ONE CONSERVATION LAW IN ONE SPACE VARIABLE

by

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Abstract of the Dissertation

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Shock solutions are constructed for a conservation-law partial differential equation, given initial data satisfying mild conditions; stronger conditions guarantee finite-shock solutions. Shock structure is described, and the number of shocks is estimated in terms of the initial data. The solution and its shock structure are proven stable in various ways.

Dedicated To My Parents,

Oscar and Sarah Friedel

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1. Introduction

(1.1) Motivation

The mass of a fluid, viewed as a density-function depending on space and time, represents a type of physical entity which obeys a "conservation law"; this asserts that any change of the amount within an arbitrary region of space depends solely on the net inflow (including outflow) through the boundary of that region. In particular, if the inflow were zero, the quantity inside would remain constant, i.e., be "conserved".

The conservation law is expressed as an IDE[1]

$$(1.11) \qquad \frac{d}{dt}\int_{\Omega} u(x,t)\,dx = \int_{\partial\Omega} \langle F,\vec{\eta}\rangle .$$

Here Ω is any open subset of R^n (x-space), whose boundary $\partial\Omega$ has inward unit normal $\vec{\eta}$; $u(x,t)$ is density and $F(x,t)$ is flux. (Flux is a vector indicating the direction of flow and its rate: approximately the amount passing through a small area orthogonal to the direction of flow in a short time, divided by the product of the area and the time interval.) The left side of 1.11 is the rate-of-change of the amount of the conserved quantity in Ω, while the right side gives the

[1] IDE = integro-differential equation, PDE = partial differential equation.

rate at which it enters Ω.

If u and F are C^1 on $\Omega \times \{t\}$, then bring the derivative on the left inside the integral and apply the divergence theorem on the right to get

(1.12) $$\int_\Omega u_t dx = -\int_\Omega \underset{x}{\mathrm{div}}(F)dx \quad .$$

Thus, for $(x,t) \in \Omega \times \{t\}$ we have

(1.13) $$u_t + \underset{x}{\mathrm{div}}(F) = 0$$

because 1.12 holds when Ω is replaced by any small neighborhood of x.

In some physical processes[2] several quantities (with densities $(u^1,u^2,\ldots,u^m) = \vec{u}$) are conserved and it may be possible to replace the unknown flux-functions $F^i(x,t)$ by known functions $f^i(x,t,\vec{u})$. (Here flux $f^i(x,t,\vec{u}(x,t))$ makes no sense if $\vec{u}$ is discontinuous at (x,t).) The conservation law becomes a system of IDE's

(1.14) $$\frac{d}{dt}\Big|_{t_o} \int_\Omega u^i dx = \int_{\partial\Omega} \langle f^i,\vec{\eta}\rangle \qquad (i = 1,\ldots,m)$$

which $\vec{u}$ must satisfy whenever continuous on $\partial\Omega \times \{t_o\}$ minus a set of R^{n-1}-measure zero (so the integral over $\partial\Omega$ makes sense). Where C^1, $\vec{u}$ solves

[2] The basic example is isentropic fluid flow, in which mass and momentum are conserved.

$$(1.15) \qquad u^i_t + \operatorname*{div}_x(f^i(x,t,u(x,t))) = 0 \qquad (i = 1,\ldots,m).$$

An initial-value problem (IVP) for a nonlinear PDE such as 1.15 need not have a differentiable solution for all time $t \geq 0$. But because physical processes continue indefinitely, we would like some sort of global solution to an IVP for the conservation law 1.14. Various types of solution have been defined, the most desirable being the "shock solution".

(1.2) Results

We study the simplest nonlinear case: a single conservation law (i.e., $u(x,t)$ is scalar-valued) in one space-variable, with $f(x,t,u) = f(u)$ not explicitly dependent on (x,t). Probably such an oversimplification cannot accurately represent any physical process (it is unlikely for flux to depend only on density) but it is a model for generalization.

We give an algorithm (3.42) for constructing shock solutions using the method of characteristics. To each PDE-IVP there corresponds an "integral surface" in (x,t,u)-space, formed from those characteristics (integral curves to a certain vector field) which pass through the graph of the initial data. Locally this surface is the graph of a solution to the PDE, but globally it may not represent a single-valued function $u(x,t)$. The Rankine-Hugoniot jump condition for shocks (discontinuity-paths) is an ordinary differential equation for which a sequence of initial-value problems tells how to carve the graph of a global shock-solution out of the integral surface.

Assuming f' increasing, we show existence of shock solutions (3.43) for a class of initial data residual in C^3 modulo mild conditions at infinity (5.1), with finite-shock solutions for a large subclass.

Following Lax [2], Schaeffer [3] used the theory of singularities to prove a similar result for rapidly-decaying C^∞ initial data assuming $f'' > \varepsilon > 0$. We use only basic analysis.

Our construction yields a description of the solution's discontinuities (4) and an estimate of their number (4.1). As initial data is perturbed in C^3 the shocks maintain their mutual intersection-relationships; they vary continuously, as does the solution locally in measure and in L^1 (6.1). Golubitsky and Schaeffer [4] proved a different form of shock stability.

2. Definition of Shock Solution

We establish notation, cf. Lax [2].

(2.1) Various types of solution

Consider a single conservation law in one space variable with flux $f(u)$ depending (C^1) only on the unknown variable

$u(x,t)$. The IDE 1.14 here becomes

$$(2.11) \qquad \frac{d}{dt}\Big|_{t_o} \int_{x_1}^{x_2} u(x,t)dx = \int_{\partial(x_1,x_2)} \langle f \circ u, \vec{\eta} \rangle$$

if u is C^o at $\{x_1,x_2\} \times \{t_o\}$. The inward normal $\vec{\eta}$ is $+1$ at x_1 and -1 at x_2 so

$$(2.12) \qquad \int_{\partial(x_1,x_2)} \langle f \circ u, \vec{\eta} \rangle = f \circ u\Big]_{(x_2,t_o)}^{(x_1,t_o)} = -\int_{u(x_1,t_o)}^{u(x_2,t_o)} f' \ .$$

Set $f' = a$. Then 2.12 changes 2.11 to

$$(2.13) \qquad \frac{d}{dt}\Big|_{t_o} \int_{x_1}^{x_2} u \; dx + \int_{u(x_1,t_o)}^{u(x_2,t_o)} a = 0$$

so where u is C^1,

$$(2.14) \qquad u_t + a(u)u_x = 0.$$

<u>Note:</u> The IDE form of the conservation law, 2.13, may be recovered from the simpler form 2.14. Thus we regard the PDE 2.14 as the central object of study, for which we seek global generalized solutions. Any definition of "solution" ought not to depend on the particular antiderivative of (a) that f happens to be, because 2.14 (also 2.13) involves only $f' = a$, not f.

<u>Notation:</u> (i) $H = R \times R_{\geq 0}$ = space-time (x,t). A function $u(x,t)$ is "global" iff its domain is H.

(ii) $\mathcal{J}$ denotes the set of C^1 test-functions $\varphi : H \to R$ which vanish outside a bounded subset of $R \times R_{>0} \subset H$.

(iii) $u : H \to R$ is "piece-wise C^1" iff for each bounded subset $B \subset H$ we have $u(B)$ is bounded, and u restricted to B is C^1 on the complement of a finite union of graphs of C^1 paths.

(2.15) Definition. Let $\psi \in C^1(R)$. "IVP(ψ)" denotes the problem of finding $u : H \to R$ such that $u(x,0) = \psi(x)$ and u is a "generalized solution" of 2.14 in one of these four ways: u is a

(i) "(global) C^1 solution" iff u is C^1 and solves 2.14.

(ii) "IDE solution" iff u solves 2.13.

(iii) "weak solution"[3] iff $\iint_H (\varphi_t u + \varphi_x f \circ u)dxdt = 0$ for all $\varphi \in \mathfrak{J}$.

(iv) "piece-wise C^1 solution" iff u is piece-wise C^1 and, where C^1, solves 2.14.

(2.2) Relative status of definitions

Notation: Let $u(x,t)$ be piece-wise C^1, with a discontinuity-path γ. (i) Let g be a function of u : $g = f$ or $g = a$ or g = identity. Define the "jump function" [g] on graph(γ) as $g]^{u(\gamma(t)+0,t)}_{u(\gamma(t)-0,t)}$.

(ii) The average of a function $a(u)$ for u between u_1 and u_2 is

[3] Integrate by parts $\varphi(u_t + (f \circ u)_x) = 0$. "Weak solution" is independent of the particular antiderivative f since $\int_R \varphi_x(x,t)dx = 0$.

$$a\Big]_{u_1}^{u_2} = \begin{cases} \dfrac{\int_{u_1}^{u_2} a(u)du}{u_2-u_1} & \text{if } u_1 \neq u_2 \\ a(u_1) = a(u_2) & \text{if } u_1 = u_2 \end{cases}$$

Define $a]$ along γ as $a\Big]_{u(\gamma(t)-0,t)}^{u(\gamma(t)+0,t)}$

(iii) The "Rankine-Hugoniot condition" is $\gamma' = a]$ at interior points of γ, for all discontinuity paths γ.

<u>Note</u>: $a] = \dfrac{[f]}{[u]}$.

(2.21) <u>Proposition</u>

(2.211) A C^1 solution is a solution in all 4 ways 2.15.

(2.212) A solution in any of the 4 ways 2.15 satisfies the PDE 2.14 on any open set on which it is C^1.

(2.213) Let u be a piece-wise C^1 solution. Then the following 3 statements are equivalent:

(i) u obeys the Rankine-Hugoniot condition.

(ii) u is an IDE solution.

(iii) u is a weak solution.

<u>Proof</u>. See Appendix.

(2.3) <u>The Shock Solution</u>

<u>Definition</u>. A piece-wise C^1 solution obeys the "shock condition" iff $[a] < 0$ on the interior of each discontinuity-path ("shock"). A "shock solution" is a piece-wise C^1 solution which obeys both the Rankine-Hugoniot condition and

the shock condition.

(2.31) Note. If (a) is increasing, then the shock condition is equivalent to $[u] < 0$.

Recalling that for the PDE 2.14 the characteristic speed is $\frac{dx}{dt} = a \circ u(x,t)$, we see the shock condition says every shock is forced by the intersection of characteristics emanating from either side, carrying conflicting values for the solution.

A shock solution (if it exists) seems the best alternative to a global C^1 solution. It is a generalized solution in all senses except C^1 (by 2.213). The discontinuities are simple (paths) and minimal ("forced"). Oleinik [1] proved that a shock solution is unique among weak solutions up to redefinition on a set of measure zero.

3. Construction of Solution

(3.1) The integral surface

Assume hence $a(u)$ is C^1. The method of characteristics tries to build the graph of a C^1 solution to IVP(ψ) out of integral curves to the characteristic vector field $(x,t,u) \to (a(u),1,0)$ in $H \times R_u$ which pass through graph (ψ). Thus we seek functions $(s,\tau) \to (\bar{x},\bar{t},\bar{u})$ solving

$$(3.11) \qquad \begin{cases} \bar{x}_\tau = a(\bar{u}) & , \ \bar{x}(s,0) = s \\ \bar{t}_\tau = 1 & , \ \bar{t}(s,0) = 0 \\ \bar{u}_\tau = 0 & , \ \bar{u}(s,0) = \psi(s) \end{cases} .$$

Here s parametrizes the x-axis and τ parametrizes each characteristic curve, i.e., $\tau \to (\bar{x},\bar{t},\bar{u})\big|_{(s,\tau)}$ gives the characteristic through $(s,0,\psi(s))$.

$\bar{t}_\tau = 1$ and $\bar{t}(s,0) = 0$ imply $\bar{t}(s,\tau) = \tau$, so we identify τ with "time" t. Then 3.11 simplifies to

$$\begin{cases} \bar{x}_t = a(\bar{u}) & , \quad x(s,0) = s \\ \bar{u}_t = 0 & , \quad u(s,0) = \psi(s) \end{cases}$$

which has solution

$$\bar{u}(s,t) = \psi(s), \qquad \bar{x}(s,t) = t\cdot a\circ\psi(s) + s.$$

(3.12) <u>Note</u>: $\bar{u}_s = \psi'(s)$, $\bar{x}_s = t\cdot(a\circ\psi)'(s)+1 =$

$t\cdot a'(\psi(s))\cdot\psi'(s)+1$, and $\bar{x}_t = a(\bar{u}) = a\circ\psi(s)$, $\bar{u}_t = 0$.

<u>Definition</u>. $P = \{(s,t) \in R\times R_{\geq 0}\}$, parameter-space.
$\Phi : P \to H\times R$, $\Phi(s,t) = (\bar{x},t,\bar{u})$; $\wp = \Phi(P)$ (parametrized integral surface).

(3.13) <u>Proposition</u>. Φ is a proper embedding.[4] Thus $\wp$ is a surface in $H\times R$ for which the relative topology from R^3 equals the topology induced via Φ from P; $\Phi^{-1} : \wp \to P$ is continuous.

---

[4]i.e., Φ is C^1, injective, has injective differential, and the pre-image of compact sets is compact. Fact: A proper embedding maps its domain diffeomorphically onto its image, which is a submanifold of the range space.

Proof. Φ is injective because integral curves of a C^1 vector field do not intersect. 3.12 gives Φ is C^1, with

$$d\Phi = \begin{pmatrix} t\cdot a'(\psi(s))\cdot\psi'(s)+1 & a\circ\psi(s) \\ 0 & 1 \\ \psi'(s) & 0 \end{pmatrix} .$$

$d\Phi$ has rank 2, because the first 2 or the last 2 rows are independent (corresponding to $\psi'(s) = 0$ or $\neq 0$).

We show $\Phi^{-1}(K)$ is compact for compact $K \subset H\times R$. K is closed and Φ is C^0, so $\Phi^{-1}(K)$ is closed. K is bounded, so the last 2 slots $(t,\psi(s))$ are bounded, implying $t\cdot a\circ\psi(s)$ is bounded for $(s,t) \in \Phi^{-1}(K)$; but the first slot $(t\cdot a\circ\psi(s)+s)$ also is bounded, so that s is bounded in $\Phi^{-1}(K)$. Thus $\Phi^{-1}(K)$ is bounded and closed, hence compact.

QED 3.13.

Definition. $\mathcal{V} = \{p\in\mathcal{P} : (0,0,1) \text{ is tangent to } \mathcal{P} \text{ at } p\}$, $V = \Phi^{-1}(\mathcal{V})$, "vertical sets" in $\mathcal{P}$ and in P.

(3.14) Lemma. $V = \{(s,t)\in P : \bar{x}_s(s,t) = 0\} =$

$$\{(s,t)\in P : t = \frac{-1}{(a\circ\psi)'(s)}\} =$$

$= \{(s,t) : (0,0,1) \text{ is tangent to curve } \Phi(R\times\{t\}) \text{ at } \Phi(s,t)\}$.

Proof. The tangent plane to $\mathcal{P}$ at $\Phi(s,t)$ is spanned by Φ_s (tangent to $\Phi(R\times\{t\})$) and Φ_t, so $(s,t) \in V$ iff

$$\begin{vmatrix} \bar{x}_s & \bar{x}_t & 0 \\ 0 & 1 & 0 \\ \bar{u}_s & \bar{u}_t & 1 \end{vmatrix} = 0$$

iff $0 = \bar{x}_s = t\cdot(a\circ\psi)'(s)+1$. QED 3.14.

Definition. $\pi : \mathcal{P} \to H$, $\pi(x,t,u) = (x,t)$. A subset $\mathcal{S} \subset \mathcal{P}$ is "1-valued" if $\pi|_{\mathcal{S}}$ is injective, i.e., $\mathcal{S}$ is the graph of a function $u^{\mathcal{S}} : \pi(\mathcal{S}) \to R$.

(3.15) Proposition.

(3.151) Let $p \in \mathcal{P}$. $\mathcal{P}$ contains a 1-valued neighborhood of p iff $p \in \mathcal{P} - \mathcal{V}$.

(3.152) Let $\mathcal{O} \subset \mathcal{P}$ be 1-valued and open. Then $u^{\mathcal{O}}$ is a C^1 solution of 2.14 with open domain $\pi(\mathcal{O})$.

(3.153) IVP(ψ) has a global C^1 solution iff $\mathcal{V} = \emptyset$, iff $(a\circ\psi)' \geq 0$.

Proof. Suppose no neighborhood of p in $\mathcal{P}$ is 1-valued. Then write $\Phi^{-1}(p) = (s_o,t_o)$ and use 3.13 to ensure sequences $\{(s_n,t_n)\}_1^\infty$, $\{(\hat{s}_n,\hat{t}_n)\}_1^\infty$ in P converging to (s_o,t_o), with $\pi\ \Phi]_{(\hat{s}_n,\hat{t}_n)}^{(s_n,t_n)} = 0$ for all n. Then $t_n = \hat{t}_n$, and $\bar{x}]_{(\hat{s}_n,\hat{t}_n)}^{(s_n,t_n)} = 0$, so the mean value theorem guarantees $\check{s}_n \in (s_n,\hat{s}_n)$ such that $\bar{x}_s(\check{s}_n,t_n) = 0$. Now $\lim\limits_{n\to\infty} \check{s}_n = s_o$ (because $\lim s_n=s_o=\lim \hat{s}_n$)

so $\lim (\check{s}_n, t_n) = (s_o, t_o)$. Thus $\bar{x}_s(s_o, t_o) = 0$, and 3.14 gives $p \in \mathcal{V}$.

Now let $p \in \mathcal{P} - \mathcal{V}$. Jacobian $(\pi \circ \Phi) = \bar{x}_s$, nonzero at $\Phi^{-1}(p)$ (by 3.14), so the inverse-function theorem promises a neighborhood N_p of $\Phi^{-1}(p)$ in P, which $\pi \circ \Phi$ maps diffeomorphically onto $\pi \circ \Phi(N_p)$.

In particular, π is injective on $\Phi(N_p)$, a neighborhood of p (by 3.13).

QED 3.151.

For $p \in \mathcal{O}$, let N_p as above; we may assume $N_p \subset \Phi^{-1}(\mathcal{O})$ since $\Phi^{-1}(\mathcal{O})$ is open in P-V. Then $\pi \circ \Phi(N_p)$ is open, so $\pi(\mathcal{O}) = \pi \circ \Phi(\Phi^{-1}(\mathcal{O})) = \pi \circ \Phi(\bigcup_{p \in \mathcal{O}} N_p) = \bigcup \pi \circ \Phi(N_p)$ is open.

Let $\tilde{s}_p : \pi \circ \Phi(N_p) \to N_p$ denote the inverse function such that $\pi \circ \Phi(\tilde{s}(x,t),t) = (x,t)$; then $\tilde{s}_p$ is C^1, since so is $\pi \circ \Phi$. Note $u^{\mathcal{O}}(x,t) = \bar{u}(\tilde{s}_p(x,t),t)$ for $(x,t) \in \pi \circ \Phi(N_p)$, so $u^{\mathcal{O}}$ is C^1 near $\pi(p)$.

$\mathcal{O}$ has normal vector field $(u^{\mathcal{O}}_x, u^{\mathcal{O}}_t, -1)$. But $\mathcal{P}$ is tangent to the vector field $(a(u),1,0)$. $0 = \langle \text{tangent, normal} \rangle = \langle (a(u),1,0),(u_x,u_t,-1) \rangle = u_t + a(u)u_x$. (Standard proof.)

QED 3.152.

3.14 shows $\mathcal{V} = \emptyset$ iff $(a \circ \psi)' \geq 0$.

If a global C^1 solution exists, then its graph is $\mathcal{P}$. Suppose $(s,t) \in V \neq \emptyset$. Then $0 = \bar{x}_s(s,t) = t \cdot a'(\psi(s)) \cdot \psi'(s) + 1$

(by 3.14), and $0 \neq \bar{u}_s(s,t) = \psi'(s)$ (else $\bar{x}_s(s,t) = 1$), so $\frac{\partial u}{\partial x}(\pi \circ \Phi(s,t)) = \frac{\bar{u}_s}{\bar{x}_s}(s,t) = \pm\infty$, contradicting u is C^1.

Finally, suppose $V = \emptyset$. ρ is 1-valued (else the mean-value theorem yields a zero for $\bar{x}_s$, contradicting $V = \emptyset$) and a C^1 solution where defined (by 3.152). But $V = \emptyset$ implies $(a\circ\psi)' \geq 0$ so $t\cdot(a\circ\psi)'(s)+1 = \bar{x}_s > 1$, so that $\{\bar{x}(s,t) : s \in R\} = R$ for each $t \geq 0$; thus $\pi\circ\Phi(P) = H$, and u^{ρ} is defined on all H.

QED 3.15.

(3.2) Sheets

We divide the integral surface into 1-valued subsets.

Definition. $\tau : \{s:(a\circ\psi)'(s) < 0\} \to R$, $\tau(s) = \dfrac{-1}{(a\circ\psi)'(s)}$.

Note. $V = \text{graph}(\tau)$ (by 3.14).

Definition. $P^+ = \{(s,t) \in P : t < \tau(s)\}$. $\rho^+ = \Phi(P^+)$.

Note. $P^+ = \{(s,t) : \bar{x}_s(s,t) > 0\}$ (by 3.14).

Assume hence $a, \psi \in C^2$. $\{s : (a\circ\psi)'(s)<0, (a\circ\psi)''(s) = 0\}$ is the critical set of τ; assume here it is finite.

Definition. Suppose the local minima of τ occur at $s^1 < s^2 <\ldots< s^n$, with $t^i = \tau(s^i)$. Call each t^i a "bifurcation-time", and (s^i,t^i) a "bifurcation-point". Write

$$s^i_+ = \begin{cases} \min\{s : s>s^i, \tau(s) = t^i\}, & \text{if this set is nonempty} \\ \infty & \text{, otherwise} \end{cases}$$

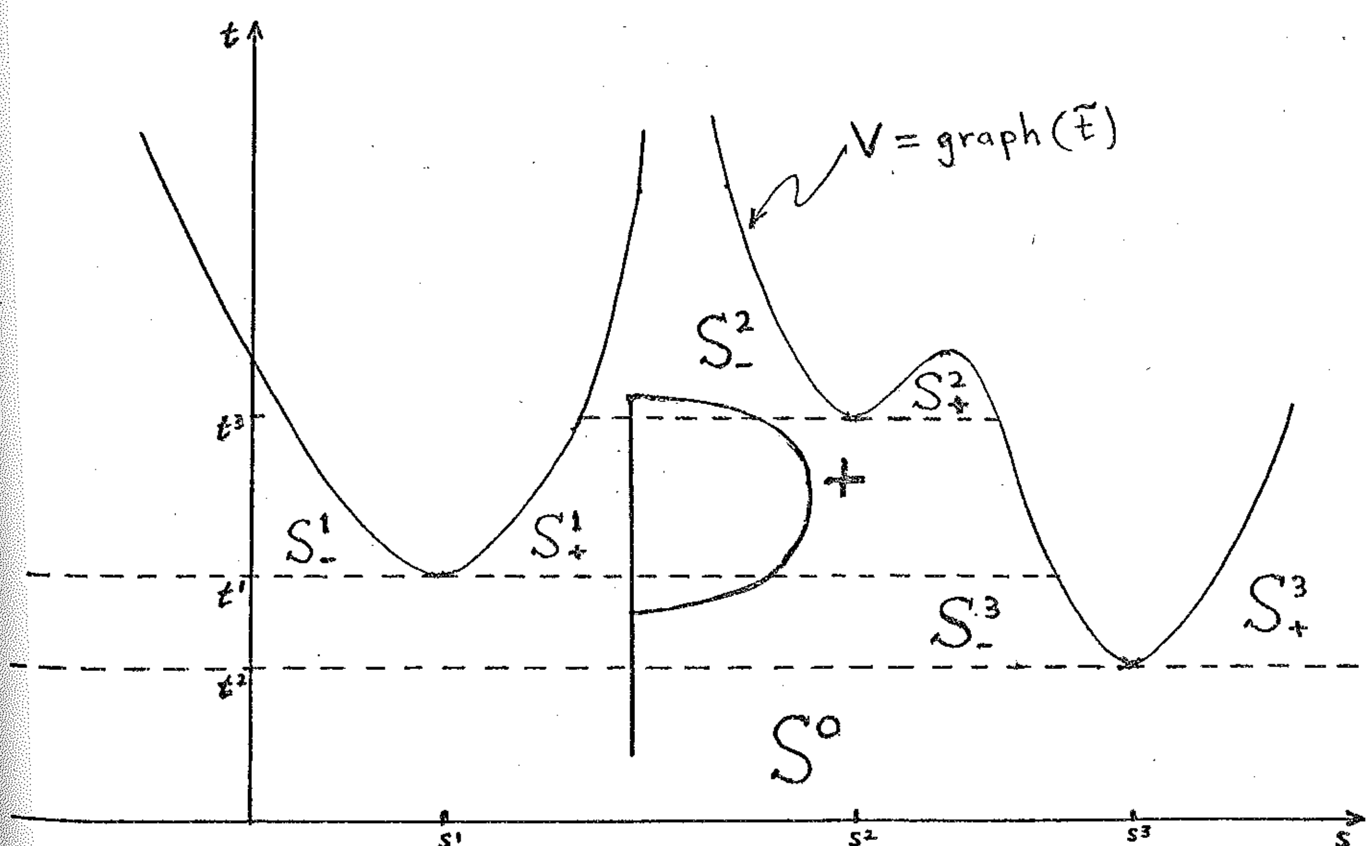
$$s_-^i = \begin{cases} \max\{s : s<s^i, \tilde{t}(s) = t^i\}, & \text{if this set is nonempty} \\ -\infty & \text{, otherwise} \end{cases}.$$

Define the "base" $B_+^i = (s^i, s_+^i) \times \{t^i\}$ and the base $B_-^i = (s_-^i, s^i) \times \{t^i\}$. Also view the s-axis $\underset{s}{R} \times \underset{t}{\{0\}}$ as a base, B^0.

A point $p = (s,t) \in P^+$ is said to "lie over" a base $(s_L, s_R) \times \{t_o\}$ iff $s_L < s < s_R$ and $t_o \leq t$. Among the bases over which P lies, there is one base B with largest t-coordinate and we say p "lies just over" B.

Define the "sheet" over the base $B_\pm^i$ as $S_\pm^i = \{p \in P^+ : p \text{ lies just over } B_\pm^i\}$.

Order the sheets (except S^0) by $S_-^1 < S_+^1 < \ldots < S_-^n < S_+^n$. Extend this terminology to $\mathcal{P}$ and to H via Φ and π.



<u>Notation</u>. Fix sheets S, S_1. Write $\mathcal{S} = \Phi(S)$, $\sigma = \pi(\mathcal{S})$ and $\mathcal{S}_1 = \Phi(S_1)$, $\sigma_1 = \pi(\mathcal{S}_1)$.

(3.21) <u>Lemma</u>. (Sheet structure in P).

(3.211) There exist t_o, t_1 $(0 \leq t_o < t_1 \leq \infty)$ such that $[t_o, t_1)$ is domain for C^o paths $s_L(t) < s_R(t)$ (except we allow $s_L \equiv -\infty$ or $s_R \equiv +\infty$) with $S = \{(s,t) : t_o \leq t < t_1, s_L(t) < s < s_R(t)\}$.

Call s_L the "left side" of s, and s_R the "right side".

(3.212) If s_L (or s_R) is finite-valued, then it is C^1 on the complement of a finite set, with graph in V.

(3.213) t_o is the bifurcation-time of S. Suppose $S_1 < S$ share a bifurcation-point (s_o, t_o), and $\hat{s}_R : [t_o, t_2) \to R$ is the right side of S_1; then $s_L(t_o) = s_o = \hat{s}_R(t_o)$ and $\bar{x}(s_L(t),t) < \bar{x}(\hat{s}_R(t),t)$ for $t_o < t < \min\{t_1, t_2\}$.

(3.214) t_1 is a bifurcation-time, or $t_1 = \infty$, or $s_L(t_1-0) = s_R(t_1-0)$.

<u>Proof</u>. If $S = S^o$, then $t_o = 0$, $t_1 = \min\{t^i\}_1^n$, and $s_L \equiv -\infty$, $s_R \equiv +\infty$. Hence, assume $S = S_+^i$ for some $i > 0$ (similar proof for $S = S_-^i$). Recall $B_+^i = (s^i, s_+^i) \times \{t^i\}$, and set $t_o = t^i$ (bifurcation-time).

Set $\alpha = \sup\{s > s^i : \mathfrak{T}$ increasing on $(s^i, s)\}$

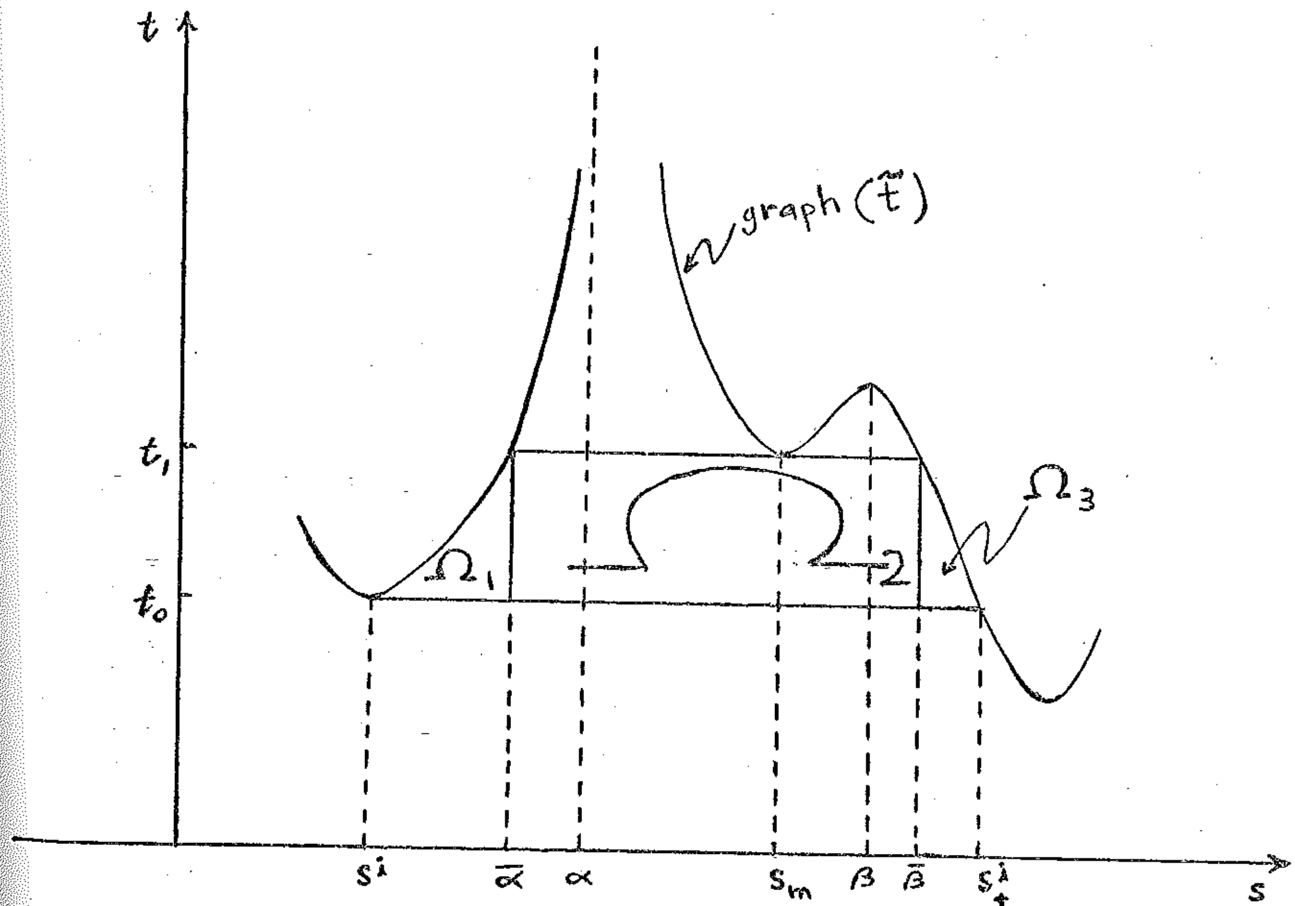
$$\beta = \begin{cases} \inf\{s < s_+^i : \mathfrak{T} \text{ decreasing on } (s, s_+^i)\}, & \text{if } s_+^i < \infty \\ \infty, & \text{if } s_+^i = \infty \end{cases}$$

$$t_1 = \begin{cases} \inf\{\,\tilde{t}(s) : \alpha \le s \le \beta\}, & \text{if } \operatorname{domain}(\tilde{t}) \cap [\alpha,\beta] \ne \emptyset \\ \infty & \text{, otherwise} \end{cases}$$

$$\bar{\alpha} = \begin{cases} \inf\{s : s^i \le s \le \alpha,\ \tilde{t}(s) = t_1\}, & \text{if } t_1 < \infty \\ \alpha & \text{, if } t_1 = \infty \end{cases}$$

$$\bar{\beta} = \begin{cases} \sup\{s : \beta \le s \le s_+^i,\ \tilde{t}(s) = t_1\}, & \text{if } s_+^i < \infty \\ \infty & \text{, if } s_+^i = \infty \end{cases}$$

Note $s^i < \bar{\alpha} \le \alpha \le \beta \le \bar{\beta} \le s_+^i \le \infty$, and $t_o < t_1$.



$\mathcal{T}$ is monotone on $[s^i, \bar{\alpha})$, hence has a C^o inverse, $s_L(t)$ for $t_o \le t < t_1$. If $s^i_+ < \infty$, then also $\bar{\beta} < s^i_+$, and $\mathcal{T}$ is monotone on $(\bar{\beta}, s^+_i]$, with a C^o inverse $s_R(t)$ for $t_o \le t < t_1$; if $s^+_1 = \infty$, set $s_R(t) \equiv \infty$ $(t_o \le t < t_1)$.

If s_L (or s_R) is finite-valued, then its graph is part of graph$(\mathcal{T}) = V$. It is C^1 except possibly at critical values of $\mathcal{T}$ (a finite set) by the inverse-function theorem.

QED 3.212.

Define 3 regions

$$\Omega_1 = \{(s,t) : s^i < s < \bar{\alpha},\ t_o \le t < \mathcal{T}(s)\}$$

$$\Omega_2 = \{(s,t) : \bar{\alpha} \le s \le \bar{\beta},\ t_o \le t < t_1\}$$

$$\Omega_3 = \{(s,t) : \bar{\beta} < s < s^i_+,\ t_o \le t < \mathcal{T}(s)\}.$$

It is easy to check $(k = 1,2,3)$ that $p \in \Omega_k$ implies $p \in P^+$ and p lies just over B^i_+; thus, $\bigcup\limits_1 \Omega_k \subset S$. Also note we may rewrite the sets Ω_k as

$$\Omega_1 = \{(s,t) : t_o \le t < t_1,\ s_L(t) < s < \bar{\alpha}\}$$

$$\Omega_2 = \{(s,t) : t_o \le t < t_1,\ \bar{\alpha} \le s \le \bar{\beta}\}$$

$$\Omega_3 = \{(s,t) : t_o \le t < t_1,\ \bar{\beta} < s < s_R(t)\},$$

which shows that our advertised description of S is equivalent to "$S = \bigcup\limits_1 \Omega_k$", so we prove $S \subset \bigcup\limits_1 \Omega_k$.

Let $p = (\hat{s}, \hat{t}) \in P^+$, p lies just over B^i_+ (i.e., $p \in S$); we suppose $p \notin \cup\, \Omega_k$ and derive a contradiction. "p lies over B_+^i" implies $s^i < \hat{s} < s_+^i$, $\hat{s} \notin (s^i, \bar{\alpha})$, since otherwise $t_o \leq \hat{t} < \tau(\hat{s})$ (which follows from $p \in P^+$ and p lying over B_+^i) contradicts $p \notin \Omega_1$. Similarly $\hat{s} \notin (\bar{\beta}, s_+^i)$, so we must have $\hat{s} \in [\bar{\alpha}, \bar{\beta}]$. Then $p \notin W_2$ implies $t_1 < \hat{t}$, so $t_1 < \infty$. Also note $\bar{\alpha} \neq \bar{\beta}$; otherwise ($\bar{\alpha} = \bar{\beta} = \hat{s}$) τ achieves a local maximum (t_1) at $\hat{s}$, and $\tau(\hat{s}) = t_1 < \hat{t}$ contradicts $(\hat{s}, \hat{t}) \in P^+$.

Thus we are left with $\bar{\alpha} < \bar{\beta}$ and $t_1 < \infty$. One may check that $\tau|_{[\bar{\alpha},\bar{\beta}]}$ assumes the minimum t_1 at an interior point $s_m \in (\bar{\alpha}, \bar{\beta})$, and (s_m, t_1) is a bifurcation-point generating 2 bases. p must lie over 1 of these bases, contradicting "p lies just over B_+^i".

QED 3.211.

If $\bar{\alpha} < \bar{\beta}$ and $t_1 < \infty$, then we just saw t_1 is a bifurcation-time. If $\bar{\alpha} = \bar{\beta}$, then $s_L(t_1 - 0) = s_R(t_1 - 0)$ (if τ continuous at $\bar{\alpha} = \bar{\beta}$) or $t_1 = \infty$ (if τ discontinuous there).

QED 3.214.

Recall $S = S_+^i$. Suppose now $S_1 = S_-^i$. (We have written the bifurcation-point (s^i, t^i) as (s_o, t_o).) A mirror-image of the previous development for S applies to S_1, so $\hat{s}_R$ is the inverse of τ on an interval $(\hat{\alpha}, s^i] = (\hat{\alpha}, s_o]$ where τ is decreasing. Thus $\hat{s}(t_o) = s_o = s_L(t_o)$. Also, if $\hat{s}_R$, s_L both exist at

$t_3 > t_o$, then τ is decreasing on $[\hat{s}_R(t_3), s_o]$, increasing on $[s_o, s_L(t_3)]$, so

$(\hat{s}_R(t_3), s_L(t_3)) \times \{t_3\} \subset \{(s,t) \in P : t > \tau(s)\} = P - (P^+ \cup V) =$

$$\{(s,t) : \bar{x}_s(s,t) < 0\}, \text{ implying}$$

$\bar{x}(\hat{s}_R(t_3), t_3) > \bar{x}(s_L(t_3), t_3)$.

QED 3.213, 3.21.

(3.22) Lemma. $\mathcal{S}$ is 1-valued.

Proof. Otherwise $\pi \circ \Phi]_q^p = 0$ for some $p, q \in S$, $p \neq q$. $\pi \circ \Phi$ preserves the t-coordinate, so $p = (s_1, t_1)$, $q = (s_2, t_1)$ with $s_1 \neq s_2$ (say $s_1 < s_2$). 3.211 shows $[s_1, s_2] \times \{t_1\} \subset S \subset P^+ = \{(s,t) : \bar{x}_s > 0\}$, so $\bar{x}(s_1, t_1) < \bar{x}(s_2, t_1)$, contradicting $\pi \circ \Phi]_{(s_2,t_2)}^{(s_1,t_1)} = 0$.

QED 3.22.

(3.23) Lemma. (Sheet Structure in H).

(3.231) There exist t_o, t_1 $(0 \le t_o < t_1 \le \infty)$ such that $[t_o, t_1)$ is domain for C^1 paths $x_L(t) < x_R(t)$ (except we allow $x_L \equiv -\infty$ or $x_R \equiv +\infty$) with

$$\sigma = \{(x,t) : t_o \le t < t_1, \ x_L(t) < x < x_R(t)\}.$$

Call x_L the "left side" of σ, and x_R the "right side".

(3.232) Extend $u^{\mathcal{S}}$ by continuity to $\text{graph}(x_L) \cup \text{graph}(x_R)$.

Then $x_L'(t) = a \circ u^{\mathcal{S}}(x_L(t),t)$ and $x_R'(t) = a \circ u^{\mathcal{S}}(x_R(\mathcal{S}),t)$.

(3.233) t_o is the bifurcation-time of σ. Suppose $\sigma_1 < \sigma$ share a bifurcation-point (x_o,t_o), and $\hat{x}_R : [t_o,t_2) \to R$ is the right side of σ_1; then $x_L(t_o) = x_o = \hat{x}_R(t_o)$ and $x_L(t) < \hat{x}_R(t)$ for $t_o < t < \min\{t_1,t_2\}$.

(3.234) t_1 is a bifurcation-time, or $t_1 = \infty$, or $x_L(t_1-0) = x_R(t_1-0)$.

Proof. Use notation of 3.21. Define $x_L \equiv -\infty$ if $s_L \equiv -\infty$, and $x_R \equiv \infty$ if $s_R \equiv \infty$; otherwise $x_L(t) = \bar{x}(s_L(t),t)$ and $x_R(t) = \bar{x}(s_R(t),t)$. Note for $t_3 \in (t_o,t_1]$ that $\bar{x}(\{(s,t_3) : s_L(t_3) < s < s_R(t_3)\}) = \{(x,t_3) : x_L(t_3) < x < x_R(t_3)\}$ since $\bar{x}_s > 0$ here. Thus

$$\sigma = \pi \circ \Phi(S) = \pi \circ \Phi\Big(\bigcup_{t_3 \in [t_o,t_1)} \{(s,t_3) : s_L(t_3) < s < s_R(t_3)\}\Big) =$$

$$\bigcup_{t_3} \pi \circ \Phi(\{(s,t_3) : s_L(t_3) < s < s_R(t_3)\}) =$$

$$\bigcup_{t_3} (\bar{x}(\{(s,t_3) : s_L(t_3) < s < s_R(t_3)\}), t_3) = \bigcup_{t_3} \{(x,t_3) : x_L(t_3) < x < x_R(t_3)\} =$$

$$\{(x,t) : t_o \le t < t_1,\ x_L(t) < x < x_R(t)\}, \text{ as promised.}$$

Except for a finite set $\{\tau_i\}_1^N$ where $s_L(t)$ is not differentiable, the chain rule gives

$x_L'(t) = \bar{x}_s(s_L(t),t) \cdot s_L'(t) + \bar{x}_t(s_L(t),t)$.

$\bar{x}_s(s_L(t),t) = 0$ (by 3.212 and 3.14), and 3.12 implies

$\bar{x}_t(s_L(t),t) = a\circ\bar{u}(s_L(t),t) = a\circ u^{\mathcal{S}}(\bar{x}(s_L(t),t),t) =$

$a\circ u^{\mathcal{S}}(x_L(t),t)$. Thus

$x_L'(t) = a\circ u^{\mathcal{S}}(x_L(t),t)$ for $t \in [t_o,t_1) - \{\tau_i\}_1^N$.

But $x_L(t)$ is C^o (being composed of C^o functions) and $\lim_{t\to\tau_i} x_L'(t) = a\circ u^{\mathcal{S}}(x_L(\tau_i),\tau_i)$ for each τ_i, so that x_L' exists and is continuous at τ_i.

QED 3.231, 3.232.

3.213 implies 3.233, and 3.214 implies 3.234.

QED 3.23.

(3.24) Lemma. Assume (a) increasing. $\mathcal{S}_1 < \mathcal{S}$ iff $u^{\mathcal{S}_1} > u^{\mathcal{S}}$ wherever both defined, except a common bifurcation-point. If σ_1, σ share a bifurcation-point p, then $u^{\mathcal{S}}(p) = u^{\mathcal{S}_1}(p)$.

Proof. Let x_L, x_R be the left and right sides of σ, and analogously $\hat{x}_L$, $\hat{x}_R$ for σ_1. Recall $\text{domain}(u^{\mathcal{S}}) = \sigma \cup \text{graph}(x_L) \cup \text{graph}(x_R)$, and similarly for $\mathcal{S}_1$.

Claim 1: Let $p \in \sigma_1 \cap \sigma$. Then $\mathcal{S}_1 < \mathcal{S}$ iff $u^{\mathcal{S}_1}(p) > u^{\mathcal{S}}(p)$.

Proof. Let $q \in S$, $q_1 \in \mathcal{S}_1$ be the unique points such that $\pi\circ\Phi(q) = p = \pi\circ\Phi(q_1)$. These have the same t-coordinate: $q = (s_o,t_o)$, $q_1 = (s_1,t_o)$. Note $S_1 < S$ iff $s_1 < s_o$; also

$u^{\mathcal{S}}(p) = \bar{u}(q) = \psi(s_o)$ and $u^{\mathcal{S}_1}(p) = \bar{u}(q_1) = \psi(s_1)$. Thus, it suffices to show $s_1 < s_o$ iff $\psi]_{s_o}^{s_1} > 0$.

$$\pi\circ\Phi]_q^{q_1} = 0, \text{ so } \bar{x}]_q^{q_1} = 0, \text{ so } t \cdot a\circ\psi(s_1) + s_1 =$$

$t_o \cdot a\circ\psi(s_o) + s_o$, so $t_o \cdot a\circ\psi]_{s_o}^{s_1} = s_o - s_1$. Thus $s_1 < s_o$ iff $t_o \cdot a\circ\psi]_{s_o}^{s_1} > 0$, iff $\psi]_{s_o}^{s_1} > 0$ (use $t_o > 0$ and (a) increasing).

Claim 2: Suppose $\mathcal{S}_1 < \mathcal{S}$. Then $u^{\mathcal{S}_1}(p) = u^{\mathcal{S}}(p)$ iff p is the bifurcation point of both σ_1, σ.

Proof. If $p \in H$ is a common bifurcation-point, then (letting p_o be the corresponding bifurcation-point in P) $u^{\mathcal{S}}(p) = \bar{u}(p_o) = u^{\mathcal{S}_1}(p)$. Conversely, suppose $p = (x_o, t_o)$ with $u^{\mathcal{S}_1}(p) = u^{\mathcal{S}}(p) = u_o$. Then

$\Phi^{-1}(x_o,t_o,u_o) \in (\text{graph}(s_L) \cup \text{graph}(s_R)) \cap (\text{graph}(\hat{s}_L) \cup \text{graph}(\hat{s}_R))$,

which is possible only if $\Phi^{-1}(x_o,t_o,u_o)$ is a common bifurcation-point (in P) for S_1, S.

QED 3.24.

(3.3) Shocks

The Rankine-Hugoniot condition will choose the discontinuity-path to mark off separate domains for 2 sheets which disagree about $u(x,t)$.

Definition. Suppose $(x_o,t_o) = p \in \sigma_1 \cap \sigma$, or p is the common bifurcation-point (in H) of σ_1, σ. Let $\gamma(t)$ be the unique maximal path solving

$$\gamma'(t) = a\rceil_{u^{s_1}}^{u^{s}}(\gamma(t),t) \quad \text{and} \quad \gamma(t_o) = x_o \ .$$

We say γ "solves (s_1, s, p)".

(3.31) Lemma. Use above notation. Assume (a) increasing and $s_1 < s$. Write x_L, x_R as left and right sides of σ, and $\hat{x}_L$, $\hat{x}_R$ for σ_1.

(3.311) $\gamma(t)$ exists for small $t-t_o > 0$.

(3.312) $x_L(t) < \gamma(t) < \hat{x}_R(t)$ for $t > t_o$ when $\gamma(t)$ exists.

(3.313) Suppose also $s < s_2$, $\tilde{\gamma}$ solves $(s, s_2, (x_1, t_o))$, and $x_o < x_1$. If $\gamma(t_1) = x_R(t_1)$ or $\tilde{\gamma}(t_1) = x_L(t_1)$ for $t_1 > t_o$, then $\gamma(t_2) = \tilde{\gamma}(t_2)$ for some $t_2 \in (t_o, t_1)$.

Proof. Let domain$(x_R) = [\alpha, \beta)$ and domain$(\hat{x}_L) = [\alpha_1, \beta_1)$. Note $t_o \in [\alpha,\beta) \cap [\alpha_1,\beta_1) = [\max(\alpha,\alpha_1), \min(\beta,\beta_1)) = [\bar{\alpha},\bar{\beta})$.

Claim 1: $\sigma_1 \cap \sigma = \{(x,t) : \bar{\alpha} \leq t < \bar{\beta},$

$$\hat{x}_L(t) < x < \hat{x}_R(t), x_L(t) < x < x_R(t)\}.$$

Proof. 3.231.

Claim 2: If $p \in \sigma_1 \cap \sigma$, then 3.311 holds.

Proof. The vector $(dx, dt) = (a\rceil, 1)$ at p points into $\sigma_1 \cap \sigma$ (by Claim 1) so the vector field $(a\rceil, 1)$ has an integral curve through p which runs for some time past t_o.

Proof. Define $W = \{(x,t) : t_o < t \leq t_2, x_L(t) < x < \hat{x}_R(t)\}$.

Claim 2 shows $x_L < \hat{x}_R$ on $(t_o, t_2]$; Claims 2 and 3 give $W \subset \sigma_1 \cap \sigma_2$. For $n \in Z_+$, $n > \frac{1}{t_2 - t_o}$, define

$$p_n = \left(\frac{x_L(t_o + \frac{1}{n}) + \hat{x}_R(t_o + \frac{1}{n})}{2}, \; t_o + \frac{1}{n}\right) \in W.$$ Claim 5 shows

$$\lim_{n\to\infty} p_n = p.$$

$(\mathcal{S}_1, \mathcal{S}, p_n)$ has a solution γ_n in W, which (by claim 2) runs for some $t > t_o + \frac{1}{n}$, until it intersects the boundary, $\partial W \subset \text{graph}(x_L) \cup \text{graph}(\hat{x}_R) \cup R \times \{t_2\}$. Claim 4 shows γ_n intersects neither x_L nor $\hat{x}_R$, hence must exist until t_2. Now define $\gamma : [t_o, t_2] \to R$ by

$$\gamma(t) = \begin{cases} x_o, & \text{if } t = t_o \\ \lim_{n\to\infty} \gamma_n(t), & \text{if } t_o < t \le t_2 \end{cases}.$$

It is easy to prove γ is C^1, solves $(\mathcal{S}_1, \mathcal{S}, p)$, and $x_L(t) \le \gamma(t) \le \hat{x}_R(t)$.

We show $x_L(t) < \gamma(t) < \hat{x}_R(t)$ for $t_o < t \le t_2$. Suppose not, say $x_L(t_3) = \gamma(t_3)$, $t_o < t_3 \le t_2$. Then Claim 4 implies $x_L(t) = \gamma(t)$ for all $t \in [t_o, t_3]$, so

$$\gamma'(t_3) = x_L'(t_3) = a \circ u^{\mathcal{S}}(x_L(t_3), t_3) < a] = \gamma'(t_3),$$

contradiction.

QED 3.311, 3.312.

Apply the intermediate-value theorem to $\gamma - \tilde{\gamma}$ on $[t_o, t_1]$. $(\gamma - \tilde{\gamma})(t_o) < 0$ since $x_o < x_1$. In case $\gamma(t_1) = x_R(t_1)$, then $(\gamma - \tilde{\gamma})(t_1) = (x_R - \tilde{\gamma})(t_1) > 0$ by 3.312 applied to $(\mathcal{S}, \mathcal{S}_2, (x_1, t_o))$.

QED 3.31.

<u>Claim 3</u>: $a \circ u^{s_1} > a\big]_{u^{s_1}}^{u^{s}} > a \circ u^{s}$, except equality holds at a common bifurcation-point.

<u>Proof</u>: 3.24 and (a) increasing.

<u>Claim 4</u>: If $p \in \sigma_1 \cap \sigma$, then 3.312 holds.

<u>Proof</u>. We show $x_L < \gamma$ ($\gamma < \hat{x}_R$ is similar). Suppose not; let $t_1 = \min\{t \geq t_o : x_L(t) = \gamma(t)\} > t_o$. Note $x_L(t) < \gamma(t)$ for $t_o \leq t < t_1$, by Claim 1. Thus

$$\gamma'(t_1) = \lim_{t \to t_1 - 0} \frac{\gamma(t_1)-\gamma(t)}{t_1 - t} \leq \lim \frac{x_L(t_1)-x_L(t)}{t_1-t} =$$

$$x_L'(t_1) = a \circ u^{s}(x_L(t_1),t_1) < a\big]_{u^{s_1}}^{u^{s}}(x_L(t_1),t_1) = \gamma'(t_1),$$

contradiction! (We used 3.232 and Claim 3.)

<u>Claim 5</u>: Suppose p is a bifurcation-point for both σ_1, σ. Then $x_L(t_o) = x_o = \hat{x}_R(t_o)$. Also there exists $t_2 > t_0$ such that on $[t_o,t_2]$ all $\{x_L, x_R, \hat{x}_L, \hat{x}_R\}$ are defined and on $(t_o,t_2]$ they satisfy $\hat{x}_L < x_L < \hat{x}_R < x_R$.

<u>Proof</u>. 3.233 gives $x_L(t_o) = x_o = \hat{x}_R(t_o)$, and 3.231 gives $x_L < x_R$, $\hat{x}_L < \hat{x}_R$; thus $\hat{x}_L(t_o) < x_L(t_o)$, $\hat{x}_R(t_o) < x_R(t_o)$. By continuity these inequalities persist for small time past t_o. Finally, 3.233 gives $x_L < \hat{x}_R$.

<u>Claim 6</u>: Suppose p is a bifurcation-point for both σ_1,σ. Let t_2 as in Claim 5. Then (s_1, s, p) has a solution γ

with $[t_0, t_2] \subset \text{domain}(\gamma)$ and $x_L(t) < \gamma(t) < \hat{x}_R(t)$ for $t_0 < t \le t_2$.

(3.4) Algorithm for Constructing Solution

(3.41) Lemma. $\liminf\limits_{|s|\to\infty} \frac{a\circ\psi(s)}{s} \ge 0$[5] implies $\pi\circ\Phi : P \to H$ is onto and proper.

Proof. $\bar{x}(s,t) = s(t\cdot \frac{a\circ\psi(s)}{s} + 1)$, so for t fixed $\lim\limits_{s\to\infty} \bar{x}(s,t) = \infty$, $\lim\limits_{s\to -\infty} \bar{x}(s,t) = -\infty$. Thus $\pi\circ\Phi(P) = H$.

Showing $\pi\circ\Phi$ proper reduces to proving that for $M > 0$ there exists $\tilde{M}$ such that if $t < M$ and $|s| > \tilde{M}$, then $|\bar{x}(s,t)| > M$. Indeed, choose $\tilde{M} > 2M$ such that $|s| > \tilde{M}$ implies $\frac{a\circ\psi(s)}{s} > \frac{-1}{2M}$. If $t < M$ and $|s| > \tilde{M}$, then $\frac{-1}{2M} > \frac{-1}{2t}$, so $\frac{a\circ\psi(s)}{s} > \frac{-1}{2t}$, and $t\frac{a\circ\psi(s)}{s} + 1 > \frac{1}{2}$; thus $|\bar{x}(s,t)| = |s|\ |t\frac{a\circ\psi(s)}{s} + 1| > \tilde{M}\cdot\frac{1}{2} > M$.

QED 3.41.

Fix $a \in C^2$, increasing.

Definition. $\mathfrak{F}_0$ (respectively, $\mathfrak{F}$) is the set of $\psi \in C^2$ such that $\liminf\limits_{|s|\to\infty} \frac{a\circ\psi(s)}{s} \ge 0$ and

---

[5] Given $\epsilon > 0$, there exists M such that $|s| > M$ implies $\frac{a\circ\psi(s)}{s} > -\epsilon$.

$\{s : (a\ \psi)'(s) < 0,\ (a\circ\psi)''(s) = 0\}$ is finite (resp., discrete).

(3.42) Theorem. IVP(ψ) has a finite-shock solution for $\psi \in \mathfrak{F}_0$, generated in finitely many steps by the following algorithm.

Define inductively a sequence ($n = 0,1,\ldots$) of pairs

$$\begin{cases} T_n \in [0,\infty] \\ \text{a finite-shock solution } u(x,t) \text{ for } t < T_n \text{ (for } t \le T_n \\ \qquad \text{if } T_n < \infty) \text{ with graph in } \mathcal{P}^+. \end{cases}$$

$T_0 = 0$, $u(x,0) = \psi(x)$. Given the n^{th} pair with $T_n < \infty$, construct the $(n+1)^{st}$ as follows:

Let $x_1 < x_2 < \ldots < x_k$ be the points at which $u(\cdot,T_n)$ discontinuous or $u_x(\cdot,T_n) = \pm\infty$; (x_i,T_n) belongs to one or more previous shocks, or is a bifurcation-point in H. For each $i \in \{0,1,\ldots,k\}$, the restriction $u(\cdot,T_n)|_{(x_i,x_{i+1})}$ has graph belonging to a unique sheet $\mathcal{S}_i$ (we used $x_0 = -\infty$, $x_{k+1} = \infty$). Let the path γ_i solve $(\mathcal{S}_{i-1},\mathcal{S}_i,(x_i,T_n))$, for $1 \le i \le k$. Set

$$T_{n+1} = \begin{cases} \min\left(\bigcup_{i=1}^{k}\{t > T_n : \gamma_i(t) = \gamma_{i-1}(t)\} \cup \{\text{bifurcation-times} > T_n\}\right) \\ \infty, \quad \text{if above set is } \emptyset. \end{cases}$$

Take $u(x,t)$ as defined for $t \le T_n$. For $T_n < t < T_{n+1}$, set

$$u(x,t) = \begin{cases} u^{s_0}(x,t), & \text{if } x < \gamma_1(t) \\ u^{s_1}(x,t), & \text{if } \gamma_1(t) < x < \gamma_2(t) \\ \vdots & \\ u^{s_k}(x,t), & \text{if } \gamma_k(t) < x \end{cases}$$

If $T_{n+1} < \infty$, define $u(x,T_{n+1}) = u(x,T_{n+1}-0)$ for all but finitely many x where this limit or $u_x(x,T_{n+1})$ doesn't exist.

Proof. Suppose given the n^{th} solution-pair with $T_n < \infty$, "$x_1 < x_2 < \ldots < x_k$" is really defined since $u : R \times [0,T_n] \to R$ is a finite-shock solution so only finitely many shocks hit $\underset{x}{R} \times \{T_n\}$. Also, because graph$(u) \subset \mathcal{P}^+$, by 3.24 see (x_i,T_n) is a bifurcation-point if not part of a shock.

Claims $1 \to 7$ show the algorithm does produce the $(n+1)^{st}$ solution-pair.

Claim 1: $\{s_i\}_1^k$ are well-defined.

Proof. Let $G = \text{graph}(u|_{(x_i,x_{i+1})})$. G is connected, and so is $\Phi^{-1}(G)$, by 3.13. Note $\Phi^{-1}(G) \subset \underset{s}{R} \times \{T_n\}$. A connected subset of $R \times \{T_n\}$ is an interval: $\Phi^{-1}(G) = I \times \{T_n\}$. $G \subset \mathcal{P}^+$ (use 3.14) so $I \times \{T_n\} \subset P^+$ and belongs to a single sheet (use 3.21).

Claim 2: $s_{i-1} < s_i$.

<u>Proof</u>. The shock condition 2.31 holds for $u : R \times [0,T_n] \to R$, implying $u(x-0,T_n) \geq u(x+0,T_n)$ for all x, in particular $u^{\mathcal{S}_{i-1}}(x_i,T_n) \geq u^{\mathcal{S}_i}(x_i,T_n)$. If $u^{\mathcal{S}_{i-1}}(x_i,T_n) > u^{\mathcal{S}_i}(x_i,T_n)$, then 3.24 gives $\mathcal{S}_{i-1} < \mathcal{S}_i$.

There remains $u^{\mathcal{S}_{i-1}}(x_i,T_n) = u^{\mathcal{S}_i}(x_i,T_n) = \hat{u}$. Then $u_x(x_i,T_n) = \pm\infty$, so $(x_i,T_n,\hat{u}) \in \mathcal{V}$. Write $(\hat{s},T_n) = \Phi^{-1}(x_i,T_n,\hat{u}) \in V$. Because $S_{i-1},S_i \subset P^+ = \{(s,t) : \bar{x}_s(s,t) > 0\}$, we see for small $\epsilon > 0$ that $(\hat{s}-\epsilon,T_n) \in S_{i-1}$, $(\hat{s}+\epsilon,T_n) \in S_i$. Thus S_{i-1},S_i are distinct sheets (use 3.211 and $(\hat{s},T_n) \in V$). $(\hat{s},T_n)$ is the bifurcation-point of both S_{i-1},S_i (by 3.24); S_{i-1},S_i are respectively the minus-sheet and plus-sheet attached to this bifurcation-point, so $S_{i-1} < S_i$.

QED Claim 2.

Let $\tau = \sup(\text{domain}(\gamma_i))$. Use 3.23 to write

$$\sigma_i = \pi(\mathcal{S}_i) = \{(x,t) : t_o \leq t < t_1,\ x_L(t) < x < x_R(t)\}$$

$$\sigma_{i-1} = \pi(\mathcal{S}_{i-1}) = \{(x,t) : \hat{t}_o \leq t < \hat{t}_1,\ \hat{x}_L(t) < x < \hat{x}_R(t)\}$$

$$\sigma_{i-1}\cap\sigma_i = \{(x,t): \alpha < t \leq \beta, x_L(t) < x < x_R(t), \hat{x}_L(t) < x < \hat{x}_R(t)\}$$

where $\alpha = \max(t_o,\hat{t}_o)$, $\beta = \min(t_1,\hat{t}_1)$. Note $\sigma_{i-1}\cap\sigma_i \neq \emptyset$, since $(x_i,T_n) \in \sigma_{i-1}\cap\sigma_i$, or is a bifurcation-point for both σ_{i-1}, σ_i.

Claim 3: If $\tau < \infty$, then $\sigma_{i-1} \cap \sigma_i \cap R \times [T_n, \tau)$ is bounded.

Proof. Otherwise both $\{x_L, \hat{x}_L\}$ or both $\{x_R, \hat{x}_R\}$ are unbounded on $[T_n, \tau)$. Suppose $x_L, \hat{x}_L$ unbounded on $[T_n, \tau)$; at least one is finite-valued, say x_L. Recall $x_L(t) = \bar{x}(s_L(t), t)$, where s_L is inverse to a monotone portion of $\mathfrak{T}_o$. $s_L([T_n, \tau))$ is an unbounded interval I. $\mathfrak{T}$ is monotone on I and $\mathfrak{T}(I) = [T_n, \tau)$; but then

$I \times \{\tau\} \subset \{(s,t): t > \mathfrak{T}(s)\} = \{(s,t): \bar{x}_s(s,t) < 0\}$,

violating 3.42.

Claim 4: $T_{n+1} \leq \tau$.

Proof. True if $\tau = \infty$, hence assume $\tau < \infty$. Claim 4 shows $(\gamma_i(\tau - 0), \tau) \in \text{boundary}(\sigma_{i-1} \cap \sigma_i)$, so three cases arise.

Case 1: $\tau = t_1$ or $\hat{t}_1$.

Use 3.234. If τ is a bifurcation-time, then $T_{n+1} \leq \tau$. Else $x_L(\tau - 0) = \gamma_i(\tau - 0) = x_R(\tau - 0)$, or $\hat{x}_L(\tau - 0) = \gamma_i(\tau - 0) = \hat{x}_R(\tau - 0)$; but $x_L(\tau - 0) < \gamma_i(\tau - 0) < \hat{x}_R(\tau - 0)$ by the same proof as for 3.312 (use Claim 3).

Case 2: $\gamma_i(\tau) = x_L(\tau)$ or $\hat{x}_R(\tau)$.

Impossible by 3.312.

Case 3: $\gamma_i(\tau) = x_R(\tau)$ or $\hat{x}_L(\tau)$.

3.313 shows γ_i intersects γ_{i-1} or γ_{i+1} at some time before τ.

Claim 5: u is defined on $R \times (T_n, T_{n+1})$.

Proof. 3.22 and Claim 4 show u defined on $\{(x,t) : T_n < t < T_{n+1},\ \gamma_1(t) < x < \gamma_k(t)\}$; 3.41 shows $\mathcal{S}_0, \mathcal{S}_k$ cover (x,t) for $|x|$ large.

Claim 6: Suppose $T_{n+1} < \infty$. $u(x,T_{n+1})$ is defined and $(x,T_{n+1},u(x,T_{n+1})) \in \mathcal{P}^+$, for all but finitely many x.

Proof. $u(x,T_{n+1}-0)$ exists for $x \in R - \{\gamma_i(T_{n+1}-0)\}_1^k$. $\mathcal{P}^+$ is open in $\mathcal{P}$ with boundary $\mathcal{V}$, and graph$(u : R \times [0,T_n) \to R) \subset \mathcal{P}^+$, so that (if defined) $(x,T_{n+1},u(x,T_{n+1}-0)) \in \mathcal{P}^+ \cup \mathcal{V}$. 3.24 shows that, this limit is in $\mathcal{V}$ (i.e., $u_x = \pm\infty$) iff it is a bifurcation-point in $\mathcal{P}$; there are only finitely many such points.

Claim 7: u is a finite-shock solution for $t < T_{n+1}$ (for $t \le T_{n+1}$ if $T_{n+1} < \infty$) with graph in $\mathcal{P}^+$.

Proof. Each sheet $\mathcal{S}_i \subset \mathcal{P}^+$, and Claim 6, give graph$(u) \subset \mathcal{P}^+$. graph(u) was constructed as an open subset of $\mathcal{P}^+$, so use 3.152 off the discontinuity set (a finite union of graphs of paths) to see u is a piecewise C^1 solution. By construction, each discontinuity path obeys the Rankine-Hugoniot condition ($\gamma' = a]$). The shock condition 2.31 holds, by Claim 2 and 3.24.

Claim 8: Algorithm ends ($T_N = \infty$ for some $N \in Z_+$) yielding a global, finite-shock solution.

Proof. The number of bifurcation points (and bifurcation-times) is finite, by 3.411. A shock may start only at a bifurcation-point, or at the intersection of 2 shocks. No shocks exist before the first (smallest) bifurcation-time and only finitely many appear at the first such time. Between one bifurcation-time and the next, only finitely many shocks arise by intersection because only 1 shock emerges from the intersection of several. Similarly, only finitely many arise after the last bifurcation-time.

QED 3.42.

(3.43) Corollary. IVP(ψ) has a shock solution for $\psi \in \mathfrak{F}$.

Proof. We define the solution on each compact subset $K_n = \{(x,t) : |x| = n, t \le n\} \subset H$, $n > 0$. By 3.11 $(\pi \circ \Phi)^{-1}(K_n)$ is compact; the algorithm 3.42 may be modified to apply only to this set, producing a solution over K_n with only finitely many shocks there.

4. Shock Structure

Let u solve IVP(ψ), $\psi \in \mathfrak{F}$.

By construction, u is as smooth off shocks as $a \circ \psi$; shocks are also this smooth. All directional derivatives exist at each $p \in H$, except possibly in the direction $\frac{dx}{dt} = \gamma'$ if p belongs to a shock γ. The solution and its derivatives are bounded on bounded sets, except $u_x = \pm\infty$ at isolated bifurcation points.

A shock begins at a bifurcation point or at the intersection of 2 previous shocks. When 2 or more shocks meet, just 1 new shock emerges. Thus a shock may be identified by its first endpoint.

Definition. "Shock structure" is the partially ordered collection of shocks: $\gamma < \hat{\gamma}$ if there is a sequence of shocks $\gamma = \gamma_1, \gamma_2, \ldots, \gamma_n = \hat{\gamma}$ such that the last endpoint of γ_i is the first endpoint of γ_{i+1} $(1 \le i < n)$. Shock structure is "normal" if no 3 shocks share a last endpoint.

(4.1) Theorem. If $\psi \in \mathfrak{F}_0$, define

$d(t) = \#\{x : u(\cdot,t) \text{ or } u_x(\cdot,t) \text{ discontinuous at } x\}$

$b(t) = \#\{s : (a\circ\psi)'(s) \le \frac{-1}{t}, (a\circ\psi)''(s) = 0\}$ $(b(0) = 0)$, and

$\bar{b} = \#\{s : (a\circ\psi)'(s) < 0, (a\circ\psi)''(s) = 0\}$. Then

$d(t) \le b(t) \le \bar{b}$, with $d(t)$ eventually constant. The total number of shocks is $< 2^{\bar{b}}$.

Proof. $b(t)$ counts the bifurcation-points occuring before t, eventually constant at $\bar{b}$, the total number of such points.

By induction on n we show $d(t) \le b(t)$ for $0 \le t \le T_n$ (notation of 3.42). True for $n = 0$. Assume it for n, we prove it for $n+1$. $d(t)$, $b(t)$ both constant for $T_n \le t < T_{n+1}$, by our algorithm. T_{n+1} is a bifurcation-time or the intersection-time of 2 shocks. For each bifurcation-point appearing then, $b(t)$ grows by 1 while $d(t)$ grows by 1 or 0. Each shock-intersection doesn't change $b(t)$ but decreases $d(t)$ by 1.

$d(t)$ becomes constant after all bifurcation-times and shock intersections.

The minimal elements of the shock structure are those shocks emerging from bifurcation-points. Each shock determines a set of minimal elements $\leq$ that shock; distinct shocks determine distinct sets of minimal elements. There are $\leq \bar{b}$ minimal elements, hence $\leq 2^{\bar{b}}$ possible sets, which is a gross upper bound on the number of shocks.

QED 4.1.

5. Genericity

Notation. (i) A subset of a topological space is "generic" if it contains a dense open set; it is "residual" if it is a countable intersection of generic sets.
(ii) C^3 is topologized by uniform convergence on compacta of the function and 3 derivatives. C^3_c (compact support) and $C^3_{[\alpha,\beta]}$ (support in $[\alpha,\beta]$) are subspaces.

Note. Residual subsets of such function-spaces are dense, by Baire's theorem.

(5.1) Theorem. Let $a \in C^3$, increasing. IVP(ψ) has a

(5.11) finite-shock solution for generic $\psi \in C^3_{[\alpha,\beta]}$, with normal shock-structure for a residual subset.

(5.12) normal, finite-shock solution for residual $\psi \in C^3_c$.

(5.13) normal, shock solution for residual

$\psi \in \{\tilde{\psi} \in C^3 : \liminf_{|s|\to\infty} \frac{a\circ\tilde{\psi}(s)}{s} \geq 0\}$.

Proof. $\mathfrak{n} = \{\psi \in \mathfrak{F} : \text{the solution to IVP}(\psi) \text{ is normal}\}$ is residual in $\mathfrak{F}$. Proof: $\mathfrak{n} = \bigcap_{j=1}^{\infty} \mathfrak{n}_j$, where $\mathfrak{n}_j =$ $\{\psi \in \mathfrak{F} : \psi|_{[-j,j]}$ generates normal shock structure$\}$, generic in $\mathfrak{F}$.

The map A, $A(\psi) = a\circ\psi$, is a homeomorphism onto its image, if domain(A) is any subspace of C^2.

proof: A is continuous. So is A^{-1}, because if $\{\psi_n\}_1^\infty$ is such that $a\circ\psi_n$ approaches $a\circ\psi$ in C^2 (or C^3), then ψ_n approaches ψ in C^2 (or C^3); use a' positive almost everywhere, and $(a\circ\psi_n)'(s) = a'\circ\psi_n(s)\cdot\psi_n'(s)$ and similarly for $(a\circ\psi_n)''$, $(a\circ\psi_n)'''$.

$S = \{g:(g'')^{-1}(0) \cap [\alpha,\beta] \text{ is finite}\}$ is generic in C^3, hence $A^{-1}(S)$ is generic in C^3, and in $C^3_{[\alpha,\beta]}$. But $A^{-1}(S) \subset \mathfrak{F}_o$.

QED 5.11.

$S_n = \{g:(g'')^{-1}(0) \cap [-n,n] \text{ is finite}\}$ is generic in C^3, hence $A^{-1}(S_n)$ is generic in C^3_c, and $\bigcap_{n=1}^{\infty} A^{-1}(S_n)$

residual in C_c^3. But $A^{-1}(S_n) \subset \mathcal{F}_0$, all n.

QED 5.12.

Similarly, $A^{-1}(S_n)$ is generic in C^3, and

$\bigcap_1^\infty A^{-1}(S_n)$ residual; but

$$\{\psi \in A^{-1}(S_n) : \liminf_{|s|\to\infty} \frac{a\circ\psi(s)}{s} \geq 0\} \subset \mathcal{F}.$$

QED 5.13.

6. Stability

(6.1) Theorem. Let $a \in C^3$, increasing. Let $\{\psi_n\}_1^\infty$, $\psi \in \mathcal{F} \cap C^3$ with solutions u_n, u to IVP(ψ_n), IVP(ψ). Suppose ψ_n approaches ψ uniformly, with derivatives converging uniformly on compacta. Fix compact $K = [-k,k] \times [0,k] \subset H$.

(6.11) Let $\varepsilon > 0$. For large n, shock structures of u, u_n restricted to K are isomorphic, i.e., shocks of u, u_n intersecting K correspond one-to-one $\{\gamma_i\} \longleftrightarrow \{\gamma_i^n\}$, and $\gamma_i < \gamma_j$ iff $\gamma_i^n < \gamma_j^n$. Also corresponding shock-endpoints are less than ε distance apart in K, and $|\gamma_i(t) - \gamma_i^n(t)| < \varepsilon$ where defined.

(6.12) $\lim_{n\to\infty}$ (measure$\{(x,t) \in K : |(u-u_n)(x,t)| > \varepsilon\}$) = 0.

(6.13) $\lim_{n\to\infty} \iint_K |u-u_n| dxdt = 0$.

Proof. Because $\psi_n \to \psi$ uniformly, there exists M such that for large n, we have $u_n|_K$, $u|_K$ depending only on $\psi_n|_{(-M,M)}$, $\psi|_{(-M,M)}$ (recall characteristic speed is $a\circ\psi_n$, $a\circ\psi$). List

$$\{s^i\}_1^j = \{s : |s| < M,\ (a\circ\psi)'(s) < 0,\ (a\circ\psi)''(s) = 0\}$$

with $s^i < s^{i+1}$; similarly list $\{s_n^i\}_{i=1}^{j_n}$ for ψ_n. $j = j_n$ for n large, and the relevant bifurcation points $\{(s^i, \frac{-1}{(a\circ\psi)'(s^i)})\}_1^j$, for ψ correspond to those for ψ_n, as do their sheets (which cover K). Corresponding sheets cover almost the same subset of K, and give almost equal u-values where both defined. The algorithm 3.42 for IVP(ψ_n), IVP(ψ) gives initial-value problems for ordinary differential equations which (over K) correspond, with initial data and equation coefficients almost equal. 6.11, 6.12 follow from stability of ODE's. 6.13 follows from 6.12 and $\{u_n\}$, u being uniformly bounded on K (since $\{\psi_n\}$, ψ uniformly bounded on $(-M,M)$).

QED 6.1.

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Appendix: Proof of 2.21

2.211 and 2.212 are easy. We prove 2.213. Hence, let u denote a piece-wise C^1 solution.

The discontinuity-set of u is a countable union of graphs of discontinuity paths and their endpoints:

$D = (\bigcup_i \text{graph}(\gamma_i)) \cup (\bigcup_i \{\text{endpoints of } \gamma_i\})$.

For each γ_i, let

$W_i = \{(\gamma_i(t),t) : \text{sign } (\gamma') \text{ is not constant in any neighborhood of } t\}$.

Define

$E = (\bigcup_i W_i) \cup \{\text{endpoints}\}$. Note E is countable.

Temporarily we say a set of the form $B = (x_1,x_2) \times (t_1,t_2) \subset H$ is a "1-box" if at most one discontinuity path (γ) intersects B, with γ defined on all (t_1,t_2), $\text{sign}(\gamma')$ constant on (t_1,t_2), and $\gamma((t_1,t_2)) = (x_1,x_2)$ unless $\gamma' = 0$ on (t_1,t_2). Note every point in H - E has a 1-box containing it.

Claim 1: Let B as above and $\varphi \in \mathfrak{J}$.

$$\iint_B (\varphi_t u + \varphi_x f\circ u)\,dxdt =$$

$$\int_{\gamma|_{(t_1,t_2)}} \varphi([u]dx - [f]dt) + \int_{x_1}^{x_2} (\varphi u]_{(x,t_1)}^{(x,t_2)})dx +$$

$$+ \int_{t_1}^{t_2} (\varphi f\circ u]_{(x_1,t)}^{(x_2,t)})dt .$$

Proof. Assume $\gamma' > 0$, the other cases are similar. Note γ has a local inverse $\gamma^{-1} : (x_1,x_2) \to (t_1,t_2)$. If $(x,t) = (\gamma(t),t) = (x,\gamma^{-1}(x))$, then $u(\gamma(t) \pm 0,t) = u(x,\gamma^{-1}(x) \mp 0)$. We calculate

$$\iint_B \varphi u_t dxdt = \int_{x_1}^{x_2} \left(\int_{t_1}^{t_2} \varphi u_t dt\right) dx =$$

$$\int_{x_1}^{x_2} \left(\int_{t_1}^{\gamma^{-1}(x)} \varphi u_t dt + \int_{\gamma^{-1}(x)}^{t_2} \varphi u_t dt\right) dx =$$

$$\int_{x_1}^{x_2} \left(\varphi u\right]_{(x,t_1)}^{(x,\gamma^{-1}(x)-0)} - \int_{t_1}^{\gamma^{-1}(x)} \varphi_t u dt + \varphi u\Big]_{(x,\gamma^{-1}(x)+0)}^{(x,t_2)} -$$

$$- \int_{\gamma^{-1}(x)}^{t_2} \varphi_t u dt) dx =$$

$$\int_\gamma \varphi[u] dx - \iint_B \varphi_t u dxdt + \int_{x_1}^{x_2} \varphi u\Big]_{(x,t_1)}^{(x,t_2)} dx. \quad \text{Also}$$

$$\iint_B (f\circ u)_x dxdt = \int_\gamma -\varphi[f] dt - \iint_B \varphi_x f\circ u dxdt + \int_{t_1}^{t_2} \varphi f\circ u\Big]_{(x_1,t)}^{(x_2,t)} dt.$$

Add these two equations and recall $u_t + (f\circ u)_x = 0$, to get Claim 1. An immediate consequence is

Claim 2: If u obeys the Rankine-Hugoniot condition, then

$$\iint_B (\varphi_t u + \varphi_x f\circ u) dxdt = \int_{x_1}^{x_2} \varphi u\Big]_{(x,t_1)}^{(x,t_2)} dx + \int_{t_1}^{t_2} \varphi f\circ u\Big]_{(x_1,t)}^{(x_2,t)} dt.$$

Claim 3: If u obeys the Rankine-Hugoniot condition, then u is a weak solution.

Proof. Fix $\varphi \in \mathfrak{J}$. There exists $\alpha > 0$ such that $\text{support}(\varphi) \subset [-\alpha,\alpha] \times [0,\alpha] = K$. We show

$\iint_K (\varphi_t u + \varphi_x f\circ u) = 0.$

Choose $\epsilon > 0$. Write $E \cap K = \{(s_i, t_i) : i \in Z_+\}$. For $i \in Z_+$ let

$$B_i = (s_i - \frac{\epsilon}{2^i}, s_i + \frac{\epsilon}{2^i}) \times (t_i - \frac{\epsilon}{2^i}, t_i + \frac{\epsilon}{2^i}).$$

For $p \in K - E$, let B^p be a 1-box with $p \in B^p$.

The open sets $\{B_i : i \in Z_+\} \cup \{B^p : p \in K\text{-}E\}$ cover the compact set K, so there exist finite sets $F_1 \subset Z_+$ and $F_2 \subset K\text{-}E$ such that $\{B_i : i \in F_1\} \cup \{B^p : p \in F_2\}$ cover K. It is easy to see that there is a finite sequence of disjoint 1-boxes $\{R_j : 1 \le j \le N\}$ and a set z of measure zero such that $\bigcup_j R_j \cup z = K - \bigcup_{F_1} B_i$. Thus

$$\iint_K (\varphi_t u + \varphi_x f\circ u) = \iint_{\bigcup_{F_1} B_i} + \iint_{\bigcup_j R_j} .$$

Set $M = \sup_K (|\varphi_t u + \varphi_x f\circ u| + |\varphi u| + |\varphi f\circ u|) < \infty$. Thus

$$\left|\iint_{\bigcup_{F_1} B_i} (\varphi_t u + \varphi_x f\circ u)\right| < M \sum_{i=1}^{\infty} \text{area}(B_i) = O(\epsilon).$$ Also

$\iint_{\bigcup_j R_j} (\varphi_t u + \varphi_x f \circ u) = \sum_j \iint_{R_j}$ may be evaluated by Claim 2; cancellation occurs except along the boundary of $\bigcup_{F_i} B_i$, a rectilinear curve with length $O(\epsilon)$, so that the integral along this curve of a function, whose absolute value is less than M, is $O(\epsilon)$.

Thus $\iint_K (\varphi_t u + \varphi_x f \circ u) = O(\epsilon)$. Since we can choose ϵ arbitrarily small, Claim 3 holds.

<u>Claim 4</u>: If u is a weak solution, then u obeys the Rankine-Hugoniot condition.

<u>Proof</u>. Let $p = (x_0, t_0)$ be an interior point of a discontinuity path γ.

Assume first that $p \in D\text{-}E$. Let $B = (x_1, x_2) \times (t_1, t_2)$ be a 1-box containing p. For any $\varphi \in \mathfrak{J}$ with support in B, apply Claim 1 to get

$$0 = \int_{\gamma|_{(t_1,t_2)}} \varphi([u]dx - [f]dt) = \int_{t_1}^{t_2} \varphi([u]\gamma' - [f])dt.$$

The arbitrariness of φ gives $([u]\gamma' - [f])\big|_{t=t_0} = 0$.

Because γ' is C^0 and E is only countable, we see $\gamma' = \dfrac{[f]}{[u]}$ at all interior points of γ.

<u>Claim 5</u>: If u obeys the Rankine-Hugoniot condition, then

u is an IDE solution.

Proof. Let $(x_1,x_2) \times \{t_o\} = I$ be given. Finitely many discontinuity-paths intersect I; let $\{\gamma_i : 1 \le i \le N\}$ (respectively, $\{\delta_i : 1 \le i \le N'\}$) parametrize those defined for some $t > t_o$ (resp., $t < t_o$) and assume that for small $\varepsilon > 0$ (resp. $\varepsilon < 0$) the paths are ordered at $(t_o+\varepsilon)$ by $\gamma_i < \gamma_{i+1}$ (resp. $\delta_i < \delta_{i+1}$). Let $(\frac{d}{dt})^+$ (resp., $(\frac{d}{dt})^-$) denote the right (resp., left) derivative. Then

$$(\frac{d}{dt})^+\Big|_{t_o} \int_{x_1}^{x_2} u(x,t)dx = (\frac{d}{dt})^+ (\int_{x_1}^{\gamma_1(t)} + \int_{\gamma_1}^{\gamma_2} + \dots \int_{\gamma_N}^{x_2}) =$$

$$\int_{x_1}^{\gamma_1(t_o)} u_t(x,t_o)dx + \gamma_1'(t_o)u(\gamma_1(t_o)-0,t_o) +$$

$$\int_{\gamma_1(t_o)}^{\gamma_2(t_o)} u_t dx + \gamma_2'(t_o)\, u\, (\gamma_2(t_o)-0,t_o) - \gamma_1'(t_o)u(\gamma_1(t_o)+0,t_o) +$$

$$\dots \int_{\gamma_N(t_o)}^{x_2} u_t dx - \gamma_N'(t_o)u(\gamma_N(t_o)+0,t_o) =$$

$$\int_{x_1}^{\gamma_1(t_o)} -a(u)u_x dx + \int_{\gamma_1(t_o)}^{\gamma_2(t_o)} -au_x dx + \dots \int_{\gamma_N(t_o)}^{x_2} -au_x dx -$$

$$- \sum_{i=1}^{N} \gamma_i'(t_o)u\Big]_{(\gamma_i(t_o)-0,t_o)}^{(\gamma_i(t_o)+0,t_o)} =$$

$$-f\Big]_{u(x_1,t_o)}^{u(\gamma_1(t_o)-0,t_o)} - f\Big]_{u(\gamma_1(t_o)+0,t_o)}^{u(\gamma_2(t_o)-0,t_o)} - \cdots - f\Big]_{u(\gamma_n(t_o)+0,t_o)}^{u(x_2,t_o)} +$$

$$+ \sum_{i=1}^{N} f\Big]_{u(\gamma_i(t_o)-0,t_o)}^{u(\gamma_i(t_o)+0,t_o)} =$$

$$- f\Big]_{u(x_1,t_o)}^{u(x_2,t_o)} = - \int_{u(x_1,t_o)}^{u(x_2,t_o)} a .$$

A similar calculation with the paths $\{\delta_i\}_1^{N'}$ gives

$$\left(\frac{d}{dt}\right)^{-}\Big|_{t_o} \int_{x_1}^{x_2} u(x,t)\,dx = - \int_{u(x_1,t_o)}^{u(x_2,t_o)} a.$$

Thus (2.13) holds, so u is an IDE solution.

<u>Claim 6</u>: If u is an IDE solution, then u satisfies the Rankine-Hugoniot condition.

<u>Proof</u>. Let (x_o,t_o) be an interior point of the discontinuity-path γ. For small $\varepsilon > 0$ we have

$$\int_{u(x_o-\varepsilon,t_o)}^{u(x_o+\varepsilon,t_o)} a = - \frac{d}{dt}\Big|_{t_o} \int_{x_o-\varepsilon}^{x_o+\varepsilon} u(x,t)\,dx = - \frac{d}{dt}\Big|_{t_o} \left(\int_{x_o-\varepsilon}^{\gamma(t)} + \int_{\gamma(t)}^{x_o+\varepsilon}\right) =$$

$$\gamma'(t_o)\cdot[u] - \int_{x_o-\varepsilon}^{x_o+\varepsilon} u_t(x,t_o)\,dx.$$

Let ε approach 0 to get $[f] = \gamma'\cdot[u]$.

QED 2.21.

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