\|=1 \\ X_n^*(x_n) \ge 0}} X_n^*(x_n).$$

Hence

$$\sup_{\substack{\max(\|x_1\|, \ldots, \|x_n\|)=1 \\ X_j^*(x_j) \ge 0, \text{ all } j}} (X_1^*(x_1) + \cdots + X_n^*(x_n))$$

$$\le \sup_{\substack{\|x_1\|=1 \\ X_1^*(x_1) \ge 0}} X_1^*(x_1) + \cdots + \sup_{\substack{\|x_n\|=1 \\ X_n^*(x_n) \ge 0}} X_n^*(x_n)$$

but not with strict inequality, for
$\sup(X_1^*(x_1) +\dots+ X_n^*(x_n)) < \sup X_1^*(x_1) +\dots+ \sup X_n^*(x_n)$ implies there exists $\tilde{x}_j \in B_j$ such that $\|\tilde{x}_j\| = 1$, $X_j^*(\tilde{x}_j) \geq 0$, $1 \leq j \leq n$, such that

$$\sup_{\substack{\max(\|x_1\|,\dots,\|x_n\|)=1 \\ X_j^*(x_j)\geq 0,\ \text{all } j}} (X_1^*(x_1) +\dots+ X_n^*(x_n))$$

$$< X_1^*(\tilde{x}_1) +\dots+ X_n^*(\tilde{x}_n),$$

a contradiction. Hence

$$\|(X_1^*,\dots,X_n^*)\| = \sup_{\substack{\max(\|x_1\|,\dots,\|x_n\|)=1 \\ X_j^*(x_j)\geq 0,\ \text{all } j}} (X_1^*(x_1) +\dots+ X_n^*(x_n))$$

$$= \sup_{\substack{\|x_1\|=1 \\ X_1^*(x_1)\geq 0}} X_1^*(x_1) +\dots+ \sup_{\substack{\|x_n\|=1 \\ X_n^*(x_n)\geq 0}} X_n^*(x_n)$$

$$= \|X_1^*\|^* +\dots+ \|X_n^*\|^*.$$

Since the norm on the tangent space over a point in Teichmüller space is the L^∞ norm as given by the Finsler structure, and the cotangent and tangent spaces are dual, one has that:

<u>Corollary</u>. The dual norm on the direct sum of cotangent spaces over a point in a cross-product of Teichmüller spaces is the sum of the L^1 norms on each cotangent space in the direct sum.

CHAPTER V

AUTOMORPHISMS OF CROSS-PRODUCTS OF TEICHMÜLLER SPACES

§1. Complex linear isometries between cotangent spaces

<u>Definition</u>. Let $x, x' \in \overset{n}{\underset{i=1}{\oplus}} B_i$, where the B_i are Banach spaces, $x = (x_1, \ldots, x_n)$, $x' = (x_1', \ldots, x_n')$. x is said to be perpendicular to x' (written $x \perp x'$) if for each i, either x_i or x_i' is zero.

<u>Lemma I</u>. Let $f : Q(X_1) \oplus \ldots \oplus Q(X_n) \to Q(Y_i) \oplus \ldots \oplus Q(Y_n)$ be a complex linear isometry between direct sums of the complex Banach spaces of holomorphic quadratic differentials on compact Riemann surfaces with (possibly) punctures X_i, Y_i in the norm $\|\cdot\| + \ldots + \|\cdot\|$, where $\|\varphi\| = \frac{1}{2} \int |\varphi|$, φ a holomorphic quadratic differential. Let $x = (x_1, \ldots, x_n)$ and $x' = (x_1', \ldots, x_n')$ be such that $x \perp x'$. Let

$$(x_1, \ldots, x_n) \overset{f}{\mapsto} (y_1, \ldots, y_n)$$

$$(x_1', \ldots, x_n') \overset{f}{\mapsto} (y_1', \ldots, y_n').$$

Then for each i, $1 \leq i \leq n$, one of the following holds.

i) $y_i = 0$,

ii) $y_i' = 0$, or

iii) $y_i' = r_i y_i$ for some $r_i > 0$.

<u>Remark</u>. It turns out by Lemma II that case (iii) never occurs.

Proof. Let $x = (x_1,\ldots,x_n)$ and $x' = (x_1',\ldots,x_n')$ be such that $x \perp x'$. Let

$$(x_1,\ldots,x_n) \overset{f}{\mapsto} (y_1,\ldots,y_n)$$

$$(x_1',\ldots,x_n') \overset{f}{\mapsto} (y_1',\ldots,y_n').$$

Since f is an isometry $\|x_1\| +\ldots+ \|x_n\| = \|y_1\| +\ldots+ \|y_n\|$ and $\|x_1'\| +\ldots+ \|x_n'\| = \|y_1'\| +\ldots+ \|y_n'\|$. Since f is linear, $(x_1+x_1',\ldots,x_n+x_n') \overset{f}{\mapsto} (y_1+y_1',\ldots,y_n+y_n')$ hence $\|x_1+x_1'\| +\ldots+ \|x_n+x_n'\| = \|y_1+y_1'\| +\ldots+ \|y_n+y_n'\|$. Now $\|x_1+x_1'\| +\ldots+ \|x_n+x_n'\| = \|x_1\| +\ldots+ \|x_n\| + \|x_1'\| +\ldots+ \|x_n'\|$ since $x \perp x'$. By substitution $\|y_1+y_1'\| +\ldots+ \|y_n+y_n'\| = \|y_1\| + \|y_1'\| +\ldots+ \|y_n\| + \|y_n'\|$. Since $\|y_i+y_i'\| \leq \|y_i\| + \|y_i'\|$ for norms, all i, it follows that $\|y_i+y_i'\| = \|y_i\| + \|y_i'\|$, all i. Let $y_i = \varphi_i$, $y_i' = \eta_i$, then equivalently

$$\tfrac{1}{2} \int_{Y_i} |\varphi_i+\eta_i| = \tfrac{1}{2} \int_{Y_i} |\varphi_i| + \tfrac{1}{2} \int_{Y_i} |\eta_i|, \text{ all } i$$

or $\int_{Y_i} |\varphi_i| + |\eta_i| - |\varphi_i+\eta_i| = 0$ which implies that $|\varphi_i+\eta_i| = |\varphi_i| + |\eta_i|$ a.e., since the integrand is non-negative by the triangle inequality. Now $|\varphi_i+\eta_i| = |\varphi_i| + |\eta_i|$ a.e. implies that $\frac{\eta_i}{\varphi_i}$ can take on a.e. (that is, a.e. in every parametric disk) only values that are non-negative or ∞.

Now $\frac{\eta_i}{\varphi_i}$ is a meromorphic function on a compact Riemann surface and hence must take on all values in $C \cup \{\infty\}$ unless it is constant [27]. Let $g_i = \frac{\eta_i}{\varphi_i} : Y_i \to C \cup \{\infty\}$, and assume non-constant for all i. Since the following argument applies for any i, we take just $g_1 = \frac{\eta_1}{\varphi_1} : Y_1 \to C \cup \{\infty\}$. Let D be a disk in $C \cup \{\infty\}$ not containing any points of the non-negative real axis or infinity. Let $P \in g_1^{-1}(D) \subset Y_1$. Choose a coordinate neighborhood $P \in U \subset g_1^{-1}(D)$, then meas$\Delta \neq 0$ (Δ a disk in C via the local coordinate ξ) and yet $g_1 \circ \xi^{-1}(\Delta)$ takes on different values than the above, a contradiction, since g_1 may take on values different than the above only on a set of measure zero. Therefore g_1 is constant which is either 0, ∞, or $r_1 > 0$. Thus we conclude that for each i, one of the following is true:

i) $\varphi_i = 0$,

ii) $\eta_i = 0$, or

iii) $\eta_i = r_i \varphi_i$ for some $r_i > 0$.

If Y_i is compact with punctures, then φ_i, η_i project to meromorphic quadratic differentials on the compact surface (by filling in the punctures), then the above implies the same conclusion.

<u>Lemma II.</u> Let $f : B_1 \oplus .. \oplus B_n \to B_1' \oplus .. \oplus B_n'$ be a complex linear isometry (where the B_i, B_i' are complex Banach spaces) in the norm $\|\cdot\| + ... + \|\cdot\|$, where $\|\cdot\|$ is a norm for each Banach space. Let $x = (x_1, ..., x_n)$ and $x' = (x_1', ..., x_n')$ be such that $x \perp x'$. Let

$$(x_1, ..., x_n) \overset{f}{\mapsto} (y_1, ..., y_n)$$
$$(x_1', ..., x_n') \overset{f}{\mapsto} (y_1', ..., y_n')$$

and suppose that $y_i = y_i'$ for some i, then $y_i = y_i' = 0$.

Proof. Let $x = (x_1,\ldots,x_n) \overset{f}{\mapsto} (y_1,\ldots,y_n)$

$$x' = (x_1',\ldots,x_n') \overset{f}{\mapsto} (y_1',\ldots,y_n')$$

be such that $x \perp x'$ and suppose $y_i = y_i'$ for some i. Since f is an isometry $\|x_1\| +\ldots+ \|x_n\| = \|y_1\| +\ldots+ \|y_n\|$ and $\|x_1'\| +\ldots+ \|x_n'\| = \|y_1'\| +\ldots+ \|y_n'\|$. Since f is linear $(x_1-x_1',\ldots,x_n-x_n') \overset{f}{\mapsto} (y_1-y_1',\ldots,y_n-y_n')$ and therefore $\|x_1-x_1'\| +\ldots+ \|x_n-x_n'\| = \|y_1-y_1'\| +\ldots+ \|y_n-y_n'\|$. The term $\|y_i-y_i'\| = 0$ by hypothesis. Now $\|x_1-x_1'\| +\ldots+ \|x_n-x_n'\| = \|x_1\| +\ldots+ \|x_n\| + \|x_1'\| +\ldots+ \|x_n'\|$ since $x \perp x'$.

By substitution,

$$\|y_1-y_1'\| +\ldots+ \|y_{i-1}-y_{i-1}'\| + \|y_{i+1}-y_{i+1}'\| +\ldots+ \|y_n-y_n'\|$$

$$= \|y_1\| +\ldots+ \|y_n\| + \|y_1'\| +\ldots+ \|y_n'\|$$

$$= 2\|y_i\| + (\|y_1\| + \|y_1'\| +\ldots+ \|y_{i-1}\| + \|y_{i-1}'\|$$

$$+ \|y_{i+1}\| + \|y_{i+1}'\| +\ldots+ \|y_n\| + \|y_n'\|).$$

Since by the triangle inequality, $\|y_j-y_j'\| \le \|y_j\| + \|y_j'\|$ all j, it follows that $\|y_i\| = 0$, thus $y_i = 0$. Hence $y_i = y_i' = 0$.

Theorem. Let $f : Q(X_1) \oplus\ldots\oplus Q(X_n) \to Q(Y_1) \oplus\ldots\oplus Q(Y_n)$ be a complex linear isometry between direct sums of the complex Banach spaces of holomorphic quadratic differentials on compact Riemann surfaces with (possibly) punctures X_i, Y_i, in the norm $\|\cdot\| +\dots+ \|\cdot\|$, where $\|\varphi\| = \frac{1}{2}\int |\varphi|$, φ a holomorphic quadratic differential. Then there exists a permutation $\sigma \in S_n$ (symmetric group on n letters), such that

$$f|_{Q(X_i)} : Q(X_i) \xrightarrow{\text{onto}} Q(Y_{\sigma(i)}), \quad 1 \le i \le n.$$

Proof. The proof is by induction on n.

$n = 1$: $f : Q(X_1) \to Q(Y_1)$. There is nothing to prove.

Assume that the theorem is true for $n = k-1$. We must prove that the theorem is true for $n = k$, that is, for $f : Q(X_1) \oplus\dots\oplus Q(X_k) \to Q(Y_1) \oplus\dots\oplus Q(Y_k)$. Let $0 \ne (x_1,0,\dots,0) \overset{f}{\mapsto} (y_1,\dots,y_k)$ be fixed and $(0,x_2',\dots,x_k') \overset{f}{\mapsto} (y_1',\dots,y_k')$ be arbitrary. (Note that if $Q(X_1) = \{0\}$, we cannot pick an $x_1 \ne 0$. In that case, choose any point $(0,\dots,x_i,0,\dots,0) \in \oplus Q(X_i)$, $x_i \ne 0$, otherwise $\underset{i}{\oplus} Q(X_i) = \{0\}$ and there is nothing to prove. Let $(x_1',x_2',\dots,0,x_{i+1}',\dots,x_k')$ be the other arbitrary point.)

Since $x_1 \ne 0$ and f is an isometry, then there exists a $y_i \ne 0$. By a permutation, say $y_1 \ne 0$, then by Lemma I either $y_1' = 0$ or $y_1' = r_1 y_1$, $r_1 > 0$. If $y_1' = r_1 y_1$, $r_1 > 0$, then by linearity,

$$(0,\frac{x_2'}{r_1},\dots,\frac{x_n'}{r_1}) \overset{f}{\mapsto} (y_1,\frac{y_2'}{r_1},\dots,\frac{y_k'}{r_1})$$

so that by Lemma II, $y_1 = 0$, which is a contradiction. Hence $y_1' = 0$. Since $(0,x_2',\dots,x_n')$ is arbitrary, we see that $f : (0,x_2',\dots,x_k') \mapsto (0,y_2',\dots,y_k')$ that is, $f|_{\text{rest.}} : Q(X_2) \oplus\dots\oplus Q(X_k) \to Q(Y_2) \oplus\dots\oplus Q(Y_k)$. We claim this mapping is onto, for if not, then there exist points

$(x_1, x_2, \ldots, x_k)$ $x_1 \neq 0$, $(0, y_2, \ldots, y_k)$ such that $f(x_1, x_2, \ldots, x_k) = (0, y_2, \ldots, y_k)$. Since f is onto and we have already assumed $Q(Y_1) \neq \{0\}$, then there exist points $(x_1', \ldots, x_k') \in \oplus_i Q(X_i)$, $(y_1, 0, \ldots, 0) \in \oplus Q(Y_i)$, $y_1 \neq 0$ such that $f(x_1', \ldots, x_k') = (y_1, 0, \ldots, 0)$. Since $(0, y_2, \ldots, y_k) \perp (y_1, 0, \ldots, 0)$, We can apply Lemma I to f^{-1} and get that $x_1' = 0$ or $x_1' = r_1 x_1$ since $x_1 \neq 0$. If $x_1' = r_1 x_1$, then $f^{-1}(\frac{y_1}{r_1}, 0, \ldots, 0) = (x_1, \frac{x_2'}{r_1}, \ldots, \frac{x_k'}{r_1})$ and thus by Lemma II applied to f^{-1}, $x_1 = 0$ a contradiction. Hence $x_1' = 0$ and thus $f(0, x_2', \ldots, x_k') = (y_1, 0, \ldots, 0)$. Since $y_1 \neq 0$ this, however, contradicts the above. Hence $f_{\text{rest.}}$ is onto. Now by induction, there exists a $\tau \in S_{k-1}$ such that

$f|_{Q(X_i)} : Q(X_i) \xrightarrow{\text{onto}} Q(Y_{\tau(i)})$, $2 \leq i \leq k$. We claim that $f|_{Q(X_1)} : Q(X_1) \xrightarrow{\text{onto}} Q(Y_1)$ and hence the theorem, since there would exist $\sigma \in S_k$ such that

$$f|_{Q(X_i)} : Q(X_i) \xrightarrow{\text{onto}} Q(Y_{\sigma(i)}), \ 1 \leq i \leq k.$$

For let $(x_1, \ldots, x_k)$ be arbitrary and let x_1 vary while $x_2, \ldots, x_k$ are fixed. By linearity, $f(x_1, \ldots, x_k) = f(x_1, 0, \ldots, 0) + f(0, x_2, \ldots, x_k)$. It is clear that $f(0, x_2, \ldots, x_k)$ remains invariant as x_1 is allowed to vary. We will show that $f(x_1, 0, \ldots, 0) = (y_1, 0, \ldots, 0)$ and hence it follows that $f(x_1, \ldots, x_k)$ only varies in the first component of the image direct sum as x_1 is allowed to to vary which implies the claim. So suppose there exists some $y_i \neq 0$, $2 \leq i \leq k$. Then since $f|_{rest.}$ is onto, there exists a point $(0,x_2',\ldots,x_k')$ such that $f(0,x_2',\ldots,x_k') = (0,\ldots,y_i,0,\ldots,0)$. But then by Lemma II, $y_i = 0$, a contradiction. Thus $f(x_1,0,\ldots,0) = (y_1,0,\ldots,0)$.

<u>Theorem</u>. Let $f : Q(X_1) \oplus\ldots\oplus Q(X_m) \to Q(Y_1) \oplus\ldots\oplus Q(Y_n)$ where $m > n$ and f and the $Q(X_i)$, $Q(Y_i)$ are as in the previous theorem. Then at least m-n of the $Q(X_i)$ are $\{0\}$, and if we eliminate m-n trivial spaces from $\overset{m}{\underset{i=1}{\oplus}} Q(X_i)$, and possibly renumber the remaining spaces, then

$f : Q(X_1) \oplus\ldots\oplus Q(X_n) \xrightarrow{\text{onto}} Q(Y_1) \oplus\ldots\oplus Q(Y_n)$ and splits componentwise up to a permutation (as in the previous theorem).

<u>Proof</u>. Add m-n trivial spaces $\{0\}$ onto $\overset{n}{\underset{i=1}{\oplus}} Q(Y_i)$. Then $f : Q(X_1) \oplus\ldots\oplus Q(X_m) \to Q(Y_1) \oplus\ldots\oplus Q(Y_n) \oplus \{0\} \oplus\ldots\oplus \{0\}$. Since f is a complex linear and we now have the same number of components in each direct sum, the previous theorem implies that there exists a $\sigma \in S_m$ such that

$f|_{Q(X_i)} : Q(X_i) \xrightarrow{\text{onto}} Q(Y_{\sigma(i)})$, $1 \leq i \leq m$. Since we have added on m-n trivial spaces, there must exist at least m-n trivial spaces among $\overset{m}{\underset{i=1}{\oplus}} Q(X_i)$. Eliminate precisely m-n of them. Then after a possible renumbering of the remaining spaces, we have $f : Q(X_1) \oplus\ldots\oplus Q(X_n) \to Q(Y_1) \oplus\ldots\oplus Q(Y_n)$ and the previous theorem implies the rest.

§2. The Modular group of a cross-product

Let $T(\Gamma_1) \times \dots \times T(\Gamma_n)$ be a cross-product of Teichmüller spaces of the Fuchsian groups Γ_i, $1 \leq i \leq n$. We define the Modular group of this cross-product to be $\mathrm{Mod}(\Gamma_1) \times \dots \times \mathrm{Mod}(\Gamma_n)$ where $\mathrm{Mod}(\Gamma_i)$ is the Modular group of $T(\Gamma_i)$, $1 \leq i \leq n$, (see Chapter III, §1.). Let $x = (x_1, \dots, x_n) \in T(\Gamma_1) \times \dots \times T(\Gamma_n)$ and $\theta = (\theta_1, \dots, \theta_n) \in \mathrm{Mod}(\Gamma_1) \times \dots \times \mathrm{Mod}(\Gamma_n)$. Then $\theta(x) = (\theta_1(x_1), \dots, \theta_n(x_n)) \in T(\Gamma_1) \times \dots \times T(\Gamma_n)$.

Proposition. The Modular group acts on the cross-product $T(\Gamma_1) \times \dots \times T(\Gamma_n)$:

i) as a group of biholomorphic automorphisms,

ii) as a group of isometries in the Kobayashi metric,

iii) as a properly discontinuous group,

iv) effectively, if type $\Gamma_i = (g_i, n_i)$ satisfies $2g_i + n_i > 4$, all i.

Proof. i) Since $\mathrm{Mod}(\Gamma_1) \times \dots \times \mathrm{Mod}(\Gamma_n)$ acts componentwise on the cross-product, and it is known that $\mathrm{Mod}(\Gamma_i)$ is a group of biholomorphic automorphisms of $T(\Gamma_i)$ (see Chapter I, §5.) the result follows.

ii) Since the cross-product metric is the Kobayashi metric and from i) each element of $\mathrm{Mod}(\Gamma_1) \times \dots \times \mathrm{Mod}(\Gamma_n)$ acts biholomorphically, it follows that each such element is an isometry (see Chapter II, §2.)

iii) Since $\mathrm{Mod}(\Gamma_i)$ acts as a properly discontinuous group on $T(\Gamma_i)$ (see Chapter I, §5.), it is clear from the definition of proper discontinuity and the action of $\mathrm{Mod}(\Gamma_1) \times\ldots\times \mathrm{Mod}(\Gamma_n)$ on the cross-product that $\mathrm{Mod}(\Gamma_1) \times\ldots\times \mathrm{Mod}(\Gamma_n)$ acts properly discontinuously.

iv) If type $\Gamma_i = (g_i, n_i)$ satisfies $2g_i + n_i > 4$, then $\mathrm{Mod}(\Gamma_i)$ acts effectively on $T(\Gamma_i)$ (see Chapter I, §5.). If $\mathrm{Mod}(\Gamma_1) \times\ldots\times \mathrm{Mod}(\Gamma_n)$ were not effective, then there would exist elements $\theta = (\theta_1, \ldots, \theta_n) \in \mathrm{Mod}(\Gamma_1) \times\ldots\times \mathrm{Mod}(\Gamma_n)$ and $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathrm{Mod}(\Gamma_1) \times\ldots\times \mathrm{Mod}(\Gamma_n)$, $\theta \neq \gamma$ such that $\theta(x) = \gamma(x)$ for each $x = (x_1, \ldots, x_n) \in T(\Gamma_1) \times\ldots\times T(\Gamma_n)$. Since the elements of $\mathrm{Mod}(\Gamma_1) \times\ldots\times \mathrm{Mod}(\Gamma_n)$ act componentwise, $\theta \neq \gamma$ implies there exists an i such that $\theta_i \neq \gamma_i$ and $\theta(x) = \gamma(x)$ implies $\theta_i(x_i) = \gamma_i(x_i)$ which together contradict the fact that $\mathrm{Mod}(\Gamma_i)$ acts effectively on $T(\Gamma_i)$.

Corollary. If type $\Gamma_i = (g_i, n_i)$ satisfies $2g_i + n_i > 4$, all i, then statements i) and iv) from the above Proposition implies that

$$\mathrm{Mod}(\Gamma_1) \times\ldots\times \mathrm{Mod}(\Gamma_n) \subseteq \mathrm{Aut}\, T(\Gamma_1) \times\ldots\times T(\Gamma_n).$$

Lemma. Let M be a complex analytic manifold. Let G be a (properly) discontinuous group of biholomorphic self-mappings of M. Assume there exists an $x_0 \in M$ such that $\gamma(x_0) = x_0$ for some element $\gamma \in G$ implies that $\gamma = \mathrm{id}$. Let θ be any biholomorphic self-mapping of M. If for each $x \in M$ there exists an element $\gamma_x \in G$ such that $\theta(x) = \gamma_x(x)$, then $\theta \in G$.

Proof. [13] Let $x_0 \in M$ be as in the hypothesis and $\gamma_{x_0} \in G$

such that $\theta(x_0) = \gamma_{x_0}(x_0)$. Let $\gamma = \gamma_{x_0}$. We will show that $f = \gamma^{-1} \circ \theta = id$ on M so that $\theta = \gamma$ and hence $\theta \in G$.

Since f is a biholomorphic self-mapping of M and G is a (properly) discontinuous group, there exists a neighborhood U of x_0 in M such that $\gamma(U) \cap U = \phi$ and $\gamma(f(U)) \cap f(U) = \phi$ for all $\gamma \in G-\{id\}$. If $x \in U \cap f(U)$, then $f(x) = x$; otherwise there would exist an element $\beta \in G-\{id\}$ such that $\beta(f(x)) = x$ (by hypothesis). But then $\beta(f(U)) \cap f(U) \neq \phi$, a contradiction. Hence $f_{|U \cap f(U)} = id$ and clearly $U \cap f(U) \neq \phi$ since $x_0 \in U \cap f(U)$. Since U is an open subset of M, $f = id$ on M by the identity theorem for holomorphic functions on complex analytic manifolds, and hence the result.

<u>Corollary I.</u> Let $M = T(\Gamma)$, the Teichmüller space of the Fuchsian group Γ, where type $\Gamma = (g,n)$ satisfies $2g+n > 4$ (that is, non-exceptional). Let $G = Mod(\Gamma)$, the Modular group. Then the Lemma applies.

<u>Proof.</u> It is well-known that $Mod(\Gamma)$ operates properly discontinuous on $T(\Gamma)$ [11], and that if type $\Gamma = (g,n)$ satisfies $2g+n > 4$ that there exists a $x \in T(\Gamma)$ such that $\gamma(x) = x$, for a $\gamma \in Mod(\Gamma)$ implies that $\gamma = id$ [24].

<u>Corollary II.</u> Let $M = T(\Gamma_1) \times \ldots \times T(\Gamma_n)$, a cross-product of Teichmüller spaces of Fuchsian groups Γ_i, $1 \leq i \leq n$, where type $\Gamma_i = (g_i, n_i)$ satisfies $2g_i+n_i > 4$, all i. Let $G = Mod(\Gamma_1) \times \ldots \times Mod(\Gamma_n)$, the Modular group. Then the Lemma applies.

<u>Proof.</u> We have stated that $\mathrm{Mod}(\Gamma_1) \times \dots \times \mathrm{Mod}(\Gamma_n)$ operates properly discontinuously on $T(\Gamma_1) \times \dots \times T(\Gamma_n)$ (see §2.). Since $2g_i+n_i > 4$, all i, there exists an $x_i \in T(\Gamma_i)$ such that $\gamma_i(x_i) = x_i$ for some $\gamma_i \in \mathrm{Mod}(\Gamma_i)$ implies that $\gamma_i = \mathrm{id}$, all i. Hence, clearly $x = (x_1, \dots, x_n) \in T(\Gamma_1) \times \dots \times T(\Gamma_n)$ has the property that $\gamma(x) = x$ for some $\gamma \in \mathrm{Mod}(\Gamma_1) \times \dots \times \mathrm{Mod}(\Gamma_n)$ implies that $\gamma = \mathrm{id}$.

§3. Automorphisms of cross-products

<u>Theorem</u>. Suppose $\Gamma_1,\ldots,\Gamma_n$ are Fuchsian groups which have no elliptic elements and the type $\Gamma_i = (g_i,n_i)$ satisfies $2g_i+n_i > 4$, all i (i.e., Γ_i is non-exceptional). Then Aut $T(\Gamma_1)$ x...x $T(\Gamma_n)$ is the semidirect product of $Mod(\Gamma_1)$ x...x $Mod(\Gamma_n)$ by the finite subgroup H, where H is defined as follows: Let $N = \{(i,j)/1 \leq i < j \leq n$ and type Γ_i = type $\Gamma_j\}$. For each $(i,j) \in N$ define $f_{ij} : T(\Gamma_i) \to T(\Gamma_j)$ to be a chosen biholomorphic mapping (such mappings exist between Teichmüller spaces of groups of the same type [12]). For each $(i,j) \in N$, let $h_{ij} = (k_1,\ldots,k_n)$, where $k_r = id$ if $r \neq i,j$, $k_i = f_{ij}$, and $k_j = f_{ij}^{-1}$. Then H is the subgroup generated by the elements h_{ij}.

<u>Lemma</u>. Let the product $Mod(\Gamma_1)$ x...x $Mod(\Gamma_n)$ by H (as above) be the set of all elements of the form $g \circ h$, where $g \in Mod(\Gamma_i)$ x...x $Mod(\Gamma_n)$ and $h \in H$. Then the product $Mod(\Gamma_i)$ x...x $Mod(\Gamma_n)$ by H is a group under composition and operates properly discontinuously on $T(\Gamma_1)$ x...x $T(\Gamma_n)$. Furthermore, there exists a point $x \in T(\Gamma_1)$ x...x $T(\Gamma_n)$ such that $\theta(x) = x$ for some element θ belonging to the product $Mod(\Gamma_1)$ x...x $Mod(\Gamma_n)$ by H implies that $\theta = id$.

<u>Proof</u>. Clearly the identity element belongs to the set and the associative property holds. We need to show that every element has an inverse and the closure property. We first

remark that if $g \in \mathrm{Mod}(\Gamma_1) \times \ldots \times \mathrm{Mod}(\Gamma_n)$ and $h \in H$ are arbitrary, then $h \circ g \circ h^{-1} \in \mathrm{Mod}(\Gamma_1) \times \ldots \times \mathrm{Mod}(\Gamma_n)$ since h permutes the factors of $T(\Gamma_1) \times \ldots \times T(\Gamma_n)$, $g \in \mathrm{Aut}\, T(\Gamma_1) \times \ldots \times \mathrm{Aut}\, T(\Gamma_n)$ hence $h \circ g \circ h^{-1} \in \mathrm{Aut}\, T(\Gamma_1) \times \ldots \times \mathrm{Aut}\, T(\Gamma_n) = \mathrm{Mod}(\Gamma_1) \times \ldots \times \mathrm{Mod}(\Gamma_n)$ (since $\mathrm{Aut}\, T(\Gamma_i) = \mathrm{Mod}(\Gamma_i)$; Royden [25]). Thus if $g_1 \circ h_1$ and $g_2 \circ h_2$ are any two elements, $(g_1 \circ h_1) \circ (g_2 \circ h_2) = (h_1 \circ g_3) \circ (g_2 \circ h_2) = (h_1 \circ g_4) \circ h_2 = (g_5 \circ h_1) \circ h_2 = g_5 \circ h_3$, where the $g_i \in \mathrm{Mod}(\Gamma_1) \times \ldots \times \mathrm{Mod}(\Gamma_n)$ and the $h_i \in H$. If $g \circ h$ belongs to the product, then let $(g \circ h)^{-1} = (h^{-1} \circ g^{-1} \circ h) \circ h^{-1}$, and $(h^{-1} \circ g^{-1} \circ h) \circ h^{-1}$ belongs to the product by the above remark.

By definition, The group H is clearly finite, hence $\mathrm{Mod}(\Gamma_1) \times \ldots \times \mathrm{Mod}(\Gamma_n)$ is of finite index in the product group $\mathrm{Mod}(\Gamma_1) \times \ldots \times \mathrm{Mod}(\Gamma_n)$ by H and since $\mathrm{Mod}(\Gamma_1) \times \ldots \times \mathrm{Mod}(\Gamma_n)$ operates properly discontinuously on $T(\Gamma_1) \times \ldots \times T(\Gamma_n)$, it follows that the product $\mathrm{Mod}(\Gamma_1) \times \ldots \times \mathrm{Mod}(\Gamma_n)$ by H operates properly discontinuously on $T(\Gamma_1) \times \ldots \times T(\Gamma_n)$.

Next we claim there exists a point $x \in T(\Gamma_1) \times \ldots \times T(\Gamma_n)$ such that $\theta(x) = x$ for some element $\theta \in \mathrm{Mod}(\Gamma_1) \times \ldots \times \mathrm{Mod}(\Gamma_n)$ by H implies that $\theta = \mathrm{id}$, that is, x is not fixed by a non-trivial element. Suppose not, then let $x_0 \in T(\Gamma_1) \times \ldots \times T(\Gamma_n)$ be arbitrarily chosen. By proper discontinuity, there exists only a finite number of non-trivial elements $\theta_1, \ldots, \theta_n$ $(n \geq 1)$ belonging to the product $\mathrm{Mod}(\Gamma_1) \times \ldots \times \mathrm{Mod}(\Gamma_n)$ by H such that $\theta_i(x_0) = x_0$, $\theta_i \neq \mathrm{id}$, and a neighborhood $x \in U \subset T(\Gamma_1) \times \ldots \times T(\Gamma_n)$ such that $\theta_i(U) = U$, $1 \leq i \leq n$. Since $\theta_1 \neq \mathrm{id}$, there must exist a subset $D_1 \subseteq U$ dense in U such that θ_1 has no fixed points in D_1, otherwise by continuity and the identity theorem for holomorphic functions on complex manifolds, $\theta_1 = id$. Similarly there must exist a subset D_2 of D_1 which is dense in U such that θ_2 has no fixed points in D_2. Completing the process, we arrive at D_n a dense subset of U such that θ_n has no fixed points in D_n. Since $D_n \subseteq D_i$, $1 \leq i \leq n$, $D_n \neq \phi$, any $x \in D_n$ has the property that $\theta_i(x) \neq x$ all i, and since all $\theta \in (\text{Mod}(\Gamma_1) \times \ldots \times \text{Mod}(\Gamma_n))\, H - \{\theta_1, \ldots, \theta_n, id\}$ are such that $\theta(U) \cap U = \phi$, it follows that x is not fixed by a non-trivial element, hence a contradiction.

Proof. (of Theorem) From the Corollary at the end of the last section and the definition of H, it is clear that the product $\text{Mod}(\Gamma_1) \times \ldots \times \text{Mod}(\Gamma_n)$ by H is contained in Aut $T(\Gamma_1) \times \ldots \times T(\Gamma_n)$. Hence it suffices to prove containment in the other direction.

Let $h \in \text{Aut}\, T(\Gamma_1) \times \ldots \times T(\Gamma_n)$. Since the Kobayashi metric on $T(\Gamma_1) \times \ldots \times T(\Gamma_n)$ is induced by a Finsler structure on the tangent space invariant under biholomorphic mappings (see Chapter IV, § 3.), h induces a complex linear isometry between the tangent spaces (in the max norm given by the Finsler structure on the cross-product, see Chapter IV, § 3.) over each pair of points x and h(x) in $T(\Gamma_1) \times \ldots \times T(\Gamma_n)$. Since the cotangent space is dual to the tangent space with dual norm, the induced mapping h^* between cotangent spaces over x and h(x) is a complex linear isometry in the dual norm, which by a previous theorem is the sum of L^1 norms (see Chapter V, § 4.). Hence by a previous theorem (see Chapter V, § 1.) h^* splits componentwise up to a permutation, that is, $h^*_{Q(X_i)} : Q(X_i) \to Q(Y_{\sigma(i)})$ for some $\sigma \in S_n$. Since $\Gamma_1,\ldots,\Gamma_n$ are assumed to be of non-exceptional types, this implies by a theorem of Earle and Kra (see Chapter III, §3.) that Γ_i and $\Gamma_{\sigma(i)}$ are of the same type (g_i,n_i). Hence by definition of H, there exists an element $g \in H$ such that $g_{|T(\Gamma_i)} : T(\Gamma_i) \to T(\Gamma_{\sigma(i)})$ all i, and hence the induced mapping $g_{|Q(Y'_{\sigma(i)})} : Q(Y'_{\sigma(i)}) \to Q(X'_i)$ all i. Then one has that $(h\circ g)^*_{|Q(X_i)} : Q(X_i) \to Q(X'_i)$ all i. By the results of Royden, generalization of Earle and Kra (see Chapter III, §3.), X_i is conformally equivalent to X'_i, for all i. This implies the existence of $\Psi_i \in \mathrm{Mod}(\Gamma_i)$ such that $h\circ g(x) = (\Psi_1(x_1),\ldots,\Psi_n(x_n))$, where $x = (x_1,\ldots,x_n) \in T(\Gamma_1) \times\ldots\times T(\Gamma_n)$ (see Chapter I, §5.), or $h(x) = \Psi\circ g^{-1}(x)$, $\Psi \in \mathrm{Mod}(\Gamma_1) \times\ldots\times \mathrm{Mod}(\Gamma_n)$, $g^{-1} \in H$. Since Since $x \in T(\Gamma_1) \times\ldots\times T(\Gamma_n)$ is arbitrary, there exists an element $\theta \in (\mathrm{Mod}(\Gamma_1) \times\ldots\times \mathrm{Mod}(\Gamma_n))\, H$ such that $h(x) = \theta(x)$ for all x (see above Lemma and Chapter V, §2.). Hence $h \in \mathrm{Aut}\, T(\Gamma_1) \times\ldots\times T(\Gamma_n)$ implies that $h \in (\mathrm{Mod}(\Gamma_1) \times\ldots\times \mathrm{Mod}(\Gamma_n))\, H$.

We only have left to show that $\mathrm{Mod}(\Gamma_1) \times\ldots\times \mathrm{Mod}(\Gamma_n)$ is a normal subgroup of $\mathrm{Aut}\, T(\Gamma_1) \times\ldots\times T(\Gamma_n)$. It suffices to show $f(\mathrm{Mod}(\Gamma_1) \times\ldots\times \mathrm{Mod}(\Gamma_n))f^{-1} \subseteq \mathrm{Mod}(\Gamma_1) \times\ldots\times \mathrm{Mod}(\Gamma_n)$, because by using inverses, $f(\mathrm{Mod}(\Gamma_1) \times\ldots\times \mathrm{Mod}(\Gamma_n))f^{-1} \supseteq \mathrm{Mod}(\Gamma_1) \times\ldots\times \mathrm{Mod}(\Gamma_n)$ as well and hence the claim. Let $f \in \mathrm{Aut}\, T(\Gamma_1) \times\ldots\times T(\Gamma_1)$ and $g \in \mathrm{Mod}(\Gamma_1) \times\ldots\times \mathrm{Mod}(\Gamma_n)$. If $f \in H$, then $f\circ g\circ f^{-1}$ maps each component $T(\Gamma_i)$ onto itself. Thus $f\circ g\circ f^{-1} \in \mathrm{Mod}(\Gamma_1)\times\ldots\times \mathrm{Mod}(\Gamma_n)$ (since $\mathrm{Aut}\, T(\Gamma_i) = \mathrm{Mod}(\Gamma_i)$; Royden [25]). Then if $f \in \mathrm{Aut}\, T(\Gamma_1) \times\ldots\times T(\Gamma_n)$ is arbitrary, then $f = f_2\circ f_1$ where

$f_1 \in H$, $f_2 \in \mathrm{Mod}(\Gamma_1) \times \ldots \times \mathrm{Mod}(\Gamma_n)$ and $f \circ g \circ f^{-1} = f_2 \circ (f_1 \circ g \circ f_1^{-1}) \circ f_2^{-1} = f_2 \circ g_1 \circ f_2^{-1} \in \mathrm{Mod}(\Gamma_1) \times \ldots \times \mathrm{Mod}(\Gamma_n)$ since g_1, $f_2 \in \mathrm{Mod}(\Gamma_1) \times \ldots \times \mathrm{Mod}(\Gamma_n)$. By definition of H, clearly $(\mathrm{Mod}(\Gamma_1) \times \ldots \times \mathrm{Mod}(\Gamma_n)) \cap H = \{id\}$. Hence Aut $T(\Gamma_1) \times \ldots \times T(\Gamma_n)$ is the semidirect product of $\mathrm{Mod}(\Gamma_1) \times \ldots \times \mathrm{Mod}(\Gamma_n)$ by H.

Theorem. Suppose $\Gamma_1, \ldots, \Gamma_n$ are Fuchsian groups which have no elliptic elements and type Γ_i is not (1,1) or (0,4), all i. Then Aut $T(\Gamma_1) \times \ldots \times T(\Gamma_n)$ is the semidirect product of Aut $T(\Gamma_1) \times \ldots \times$ Aut $T(\Gamma_n)$ by the finite subgroup H, where H is defined as follows: Let $N = \{(i,j)/1 \leq i < j \leq n$ and $\exists\ f_{ij} : T(\Gamma_i) \to T(\Gamma_j)$ biholomorphic$\}$. For each $(i,j) \in N$, let $h_{ij} = (k_1, \ldots, k_n)$, where $k_r = id$ if $r \neq i,j$, $k_i = f_{ij}$, and $k_j = f_{ij}^{-1}$. Then H is the subgroup generated by the h_{ij}.

Proof. Clearly the semidirect product of Aut $T(\Gamma_1) \times \ldots \times$ Aut $T(\Gamma_n)$ by H is contained in Aut $T(\Gamma_1) \times \ldots \times T(\Gamma_n)$. Hence we need to prove only containment in the other direction. We remark that if type $(\Gamma_i) = (0,3)$ for some i, then $T(\Gamma_i)$ = pt. since $\dim_C T(\Gamma_i) = 0$. Thus Aut $T(\Gamma_i) = \{id\}$ and neither effects nor contributes anything to the result.

If type (Γ_i) is exceptional, not (0,3), and by hypothesis not (1,1) or (0,4), then there exists a Fuchsian group $\tilde{\Gamma}_i$ of non-exceptional type such that $T(\Gamma_i)$ is biholomorphically equivalent to $T(\tilde{\Gamma}_i)$ [12]. It is then obvious that one can replace the original cross-product $T(\Gamma_1) \times \ldots \times T(\Gamma_n)$ by the biholomorphically equivalent cross-product $T(\tilde{\Gamma}_1) \times \ldots \times T(\tilde{\Gamma}_n)$, where $\Gamma_i = \tilde{\Gamma}_i$ if type Γ_i is non-exceptional or (0,3), via a biholomorphic mapping $f : T(\Gamma_1) \times \ldots \times T(\Gamma_n) \to T(\tilde{\Gamma}_1) \times \ldots \times T(\tilde{\Gamma}_n)$ which splits componentwise, that is $f_{|T(\Gamma_i)} : T(\Gamma_i) \to T(\tilde{\Gamma}_i)$, $1 \le i \le n$. Let $h \in \text{Aut}\ T(\Gamma_1) \times \ldots \times T(\Gamma_n)$. Then $g = f \circ h \circ f^{-1} : T(\tilde{\Gamma}_1) \times \ldots \times T(\tilde{\Gamma}_n) \to T(\tilde{\Gamma}_1) \times \ldots \times T(\tilde{\Gamma}_n)$ is biholomorphic, and by the previous theorem and the triviality of type (0,3) (if it appears) g splits componentwise up to a permutation, that is, there exists a $\sigma \in S_n$ such that $g_{|T(\tilde{\Gamma}_i)} : T(\tilde{\Gamma}_i) \to T(\tilde{\Gamma}_{\sigma(i)})$. Since f splits componentwise, this implies that $h_{|T(\Gamma_i)} : T(\Gamma_i) \to T(\Gamma_{\sigma(i)})$. Since h evidently permutes the factors, and $\tilde{H}$ by definition is a group of transpositions of the factors, there exists $g \in \tilde{H}$ such that $\theta = h \circ g^{-1} \in \text{Aut}\ T(\Gamma_1) \times \ldots \times \text{Aut}\ T(\Gamma_n)$. Hence $h = \theta \circ g$ where $\theta \in \text{Aut}\ T(\Gamma_1) \times \ldots \times \text{Aut}\ T(\Gamma_n)$ and $g \in \tilde{H}$.

Analogous to the previous theorem, $\text{Aut}\ T(\Gamma_1) \times \ldots \times \text{Aut}\ T(\Gamma_n)$ is a normal subgroup of $\text{Aut}\ T(\Gamma_1) \times \ldots \times T(\Gamma_n)$. Clearly $\text{Aut}\ T(\Gamma_1) \times \ldots \times \text{Aut}\ T(\Gamma_n) \cap \tilde{H} = \{id\}$. Hence $\text{Aut}\ T(\Gamma_1) \times \ldots \times T(\Gamma_n)$ is the semidirect product of $\text{Aut}\ T(\Gamma_1) \times \ldots \times \text{Aut}\ T(\Gamma_n)$ by $\tilde{H}$.

<u>Remark.</u> Since Teichmüller spaces of Fuchsian groups of the same type are biholomorphically equivalent, Bers-Greenberg [12] and Teichmüller spaces of Fuchsian groups of distinct types are biholomorphically equivalent if and only if their respective types are (2,0) and (0,6), (1,2) and (0,5), or (1,1) and (0,4), Patterson [23], one has that $(i,j) \in \tilde{N}$ implies that either

type (Γ_i) = type (Γ_j) or of types (2,0) and (0,6), or (1,2) and (0,5). The remaining possibility was excluded by hypothesis.

<u>Conjecture.</u> The above theorem holds if one allows type (1,1) and type (0,4) to occur. (Note: If the theorem holds allowing one of these types, it also holds for both.)

We combine the above theorem with the following deep result of Royden [25] and generalization by Earle and Kra [14]:

Let $T(g,n) = T(\Gamma)$, $\mathrm{Mod}(g,n) = \mathrm{Mod}(\Gamma)$ where type $(\Gamma) = (g,n)$ then

Aut $T(g,n) = \mathrm{Mod}(g,n)$ if type $(\Gamma) = (g,n)$ is non-exceptional, that is, satisfies $2g+n > 4$.

Aut $T(2,0) \cong \mathrm{Mod}(0,6) \cong \mathrm{Mod}(2,0)/Z_2$

Aut $T(1,2) \cong \mathrm{Mod}(0,5)$

Aut $T(1,1) \cong$ Aut $T(0,4) \cong \mathrm{M\ddot{o}b}_R$ (since $T(1,1)$ and $T(0,4)$ are conformally equivalent to U.)

Aut $T(0,3) = \{id\}$

($\mathrm{M\ddot{o}b}_R = \{z \mapsto \frac{az+b}{cz+d} \mid ad-bc = 1,\ a,b,c,d \in R\}$ = Aut U.)

<u>Corollary I.</u> If type (Γ_i) is non-exceptional, then Aut $T(\Gamma_i) = \mathrm{Mod}(\Gamma_i)$ and $\tilde{H} = H$ (by above remark). Hence the above theorem is a generalization of the corresponding theorem for non-exceptional types.

<u>Corollary II.</u> Using the results of Royden, Earle and Kra, and

the above theorem, one explicitly knows

Aut $T(\Gamma_1) \times \ldots \times T(\Gamma_n)$, type $(\Gamma_i) \neq (1,1)$ or $(0,4)$ all i.

§4. Biholomorphic mappings between cross-products.

Theorem. Let $h : T(\Gamma_1) \times \ldots \times T(\Gamma_m) \to T(\tilde{\Gamma}_1) \times \ldots \times T(\tilde{\Gamma}_n)$, $m \geq n$ be a biholomorphic mapping between cross-products of Teichmüller spaces. Then at least m-n of the $T(\Gamma_i)$ are trivial, and if we eliminate m-n trivial spaces and possibly renumber the remaining spaces, then

$h : T(\Gamma_1) \times \ldots \times T(\Gamma_n) \to T(\tilde{\Gamma}_1) \times \ldots \times T(\tilde{\Gamma}_n)$.

Proof. Let $h : T(\Gamma_1) \times \ldots \times T(\Gamma_m) \to T(\tilde{\Gamma}_1) \times \ldots \times T(\tilde{\Gamma}_n)$, $m \geq n$ be biholomorphic. Then for each pair of points $x = (x_1, \ldots, x_m) \in T(\Gamma_1) \times \ldots \times T(\Gamma_m)$ and $h(x) \in T(\tilde{\Gamma}_1) \times \ldots \times T(\tilde{\Gamma}_n)$ there is an induced complex linear isometry $f : Q(X_1) \oplus \ldots \oplus Q(X_m) \to Q(Y_1) \oplus \ldots \oplus Q(Y_n)$ on the corresponding cotangent spaces over x and h(x). If $m > n$, then there exist at least m-n $Q(X_i) = \{0\}$ by a previous theorem (see Chapter V, §1). Since $\dim_C T(\Gamma_i) = \dim_C Q(X_i)$, the result follows. By eliminating m-n trivial spaces and possibly renumbering the remaining spaces, then $h : T(\Gamma_1) \times \ldots \times T(\Gamma_n) \to T(\tilde{\Gamma}_1) \times \ldots \times T(\tilde{\Gamma}_n)$. If $m = n$, the statement holds vacuously.

Corollary I. There does not exist a biholomorphic mapping between an n-fold product of Teichmüller spaces and an m-fold product, unless the number of non-trivial spaces in each product is the same. If the number of non-trivial spaces is the same and in addition we assume all non-trivial types non-exceptional in both cross-products, then a biholomorphic mapping exists if and only if the non-trivial types in one product are precisely those appearing among the non-trivial types of the other (counting multiplicities).

Remark. Exceptional types, if any, must appear in a prescribed manner.

Proof. The first statement follows directly from the theorem. Assume that the number of non-trivial spaces in each product are the same and each non-trivial type is also non-exceptional in both cross-products. If the non-trivial types appearing in one product are precisely those appearing in the other (counting multiplicities), clearly there exists a biholomorphic mapping between products, since Teichmüller spaces of the same type are biholomorphically equivalent [12].

Conversely, if h is biholomorphic under the above assumptions, the induced mapping on cotangent spaces over corresponding points splits componentwise up to a permutation (see Chapter V, §1), and using the result of Earle and Kra (see Chapter III, §3), the non-trivial types in one product of Teichmüller spaces are precisely those in the other product (counting multiplicities).

Corollary II. A cross-product of non-trivial Teichmüller spaces is never a Teichmüller space.

Proof. Follows directly from Theorem.

Corollary III. If $h : T(\Gamma_1) \times \ldots \times T(\Gamma_n) \to T(\tilde{\Gamma}_1) \times \ldots \times T(\tilde{\Gamma}_n)$ is biholomorphic and Γ_i, $\tilde{\Gamma}_i$ are non-exceptional, then h splits componentwise up to a permutation, that is, there exists a $\sigma \in S_n$ such that $h|_{T(\Gamma_i)} : T(\Gamma_i) \xrightarrow{\text{onto}} T(\tilde{\Gamma}_{\sigma(i)})$, all i.

Proof. By the above theorem, $\exists\ \sigma \in S_n$ such that type Γ_i = type $\tilde{\Gamma}_{\sigma(i)}$. Since Teichmüller spaces of the same type are biholomorphically equivalent, there exists

$$f : T(\tilde{\Gamma}_1) \times \ldots \times T(\tilde{\Gamma}_n) \to T(\Gamma_1) \times \ldots \times T(\Gamma_n)$$

biholomorphic, f splitting componentwise up to a permutation. Then $f \circ h : T(\Gamma_1) \times \ldots \times T(\Gamma_n) \to T(\Gamma_1) \times \ldots \times T(\Gamma_n)$, so that by a previous theorem (see Chapter V, § 3), $f \circ h$ belongs to the semidirect product of $\mathrm{Mod}(\Gamma_1) \times \ldots \times \mathrm{Mod}(\Gamma_n)$ by H. Therefore, $f \circ h$ splits componentwise up to a permutation, and hence h splits componentwise up to a permutation.

CHAPTER VI

THE DEFORMATION SPACE OF A KLEINIAN GROUP*

§1. The Deformation space

Let G be a subgroup of M, the group of fractional linear transformations of the Riemann sphere $C \cup \{\infty\}$, that is, $M = \{z \mapsto \frac{az+b}{cz+d} / a,b,c,d \in C,\ ad-bc = 1\}$. Let $\Omega(G) \subseteq C \cup \{\infty\}$ denote the <u>region of (proper) discontinuity</u> ($z \in \Omega(G) \Leftrightarrow G$ is (properly) discontinuous at z; see Chapter III, §1). Let $\Lambda(G) = C \cup \{\infty\} - \Omega(G)$, called the <u>limit set</u> of G. G is called a <u>Kleinian</u> group if $\Omega(G) \neq \phi$. From now on we will assume G is a Kleinian group which is finitely generated and non-elementary, that is, $\Lambda(G)$ contains more than two points.

Let f be a quasiconformal automorphism of $C \cup \{\infty\}$. f is said to be <u>compatible</u> with the group G if fGf^{-1} is again a Kleinian group and f is conformal on $\Lambda(G)$. (It is not known if this latter condition is automatically satisfied by all finitely generated Kleinian groups.) f is called <u>normalized</u> if f fixes 0,1 and ∞. One defines an equivalence relation on the set of normalized compatible quasiconformal mappings as follows: f_1 is equivalent to f_2 if $f_1(z) = f_2(z)$ for all $z \in \Lambda(G)$. It is not difficult to show that f_1 is equivalent to f_2 if and only if $f_2^{-1} \circ f_1$ is the identity conjugation of G onto itself [19].

*<u>Remark</u>. This is essentially an exposition of the Deformation space of a Kleinian group as defined in [19].

The Deformation space of the Kleinian group G is the set of equivalence classes of normalized compatible quasiconformal automorphisms of $C \cup \{\infty\}$ and will be denoted by $T(G,\Omega)$, where $\Omega = \Omega(G)$. Let Δ be a connected component of the region of discontinuity Ω of the group G. Let G denote the stability subgroup of Δ, that is, $G_\Delta = \{ g \in G/g(\Delta) = \Delta\}$. Ω is the union of the connected components Δ_i. Two components Δ_1 and Δ_2 are called equivalent if there exists an element $g \in G$ such that $g(\Delta_1) = \Delta_2$. Let $\{\Delta_i\}$ be a maximal collection of non-equivalent components of Ω. Since G is finitely generated, and non-elementary, a theorem of Ahlfors [3] implies the collection is finite, $\Delta_1, \dots \Delta_n$. Let $G_{\Delta_1}, \dots, G_{\Delta_n}$ be the corresponding stability subgroups. By Ahlfors' theorem each Δ_i/G_{Δ_i} is a compact Riemann surface with possibly finitely many points deleted.

Analogously, there is defined a Deformation space of G_Δ, denoted by $T(G_\Delta, \Delta)$ of equivalence classes of normalized compatible quasiconformal deformations of G_Δ, except the deformations f are required to be conformal on $C \cup \{\infty\} - \Delta$.

Let Δ be a component of Ω. Since G is non-elementary, there exists a holomorphic universal covering mapping $h : U \to \Delta$ (U the upper half plane). Let Γ be the group of conformal automorphisms of U such that $\gamma \in \Gamma$ if there exists an element $\chi(\gamma) \in G$ such that $h \circ \gamma = \chi(\gamma) \circ h$. There is an exact sequence of Kleinian groups

$$\{1\} \longrightarrow H \xrightarrow{inj} \Gamma \longrightarrow G_\Delta \longrightarrow \{1\},$$

where H is the covering group of $h : U \to \Delta$. Γ is Fuchsian and U/Γ is conformally equivalent to Δ/G_Δ. Γ is called the <u>Fuchsian</u> <u>model</u> of G over Δ [19]. Since Γ is determined by h, which is unique up to a fractional linear transformation with real coefficients, Γ is determined uniquely up to conjugation by a fractional linear transformation with real coefficients.

§2. Isomorphism Theorems

Let G be a finitely generated non-elementary Kleinian group. Let $\Delta_1,\ldots,\Delta_n$ be a maximal collection of non-equivalent components of the region of discontinuity Ω. Let $G_{\Delta_1},\ldots,G_{\Delta_n}$ be the corresponding stability subgroups. Then

<u>Theorem.</u> (Kra) $T(G,\Omega) \cong T(G_{\Delta_1},\Delta_1) \times\ldots\times T(G_{\Delta_n},\Delta_n)$ (biholomorphically).

<u>Proof.</u> See the literature [19].

From the previous section, there exist holomorphic universal covering mappings $h_j : U \to \Delta_j$ and Fuchsian models $\Gamma_1,\ldots,\Gamma_n$. Then

<u>Theorem.</u> (Maskit) $T(\Gamma_1) \times\ldots\times T(\Gamma_n)$ is the holomorphic universal covering space of $T(G,\Omega)$ with holomorphic covering group

$$\mathrm{Mod}_0(\Gamma_1) \times\ldots\times \mathrm{Mod}_0(\Gamma_n) \subset \mathrm{Mod}(\Gamma_1) \times\ldots\times \mathrm{Mod}(\Gamma_n)$$

for some subgroups $\mathrm{Mod}_0(\Gamma_i) \subset \mathrm{Mod}(\Gamma_i)$. Hence

$$T(G,\Omega) \cong \frac{T(\Gamma_1) \times\ldots\times T(\Gamma_n)}{\mathrm{Mod}_0(\Gamma_1) \times\ldots\times \mathrm{Mod}_0(\Gamma_n)} = \frac{T(\Gamma_1)}{\mathrm{Mod}_0(\Gamma_1)} \times\ldots\times \frac{T(\Gamma_n)}{\mathrm{Mod}_0(\Gamma_n)}.$$

<u>Proof.</u> See the literature [20].

<u>Corollary I.</u> $T(G,\Omega)$ is a complex analytic manifold.

<u>Proof.</u> $T(\Gamma_i)$ is a complex analytic manifold and $\mathrm{Mod}_0(\Gamma_i)$ is a properly discontinuous fixed point free group of

biholomorphic automorphisms, all i.

Corollary II. $T(G, \Omega)$ is simply connected if each component Δ_i is simply connected.

Proof. Δ_i simply connected implies $Mod_0(\Gamma_i) = \{id\}$ (this will be proven in the next section) and $T(\Gamma_i)$ is simply connected, all i.

§3. The Modular group

Let G be a finitely generated non-elementary Kleinian group with region of discontinuity Ω. Let $\Delta_1, \ldots, \Delta_n$ be a maximal collection of non-equivalent components of Ω. Let $G_{\Delta_1}, \ldots, G_{\Delta_n}$ be the corresponding stability subgroups. Let Γ_i, $1 \leq i \leq n$ be the Fuchsian model of G over Δ_i. From now on $T(G,\Omega)$, via the isomorphism (see §2), is

$$\frac{T(\Gamma_1)}{\mathrm{Mod}_0(\Gamma_1)} \times \cdots \times \frac{T(\Gamma_n)}{\mathrm{Mod}_0(\Gamma_n)} .$$

For each component Δ_i, let $h_i : U \to \Delta_i$ denote the holomorphic universal covering mapping with holomorphic covering group H_i. It is clear from the definition of Γ_i that H_i is a normal subgroup of Γ_i. Let $\mathrm{Mod}^{H_i}(\Gamma_i)$ denote the subgroup of $\mathrm{Mod}(\Gamma_i)$ induced by all $w : U \to U$ quasiconformal such that $w\Gamma_i w^{-1} = \Gamma_i$ and $wH_iw^{-1} = H_i$. Each such w induces a quasiconformal automorphism $f : \Delta_i \to \Delta_i$ such that $f \circ h_i = h_i \circ w$ and $fG_{\Delta_i}f^{-1} = G_{\Delta_i}$. Further, every quasiconformal automorphism $f : \Delta_i \to \Delta_i$ such that $fG_{\Delta_i}f^{-1} = G_{\Delta_i}$ is so induced (and unique up to an element $h \in H_i$). Let $\mathrm{Mod}_{H_i}(\Gamma_i)$ denote the subgroup of $\mathrm{Mod}^{H_i}(\Gamma_i)$ induced by all $w : U \to U$ quasiconformal such that $w\Gamma_i w^{-1} = \Gamma_i$, $wH_iw^{-1} = H_i$, and $w \circ \gamma \circ w^{-1} \circ \gamma^{-1} \in H_i$, all $\gamma \in \Gamma_i$. The latter condition is equivalent to $f \circ g \circ f^{-1} = g$ for all $g \in G_\Delta$, where f is the induced quasiconformal mapping of Δ_i. We remark that $\mathrm{Mod}_0(\Gamma_i) = \mathrm{Mod}_{H_i}(\Gamma_i)$, all i in the previous theorem (see §2). If Δ_i is simply connected, then $H_i = \{id\}$ since $H_i \cong \pi_1(\Delta_i)$. Then w induces an element of

$\mathrm{Mod}_{\{id\}}(\Gamma_i)$ if $w \circ \gamma \circ w^{-1} \circ \gamma^{-1} = id$, all $\gamma \in \Gamma_i$, that is, $w \circ \gamma \circ w^{-1} = \gamma$, all $\gamma \in \Gamma_i$. Such a w clearly induces the identity element of $\mathrm{Mod}(\Gamma_i)$. Hence Δ_i simply connected implies $\mathrm{Mod}_{\{id\}}(\Gamma_i) = \{id\}$. Thus if all components are simply connected, $T(G,\Omega)$ is $T(\Gamma_1) \times \ldots \times T(\Gamma_n)$.

We define the Modular group $\mathrm{Mod}\,(G,\Omega)$* to be

$$\frac{\mathrm{Mod}^{H_1}(\Gamma_1) \times \ldots \times \mathrm{Mod}^{H_n}(\Gamma_n)}{\mathrm{Mod}_{H_1}(\Gamma_1) \times \ldots \times \mathrm{Mod}_{H_n}(\Gamma_n)} = \frac{\mathrm{Mod}^{H_1}(\Gamma_1)}{\mathrm{Mod}_{H_1}(\Gamma_1)} \times \ldots \times \frac{\mathrm{Mod}^{H_n}(\Gamma_n)}{\mathrm{Mod}_{H_n}(\Gamma_n)}.$$

This definition is consistent with the previously defined Modular group for a product of Teichmüller spaces since if all components are simply connected, then $H_i = \{id\}$ all i, and since $\mathrm{Mod}^{\{id\}}(\Gamma_i) = \mathrm{Mod}(\Gamma_i)$, $\mathrm{Mod}_{\{id\}}(\Gamma_i) = \{id\}$ all i, the Modular group reduces to $\mathrm{Mod}(\Gamma_1) \times \ldots \times \mathrm{Mod}(\Gamma_n)$.

$\mathrm{Mod}(G,\Omega)$ acts properly discontinuously on $T(G,\Omega)$ as a group of biholomorphic automorphisms [9]. Since $\mathrm{Mod}(\Gamma_1) \times \ldots \times \mathrm{Mod}(\Gamma_n)$ acts effectively on $T(\Gamma_1) \times \ldots \times T(\Gamma_n)$ if type $\Gamma_i = (g_i, n_i)$ satisfies $2g_i + n_i > 4$, all i, it is clear that under these conditions, $\mathrm{Mod}(G,\Omega)$ acts effectively on $T(G,\Omega)$. In the next section we define the Kobayashi metric on $T(G,\Omega)$ from which it follows that $\mathrm{Mod}(G,\Omega)$ acts as a group of isometries of $T(G,\Omega)$ in this metric.

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* Remark. This is not the definition of the Modular group as defined in [19], but rather that in [20].

§4. The Kobayashi metric

Since $T(G,\Omega)$ is a complex analytic manifold, we define the metric on $T(G,\Omega)$ to be the Kobayashi metric, consistent with the metrics previously defined on the Teichmüller space and cross-product of Teichmüller spaces as special cases. Since $T(G,\Omega)$ has $T(\Gamma_1) \times \ldots \times T(\Gamma_n)$ as a holomorphic covering space (see §2), more precise information about the Kobayashi metric on $T(G,\Omega)$ is known from the following general theorem.

<u>Theorem</u>. (Kobayashi) Let M be a complex manifold and $\tilde{M}$ a covering with projection mapping $\pi : \tilde{M} \to M$. Let $p,q \in M$ and $\tilde{p},\tilde{q} \in \tilde{M}$ such that $\pi(\tilde{p}) = p$ and $\pi(\tilde{q}) = q$. Let $k_{\tilde{M}}$ and k_M be respectively the Kobayashi metrics on $\tilde{M}$, M. Then

$$k_M(p,q) = \inf_{\tilde{q}} k_{\tilde{M}}(\tilde{p},\tilde{q})$$

where the infimum is taken over all $\tilde{q} \in \tilde{M}$ such that $q = \pi(\tilde{q})$.

<u>Proof</u>. See the literature [17].

If $M = T(G,\Omega)$ and $\tilde{M} = T(\Gamma_1) \times \ldots \times T(\Gamma_n)$, we obtain the result that the Kobayashi metric on $T(G,\Omega)$ is the metric induced from the Kobayashi metric on the covering space $T(\Gamma_1) \times \ldots \times T(\Gamma_n)$ via the holomorphic covering mapping.

§5. Teichmüller metric = Kobayashi metric

Let $T(G,\Omega)$ be the Deformation space of the Kleinian group G with region of discontinuity Ω. Then a point in $T(G,\Omega)$ is an equivalence class, denoted $[f]$, of quasiconformal deformations of G (see §1). The Teichmüller metric $\tau_{T(G,\Omega)}$ on $T(G,\Omega)$ is defined as follows:

$$\tau_{T(G,\Omega)}([f],[g]) = \inf_{\substack{\tilde{f}\in[f]\\ \tilde{g}\in[g]}} \log K(\tilde{f}\circ\tilde{g}^{-1})$$

(see Chapter II, §1). If Δ is a connected component of Ω and G_Δ is the subgroup of G which fixes Δ, that is, the stability subgroup of Δ, one has defined analogously a Teichmüller metric $\tau_{T(G_\Delta,\Delta)}$ on $T(G_\Delta,\Delta)$, the Deformation space of G_Δ. Let $k_{T(G,\Omega)}$ (respectively $k_{T(G_\Delta,\Delta)}$) denote the Kobayashi metric on $T(G,\Omega)$ (respectively $T(G_\Delta,\Delta)$). We prove that $\tau_{T(G,\Omega)} = k_{T(G,\Omega)}$.

We recall that $T(G,\Omega) \cong T(G_{\Delta_1},\Delta_1) \times\ldots\times T(G_{\Delta_n},\Delta_n)$ and since $T(G_{\Delta_i}\Delta_i) \cong T(\Gamma_i)/\mathrm{Mod}_{H_i}(\Gamma_i)$ that

$$T(G,\Omega) \cong \frac{T(\Gamma_1) \times\ldots\times T(\Gamma_n)}{\mathrm{Mod}_{H_1}(\Gamma_1) \times\ldots\times \mathrm{Mod}_{H_n}(\Gamma_n)}$$

(see §2, [19], and [20]).

The isomorphism $T(G,\Omega) \cong T(G_{\Delta_1},\Delta_1) \times\ldots\times T(G_{\Delta_n},\Delta_n)$ is obtained as follows: Let $[f] \in T(G,\Omega)$ and $u_f(z) = \dfrac{f_{\bar{z}}(z)}{f_z(z)}$. Define $\tilde{u}_i(z) = \begin{cases} u_f(z), & \text{if } z \in g^{-1}(\Delta_i) \text{ for some } g \in G. \\ 0 & \text{otherwise.}\end{cases}$

Let $\tilde{f}_i$ be the corresponding unique normalized Beltrami equation solution, that is, $\tilde{f}_{i_{\bar{z}}} = \tilde{u}_i \tilde{f}_{i_z}$. Then the mapping $\psi : f \mapsto (\tilde{f}_1,\dots,\tilde{f}_n)$ which projects $\psi : [f] \mapsto ([\tilde{f}_1],\dots,[\tilde{f}_n])$ is the isomorphism (for details, see [18]).

Let $h_i : U \to \Delta_i$ be a holomorphic covering mapping for the component Δ_i. Let $M(G_{\Delta_i})$ (respectively $M(\Gamma_i)$) denote the unit ball in $L^\infty(G_{\Delta_i})$ (respectively $L^\infty(\Gamma_i)$) (see Chapter III, §2). Then there is an induced mapping $h_{i_*} : M(\Gamma_i) \to M(G_{\Delta_i})$ defined by

$$(h_{i_*} u_i)(h_i(z)) = u_i(z) \frac{\overline{h_i'(z)}}{h_i'(z)}, \text{for } z \in U$$

which is a linear, norm preserving, surjective isomorphism. The mapping h_{i_*} projects to a holomorphic surjective mapping $h_{i_*} : T(\Gamma_i) \to T(G_{\Delta_i},\Delta_i)$ which is a holomorphic covering with covering group $\mathrm{Mod}_{H_i}(\Gamma_i) \subseteq \mathrm{Mod}(\Gamma_i)$, so that $T(G_{\Delta_i},\Delta_i) \cong T(\Gamma_i)/\mathrm{Mod}_{H_i}(\Gamma_i)$ [20]. Hence if $h_* = (h_{1_*},\dots,h_{n_*})$, then $h_* : T(\Gamma_1) \times\dots\times T(\Gamma_n) \to T(G_{\Delta_1},\Delta_1) \times\dots\times T(G_{\Delta_n},\Delta_n) \cong T(G,\Omega)$ is the universal holomorphic covering space of $T(G,\Omega)$ (the cross-product is simply connected), with covering group $\mathrm{Mod}_{H_1}(\Gamma_1) \times\dots\times \mathrm{Mod}_{H_n}(\Gamma_n)$ so that

$$T(G,\Omega) \cong \frac{T(\Gamma_1) \times\dots\times T(\Gamma_n)}{\mathrm{Mod}_{H_1}(\Gamma_1) \times\dots\times \mathrm{Mod}_{H_n}(\Gamma_n)} .$$

<u>Theorem.</u> Let $[x], [y] \in T(G_{\Delta_i},\Delta_i)$, where $x,y \in T(\Gamma_i)$ (via the isomorphism). Then

$$\tau_{T(G_{\Delta_i},\Delta_i)}[x],[y] = \inf_{\substack{\tilde{x}\in[x]\\ \tilde{y}\in[y]}} \tau_{T(\Gamma_i)}(\tilde{x},\tilde{y}), \text{ each } i.$$

Proof. Let $h_{i_*} : M(\Gamma_i) \to M(G_{\Delta_i})$ be as above. For $u_i \in M(\Gamma_i)$, let f_i be the unique normalized solution to $f_{i_{\bar{z}}} = u_i f_{i_z}$. Then the mapping h_{i_*} can be equivalently viewed as a mapping $h_{i_*} : f_i \to h_{i_*}(f_i)$, where $h_{i_*}(f_i)$ is the unique normalized solution to the Beltrami equation $g_{\bar{z}} = h_{i_*}(u_i) g_z$. Let $x, y \in T(\Gamma_i)$, and $f_i \in x$, $g_i \in y$. Then $\log K(f_i \circ g_i^{-1}) = \log K(h_{i_*}(f_i) \circ h_{i_*}^{-1}(g_i))$ since $h_{i_*} : M(\Gamma_i) \to M(G_{\Delta_i})$ is isometric. Hence

$$\tau_{T(\Gamma_i)}(x,y) = \inf_{\substack{f_i \in x \\ g_i \in y}} \log K(f_i \circ g_i^{-1}) = \inf_{\substack{f_i \in x \\ g_i \in y}} \log K(h_{i_*}(f_i) \circ h_*^{-1}(g_i))$$

$$\geq \tau_{T(G_{\Delta_i}, \Delta_i)}([x],[y]).$$

The above inequality comes from the fact that $\tilde{x} \in [x] \in T(G_{\Delta_i}, \Delta_i)$ (respectively $\tilde{y} \in [y] \in T(G_{\Delta_i}, \Delta_i)$) need not imply that there exist $f_i \in x \in T(\Gamma_i)$ (respectively $g_i \in y \in T(\Gamma_i)$) such that $h_{i_*}(f_i) = \tilde{x}$ (respectively $h_{i_*}(g_i) = \tilde{y}$). Thus

$$\tau_{T(G_{\Delta_i}, \Delta_i)}([x],[y]) \leq \inf_{\substack{\tilde{x} \in [x] \\ \tilde{y} \in [y]}} \tau_{T(\Gamma_i)}(\tilde{x}, \tilde{y}).$$

However, from the definition and properties of $\mathrm{Mod}_{H_i}(\Gamma_i)$ (see §3), it is not difficult to see that there exists $f_i \in \theta_1(x) \in T(\Gamma_i)$ and $g_i \in \theta_2(y) \in T(\Gamma_i)$, where $\theta_1, \theta_2 \in \mathrm{Mod}_{H_i}(\Gamma_i)$, if and only if $h_{i_*}(f_i) = \tilde{x}$ and $h_{i_*}(g_i) = \tilde{y}$. Hence

$$\tau_{T(G_{\Delta_i},\Delta_i)} = \inf_{\substack{\tilde{x}\in[x] \\ \tilde{y}\in[y]}} \tau_{T(\Gamma_i)}(\tilde{x},\tilde{y}), \text{ each } i.$$

It follows directly from the isomorphism $T(G,\Omega) \cong T(G_{\Delta_1},\Delta_1) \times\ldots\times T(G_{\Delta_n},\Delta_n)$, that $\tau_{T(G,\Omega)} = \max_i \{\tau_{T(G_{\Delta_i},\Delta_i)}\}$. With this in mind, we define the Teichmüller metric on $T(\Gamma_1) \times\ldots\times T(\Gamma_n)$ to be $\tau_{T(\Gamma_1)\times\ldots\times T(\Gamma_n)} = \max_i \{\tau_{T(\Gamma_i)}\}$. Let $h_* = (h_{1*},\ldots,h_{n*})$. Then

$$h_* : T(\Gamma_1) \times\ldots\times T(\Gamma_n) \to T(G_{\Delta_1},\Delta_1) \times\ldots\times T(G_{\Delta_n},\Delta_n) \cong T(G,\Omega)$$

is the universal holomorphic covering mapping, and by a similar argument as above,

$$\tau_{T(G,\Omega)} = \inf \tau_{T(\Gamma_1)\times\ldots\times T(\Gamma_n)} = \inf \max_i \{\tau_{T(\Gamma_i)}\},$$

where the infimum is taken over all elements equivalent under $\mathrm{Mod}_{H_1}(\Gamma_1) \times\ldots\times \mathrm{Mod}_{H_n}(\Gamma_n)$.

<u>Theorem</u>. $\tau_{T(G,\Omega)} = k_{T(G,\Omega)}$, where τ is the Teichmüller metric on $T(G,\Omega)$, k the Kobayashi metric on $T(G,\Omega)$.

<u>Proof</u>. From above, $\tau_{T(G,\Omega)} = \inf \max_i \{\tau_{T(\Gamma_i)}\}$. Now from a previous theorem $k_{T(\Gamma_i)\times\ldots\times T(\Gamma_n)} = \max_i \{\tau_{T(\Gamma_i)}\}$ (see Chapter IV, §2). Also $k_{T(G,\Omega)} = \inf k_{T(\Gamma_1)\times\ldots\times T(\Gamma_n)}$ where the infimum is taken over all elements equivalent under $\mathrm{Mod}_{H_1}(\Gamma_1) \times\ldots\times \mathrm{Mod}_{H_n}(\Gamma_n)$ (see Chapter V, §4). Hence

$$\tau_{T(G,\Omega)} = \inf \max_i \{\tau_{T(\Gamma_i)}\} = \inf k_{T(\Gamma_1) \times\ldots\times T(\Gamma_n)}$$

$$= k_{T(G,\Omega)}.$$

<u>Corollary.</u> Every biholomorphic mapping $h : T(G_1, \Omega_1) \to T(G_2, \Omega_2)$ is an isometry in the Teichmüller metric. In particular if $G_1 = G_2$, every automorphism is an isometry.

CHAPTER VII

AUTOMORPHISMS OF THE DEFORMATION SPACE OF A KLEINIAN GROUP

§1. Automorphisms of the Deformation space

Let G be a finitely generated non-elementary Kleinian group with region of discontinuity Ω. Let $\Delta_1,\ldots,\Delta_n$ be a maximal collection of non-equivalent components of Ω. Let $G_{\Delta_1},\ldots,G_{\Delta_n}$ be the corresponding stability subgroups. Then $T(G,\Omega) \cong T(G_{\Delta_1},\Delta_1) \times\ldots\times T(G_{\Delta_n},\Delta_n)$ (biholomorphically; see Chapter VI, §2).

<u>Definition</u>. We call a Kleinian group G <u>exceptional</u> if for its Fuchsian model $\Gamma_1,\ldots,\Gamma_n$, some Γ_i is of exceptional type.

<u>Theorem</u>. Let G be a finitely generated non-elementary, non-exceptional Kleinian group. Let $h : T(G,\Omega) \to T(G,\Omega)$ be a biholomorphic self-mapping of $T(G,\Omega)$. Then there exists a permutation $\sigma \in S_n$ such that $h|T(G_{\Delta_i},\Delta_i) : T(G_{\Delta_i},\Delta_i) \xrightarrow{\text{onto}} T(G_{\Delta_{\sigma(i)}},\Delta_{\sigma(i)})$ biholomorphically, $1 \leq i \leq n$. (h acts on the product via the isomorphism.)

<u>Proof</u>. The universal covering space of $T(G,\Omega)$ is $T(\Gamma_1) \times\ldots\times T(\Gamma_n)$ for some Fuchsian groups $\Gamma_1,\ldots,\Gamma_n$ (see Chapter VI, §2). The holomorphic covering mapping is $\pi = \pi_1 \times\ldots\times \pi_n$ where $\pi_i : T(\Gamma_i) \to T(G_{\Delta_i},\Delta_i)$ is the holomorphic

covering mapping for each component, $1 \leq i \leq n$. Let $h : T(G,\Omega) \to T(G,\Omega)$ be a biholomorphic self-mapping of $T(G,\Omega)$. Let $\tilde{h} : T(\Gamma_1) \times \ldots \times T(\Gamma_n) \to T(\Gamma_1) \times \ldots \times T(\Gamma_n)$ be a lifting of h to the universal covering space. Then by the theorem on cross-products of Teichmüller spaces (see Chapter V, §3), there exists a permutation $\sigma \in S_n$ such that $\tilde{h}|_{T(\Gamma_i)} : T(\Gamma_i) \xrightarrow{\text{onto}} T(\Gamma_{\sigma(i)})$ biholomorphically, $1 \leq i \leq n$. Since π is of the form $\pi = \pi_1 \times \ldots \times \pi_n$, $\pi_i : T(\Gamma_i) \to T(G_{\Delta_i},\Delta_i)$ it follows easily that

$h|_{T(G_{\Delta_i},\Delta_i)} : T(G_{\Delta_i},\Delta_i) \xrightarrow{\text{onto}} T(G_{\Delta_{\sigma(i)}},\Delta_{\sigma(i)})$ biholomorphically.

Let

$$T(G,\Omega) = \frac{T(\Gamma_1) \times \ldots \times T(\Gamma_n)}{\mathrm{Mod}_{H_1}(\Gamma_1) \times \ldots \times \mathrm{Mod}_{H_n}(\Gamma_n)}$$

be the deformation space, via the isomorphism (see §2) of the finitely generated non-elementary Kleinian group G. Let

$$\mathrm{Mod}(G,\Omega) = \frac{\mathrm{Mod}^{H_1}(\Gamma_1) \times \ldots \times \mathrm{Mod}^{H_n}(\Gamma_n)}{\mathrm{Mod}_{H_1}(\Gamma_1) \times \ldots \times \mathrm{Mod}_{H_n}(\Gamma_n)}$$

be the associated Modular group acting on $T(G,\Omega)$. Then

Theorem. Let G be a finitely generated non-elementary, non-exceptional Kleinian group with non-equivalent components $\Delta_1,\ldots,\Delta_n$ and Fuchsian model $\Gamma_1,\ldots,\Gamma_n$. Assume, in addition, that G does not contain elliptic elements and that Δ_i is simply connected for all i, $1 \leq i \leq n$. Then Aut $T(G,\Omega)$ is the semidirect product of $\mathrm{Mod}(G,\Omega)$ by the subgroup H, where

H is defined as in Chapter V, §3.

Proof. We have shown before that Δ_i simply connected implies that $H_i = \{id\}$ which in turn implies that $\mathrm{Mod}_{\{id\}}(\Gamma_i) = \{id\}$ for $1 \leq i \leq n$ (see Chapter VI, §3). Hence $T(G,\Omega) = T(\Gamma_1) \times \ldots \times T(\Gamma_n)$, a cross-product of Teichmüller spaces, where $\Gamma_1,\ldots,\Gamma_n$ is the Fuchsian model of G. Since G does not contain elliptic elements, Γ_i has no elliptic elements, $1 \leq i \leq n$. Also G non-exceptional by definition means that each Γ_i is of non-exceptional type. Hence the theorem on cross-products of Teichmüller spaces (see Chapter V, §3) yields the result.

Remark. The requirement that G not contain elliptic elements is necessary. If G contains elliptic elements then, in general, one has $\mathrm{Mod}(G,\Omega)H \subsetneq \mathrm{Aut}\ T(G,\Omega)$ (properly contained) already. We give a simple example to illustrate. Let Γ be a finitely generated Fuchsian group with signature $(2,2;2,3)$, that is U/Γ is a compact Riemann surface with two distinguished points p_1 and p_2 such that if $z_1, z_2 \in U$ are such that $\pi(z_i) = p_i$, $i = 1,2$, where $\pi : U \to U/\Gamma$ is the projection mapping, then there are elliptic elements $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1(z_1) = z_1$, $v_1 = \mathrm{order}\langle\gamma_1\rangle = 2$; $\gamma_2(z_2) = z_2$, $v_2 = \mathrm{order}\langle\gamma_2\rangle = 3$. Now $\mathrm{Mod}(\Gamma) \subsetneq \mathrm{Aut}\ T(\Gamma)$ because every quasiconformal self-mapping f of U/Γ which leaves the set $\{p_1,p_2\}$ fixed induces an element of $\mathrm{Aut}\ T(\Gamma)$, but only those f which send every

p_i into a p_j with $v_i = v_j$, $i,j = 1,2$ correspond to elements in $\mathrm{Mod}(\Gamma)$. $\Omega = C-R$, then $T(\Gamma,\Omega) = T(\Gamma) \times T(\Gamma)$. Let $h \in \mathrm{Aut}\, T(\Gamma)$, $h \notin \mathrm{Mod}(\Gamma)$. Then the mapping $h \times h \in \mathrm{Aut}\, T(\Gamma,\Omega)$, but $h \times h \notin \mathrm{Mod}(G,\Omega)H$. The same argument applies unchanged for a whole collection of finitely generated Fuchsian groups Γ with signature $(g,n;v_1,\ldots,v_n)$ such that $g \geq 2$, $n \geq 2$, $2 \leq v_i \leq \infty$ and $v_i < v_j < \infty$ for at least two indices i and j.

In the following theorem we drop the requirement that the components Δ_i of G be simply connected. Then

<u>Theorem</u>. Let G be a finitely generated non-elementary, non-exceptional Kleinian group. Assume that G does not contain elliptic elements. Then $\mathrm{Aut}\, T(G,\Omega)$ is the semidirect product of $\mathrm{Mod}(G,\Omega)$ by K, where $K = \{h \in \mathrm{Aut}\, T(G,\Omega)/h$ lifts to an $\tilde{h} \in H\}$ (H previously defined in Chapter V, §3), if and only if the normalizer of $\mathrm{Mod}_{H_i}(\Gamma_i)$ in $\mathrm{Mod}(\Gamma_i)$ is $\mathrm{Mod}^{H_i}(\Gamma_i)$, $1 \leq i \leq n$. (For notation, see Chapter VI, §3.)

<u>Remark</u>. Since $\mathrm{Mod}_{H_i}(\Gamma_i)$ is a normal subgroup of $\mathrm{Mod}^{H_i}(\Gamma_i)$, $\mathrm{Mod}^{H_i}(\Gamma_i)$ is contained in the normalizer of $\mathrm{Mod}_{H_i}(\Gamma_i)$ in $\mathrm{Mod}(\Gamma_i)$.

<u>Proof</u>. We recall that $T(\Gamma_1) \times\ldots\times T(\Gamma_n)$ is the universal covering space of $T(G,\Omega)$ and that

$$T(G,\Omega) = \frac{T(\Gamma_1) \times\ldots\times T(\Gamma_n)}{\mathrm{Mod}_{H_1}(\Gamma_1) \times\ldots\times \mathrm{Mod}_{H_n}(\Gamma_n)}$$

(Chapter VI, §2). By general covering space theory, it is not difficult to show that $f \in \text{Aut } T(G,\Omega)$ if and only if a lifting $\tilde{f}$ belongs to the normalizer of $\text{Mod}_{H_1}(\Gamma_1) \times \ldots \times \text{Mod}_{H_n}(\Gamma_n)$ in $\text{Aut } T(\Gamma_1) \times \ldots \times T(\Gamma_n)$. From a previous theorem, $\text{Aut } T(\Gamma_1) \times \ldots \times T(\Gamma_n)$ is the semidirect product of $\text{Mod}(\Gamma_1) \times \ldots \times \text{Mod}(\Gamma_n)$ by H (Chapter V, §3). From another result (in this section), every $f \in \text{Aut } T(G,\Omega)$ has the property that

$$f|T(\Gamma_i)/\text{Mod}_{H_i}(\Gamma_i) : \frac{T(\Gamma_i)}{\text{Mod}_{H_i}(\Gamma_i)} \xrightarrow{\text{onto}} \frac{T(\Gamma_{\sigma(i)})}{\text{Mod}_{H_{\sigma(i)}}(\Gamma_{\sigma(i)})}$$

biholomorphically for some $\sigma \in S_n$. Hence $f \in \text{Aut } T(G,\Omega)$ and $\sigma = \text{id}$, if and only if a lift of such an f, $\tilde{f}|T(\Gamma_i) : T(\Gamma_i) \to T(\Gamma_i)$ belongs to the normalizer of $\text{Mod}_{H_i}(\Gamma_i)$ in $\text{Mod}(\Gamma_i)$, all i. But $f \in \text{Mod}(G,\Omega)$ if and only if the normalizer of $\text{Mod}_{H_i}(\Gamma_i)$ in $\text{Mod}(\Gamma_i)$ is $\text{Mod}^{H_i}(\Gamma_i)$, all i. Now if $\sigma \neq \text{id}$, we again lift f to an $\tilde{f}$ and conclude that $\tilde{f} \in H$, hence by definition, $f \in K$. Once one realizes that all such $f \in \text{Aut } T(G,\Omega)$, $\sigma \neq \text{id}$ can arise in this fashion only, that is $\tilde{f} \in H$, the result is proven.

<u>Proposition</u>. Let G be a finitely generated non-elementary, non-exceptional Kleinian group. Then $T(G,\Omega)$ is not homogeneous, that is, if $x,y \in T(G,\Omega)$ are arbitrary, there does not exist in general an $h \in \text{Aut } T(G,\Omega)$ such that $h(x) = y$.

<u>Proof</u>. We remark that if $x = (x_1,\ldots,x_n) \in T(G,\Omega)$ then (via the isomorphism theorem, Chapter VI, §2) each x_i is an

equivalence class of points in $T(\Gamma_i)$ under $\mathrm{Mod}_{H_i}(\Gamma_i) \subseteq \mathrm{Mod}(\Gamma_i)$. Since $\mathrm{Mod}(\Gamma_i)$ identifies conformally equivalent Riemann surfaces (see Chapter I, §5), any two Riemann surfaces representing x_i are conformally equivalent. Let $x = (x_1,\ldots,x_n) \in T(G,\Omega)$ be arbitrarily chosen.

Choose $y = (y_1,\ldots,y_n) \in T(G,\Omega)$ such that a Riemann surface in the equivalence class y_1 is not conformally equivalent to a representative Riemann surface from any of the x_i, $1 \leq i \leq n$ (this can always be done assuming the hypothesis). Suppose there existed an $h \in \mathrm{Aut}\ T(G,\Omega)$ such that $h(x) = y$. Lift h to an $\tilde{h}$ on the universal covering space $T(\Gamma_1) \times\ldots\times T(\Gamma_n)$. By a previous theorem (see Chapter V, §3), $\tilde{h}$ is an element of the semidirect product of $\mathrm{Mod}(\Gamma_1) \times\ldots\times \mathrm{Mod}(\Gamma_n)$ by H. Since each element of $\mathrm{Mod}(\Gamma_i)$, each i, identifies conformally equivalent Riemann surfaces (Chapter I, §5), so does $\mathrm{Mod}(\Gamma_1) \times\ldots\times \mathrm{Mod}(\Gamma_n)$ on $T(\Gamma_1) \times\ldots\times T(\Gamma_n)$ componentwise. Similarly, any element of H identifies conformally equivalent Riemann surfaces by looking at the induced mapping on the cotangent space, using the cotangent space splitting theorem (see Chapter V, §1), and applying Royden's result (and generalization, see Chapter III, §3). Hence every element of $\mathrm{Aut}\ T(\Gamma_1) \times\ldots\times T(\Gamma_n)$ identifies conformally equivalent Riemann surfaces. In particular, $\tilde{h} \in \mathrm{Aut}\ T(\Gamma_1) \times\ldots\times T(\Gamma_n)$. But $\tilde{h}$ maps some lift of x, via the covering mapping, denoted $\tilde{x} = (\tilde{x}_1,\ldots,\tilde{x}_n)$ to a lift of y, denoted $\tilde{y} = (\tilde{y}_1,\ldots,\tilde{y}_n)$.

Then, in particular, a Riemann surface representing $\tilde{y}_1$ must be conformally equivalent to a Riemann surface representing $\tilde{x}_i$ for some i. However, by the opening remark and the way $y = (y_1, \ldots, y_n)$ was chosen, we arrive at a contradiction.

§2. Biholomorphic mappings between Deformation spaces

Theorem. Let G_1, G_2 be finitely generated, non-elementary Kleinian groups with region of discontinuity Ω_1, Ω_2. Assume that Ω_1/G_1 and Ω_2/G_2 each is a union of Riemann surfaces of non-trivial type. Let $h : T(G_1, \Omega_1) \to T(G_2, \Omega_2)$ be biholomorphic. Then there exists a one-to-one correspondence between the number of non-equivalent connected components of G_1 and those of G_2.

Proof. Let $h : T(G_1, \Omega_1) \to T(G_2, \Omega_2)$ be biholomorphic. Let $T(\Gamma_1) \times \ldots \times T(\Gamma_n)$ be the universal covering space of $T(G_1, \Omega_1)$ and $T(\tilde{\Gamma}_1) \times \ldots \times T(\tilde{\Gamma}_m)$ the universal covering space of $T(G_2, \Omega_2)$. Then h lifts to a biholomorphic mapping $h : T(\Gamma_1) \times \ldots \times T(\Gamma_n) \to T(\tilde{\Gamma}_1) \times \ldots \times T(\tilde{\Gamma}_m)$. Since Ω_1/G_1, Ω_2/G_2 each is a union of Riemann surfaces of non-trivial type, each $T(\Gamma_i)$, $T(\tilde{\Gamma}_j)$, $1 \le i \le n$, $1 \le j \le m$ is non-trivial. Hence by a previous theorem (Chapter V, §1), $n = m$ and hence the conclusion follows since the Γ_i, $\tilde{\Gamma}_j$ are Fuchsian models over the corresponding components Δ_i, $\tilde{\Delta}_j$.

Note: We do not assume here that G_1 or G_2 be non-exceptional.

Question. Does $h : T(G_1, \Omega_1) \to T(G_2, \Omega_2)$ biholomorphic imply that G_2 is a quasiconformal deformation of G_1?

Answer. No. There are examples to show that G_2 is not a quasiconformal deformation of G_1 even though the statement is true for G_1, G_2 Fuchsian without elliptic elements (and non-exceptional).

Example. There exists uncountably many purely loxodromic finitely generated non-elementary Kleinian groups with connected and simply connected region of discontinuity having the same Riemann surface and yet the groups are not quasiconformally equivalent [9]. Hence if G_1, G_2 are as above, then clearly $T(G_1,\Omega_1) \cong T(G_2,\Omega_2)$ (biholomorphically), but G_1, G_2 are not quasiconformally equivalent. However,

Theorem. Let G_1, G_2 be non-elementary finitely generated non-elementary, non-exceptional Kleinian groups and $h : T(G_1,\Omega_1) \to T(G_2,\Omega_2)$ biholomorphic. Assume G_1, G_2 does not have elliptic elements. Then there exists a Kleinian group G such that Ω/G and Ω_2/G_2 each represent a finite number of Riemann surfaces in which the surfaces of Ω/G are conformally equivalent to those of Ω_2/G_2 and $T(G_1,\Omega_1) \cong T(G_2,\Omega_2) \cong T(G,\Omega)$ (biholomorphically) and G is a quasiconformal deformation of G_1.

Proof. First we remark that from the previous result, $T(G_1,\Omega_1) \cong T(G_2,\Omega_2)$ implies that G_1 and G_2 have the same number of non-equivalent connected components. Lift the mapping h to a biholomorphic mapping $\tilde{h} : T(\Gamma_1) \times \dots \times T(\Gamma_n) \to T(\tilde{\Gamma}_1) \times \dots \times T(\tilde{\Gamma}_n)$ on the covering spaces. Then $\tilde{h}$ induces a complex linear isometry between cotangent spaces over each pair of points x and $\tilde{h}(x)$, where $x \in T(\Gamma_1) \times \dots \times T(\Gamma_n)$. By a previous result (Chapter V, §4), this mapping splits componentwise up to a permutation, and since the Γ_i, $\tilde{\Gamma}_j$ are assumed non-exceptional, a theorem of Earle and Kra (Chapter III, §3) implies the existence of a $\sigma \in S_n$ such that $X_i = \Delta_i/G_{1\Delta_i}$ has the same type as $Y_{\sigma(i)} = \Delta'_{\sigma(i)}/G_{2\Delta'_{\sigma(i)}}$ which in turn implies the existence of $f_i : X_i \to Y_{\sigma(i)}$ quasiconformal mappings, $1 \leq i \leq n$ [6]. Now $\Omega_2/G_2 = \bigcup_{i=1}^{n} Y_i$. If we pull back the Beltrami coefficients u_{f_i} to Ω_1 and extend it to $\hat{C}$ by defining $u_{f_i}|\Lambda_i \equiv 0$, then there exists a quasiconformal mapping f of C and a Kleinian group G such that $G = fG_1f^{-1}$, $\Omega/G = \bigcup_{i=1}^{n} Y_i$ [6], and then obviously $T(G_2,\Omega_2) \cong T(G_1,\Omega_1) \cong T(G,\Omega)$ and G is a quasiconformal deformation of G_1.

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