NONNEGATIVELY CURVED MANIFOLDS
DIFFEOMORPHIC TO EUCLIDEAN SPACE

A Dissertation presented

by

Douglas Everett Elerath

to

The Graduate School

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in

Mathematics

State University of New York

at

Stony Brook

August, 1977

STATE UNIVERSITY OF NEW YORK

AT STONY BROOK

THE GRADUATE SCHOOL

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Herbert Weisinger
Dean of the Graduate School

August, 1977

Abstract of the Dissertation

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In this dissertation we prove that if M^n is a complete, open nonnegatively curved manifold, and if for some p in M there exists a real number $r_o > 0$ such that $K(x) > (\pi/\lambda r_o)^2$ for each x in M with $d(x,p) < r_o$, $\lambda \cong 2.46$, then M is diffeomorphic to $\mathbb{R}^n$. Here K(x) denotes the infimum of all sectional curvatures at the point x.

To achieve this result, we (1) characterize the cut locus on complete open surfaces of revolution whose curvature is a monotone nonincreasing function of the distance from the vertex, and (2) prove a generalized version of Toponogov's comparison theorem in which the comparison surface is a surface of revolution as in (1) above.

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Acknowledgments

My most long standing mathematical debt is to Russ Butts, without whose influence I would still be an engineer. The guidance and excellent teaching provided by the faculty at LSU, and especially by Professors David Roselle, Richard Schori and Carruth McGehee, are largely responsible for my having survived my formal introduction to mathematics. My deepest thanks, however, are reserved for my advisor, Professor Jeff Cheeger. Without his guidance, advice and occasional prodding, this dissertation would most likely have remained an ill-formed idea. Finally, I thank my wife Linda for typing the manuscript.

INTRODUCTION

Conjecture (Cheeger and Gromoll): If M^n is a complete open nonnegatively curved manifold, and if all sectional curvatures at some point $x \in M$ are positive, then M is diffeomorphic to R^n.

This conjecture is known to be true if $n = 2$ (Cohn-Vossen [1936]), if $n = 3$ (Cheeger and Gromoll [1972]), or if $n = 4$ (Cheeger and Gromoll, unpublished). If the words "some point" are replaced by "every point," the conjecture follows from Gromoll and Meyer [1969] or from Cheeger and Gromoll [1972].

This thesis grew out of an attempt to prove the above conjecture in its entirety. I have had to settle for less. The main result, theorem 5.2, is the proof of a version of this conjecture with curvature conditions stronger than those stated in the above, but weaker than those in Gromoll and Meyer. On the way to this result I characterize the cut locus on certain surfaces of revolution and prove a generalized version of Toponogov's theorem.

I think that the techniques used in this thesis can be improved to achieve stronger partial results than are presented here, but that the return will not justify the investment. These techniques are not up to the task of proving the full conjecture. Known "global" comparison theorems are still too local, i.e., local irregularities in curvature can affect the entire comparison surface.

1. PRELIMINARIES

This section summarizes those elementary aspects of differential geometry which are cited in the body of this thesis, and establishes notation to be used throughout. A proof will be given only for corollary 1.4.2 which, to the best of my knowledge, is not standard. All proofs and definitions which are omitted from this section are most easily found in Cheeger and Ebin [1975].

1.1 Notation

M^n will always denote a complete n-dimensional Riemannian manifold with metric $\langle , \rangle$ and Levi-Civita connection ∇. Unless otherwise specified, all geodesics $\gamma : [0,b] \to M$ will be assumed to be normal: $\|\gamma'\| = 1$, where $\gamma'(t)$ denotes the tangent vector to γ at $\gamma(t)$. If x and y are points in M, then Cur(x,y), Geo(x,y) and $Geo_m(x,y)$ will denote respectively those curves, geodesics and globally minimal geodesics which begin at x and end at y. Elements of Cur(x,y) are required to be at least piecewise smooth. Thus if $\gamma \in Cur(x,y)$ is parameterized on [0,b], then $\gamma(0) = x$ and $\gamma(b) = y$. Similarly, if $N \subseteq M$, then Cur(N), Geo(N) and $Geo_m(N)$ will denote respectively those curves, geodesics and globally minimal geodesics which have images contained in N. Again, elements of Cur(N) are required to be at least piecewise smooth. If the parameter range of any curve is not specified, it is understood to be $[0,b]$. Finally, if $\gamma \in \mathrm{Geo}(x,y)$ and $\eta \in \mathrm{Geo}(x,z)$, we let $\measuredangle(\gamma,\eta)(x) := \arccos\langle\gamma'(0),\eta'(0)\rangle$, $0 \le \measuredangle(\gamma,\eta) \le \pi$.

If $\gamma : [0,b] \to M$, define $-\gamma : [0,b] \to M$ by $-\gamma(t) := \gamma(b-t)$. If $\gamma_1 \in \mathrm{Cur}(x,y)$ and $\gamma_2 \in \mathrm{Cur}(y,z)$ are parameterized on $[0,b_1]$ and $[0,b_2]$ respectively, define $\gamma_1 \vee \gamma_2 : [0,b_1+b_2] \to M$ by

$$\gamma_1 \vee \gamma_2 := \begin{cases} \gamma_1(t) & \text{if } t \in [0,b_1] \\ \gamma_2(t-b_1) & \text{if } t \in [b_1,b_1+b_2]. \end{cases}$$

Thus $\gamma_1 \vee \gamma_2 \in \mathrm{Cur}(x,z)$.

And sometimes we shall, without further warning, let the same symbol denote both a curve and its image.

If $N \subset M$ is any subset and $r \in \mathbb{R}$ is any positive real number, $T_r(N) := \{x \in M \mid d(x,N) < r\}$ is an (open) tubular neighborhood of N. Here, of course, d denotes distance in the metric space structure induced by the connection.

1.2 First Variation Formula

If $c : [a,b] \to M$ is a smooth curve in M, then $L[c]$ will denote the length of c, $L[c] := \int_a^b \|c'(t)\|\,dt$. This applies as well if c is only piecewise smooth by omitting the singular points of $\|c'\|$ from the range of integration.

If $\alpha : (-\epsilon,\epsilon) \times [0,b] \to M$ is a smooth variation of c, so $\alpha|(\{0\} \times [0,b]) = c$, then we let $c_s(t) := \alpha(s,t)$ and speak of the variation $\{c_s\}$. If c_0 is parameterized proportional to arclength, so that $\|c_0'\|$ is constant, and if $S := d\alpha(\partial/\partial s)$ and $T := d\alpha(\partial/\partial t)$, then the <u>first variation formula</u> is

1.2.1 $$\frac{d}{ds} L[c_s]|_{s=0} = \|c_0'\|^{-1}\left(\langle S,T\rangle|_0^b - \int_0^b \langle S,\nabla_T T\rangle \, dt \right).$$

If $c \in \mathrm{Geo}(M)$, then this becomes

1.2.2 $$\frac{d}{ds} L[c_s]|_{s=0} = \langle S,T\rangle|_0^b.$$

1.3 Curvature and Conjugate Points

If $x \in M$ and σ is a plane in M_x, then $K(\sigma)$ will denote the sectional curvature of M at x determined by any two vectors spanning σ. The notation $K(x)$ will occure only in conjunction with an inequality sign, and will denote either $\inf\{K(\sigma)\}$ or $\sup\{K(\sigma)\}$ over all planes σ in M_x according as we have $K(x) \geq$ or $K(x) \leq$.

If $\gamma \in \mathrm{Geo}(x,y)$, we will say that γ is <u>free of conjugate points</u> when we mean that $\gamma(t)$ is not conjugate to x along γ for any t in the domain of γ.

1.3.1 <u>Lemma</u>: Let M^n and M_0^{n+k} be Riemannian manifolds, let $\gamma : [0,b] \to M$ and $\gamma_0 : [0,b] \to M_0$ be normal geodesics, and suppose that $K(\gamma_0(t)) \geq K(\gamma(t))$ for all $t \in [0,b]$. Then if γ_0 is free of conjugate points, so is γ.

Proof: See either Gromoll, Klingenberg and Meyer [1975] pp 174-6, or Cheeger and Ebin [1975] p 30. This is proved in the first reference without use of, and in the second reference as a corollary of the Rauch comparison theorem. ∎

Remark: If M and M_o are both 2-dimensional this lemma follows immediately from the Sturm comparison theorem for second order ordinary differential equations. Thus, despite the application of this relatively modern result [c1950] to surfaces in section 2, the techniques there should be considered entirely classical.

1.4 Rauch-Berger Comparison Theorem

Suppose that x and y are points in M^n. If $\gamma \in \mathrm{Geo}(x,y)$, we will say that γ is free of focal points when we mean that $\gamma(t)$ is not a focal point of the $(n-1)$-dimensional embedded submanifold defined by restricting exp to a sufficiently small neighborhood of $0 \in \gamma'(0)^{\perp} \subset M_x$.

1.4.1 Theorem (Berger): Let M^n and M_o^{n+k} be Riemannian manifolds. Let $\gamma : [0,b] \to M$ and $\gamma_o : [0,b] \to M_o$ be normal geodesics with γ_o free of focal points. Assume for each $t \in [0,b]$, each $v \in M_{\gamma(t)}$ and each $v_o \in M_{o\gamma_o(t)}$ that the sectional curvatures of the sections σ spanned by $(\gamma'(t),v)$ and σ_o spanned by $(\gamma_o'(t),v_o)$ satisfy $K(\sigma_o) \geq K(\sigma)$. Let

$T(t) := \gamma'(t)$ and $T_o(t) := \gamma_o'(t)$, and let J and J_o be Jacobi fields along γ and γ_o respectively satisfying

(1) $(\nabla_T J)(0)$ and $(\nabla_{T_o} J_o)(0)$ are tangent to γ and γ_o respectively,

(2) $\|\nabla_T J\|(0) = \|\nabla_{T_o} J_o\|(0)$,

(3) $\langle T, J\rangle(0) = \langle T_o, J_o\rangle(0)$, and

(4) $\|J\|(0) = \|J_o\|(0)$.

Then for each $t \in [0,b]$, $\|J\|(t) \geq \|J_o\|(t)$.

Proof: See Cheeger and Ebin [1975], theorem 1.29. ∎

The following corollary is a slight generalization of that presented in Cheeger and Ebin.

1.4.2 Corollary: Let $\gamma : [0,b] \to M^n$ and $\gamma_o : [0,b] \to M_o^{n+k}$ be normal geodesics, and let E and E_o be parallel unit vector fields along γ and γ_o respectively with $\langle E, \gamma'\rangle = \langle E_o, \gamma_o'\rangle$. Suppose that $c : [0,b] \to M$ and $c_o : [0,b] \to M_o$ are defined by

$$c(t) := \exp_{\gamma(t)}(f(t)E(t)) \text{ and}$$
$$c_o(t) := \exp_{\gamma_o(t)}(f(t)E_o(t))$$

where $f : [0,b] \to \mathbb{R}$ is smooth. Let $\eta_t : [0,1] \to M$ and $\eta_{ot} : [0,1] \to M_o$ be defined by

$$\eta_t(s) := \exp_{\gamma(t)}(sf(t)E(t)) \text{ and}$$

$$\eta_{ot}(s) := \exp_{\gamma_o(t)}(sf(t)E_o(t)).$$

Assume that for each $(t,s) \in [0,b]x[0,1]$, $K(\eta_t(s)) \leq K(\eta_{ot}(s))$, and that for each $t \in [0,b]$, η_{ot} is free of focal points. Then $L[c] \geq L[c_o]$.

<u>Proof</u>: Since c and c_o are both parameterized on $[0,b]$ it suffices to compare the lengths of their tangent vectors.

If $t_1 \in [0,b]$ is fixed, and if $\alpha : (-\epsilon,\epsilon)x[0,1] \to M$ is a variation given by $\alpha(t,s) := \eta_{t_1+t}(s)$, then $V(s) := \partial\alpha/\partial t(0,s)$ is a Jacobi field along η_{t_1} with end values $V(0) = \gamma'(t_1)$ and $V(1) = c'(t_1)$ (cf. Milnor [1963], §14.4). Note further that since $\nabla_{\gamma'}E = 0$ and $\eta_t'(0) = f(t)E(t)$,

$$\begin{aligned}(\nabla_{\eta_{t_1}'}V)(0) &= [\nabla_{V(0)}f(t)E(t)](t_1) \\ &= [V(f(t))E(t)+f(t)\nabla_{\gamma'}E(t)](t_1) \\ &= f'(t_1)E(t_1).\end{aligned}$$

Similarly, we find that $(\nabla_{\eta_{ot_1}'}V_o)(0) = f'(t_1)E_o(t_1)$.

This shows that V and V_o satisfy condition (1) of 1.4.1. Evidently they satisfy the other conditions, and thus $\|V\|(1) \geq \|V_o\|(1)$. Hence $\|c'(t_1)\| \geq \|c_o'(t_1)\|$ for any $t_1 \in [0,b]$, and thus $L[c] \geq L[c_o]$. ∎

<u>Remark</u>: Note that $\langle E,\gamma'\rangle$ and $\langle E_o,\gamma_o'\rangle$ were <u>not</u> required to be zero in this corollary.

1.5 Cut Points and the Cut Locus

Let $x \in M^n$ and $\gamma \in Geo(M)$ with $\gamma(0) = x$. We will call $y := \gamma(t_0)$ the cut point of x (along γ) if γ is minimal between x and $\gamma(t)$ for all $t \leq t_0$ and for no $t > t_0$. The triangle inequality implies that such a y, if it exists, is unique. C(x) will denote the cut locus of x, the set of all cut points of x. Observe that for any $x \in M$, M may be viewed as C(x) with the proper boundary points identified. The following lemma characterizes cut points.

1.5.1 Lemma: Let $\gamma \in Geo(M)$. Then $\gamma(t_0)$ is the cut point of $x := \gamma(0)$ along γ if and only if one of the following holds for $t = t_0$ and neither holds for any smaller value of t:

(1) $\gamma(t_0)$ is conjugate to x along γ, or

(2) there is a geodesic $\eta \neq \gamma$ from x to $\gamma(t_0)$ such that $L[\eta] = L[\gamma]$.

Proof: See Cheeger and Ebin [1975], lemma 5.2. ∎

We also have

1.5.2 Lemma: The distance to the cut locus is a continuous function defined on an open subset of the unit sphere bundle of M. In particular, C(x) is a closed set.

Proof: See Cheeger and Ebin [1975], prop. 5.4. ∎

And finally

1.5.3 Lemma: The element of $C(x)$ which is nearest to x is either a conjugate point or lies on a nontrivial element of $Geo(x,x)$.

Proof: See Cheeger and Ebin [1975], lemma 5.6. ∎

2. THE CUT LOCUS ON NICE SURFACES OF REVOLUTION

Interest in the cut locus on surfaces dates back to Poincaré [1905] and Myers [1935],[1936]. They both proved that the cut locus of a point x on a surface S can contain no closed curve, and that the end points of the cut locus of x are conjugate to x. In addition, Myers proved that on an analytic surface the cut locus of any point is a tree with finitely many nodes in any compact subset of S.

To illustrate the notion of the cut locus, Myers describes the cut locus on several standard surfaces such as the plane, the sphere and the ellipsoid, and then states that "examples of well known simply connected surfaces on which the (cut) locus assumes a complicated, but determinable, form are naturally hard to give." Indeed. It is in fact extremely difficult to calculate the cut locus on a specific surface unless the surface is quite nice. Gluck and Singer [1976] have shown that any smooth manifold of dimension ≥ 2 can be given a Riemannian metric with a non-triangulable cut locus, and they construct such a metric on the 2-sphere.

Previously, "quite nice" has commonly meant a quadric surface, or perhaps a cylinder or torus. Even in the case of a surface of revolution, where geodesics can be explicitly exhibited in integral form, the cut locus is not generally known. This problem was, however, solved for paraboloids and hyperboloids in a beautiful paper by von Mangoldt [1881]. In this paper he sidesteps the task of calculating the elliptic integrals defining geodesics on these surfaces, and computes the conjugate locus directly. The cut locus is then immediately apparent.

In this section I will extend the notion of a nice surface of revolution, and in fact generalize the results of von Mangoldt. The techniques will not be his; but, as mentioned in section 1, the techniques are essentially classical.

Darboux [1895] and Forsyth [1920], both of whom derive 2.2.1 and 2.2.2 below, are good general references for this section. Although Spivak [1975] only derives 2.2.2, his pictures are very nice and his discussion fairly complete.

2.1 Surfaces of Revolution

For the moment, let (x,y,z) denote the standard coordinate axes in $\mathbb{R}^3$.

Suppose $f: [0,\infty) \to \mathbb{R}$ is a smooth function, with $f(0) = f'(0) = 0$, which is viewed as mapping the positive y-axis to the z-axis. Thus the graph of f is smoothly embedded in the (y,z)-plane. By revolving the graph of f about the z-axis we get a surface of revolution $S : [0,\infty)\times[0,2\pi) \to \mathbb{R}^3$ defined by

2.1.1 $$S(r,\theta) := (r \cos\theta, r \sin\theta, f(r)),$$

and we call $p := S(0,\cdot)$ the vertex of S. Representing $\mathbb{R}^2$ in polar coordinates, define $(r,\theta) : S \to \mathbb{R}^2$ by

2.1.2 $$(r,\theta)(r \cos\theta, r \sin\theta, f(r)) := (r,\theta).$$

Finally, define $\rho : S \to \mathbb{R}$ by

2.1.3 $$\rho(r \cos\theta, r \sin\theta, f(r)) := \int_0^r (1+f'^2)^{1/2}\, dz,$$

which parameterizes the graph of $f \subset S$ by arc length. Notice that (r,θ) is orthogonal projection of S into the (x,y)-plane while (ρ,θ) is a non-orthogonal projection.

This said, we abandon the use of x,y and z as coordinate lables and instead allow them to represent points on the surface S.

2.2 Geodesics on Surfaces of Revolution

With reference to the surface S defined in 2.1.1, the curves r = constant are called parallels and the curves θ = constant are called meridians. On such a surface every meridian is a geodesic and no parallel is a geodesic. If $\gamma \in \text{Geo}(S)$ and $x \in \gamma$, then $\Psi_\gamma(x)$ will denote the acute angle which γ makes with the meridian through x. If γ is itself a meridian of course $\Psi_\gamma = 0$, and conversely.

If $c : [0,b] \to S\backslash\{p\}$ is a smooth curve, we define $\theta_c : [0,b] \to \mathbb{R}$ in the following manner: $\theta_c(t)$ is the absolute value of the total angle (in polar coordinates) traced out by the curve $(r,\theta)\circ c \subset \mathbb{R}^2$ as the parameter value increases from 0 to t. In computing the total angle, clockwise travel is negative and counterclockwise travel is positive.

A brief examination of the metric on such a surface S yields firstly the following integral expression for a geodesic γ on S:

2.2.1 $$|\theta_\gamma(t_1)-\theta_\gamma(t_o)| = \tau_\gamma \int_{r(\gamma(t_o))}^{r(\gamma(t_1))} \frac{(1+f'^2)^{1/2}}{z(z^2-\tau_\gamma^2)^{1/2}}\, dz ,$$

where τ_γ is a non-negative constant of (a previous) integration; and secondly Clairaut's theorem:

2.2.2 $$r \sin\Psi_\gamma = \tau_\gamma .$$

Using these two equations we are able to give a fairly complete qualitative description of the behavior of geodesics on S.

2.2.3 Let P be any parallel on S and let $\gamma : [0,\infty) \to S$ be a normal geodesic with $\gamma(0) \in P$ and $\gamma'(0)$ tangent to P. It follows that $r(P) = \tau_\gamma$ and that γ never enters into the bounded region $r^{-1}([0,\tau_\gamma))$ which lies below P. Since r is 1-1 on the set of parallels of S, γ can be tangent to no other parallel. Thus, for $\gamma(t_o) \notin P$,

$\frac{d(r\circ\gamma)}{dt}\Big|_{t=t_0} \neq 0$. In fact $\frac{d(r\circ\gamma)}{dt} > 0$ for all $t > 0$, so that the distance of $\gamma(t)$ from the axis of revolution, and hence from the vertex of S, increases monotonically with t. This implies that furthermore

$$|\theta_\gamma(t)| = \tau_\gamma \int_{\tau_\gamma}^{r(\gamma(t))} \frac{(1+f'^2)^{1/2}}{z(z^2-\tau_\gamma^2)^{1/2}}\, dz$$

increases monotonically with t.

2.2.4 Claim: γ leaves every compact set ($\rho : S \to [0,\infty)$ is onto).

Proof: Suppose γ is condemned to revolve about the vertex in a compact region of radius less than r_0. Since f is smooth, there exists a constant $k > \tau_\gamma$ such that

$$|\theta_\gamma(t)| \le k\cdot\tau_\gamma \int_{\tau_\gamma}^{r_\rho} \frac{1}{z(z^2-\tau_\gamma^2)^{1/2}}\, dz \quad ,$$

which is finite. Since the length of $\gamma(=t)$ can increase by no more than $2\pi r_0$ in a given revolution, we see that the length of γ must be finite. This is absurd, so γ must leave every compact set. ∎

Now let γ, as defined in 2.2.3, be extended to all $\mathbb{R}$. Since the foregoing applies equally well to $\gamma|(-\infty,0]$, and since S is rotationally symmetric, we see that γ is symmetric about the point $\gamma(0)$. Furthermore, γ is, up to rotation, the unique geodesic tangent to P.

Finally, if $\eta : \mathbb{R} \to S$ is any geodesic, let P be the parallel with $r(P) = \tau_\eta$. Constructing γ as above, tangent to P, we see that γ and η are determined by the same constant $\tau_\eta = \tau_\gamma$, and so we may bring η and γ into coincidence via a rotation. Thus every geodesic is of the type discussed in the previous paragraphs.

We summarize these maunderings in the following description.

2.2.5 Description: On each geodesic $\gamma : \mathbb{R} \to S$ there is a unique point, denoted σ_γ, which is nearest to the vertex of S, about which γ is symmetric, and at which γ is tangent to a parallel of S. The two branches of γ proceed in either direction from this point and spiral in opposite senses around the axis of S heading monotonically towards infinity, so that $\rho \circ \gamma$ satisfies a maximum principle.

Ling [1946] provides a more detailed discussion of geodesics on a slightly more restricted class of surfaces, and in fact shows that their self intersections can be used to naturally partition the surface.

2.3 Nice Surfaces of Revolution

Definition: A surface of revolution

$$S(r,\theta) := (r\cos\theta, r\sin\theta, f(r))$$

will be called nice if

(1) S is smooth,

(2) $f(0) = f'(0) = 0$

(3) $dK/d\rho \leq 0$,

where K denotes the Gauss curvature of S and ρ was defined in 2.1.3.

This definition adds to our previous notion of a surface of revolution the requirement that the curvature decreases as distance from the vertex increases.

2.4 Notation

The following summary of notation on surfaces of revolution is collected here for later convenience.

(1) S := a nice surface of revolution,

$$S : [0,\infty)\times[0,2\pi) \to \mathbb{R}\ (2.1.1);$$

(2) $S_+ := S \cap (0,\infty)\times\mathbb{R}\times\mathbb{R}$, the front half of S;

(3) p := the vertex of S (2.1.1);

(4) $(r,\theta) := S^{-1}$ (2.1.2);

(5) $\rho : S \to [0,\infty)$ measures distance from the vertex (2.1.3);

(6) P_x := the parallel through a point $x \in S$;

(7) μ_x := the meridian through x with $\mu_x(0) = p$;

(8) $\overline{\mu}_x$:= the meridian opposite x with $\overline{\mu}_x(0) = p$,

i.e., $\overline{\mu}_x(t) := \exp_p(-t\mu_x'(0))$.

If $\gamma \in \mathrm{Geo}(S)$,

(9) $\tau_\gamma := \inf r(\gamma(t)),\ t \in \mathbb{R}$ (2.2.1,2.2.2);

(10) $\sigma_\gamma :=$ that unique point in S on γ for which $r(\sigma_\gamma) = \tau_\gamma$ (2.2.4);

(11) $\theta_\gamma : \mathrm{domain}(\gamma) \to \mathbb{R}$ is defined by

$$\theta_\gamma(t) := (\rho\circ\gamma)'(t)\cdot\tau_\gamma \int_{\tau_\gamma}^{r(\gamma(t))} \frac{(1+f'^2)^{1/2}}{z(z^2-\tau_\gamma^2)^{1/2}}\,dz;$$

(12) $F_\gamma : \mathrm{Im}\theta_\gamma \to \rho(\gamma)$ is defined by $F_\gamma := \rho\circ\gamma\circ\theta_\gamma^{-1}$ so that γ can be viewed as the graph of F_γ.

If $\gamma, \eta \in \mathrm{Geo}(S_+)$,

(13) we say that η <u>lies</u> <u>below</u> (<u>above</u>) γ if (i) $\theta(\gamma) \subset \theta(\eta)$, and (ii) for each $t \in \mathrm{domain}(\gamma)$, $\rho(\eta\cap\mu_{\gamma(t)}) \le (\ge)\ \rho(\gamma(t))$. The same terminology will be used if γ is simply a point. Note that if η lies below γ, it is not necessarily the case that γ lies above η.

2.5 Geodesics Revisited - Some Lemmas

This first lemma is implicit in Myers [1936], and is proved here primarily for the sake of completeness.

2.5.1 <u>Lemma</u>: Suppose M^2 is a smooth complete surface, and $x \in M$, $y \in M$. Given $g_0 \in \mathrm{Geo}_m(x,y)$, $h_0 \in \mathrm{Geo}_m(x,y)$ with $g_0 \cup h_0$

the boundary of a simply connected region D_0, then there exists $z \in Cl(D_0)$ such that x and z are conjugate along a geodesic $g \in Geo(Cl(D_0))$.

<u>Proof</u>: If x and y are not themselves conjugate, we may assume that h_0 has been chosen so that $\sphericalangle(g_0,h_0)(x) \leq \sphericalangle(g_0,h)(x)$ for all $h \in Geo_m(x,y)\backslash\{g_0\}$. Let the geodesic g_1 be chosen so that $g_1'(0)$ bisects $\sphericalangle(g_0,h_0)(x)$. Then there exists $z_1 \in D_0$ such that g_1 minimizes to, but not beyond, z_1. It is clear that $z_1 \notin (g_0 \cup h_0)$ since: (i) both g_0 and h_0 minimize to y, and (ii) $z_1 \neq y$ since $\sphericalangle(g_0,g_1)(x) < \sphericalangle(g_0,h_0)(x)$; and thus $z_1 \in Int(D_0)$.

If x and z_1 are not conjugate, we may choose $h_1 \in Geo_m(x,z_1)\backslash\{g_1\}$ so that $\sphericalangle(g_1,h_1)(x) \leq \sphericalangle(g_1,h)(x)$ for all $h \in Geo_m(x,z_1)\backslash\{g_1\}$. D_0 simply connected and g_0,h_0 both minimal imply that $h_1 \subset D_0$, so that $g_1 \cup h_1$ bounds a simply connected region $D_1 \subset D_0$. By induction, barring the appearance of some z_i conjugate to x, which would complete the proof anyway, we may generate sequences $\{z_i\}$, $\{g_i\}$. Letting $z = \lim z_i$, $\{g_i\}$ clearly converges to a geodesic $g \in Geo_m(x,z)$.

Furthermore, x and z are conjugate along g since: (i) $C(x)$ is closed implies that $z \in C(x)$, and (ii) if $h \in Geo_m(x,z)$, then $h \subset \bigcap(Cl(D_i)) = g$; and now apply lemma 1.5.2. ∎

2.5.2 Situation: In the three remaining lemmas we assume the following data:

(1) $\theta : S_+ \to (0,\pi)$;

(2) $x \in \partial S_+$ with $\theta(x) = 0$;

(3) $y \in S_+$ with $\rho(x) \leq \rho(y)$;

(4) $\gamma \in \mathrm{Geo}_m(x,y)$;

(5) $z \in P_y \cap S_+$ is such that $\theta(z) > \theta(y)$ (S_+ is open);

(6) $\eta \in \mathrm{Geo}_m(x,z)$; See fig 2.1 on p 23.

2.5.3 Lemma: $\eta \subset S_+$ and η lies below γ.

Proof: The first claim follows immediately upon observing that (i) S is symmetric with respect to reflection through the plane determined by ∂S_+, and (ii) η is minimal. Thus if h is any geodesic from x to z, we may reflect all portions of h not in S_+ through the plane determined by ∂S_+. If $h \not\subset \mathrm{Cl}(S_+)$, the result is a non-smooth curve from x to z of the same length as h, and hence h could not have been minimal. (This shows that S_+ is convex).

Since $\theta(z) > \theta(y)$, $\eta \subset S_+$ implies that $\theta(\gamma) \subset \theta(\eta)$. Since both γ and η are minimal, $\eta \cap \gamma = \{x\}$. Thus η lies either above γ or below γ. Let $H(t) := F_\eta(t+\theta_\eta(0))$ and $G(t) := F_\gamma(t+\theta_\gamma(0))$ map $\theta(\gamma)$ to $\mathbb{R}$. If η lies above γ, $H > G$, and in particular $H(\theta(y)) > G(\theta(y))$. Combining this with information from 2.4 yields $H(0) < H(\theta(z)) < H(\theta(y))$, while $0 < \theta(y) < \theta(z)$. This is of course impossible since H cannot achieve a maximum at an interior point (cf. 2.2.5). ∎

2.5.4 Lemma: $L[\gamma] < L[\eta]$.

Proof: Suppose that P_y is parameterized, not by arc length, but such that $\theta(z)-\theta(P_y(s)) = s$, and let $z_s := P_y(s)$. From $\eta'(\theta) > 0$, $P_y'(\theta) < 0$ and $P_y'(\rho) = 0$, it follows that $\langle\eta', P_y'\rangle(z) < 0$. Now choose a smooth variation $\{c_s\} : (-\epsilon,\epsilon)x[0,b] \to S$ of η with $c_s(0) = x$ and $c_s(b) = z_s$. Then $\langle\partial/\partial s\{c_s\}, c_0'\rangle|_{\substack{s=0\\t=b}} = \langle P_y', \eta'\rangle(z) < 0$, and so the first variation formula implies that $d/ds|_{s=0}L[c_s] < 0$.

Since both lemma 2.5.3 and the argument of the preceeding paragraph apply as well to each $\eta_s \in Geo_m(x,z_s)$ for all relevant s, i.e., when $\theta(y) < z_s \leq \theta(z)$, it follows that $L[\eta_s]$ decreases monotonically as s increases. Thus $L[\gamma] < L[\eta]$. ∎

2.5.5 Lemma: $\rho(\gamma(t)) > \rho(\eta(t))$ for each t in the domain of γ.

Proof: Let P be any parallel with $\tau_\gamma < \rho(P) \leq \rho(y)$. Let t_1 be such that $\gamma(t_1) \in P$ and $F_\gamma'(t_1) > 0$, and thus for all $t > t_1, \rho(\gamma(t)) > \rho(P)$. (That such a t_1 exists is clear from the data in 2.5.2 and the description 2.2.5). Let s_1

be similarly defined in terms of η. The argument given in 2.5.4 applies, mutatis mutandis, to show that $t_1 < s_1$.

If $P \cap \gamma$ is but a single point, $P \cap \eta$ will likewise be a single point, and necessarily these intersections will be the $\gamma(t_1)$ and $\eta(s_1)$ described in the previous paragraph ($\rho(y) = \rho(z) > \rho(x)$). Thus, since P lies above $\eta|[0,s_1]$, $\rho(\eta(t_1)) < \rho(\eta(s_1)) = \rho(\gamma(t_1))$.

If $P \cap \gamma$ is two points (the only other case), $P \cap \eta$ will also be two points, and it must be that $F_\gamma'(0) < 0$. Define t_o so that $\gamma(t_o) \in P$ and $F_\gamma'(t_o) < 0$, and let s_o be similarly defined in terms of η. Thus if $t_o < t < t_1$, $\rho(\gamma(t)) < \rho(P)$; while if $s_o < t < s_1$, $\rho(\eta(t)) < \rho(P)$. Let $w := \gamma(t_o)$ and let P_w be parameterized so that $\theta(w)-\theta(P_w(s)) = s$. Let $w_s := P_w(s)$ and let $\gamma_s \in Geo_m(x,w_s)$ for $0 \leq s < \theta(w)$. It is easily seen that $F_{\gamma_s}'(\theta(w_s)) < 0$: Rotate γ so that $\gamma(t_o)$ passes through w_s; it is then clear that γ_s is trapped above this rotated version of γ, and thus $F_{\gamma_s}'(\theta(w_s)) < F_\gamma'(t_o) < 0$. Thus $\langle\gamma_s', P'\rangle_{w_s} < 0$, and using a first variation argument as in 2.5.4 we see that $s_o < t_o$ (the geodesics are parameterized by arc length).

Now, given any $t \in domain(\gamma)$ we construct $P_{\gamma(t)}$, with s_o, s_1, t_o and t_1 as above. The foregoing argument implies that $s_o < t_o \leq t \leq t_1 < s_1$, and thus that $\rho(\eta(t)) < \rho(\eta(s_1)) = \rho(\gamma(t_1)) = \rho(\gamma(t))$. ∎

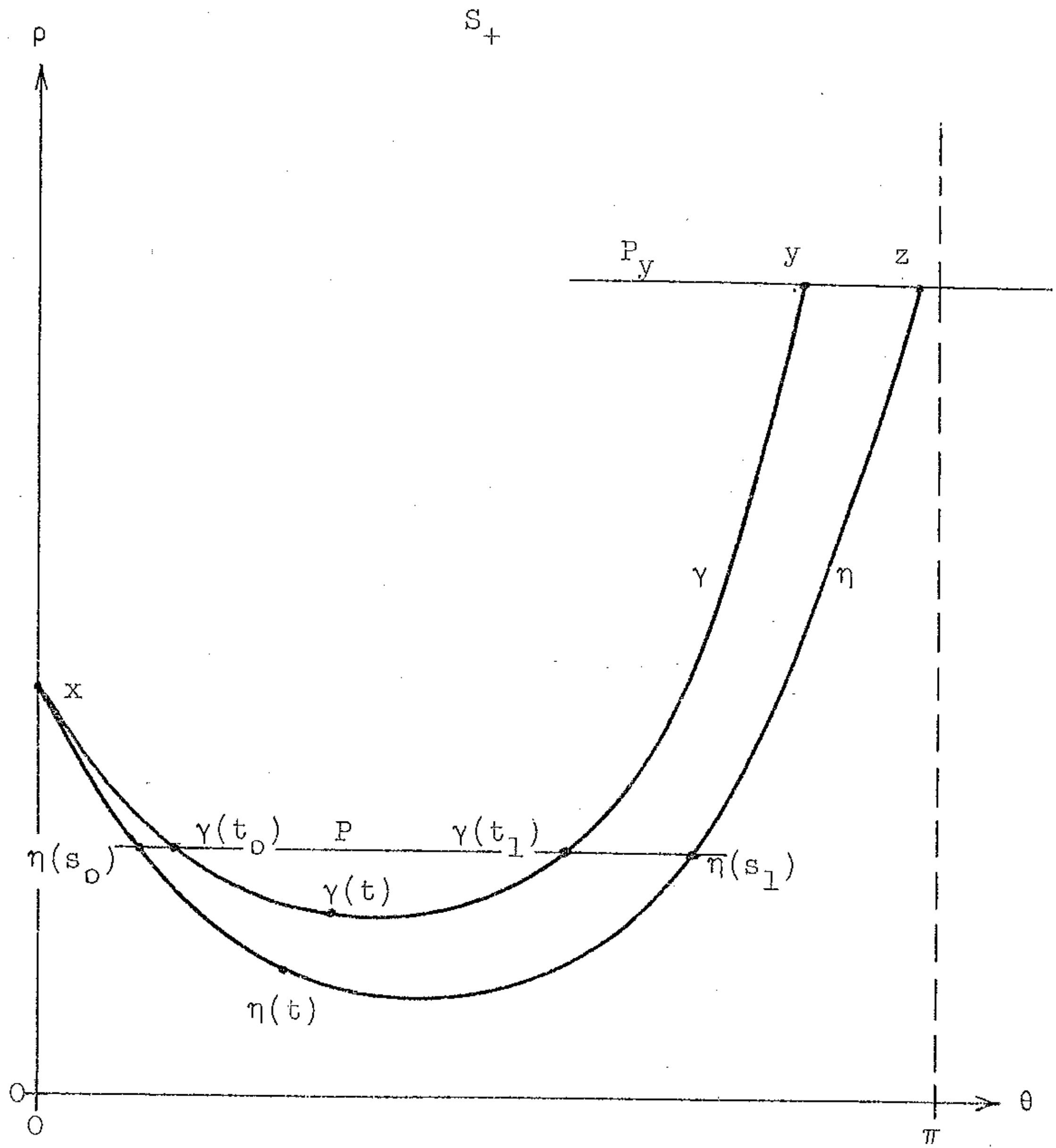


Figure 2.1

2.6 The Cut Locus on a Nice Surface

Theorem: Let S be a nice surface of revolution and $x \in S$. Then $C(x) \subset \overline{\mu}_x$.

Proof: Without loss of generality we may suppose that $x \in \partial S_+$, $\theta(S_+) = (0,\pi)$, and $\theta(x) = 0$. If $C(x) \cap S_+ \neq \emptyset$, then the convexity of S_+ (shown in 2.5.3), 1.5.1 and 2.5.1 together imply that there exists a $y \in S_+$ such that x and y are conjugate along a geodesic in S_+. By relabeling, if necessary, we may suppose that $\rho(x) < \rho(y)$. We will now see that such a situation, i.e., x conjugate to such a y in S_+, is not possible, and thus that $C(x) \cap S_+ = \emptyset$.

So, suppose we are given situation 2.5.2 on S. Then, since S is nice, $\rho(\eta(t)) < \rho(\gamma(t))$ for $t \in \text{domain}(\gamma)$ implies that $K(\eta(t)) \geq K(\gamma(t))$. Moreover, since η is minimal, x can have no conjugate points along η for a distance equal to $L[\eta] > L[\gamma]$. Thus lemma 1.3.1 shows that y is not conjugate to x along γ, and we are done. ∎

2.7 Example

This example shows that 2.6 is sharp in the following sense: Given $\epsilon > 0$, there exists a surface of revolution S such that (i) $dK/d\rho \leq 0$ except on a set $E \subset S$, and (ii) $m(E) < \epsilon$, where m denotes Lebesque measure. Equivalently, by multiplying the metric by a constant, we can replace

(ii) by (ii') $dK/d\rho|_E < \epsilon$.

We will construct a surface which is only piecewise smooth, and in fact does not satisfy our definition of a surface of revolution. But standard approximation theorems, see, z.b., Aleksandrov and Zalgaller [1967], imply that this is sufficient.

2.7.1 Let $f : [0,\infty) \to \mathbb{R}^2$ be the curve

$$f(t) := \begin{cases} (t,0) & 0 \le t \le 1 \\ (1,t-1) & 1 \le t \end{cases},$$

and let S be the surface generated by revolving f about the vertical axis. Let $\gamma : \mathbb{R} \to S$ be a geodesic with $0 < \tau_\gamma < 1$, and $\sigma_\gamma = (\tau_\gamma, 0, 0)$. Then for some $\delta > 0$, $\gamma|[-t_o, t_o] \subset S_+$, where $t_o := (1-\tau_\gamma{}^2)^{1/2}+\delta$, and both $\gamma(t_o)$ and $\gamma(-t_o)$ are in the cylindrical part of S. Thus there is another geodesic $\eta \in \mathrm{Geo}(\gamma(-t_o), \gamma(t_o)) \cap \mathrm{Geo}(S_+)$ which is distinct from γ, and so $C(\gamma(t_o)) \cap S_+ \neq \emptyset$.

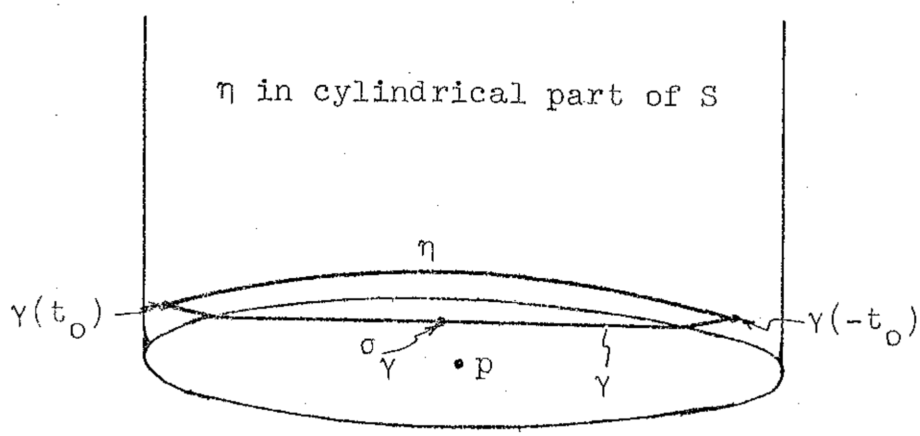


Figure 2.2

Now we can find an approximation to S for which (i) and (ii), or (i) and (ii') hold.

2.7.2 With a little more care, (i), (ii) and (ii') can all be made to hold simultaneously. Here we let $f_\theta : [0,\infty) \to \mathbb{R}^2$ be the curve

$$f_\theta(t) := \begin{cases} (1-\cos(t), \sin(t)) & 0 \le t \le \theta \le \pi/2 \\ (1-\cos(\theta), t-\theta+\sin(\theta)) & \theta \le t \end{cases},$$

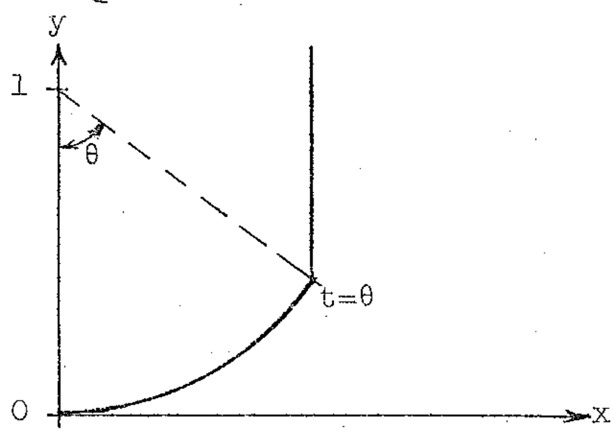


Figure 2.3

and let S_θ be the surface generated by revolving f about the vertical axis. Just as in 2.7.1 we find, for a given $\theta < \pi/2$, that there is an $x \in S_+$ with $C(x) \cap S_+ \neq \emptyset$.

If we now approximate S_θ by a smooth surface of revolution T_θ which satifies (i) and (ii') (T_θ may differ from S_θ by a conformal factor), then by increasing θ we can make T_θ satisfy (ii) while continuing to satisfy (i) and (ii').

3. HINGES, TRIANGLES AND BRANCHED COVERINGS

If we were to try to prove theorem 4.2 at this stage, we would encounter difficulties due to the fact that geodesics may wind around the axis of a surface of revolution more than once. We will resolve this potential problem in a standard fashion by use of an appropriate branched covering space (cf. Springer [1957]) with the induced geometry.

This section defines hinges and geodesic triangles in manifolds, and then examines hinges more closely on nice surfaces of revolution and their branched coverings. This examination is rather brief. But it provides the technical lemmas, of themselves fairly trivial and unimportant, which will be needed in section 4.

3.1 Notation and Definitions

Throughout this section, any geodesic written γ_i is assumed to be parameterized on $[0,b_i]$, and $e_i := \gamma_i(b_i)$. Furthermore, if $\{\gamma_i^\alpha\}$ is a family of geodesics which is parameterized by α with each γ_i^α parameterized on $[0,b_i]$, then $e_i^\alpha := \gamma_i^\alpha(b_i)$.

If (M,d) is any metric space with $A \subset M$, $B \subset M$, let $d_A(B) := \sup\{d(A,x) \mid x \in B\}$, and define the hausdorff distance between A and B by

$$hd(A,B) := \max \{d_A(B), d_B(A)\}.$$

This is clearly a metric on the power set of M.

3.2 Hinges and Triangles

Let M be a Riemannian manifold.

3.2.1 Definition: A hinge in M is a triple $(\gamma_1, \gamma_2, \alpha)$ with

(1) $\gamma_i \in \text{Geo}(M)$,

(2) $e_0 := \gamma_1(0) = \gamma_2(0)$, and

(3) $\measuredangle(\gamma_1, \gamma_2)(e_0) = \alpha$.

By a family of hinges $\{(\gamma_1{}^\alpha, \gamma_2, \alpha)\}$ we mean that for each α in some interval, $(\gamma_1{}^\alpha, \gamma_2, \alpha)$ is a hinge in M with each $\gamma_1{}^\alpha$ parameterized on $[0, b_1]$. Note that $e_1{}^\alpha := \gamma_1{}^\alpha(b_1)$ is a continuous function of α.

Sometimes, given a hinge $(\gamma_1, \gamma_2, \alpha_0)$, we will speak of increasing or decreasing α_0 by moving, say, γ_1. This proceedure may be viewed formally by constructing the family of hinges $\{(\gamma_1{}^\alpha, \gamma_2, \alpha)\}$, $\alpha \in [a,b]$, with $\alpha_0 \in [a,b]$. Then, if $\alpha_1 \in [a, \alpha_0)$, to "decrease α_0 to α_1" means to consider $\gamma_1{}^{\alpha_1}$ to be the geodesic resulting from this "movement".

3.2.2 Definition: A triple of geodesics on M, $(\gamma_0, \gamma_1, \gamma_2)$, is a geodesic triangle (on M) if $\gamma_i(b_i) = \gamma_{i+1}(0)$, $i \in Z_3$. Note that if each γ_i is minimal, then it is apparent that the γ_i satisfy the triangle inequality, i.e.,

$$L[\gamma_i] + L[\gamma_{i+1}] \geq L[\gamma_{i+2}], \quad i \in Z_3.$$

3.3 Hinges in S

Throughout 3.3, R is a fixed positive real number, and attention is restricted to the compact ball $Cl(T_{2R}(p)) \subset S$ with center at the vertex of S, S a nice surface of revolution.

The following lemma is a bit unwieldy, but not without purpose. In this lemma we define a number of constants $\delta_1, \ldots, \delta_5$ associated to a given nice surface of revolution. Having five instead of just one allows us to more easily point out exactly what hypotheses are needed in later propositions.

3.3.1 Lemma: Let S be a nice surface of revolution and fix $R \in \mathbb{R}$.

(1) There exists $\delta_1 > 0$ so that if $\gamma \in Geo(T_{2R}(p))$ with $L[\gamma] < d(\gamma(0),p) + 2\delta_1$, then γ is free of conjugate points.

(2) There exists $\delta_2 > 0$ so that if $\gamma \in Geo(T_{2R}(p))$ with $L[\gamma] < \delta_2$, then γ is free of focal points.

Let G denote the set of geodesics in $T_R(p)$ which are parameterized on $[0,b]$ such that if $\gamma \in G$, then $L[\gamma] < d(\gamma(0),p) + \delta_1$.

(3) There exists $\delta_3 > 0$ such that if $\gamma \in G$ and if η is any geodesic with $hd(\gamma,\eta) < \delta_3$, then η is unique between its end points among all those geodesics

whose hd distance from γ is less than δ_3.

(4) There exists $\delta_4 > 0$ such that if $\gamma \in G$ and if $z \in T_{\delta_4}(\gamma(0))$, then there exists $\eta \in \mathrm{Geo}(\gamma(b),z)$ with $\mathrm{hd}(\eta,\gamma) < \delta_3$ and $L[\eta] < L[\gamma] + \delta_1$. We may furthermore require that $\measuredangle(\gamma,\eta)(\eta(0)) < \pi$ unless $\gamma \vee \eta \in \mathrm{Geo}_m(\gamma(0),z)$ (this is used only in 3.3.3).

(5) Given $\epsilon > 0$ and $\varphi \in (0,\pi/2]$, there exists a $\delta_5 \in (0,\delta_4)$ such that the following holds: Let $(\gamma_1,\gamma_2,\alpha)$ be a hinge with $\gamma_1 \in G$, $L[\gamma_2] < \delta_5$ and $\alpha \in [\varphi,\pi-\varphi]$. Let η be the unique geodesic from e_1 to e_2 whose hd distance from γ_1 is less than δ_3. Recall that e_i was defined in 3.1. Then, up to reparameterization, $\eta(t) = \exp_{\gamma_1(t)} f(t)E(t)$, where E is the parallel unit vector field arrived at by parallel translating γ_2' along γ_1, and $f : [0,b] \to [0,\epsilon)$ is smooth.

Proof: In each case we shall produce the δ for an arbitrary geodesic. Then a uniform δ can be found using continuity and compactness arguments.

(1) Suppose $\gamma \in \mathrm{Geo}(x,y)$, and let μ be the meridian segment from x to p. Since x and p are not conjugate along μ, μ can be extended a small amount beyond p and still be free of conjugate points ($C(x)$ is closed). Thus, assuming that μ has been so extended, $L[\mu] > d(x,p)$ and μ contains

the meridian segment from x to p. If $p \in \gamma$, we are evidently done. Otherwise, since μ is minimal from x to p, $d(\mu(t),p) < d(\gamma(t),p)$ for $0 < t \leq d(x,p)$, and thus $d(\mu(t),p) < d(\gamma(t),p)$ holds for $0 < t < d(x,p) + 2\delta_1$ for some $\delta_1 > 0$. Since S is nice $K(\mu(t)) > K(\gamma(t))$, and the proof is completed by applying lemma 1.3.1.

(2) This is clearly true on any compact manifold.

(3)-(5) These all follow easily from the facts that γ is free of conjugate points and that $\exp_x v$ is a continuous function of both $x \in S$ and $v \in S_x$. ∎

<u>Remark</u>: Any δ_i appearing in the remainder of this thesis will be assumed to have been chosen in accordance with the above lemma.

3.3.2 <u>Lemma</u>: Let $\{(\gamma_1{}^\alpha, \gamma_2, \alpha)\}$, $\alpha \in [a,b]$, be a family of hinges in S_+. Then $d/d\alpha(d(e_1{}^\alpha, e_2)) > 0$ for all $\alpha \in [a,b]$.

<u>Proof</u>: This is a straightforward analogue to step (1) in the proof of theorem 2.2 in Cheeger and Ebin [1975]. Here it is important to observe that if $x \in Cl(S_+)$, $C(x) \cap S_+ = \emptyset$. ∎

3.3.3 <u>Lemma</u>: Let $\{(\gamma_1{}^\alpha, \gamma_2, \alpha)\}$, $\alpha \in [0,\pi]$, be a family of hinges in $T_R(p) \subset S$. Suppose that

(1) $L[\gamma_1{}^\alpha] < d(e_0,p) + \delta_1$ and

(2) $L[\gamma_2] < \delta_4$.

Let γ^α denote the unique geodesic from e_2 to $e_1{}^\alpha$ whose hd distance from $\gamma_1{}^\alpha$ is less than δ_3. Then $d/d\alpha\ L[\gamma^\alpha] > 0$ for all $\alpha \in [0,\pi]$.

Proof: This is also analogous to step (1) of theorem 2.2 in Cheeger and Ebin, but is not quite as easy as 3.3.2. It is proved here primarily for completeness sake.

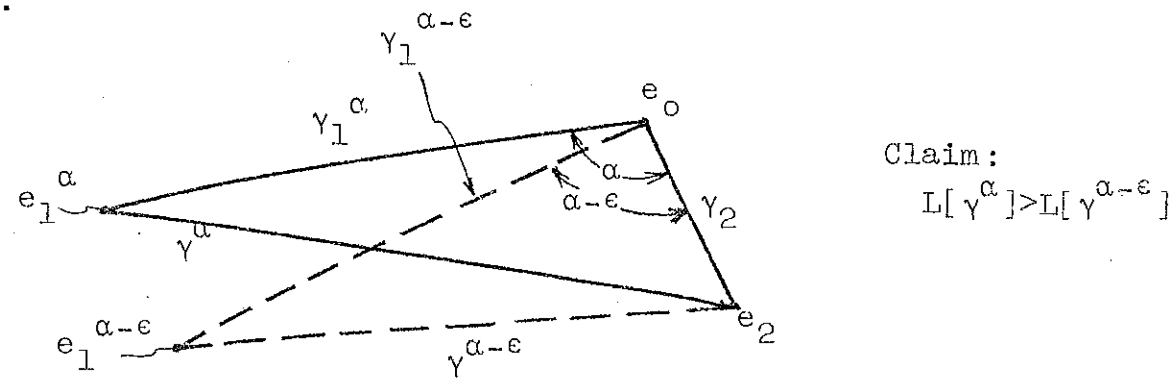


Figure 3.1

Since $\gamma_1{}^\alpha$ is free of conjugate points for $\alpha \in [0,\pi]$, $A := \{e_1{}^\alpha\}$ is a smooth submanifold (with boundary). Clearly, by choice of δ_4, γ^α is also free of conjugate points for each α. This, using the fact that $T_{\delta_3}(\gamma_1{}^\alpha)$ (in the hd metric on Geo(S)) is open, implies that $L[\gamma^\alpha]$ is a smooth function of α. It is therefore reasonable to compute $d/d\alpha\ L[\gamma^\alpha]$.

The variation vector field for the variation $\{\gamma^\alpha\}$ is zero at e_2 and tangent to A at $e_1{}^\alpha$. Since $\gamma_1{}^{\alpha\prime}(e_1{}^\alpha)$ is orthogonal to A and $\measuredangle(\gamma_1{}^{\alpha}, \gamma^{\alpha})(e_1{}^{\alpha}) < \pi$ for $\alpha \in (0,\pi)$ (by choice of δ_4), γ^{α} is never orthogonal to A for $\alpha \in (0,\pi)$. Thus, using the first variation formula, we see that $d/d\alpha\ L[\gamma^{\alpha}] \neq 0$. But it is quite clear that

$$\begin{aligned} L[\gamma^0] &= |\ L[\gamma_1] - L[\gamma_2]\ | \\ &< L[\gamma_1] + L[\gamma_2] \\ &= L[\gamma^{\pi}] \quad , \end{aligned}$$

so that $d/d\alpha\ L[\gamma^{\alpha}] > 0$. ∎

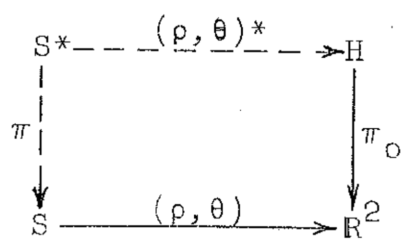
3.4 Branched Coverings

If S is a nice surface of revolution with vertex p, S* will denote the infinite-sheeted branched covering space of S, branched over p, with the induced geometry. For computational purposes we may view S* in the following manner:

Suppose $\mathbb{R}^2$ is represented in polar coordinates (r,θ), and let $H := [0,\infty)\mathrm{x}\mathbb{R}$. Define $\pi_0 : H \to \mathbb{R}^2$ by

$$\pi_0(x,y) := (x, y \bmod 2\pi) \quad ,$$

and topologize H so that π_0 is continuous. Now consider the diagram



where S* is the (ρ,θ)-pullback of H. Let the geometry of S* be the π-pullback of the geometry on S, and $p^* := \pi^{-1}(p)$.

It is clear that the topology on S^* which is induced by the geometry is the same as the topology which S^* inherits as a pullback of H. Note that S^* is not geodesically complete at p^*, but that otherwise geodesics are simply lifts of geodesics in S. Note also that minimal paths in S^* are either lifts of minimal geodesics in S, or else can be written as $\mu_1 \vee \mu_2$ where μ_1 and μ_2 are meridian segments in S^*, μ_1 ending and μ_2 beginning at p^*.

If $\rho^* := (\rho, \theta)^*$ followed by projection into the first factor of H and $\theta^* := (\rho, \theta)^*$ followed by projection into the second factor of H, then $\pi_0 \circ \rho^* = \rho \circ \pi$ and $\pi_0 \circ \theta^* = \theta \circ \pi$. Finally, if $\gamma \in \mathrm{Geo}(S^*)$, let $\theta_\gamma^* := \theta_{(\pi \circ \gamma)}$.

3.5 Technical Lemmas and Hinges in S^*

3.5.1 Lemma: On $S^* \setminus \{p^*\}$

(1) (ρ^*, θ^*) is one to one, and hence a homeomorphism between $S^* \setminus \{p^*\}$ and $H \setminus (\{0\} \times \mathbb{R})$,

(2) each simple closed path determines a bounded and an unbounded component in S^*, $p^* \notin$ bounded component, and

(3) there are no closed geodesics.

Proof: (1) Since (ρ, θ) is one to one, so is (ρ^*, θ^*). Since π and (ρ, θ) are open maps, so is (ρ^*, θ^*).

(2) This is standard, using the homeomorphism of (1).

(3) Suppose $\gamma : [0,b] \to S^*$, $\gamma \in Geo(x,x)$. Then using (2) it is clear that $\theta_\gamma^*(0) = \theta_\gamma^*(b)$, which is impossible since θ_γ^* $(= \theta_{(\pi \circ \gamma)})$ is one to one by 2.2.3. ∎

Let S_+^* denote some connected component of $\pi^{-1}(S_+)$. The following lemma along with 3.5.5 provide the primary motivation for working in S* rather than in S. This will be discussed further in section 4.

3.5.2 Lemma: If $x \in S_+^*$ and $y \in S_+^*$, then Geo(x,y) contains precisely one element, and $Geo(x,y) \subset Geo(S_+^*)$.

Proof: That there is some $\gamma \in Geo(x,y)$, $\gamma \subset S_+^*$, is clear since the geometry on S* is the π-pullback of the geometry on S, and the minimal geodesic between $\pi(x)$ and $\pi(y)$ remains in S_+. That $Geo(x,y) \cap Geo(S_+^*)$ contains only one element follows immediately from theorem 2.6, which describes the cut locus on S. If $\eta \in Geo(x,y)$, $\eta \neq \gamma$, then it must leave S_+^*. But 3.5.1(2) then implies that θ_η^* is not one to one. ∎

Much of the technical difficulty which we will encounter throughout the remainder of this thesis is due to the fact that S* is not complete. This next lemma provides us with the degree of completeness needed however. Note that we continue to apply the notation of 2.4 to S* whenever the meaning is clear.

3.5.3 <u>Lemma</u>: If $x \in S^*$, $y \in S^*$, $\gamma \in Geo(x,y)$, and γ lies below $z \in S^*$, then $Geo(x,z) \neq \emptyset$. If γ is also a minimal curve, so that $L[\gamma] = d(x,y)$, then there exists a minimal geodesic from x to z.

<u>Proof</u>: Suppose $z \neq y$ and γ is not contained in a meridian (in either case the lemma is obvious).

Let μ denote the meridian through z, and ν the meridian through x with $\nu(0) = p^*$. Let $\varphi := \measuredangle(\gamma,\nu)(x)$. Define a family of geodesics $\{\gamma_\alpha\}$, $\alpha \in [0,\varphi]$, each geodesic of which begins at x, lies above γ in some neighborhood of x, and such that $\measuredangle(\gamma,\gamma_\alpha)(x) = \alpha$. Since for $\alpha \neq \varphi$ γ_α is not a meridian, each γ_α can be extended indefinitely and in particular, at least for small α, until γ_α crosses μ. Furthermore, it is clear that $\gamma_\alpha \cap \mu$ depends continuously on α. And thus $A := \{\gamma_\alpha \cap \mu \mid \alpha \in [0,\varphi]\}$ is a connected set.

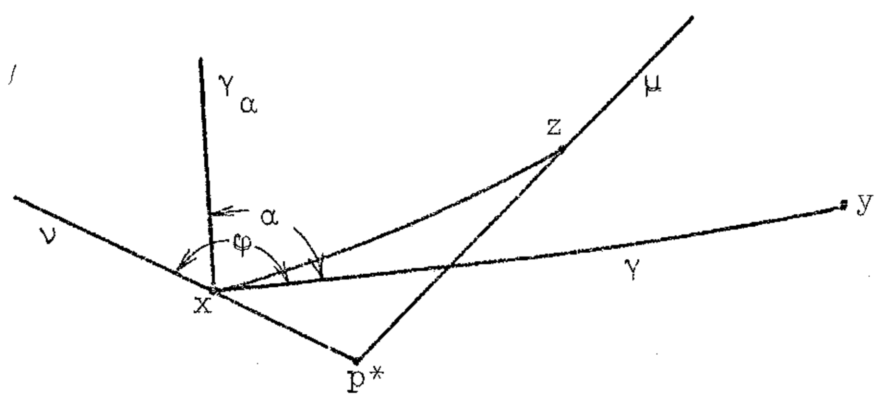


Figure 3.2

It is clear that $\gamma_\varphi \cap \mu$ is empty and, by the continuous dependence of γ_α on α, that $C := \{\alpha \mid \gamma_\alpha \cap \mu = \emptyset\}$ is closed. Let a_o be the smallest element in C, and let $\{\alpha_i\}$ be a sequence of real numbers, $0 < \alpha_i < \alpha_{i+1} < a_o$, which converge to a_o. Let $m_i := \gamma_{\alpha_i} \cap \mu$, and we claim that $\rho^*(m_i) \to \infty$: if not, $m_i \to m \in \mu$; but again by continuous dependence of γ_α on α, we see that $\gamma_{a_o} \cap \mu = m$, a contradiction.

Thus, since $\rho^*(\gamma \cap \mu) < \rho^*(z)$ and A is connected, $\gamma_\alpha \cap \mu = z$ for some $\alpha \in [0,\varphi]$.

If γ is a minimal geodesic, let c be any minimal path from x to z. If c passes through p*, then, since γ lies below z and meridians are the shortest paths from p*, c will cross γ. Since $z \neq y$, either c or γ, or both, must continue to minimize beyond $c \cap \gamma$. But it is a standard fact that this cannot happen.

Thus the minimal path from x to z cannot pass through p*, and so it must be a geodesic. That γ lies below this minimal geodesic is also quite easy to see. ∎

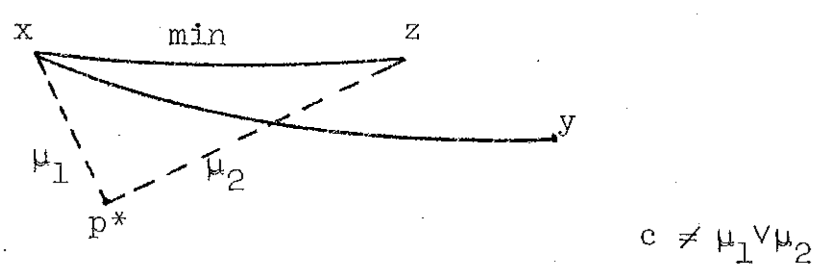


Figure 3.3

If $x \in S^*$ we call y a <u>cut point</u> of x if there is a geodesic from x to y which is minimal to, but not beyond, y. $C(x)$ will denote the collection of all cut points of x, and note that $C(x)$ may well be not-closed and/or disconnected. For example, let S be a semi-infinite cylinder capped with a hemisphere (example 2.7.2 with $\theta = \pi/2$) take out the vertex, and let x lie in the interior of the hemisphere. Then $C(x) = \{p\} \cup \overline{\mu}_x|(a,\infty)$, where $a \neq 0$. $\overline{\mu}_x(a) \notin C(x)$ since x and $\overline{\mu}_x(a)$ are conjugate on S only along a geodesic through p. See figure 3.4. The same problem arrises in S^*.

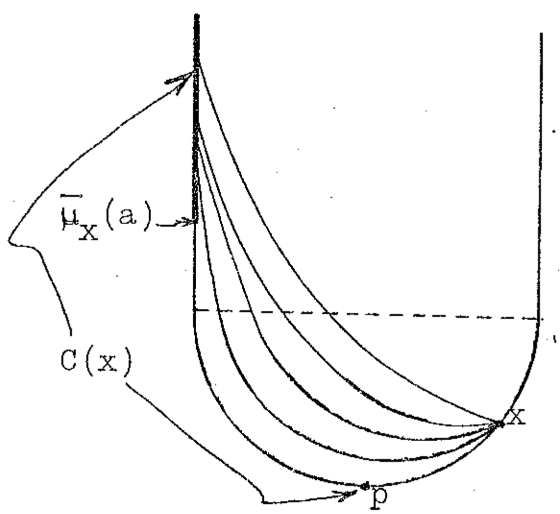


Figure 3.4

The following lemma is the natural extension of 1.5.1 to S^*.

3.5.4 <u>Lemma</u>: Let $\gamma \in \text{Geo}(S^*)$. Then $\gamma(t_o)$ is a cut point of $x := \gamma(0)$ if and only if one of the following holds for $t = t_o$, and none holds for any smaller value of t:

(1) $\gamma(t_o) = p^*$,

(2) x and $\gamma(t_o)$ are conjugate along γ, or

(3) there exists $\eta \in Cur(x,\gamma(t_o))$, $\eta \neq \gamma$, with $L[\eta] = L[\gamma]$.

Proof: Since, if γ is not a meridian nearby geodesics are also not meridians, the standard proof (cf. 1.5.1) applies. The only difference being that perhaps η passes through p^*, and is thus only necessarily piecewise geodesic. ∎

3.5.5 Lemma: Let $x \in S^*$ and $y \in C(x)\setminus\{p^*\}$. Then $d(x,y) \geq d(x,p^*)+2\delta_1$, where δ_1 is chosen for the surface S with $S^* = \pi^*(S)$ and $R >> d(x,y)$.

Proof: Let $\gamma \in Geo_m(x,y)$. If x and y are conjugate along γ, then $\pi(x)$ and $\pi(y)$ are conjugate along $\pi(\gamma)$. Thus, by 3.3.1, $L[\pi(\gamma)] \geq d(\pi(x),p)+2\delta_1$; and so $L[\gamma] \geq d(x,p^*)+2\delta_1$.

Otherwise there exists $\eta \in Cur(x,y)$, $L[\eta] = L[\gamma]$. But then $\pi(\eta)$ and $\pi(\gamma)$ are distinct elements of $Cur(\pi(x),\pi(y))$ of the same length, and thus $\pi(\gamma)$ has quit minimizing prior to $\pi(y)$. Thus, by 1.5.3 and 3.3.1, we once again see that $L[\gamma] \geq d(x,p^*)+2\delta_1$. ∎

3.5.6 Remark: Note that 3.2.2 parts (1), (2) and (3), with obvious minor modifications, apply to S^* as well as to S. The incompleteness of S^* prevents (5) from being used in S^*. Likewise 3.3.2 applies to hinges in some S^*_+, but 3.3.3 does not apply to S^*. Thus we need the following lemma on S^*.

3.5.7 Lemma: Let $\{(\gamma_1^\alpha, \gamma_2, \alpha)\}$, $\alpha \in [a.b]$, be a family of hinges in S^*. Suppose

(1) $L[\gamma_1] \leq d(e_0, p^*)$,

(2) $\gamma_2 \subset S_+^*$, S_+^* some given component of $\pi^{-1}(S_+)$, and

(3) for each $\alpha \in [a,b]$ and each $t \in [0,b_1]$ there exists a geodesic from $\gamma_1^\alpha(t)$ to e_2 without cut points.

Then $d/d\alpha\ (d(e_1^\alpha, e_2)) > 0$ for $\alpha \in (a,b)$.

Proof: This proof is quite like the proof of 3.3.3, but it is included since some of the details are different. For further comment see the remarks which follow this proof.

Let $\gamma^{\alpha,t}$ denote the unique minimal geodesic from $\gamma_1^\alpha(t)$ to e_2 whose existence is assumed. From supposition (1) and 3.5.5 it follows that $A_t := \{\gamma_1^\alpha(t)\}$, $\alpha \in [a,b]$, is a smooth submanifold for each $t \in [0,b_1]$. Since γ_2 is in S_+^* and hence unique between its end points (lemma 3.5.2), $(\gamma^{\alpha,t})'(0)$ is not orthogonal to A_t lest $\gamma_1^\alpha|[0,t] \vee \gamma^{\alpha,t}$ be another geodesic from e_0 to e_2. Furthermore, since e_2 has no cut points in $\{A_t\}$, $t \in [0,b_1]$, except perhaps p^*, it follows that $L[\gamma^{\alpha,t}]$ is a smooth function of both α and t. Thus, applying the first variation formula as in 3.3.3, $\partial/\partial\alpha\ L[\gamma^{\alpha,t}] \neq 0$.

It is easy to see that $\operatorname{sgn}(\partial/\partial\alpha\ L[\gamma^{\alpha,t}])$ is independent of t, and thus it suffices to prove the lemma for t small. But if t is sufficiently small the entire family $\{\gamma_1{}^{\alpha}\}$, $\alpha \in [0,\pi]$, is contained in S^*_+, in which case the lemma is apparent (cf. 3.3.2). ∎

3.5.8 Remarks: (1) The geodesics required in supposition (3) will always exist if $\gamma_1{}^{\alpha} \in S^*_+$ or if $\gamma_1{}^{\alpha}$ lies below e_2 (3.5.2 and 3.5.3). The uniqueness is insured if $\{\gamma_1{}^{\alpha}\} \subset S^*_+ \cup T_{d(e_2,p^*)+\delta_1}(e_2)$ (3.5.2 and 3.5.5).

(2) Note that in 3.5.7 γ_2 was allowed to be fairly long while in 3.3.3 γ_2 was required to be very short. Note however that in 3.3.3 we need not be as restrictive as in 3.5.7 to insure the existence of the γ^{α}.

4. GLOBAL COMPARISON THEOREMS

Toponogov's theorem (Toponogov [1959]; see Cheeger and Ebin for proof in English) is a beautiful and powerful global generalization of the Rauch comparison theorem. It gives distance estimates on a Riemannian manifold M by comparison with a surface of constant curvature. Specifically,

<u>Theorem</u> (Toponogov): Let M be a complete manifold with $K(x) \geq H \in \mathbb{R}$ for each $x \in M$. Let $(\gamma_1, \gamma_2, \alpha)$ be a hinge in M with γ_1 minimal and, if $H > 0$, $L[\gamma_2] \leq \pi/\sqrt{H}$. Let $(\overline{\gamma}_1, \overline{\gamma}_2, \alpha)$ be a hinge in the simply connected surface of constant curvature H with $L[\overline{\gamma}_i] = L[\gamma_i]$, and $\overline{\gamma}_i \in \mathrm{Geo}(\overline{e}_o, \overline{e}_i)$. Then $d(e_1, e_2) \leq d(\overline{e}_1, \overline{e}_2)$.

This theorem has provided estimates adequate for many important applications. It does, however, appear to be unnecessarily restrictive in the case of an open non-negatively curved manifold. Since on any such manifold the curvature must come arbitrarily near zero (Bonnet's theorem), the comparison surface must be flat or negatively curved. If, for example, M is a paraboloid and the hinge $(\gamma_1, \gamma_2, \alpha)$ has its vertex, e_o, at the vertex of M, it is obvious that distance estimates on $\mathbb{R}^2$ will not be very accurate.

A natural question is: Can we improve the estimates on such a manifold? In the following we will show that we can, and that in fact, instead of a surface of constant curvature, an appropriate nice surface of revolution may be used for the comparison surface. We actually prove two very similar such generalizations, the first serving as a lemma for the second.

4.1 Definitions and Notation

Notational conventions established here will be used throughout this section without further comment.

4.1.1 Let M^n denote a complete, open, nonnegatively curved manifold, let $\delta \geq 0$ and $p \in M$. Let $\overline{M}$ denote a nice surface of revolution with vertex $\overline{p}$. We say that M and $\overline{M}$ are <u>δ-correspondent at p</u> if, whenever $d(\overline{p},\overline{x})+\delta \geq d(p,x)$, then $K(\overline{x}) \leq K(x)$, where $x \in M$ and $\overline{x} \in \overline{M}$. That is, the curvature in $\overline{M}$ falls off with respect to distance from $\overline{p}$ faster than the curvature in M falls off with respect to distance from p. We will simply say that M and $\overline{M}$ are correspondent at p when we mean that $\delta = 0$.

4.1.2 If M and $\overline{M}$ are δ-correspondent at p and $(\gamma_1,\gamma_2,\alpha)$ is a hinge in M with $e_o = p$ or $e_1 = p$, then the hinge $(\overline{\gamma}_1,\overline{\gamma}_2,\alpha)$ in $\overline{M}$ given by specifying that

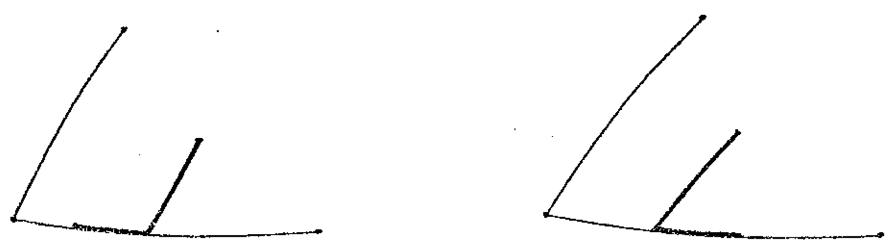
(1) $L[\overline{\gamma}_i] = L[\gamma_i]$ and

(2) $e_i = p \Rightarrow \overline{e}_i = \overline{p}$, where $\overline{\gamma}_1 \in Geo(\overline{e}_o,\overline{e}_1)$ and

$\overline{\gamma}_2 \in \mathrm{Geo}(\overline{e}_o, \overline{e}_2)$,

is said to <u>correspond</u> to $(\gamma_1, \gamma_2, \alpha)$. Since $\overline{\gamma}_1$ is a segment of a meridian in $\overline{M}$ which includes as one end point the vertex $\overline{p}$ of $\overline{M}$, this correspondence is uniquely determined up to rotation and reflection of $\overline{M}$.

4.1.3 Given a hinge $(\gamma_1, \gamma_2, \alpha)$, we call a hinge (g_1, g_2, a) a <u>subhinge</u> if $g_2 \subset \gamma_2$. We say that the subhinge <u>faces inward</u> <u>(outward)</u> if $d(e_2, \epsilon_o) < (>)\ d(e_2, \epsilon_2)$, where $g_i \in \mathrm{Geo}(\epsilon_o, \epsilon_i)$.



Inward Facing Subhinge   Outward Facing Subhinge

Figure 4.1

For the remainder of this section, a <u>hinge</u> $(\gamma_1, \gamma_2, \alpha)$ in M will be assumed to have either $e_o = p$ or $e_1 = p$ and γ_1 minimal. A <u>subhinge</u> will always be a subhinge of such a hinge, with p not necessarily an end point of one of the two geodesics making up the subhinge.

Let $h := (\gamma_1, \gamma_2, \alpha)$ and $\overline{h} := (\overline{\gamma}_1, \overline{\gamma}_2, \alpha)$ be corresponding hinges in M and $\overline{M}$ respectively. We say that the subhinges (g_1, g_2, a) of h and $(\overline{g}_1, \overline{g}_2, a)$ of $\overline{h}$ are <u>corresponding</u>

<u>subhinges</u> if:

(1) $L[g_i] = L[\overline{g}_i]$,

(2) both face in the same direction, and

(3) $d(e_0, \epsilon_0) = d(\overline{e}_0, \overline{\epsilon}_0)$, where ϵ_0 is as above.

4.1.4 Let $h := (\gamma_1, \gamma_2, \alpha)$ be a hinge in M and let $\Delta := (g_1, g_2, g_3)$ be a geodesic triangle in M with $g_2 \subset \gamma_2$. If $(\overline{\gamma}_1, \overline{\gamma}_2, \alpha)$ is a hinge in $\overline{M}$ which corresponds to h, we say that a geodesic triangle $(\overline{g}_1, \overline{g}_2, \overline{g}_3)$ in $\overline{M}$ <u>corresponds</u> to Δ if:

(1) $L[\overline{g}_i] = L[g_i]$,

(2) $\overline{g}_2 \subset \overline{\gamma}_2$ and

(3) $d(e_0, g_2(t)) = d(\overline{e}_0, \overline{g}_2(t))$ as measured along γ_2 and $\overline{\gamma}_2$ respectively.

Finally, note that all of the above notions still make sense if we are working in $\overline{M}^*$ instead of $\overline{M}$. The only situation which demands any care occurs when $(\gamma_1, \gamma_2, \alpha)$ is a hinge in $\overline{M}^*$ with $e_0 = p^*$. Then one must be sure that γ_1 and γ_2 lie on the same branch of $\overline{M}^*$.

In the following lemma we assume that M and $\overline{M}$ are δ-correspondent at $p \in M$, and that $(\gamma_1, \gamma_2, \alpha)$ and $(\overline{\gamma}_1, \overline{\gamma}_2, \alpha)$ are hinges (or subhinges) in M and $\overline{M}$ respectively with $L[\gamma_i] = L[\overline{\gamma}_i]$. In this instance we do not assume that $e_i = p$ for $i = 0$ or 1.

4.1.3 <u>Lemma</u>: Suppose that $(\gamma_1, \gamma_2, \alpha)$ and $(\overline{\gamma}_1, \overline{\gamma}_2, \alpha)$ are as above with $0 < \theta_o < \alpha < \pi - \theta_o$. Choose δ_5 with $\varphi = \theta_o$ and $\epsilon < \min\{\delta/2, \delta_2\}$, and suppose that $L[\gamma_2] < \delta_5$. Further suppose that $d(\gamma_1(t), p) \leq d(\overline{\gamma}_1(t), p)$ and let $\overline{\gamma}$ denote the unique geodesic from $\overline{e}_2$ to $\overline{e}_1$ whose hd distance from $\overline{\gamma}_1$ is less than δ_3. Then $L[\overline{\gamma}] \geq d(e_2, e_1)$.

<u>Proof</u>: It is quite straightforward to check that the hypotheses of corollary 1.4.2 are satisfied. ◻

<u>Remarks</u>: (1) Notice that if $e_1 = p$ and $d(\overline{e}_o, \overline{p}) \geq d(e_o, p)$, then the hypotheses of the lemma are satisfied. (2) This of course works as well in $\overline{M}^*$ as long as the necessary geodesics exist. Rather than check this, we occasionally make the measurement in $\overline{M}$ and then pull back up to $\overline{M}^*$.

4.2 <u>Two Global Comparison Theorems</u>

In both 4.2.1 and 4.2.2, M^n will denote an open nonnegatively curved m-dimensional Riemannian manifold, and $\overline{M}$ will denote a nice surface of revolution such that M and $\overline{M}$ correspond at $p \in M$.

4.2.1 <u>Theorem</u>: Let $(\gamma_1, \gamma_2, \alpha_o)$ and $(\overline{\gamma}_1{}^*, \overline{\gamma}_2{}^*, \alpha_o)$ be corresponding hinges in M and $\overline{M}^*$ respectively with γ_1 minimal, $e_1 = p$ and $\overline{\gamma}_2{}^* \subset \overline{M}_+{}^*$. We of course assume that $\overline{\gamma}_i{}^* \in \mathrm{Geo}(\overline{e}_o{}^*, \overline{e}_i{}^*)$. Let $\gamma_3 \in \mathrm{Geo}_m(e_2, e_1)$. Then:

(A) $d(e_1,e_2) \leq d(\overline{e}_1{}^*,\overline{e}_2{}^*)$, and

(B) There exists a triangle $(c_1,\overline{\gamma}_2{}^*,c_3)$ in $\overline{M}^*$ which corresponds to $(-\gamma_1,\gamma_2,\gamma_3)$, with both

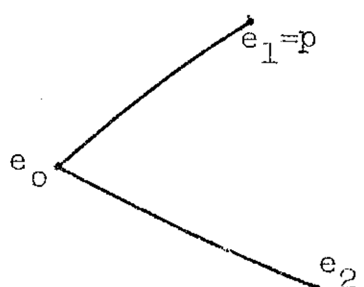
$$\measuredangle(-c_1,\overline{\gamma}_2{}^*)(\overline{e}_o{}^*) \leq \measuredangle(\gamma_1,\gamma_2)(e_o) \text{ and}$$
$$\measuredangle(\overline{\gamma}_2{}^*,c_3)(\overline{e}_2{}^*) \leq \measuredangle(\gamma_2,\gamma_3)(e_2).$$

4.2.2 Theorem: Let $(\gamma_1,\gamma_2,\alpha_o)$ and $(\overline{\gamma}_1,\overline{\gamma}_2,\alpha_o)$ be corresponding hinges in M and $\overline{M}$ respectively with γ_1 minimal and $e_o = p$. Let $\gamma_3 \in \mathrm{Geo}_m(e_2,e_1)$. Then:

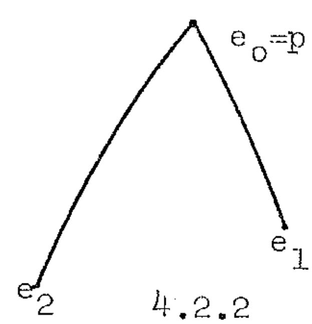
(A) $d(e_1,e_2) \leq d(\overline{e}_1,\overline{e}_2)$, and

(B) There exists a unique triangle $(c_1,\overline{\gamma}_2,c_3)$ in $\overline{M}$ which corresponds to $(-\gamma_1,\gamma_2,\gamma_3)$, with both

$$\measuredangle(-c_1,\overline{\gamma}_2)(\overline{e}_o) \leq \measuredangle(\gamma_1,\gamma_2)(e_o) \text{ and}$$
$$\measuredangle(\overline{\gamma}_2,c_3)(\overline{e}_2) \leq \measuredangle(\gamma_2,\gamma_3)(e_2).$$



4.2.1



4.2.2

Figure 4.2

Proof: The proofs of these two theorems are quite similar, and will be given simultaneously. The proof is divided into a number of steps each of which is further broken down into discussions of the difficulties particular to either 4.2.1 or 4.2.2.

Fix $R \in \mathbb{R}$, $R > 2(L[\gamma_1]+L[\gamma_2])$.

(1) We will make several simplifying assumptions, and then later show that the theorems as stated are correct.

Assume that for some $\delta > 0$, M and $\overline{M}$ are δ-correspondent. If $x \in \gamma_2$ and if $\gamma_x \in \mathrm{Geo}_m(x,p)$, assume that $\langle\gamma_x', \gamma_2'\rangle(x) \in (-1,1)$. Note that, among other things, we have assumed that $\alpha_o \in (0,\pi)$.

(2) Notation: As usual, notation established either in $\overline{M}$ or $\overline{M}$* will actually apply to both by insertion or deletion of a star (*).

Let $A := \{\arccos(\langle\gamma_2', \gamma_x'\rangle(x)) \mid x \in \gamma_2$ and $\gamma_x \in \mathrm{Geo}_m(p,x)\}$.

Let $\theta_s := \sup A$, and $\theta_i := \inf A$, as x an γ_x vary over all possibilities. Let $\theta_m := \min(\theta_i, \pi-\theta_s)$, and note that $\theta_m \in (0,\pi/2]$ by the assumptions of (1) and the compactness of γ_2. Now choose δ_5 with $\epsilon < \min\{\delta/2, \delta_2\}$ and $\varphi = \theta_m$.

Let $\{x_0=e_o, x_1, \ldots, x_{n-1}, x_n=e_2\}$ partition γ_2, $x_i = \gamma(t_i)$, such that $t_i < t_{i+1}$ and $d(x_i, x_{i+1}) < \delta_5$. Choose $\sigma_i \in \mathrm{Geo}_m(x_i, e_1)$ with $\sigma_0 = \gamma_1$, and assume that σ_i is parameterized on $[0, \lambda_i]$. Let $\{\overline{x}_i^*\}$ be a corresponding partition of $\overline{\gamma}_2^*$, with $\overline{x}_i^* = \overline{\gamma}_2^*(t_i)$. For $i < j$ define $\tau_{i,j} := \gamma_2|[t_i, t_j]$, and let $\overline{\tau}_{i,j}^*$ denote the corresponding pieces of $\overline{\gamma}_2^*$.

Call a geodesic σ in $\overline{M}$*, with $\sigma(0) = \overline{x}_i^*$, <u>interior</u>

if for all t sufficiently small the unique geodesic $\gamma \in Geo_m(e_o, \sigma(t))$ has $\measuredangle(\gamma_1, \gamma)(e_o) < \alpha_o$. Let $\{\bar{\sigma}_i^*\}$ be a

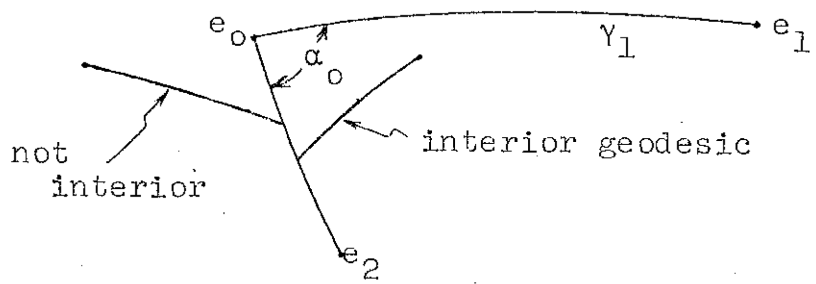


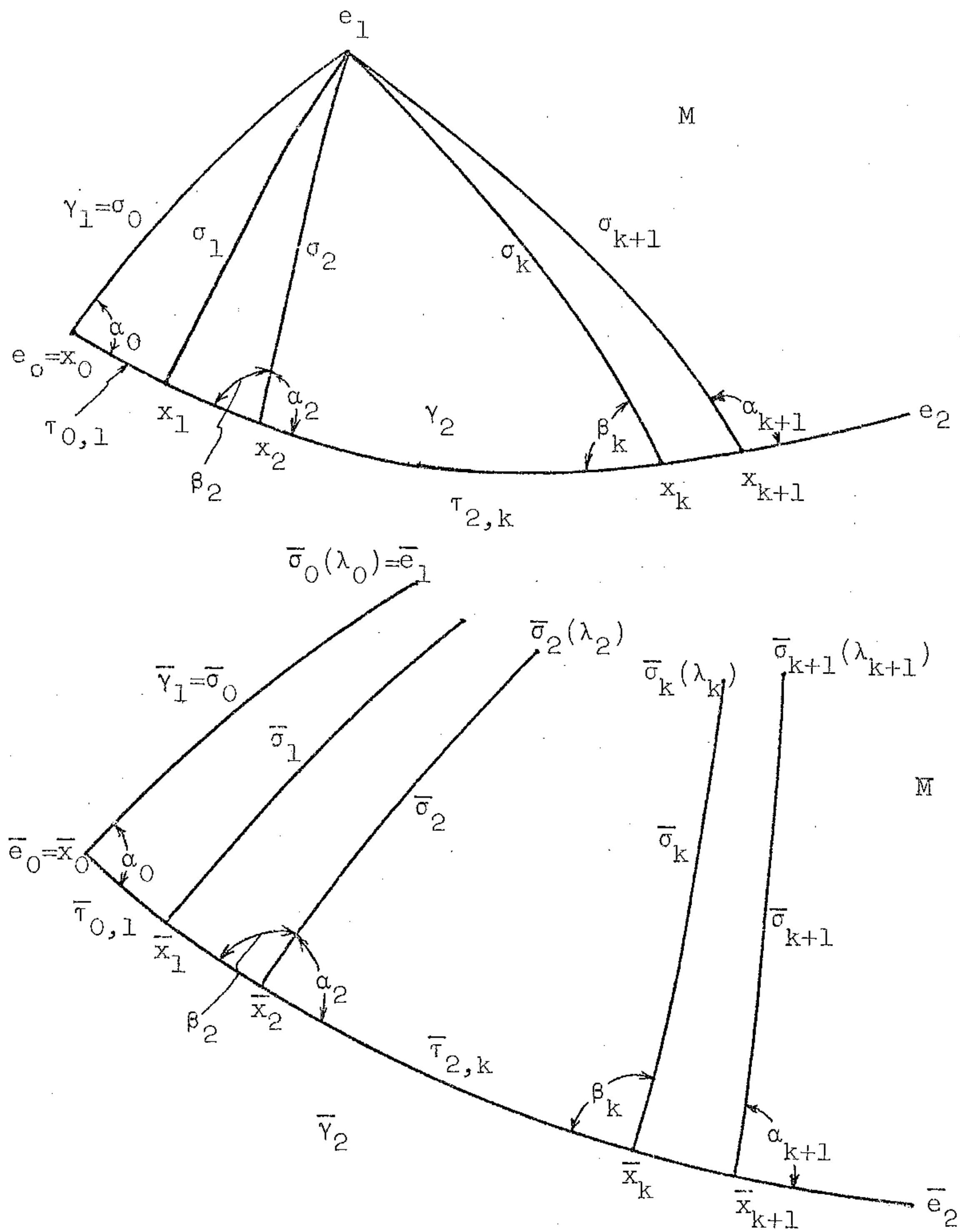
Figure 4.3

family of geodesics in $\bar{M}^*$ so that (i) $\bar{\sigma}_i^*(0) = x_i^*$, (ii) the subhinge $(\tau_{0,i}, \sigma_i, \beta_i)$ of $(\gamma_1, \gamma_2, \alpha_o)$ and the subhinge $(\bar{\tau}_{0,i}^*, \bar{\sigma}_i^*, \beta_i)$ of $(\bar{\gamma}_1^*, \bar{\gamma}_2^*, \alpha_o)$ are corresponding subhinges (note that this implicitly defines β_i), and (iii) so that each $\bar{\sigma}_i^*$ is an interior geodesic. Let $\alpha_i := \pi - \beta_i$.

Finally, without loss of generality, assume in 4.2.1 that $\bar{\gamma}_1^* \in \partial\bar{M}_+^*$ and $\bar{\gamma}_2^* \in \bar{M}_+^*$, and in 4.2.2 that $\bar{\gamma}_2 \in \partial\bar{M}_+$ and $\bar{\gamma}_1 \in \bar{M}_+$. See figure 4.4 on the following page.

(3) In addition to the assumptions made in (1), also assume for 4.2.2 that the entire construction of part (2) is contained in $\bar{M}_+$. This is of course another simplifying assumption which must be dealt with later. Furthermore, we will postpone the proof of the uniqueness of the triangle in B of 4.2.2 until step (11).

The proof will now proceed by induction.



e_1
M
$\gamma_1 = \sigma_0$
σ_1
σ_2
σ_k
σ_{k+1}
α_0
$e_0 = x_0$
$\tau_{0,1}$
x_1
α_2
γ_2
β_k
α_{k+1}
e_2
x_2
β_2
$\tau_{2,k}$
x_k
x_{k+1}
$\bar{\sigma}_0(\lambda_0) = \bar{e}_1$
$\bar{\sigma}_2(\lambda_2)$
$\bar{\sigma}_k(\lambda_k)$
$\bar{\sigma}_{k+1}(\lambda_{k+1})$
$\bar{\gamma}_1 = \bar{\sigma}_0$
$\bar{\sigma}_1$
$\bar{\sigma}_2$
$\bar{\sigma}_k$
$\bar{M}$
$\bar{e}_0 = \bar{x}_0$
α_0
$\bar{\tau}_{0,1}$
$\bar{x}_1$
$\bar{\sigma}_{k+1}$
α_2
β_2
$\bar{x}_2$
$\bar{\tau}_{2,k}$
β_k
$\bar{\gamma}_2$
α_{k+1}
$\bar{x}_k$
$\bar{x}_{k+1}$
$\bar{e}_2$

Figure 4.4

(4) Let $\bar{g}_0$ be the unique geodesic from $\bar{x}_1$ to $\bar{e}_1$ with $hd(\bar{g}_0,\bar{\sigma}_0) < \delta_3$. It follows from 4.1.3 that $L[\bar{g}_0] \geq d(e_1,x_1)$; and thus, in 4.2.1 since $\bar{g}_0$ is a meridian and in 4.2.2 since $\bar{g}_0 \subset \bar{M}_+$, that $d(\bar{e}_1,\bar{x}_1) \geq d(e_1,x_1)$. In 4.2.1 this estimate obviously applies on $\bar{M}^*$ as well. Hence (A) is true for the hinge $(\gamma_1,\tau_{0,1},\alpha_0)$ in both 4.2.1 and 4.2.2.

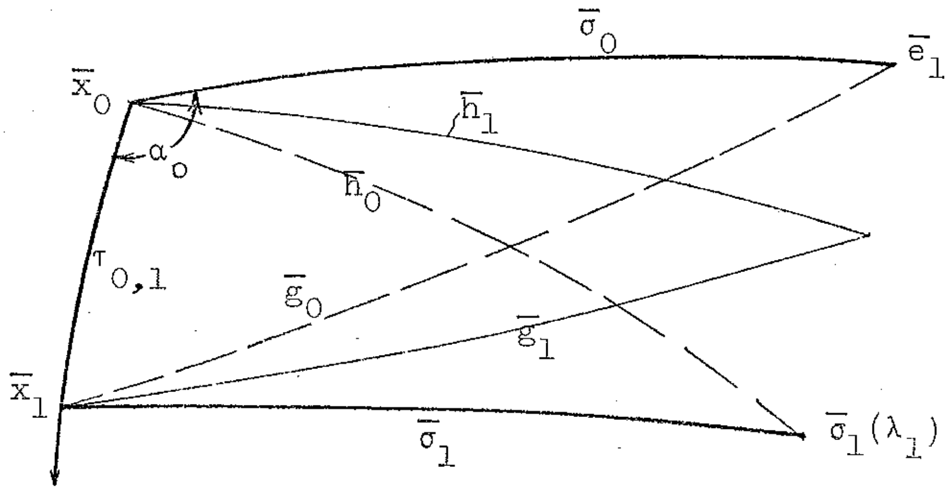


Figure 4.5

(5) Let $\bar{h}_0$ be the unique geodesic from $\bar{x}_0$ to $\bar{\sigma}_1(\lambda_1)$ with $hd(\bar{h}_0,\bar{\sigma}_1) < \delta_3$. We would now like to apply lemma 4.1.3 to conclude that $L[\bar{h}_0] \geq d(e_1,x_0)$. To this end, note that: (i) for 4.2.1, the results of part (4) insure that the hypotheses of 4.1.3 are met; and (ii) for 4.2.2, the techniques of part (4) of 4.2.1 can be used to show that $d(\bar{p},\bar{\sigma}_i(t)) \geq d(p,\sigma_i(t))$ for $t \in [0,\lambda_1]$. Thus $L[\bar{g}_0] \geq L[\bar{\sigma}_1]$ and $L[\bar{h}_0] \geq L[\bar{\sigma}_0]$ in both 4.2.1 and 4.2.2.

Now, using lemma 3.3.3, decrease α_0 by moving $\bar{\sigma}_0$

until $d(\bar{\sigma}_0(\lambda_0),\bar{x}_1)$, as measured along the unique nearby geodesic, is $L[\bar{\sigma}_1]$. Let $\bar{h}_1$ denote the geodesic resulting from this movement. Let $\bar{g}_1$ be the unique nearby geodesic from x_1 to $\bar{h}_1(\lambda_0)$. We claim that the triangle$(-\bar{h}_1,\bar{\tau}_{0,1},\bar{g}_1)$ corresponding to $(-\sigma_0,\tau_{0,1},\sigma_1)$ is the one required in part (B). See figure 4.5.

Since $\measuredangle(\bar{h}_1,\bar{\tau}_{0,1})(x_0) \leq \alpha_0$ by construction, the only thing left to check is that $\measuredangle(\bar{g}_1,-\bar{\tau}_{0,1})(\bar{x}_1) \leq \beta_1$. But it is quite easy to see that if β_1 is decreased by moving $\bar{\sigma}_1$ until $d(\bar{\sigma}_1(\lambda_1),\bar{x}_0)$, as measured along the nearby geodesic, is $L[\bar{\sigma}_0]$, then the resulting geodesic must be $\bar{g}_1$. Since $\bar{p}$ is not in the interior of the bounded region determined by $(-\bar{h}_1,\bar{\tau}_{0,1},\bar{g}_1)$, the triangle, in the case of 4.2.1, can be lifted to $\bar{M}^*$.

Hence (B) is true for the triangle $(-\sigma_0,\tau_{0,1},\sigma_1)$ in both 4.2.1 and 4.2.2.

Remark: In this step of 4.2.1 we had to work on $\bar{M}$ since there was no *a priori* guarantee that $\bar{h}_0$ could be lifted to $\bar{M}^*$.

(6) Now suppose that $d(e_0,x_k) \leq d(\bar{e}_0,\bar{x}_k)$ and that (B) is true for the triangle $(-\sigma_0,\tau_{0,k},\sigma_k)$. In the following two parts several families of hinges shall be used. To avoid a surfeit of subscripted superscripts, we introduce the following notation.

Let

$$\varphi_0 := \alpha_0, \quad \chi_0 := \alpha_k \text{ and } \Psi_0 := \alpha_{k+1} .$$

Let

$$\sigma_0(\varphi) := \bar{\sigma}_0^{(\varphi_0-\varphi)}, \quad \sigma_k(\chi) := \bar{\sigma}_k^{(\chi_0-\chi)}, \text{ and}$$

$$\sigma_{k+1}(\Psi) := \bar{\sigma}_{k+1}^{(\Psi_0-\Psi)} \qquad \text{(cf. 3.1).}$$

Further, let

$$e_0(\varphi) := \sigma_0(\varphi)(\lambda_0), \; e_k(\chi) := \sigma_k(\chi)(\lambda_k), \text{ and}$$

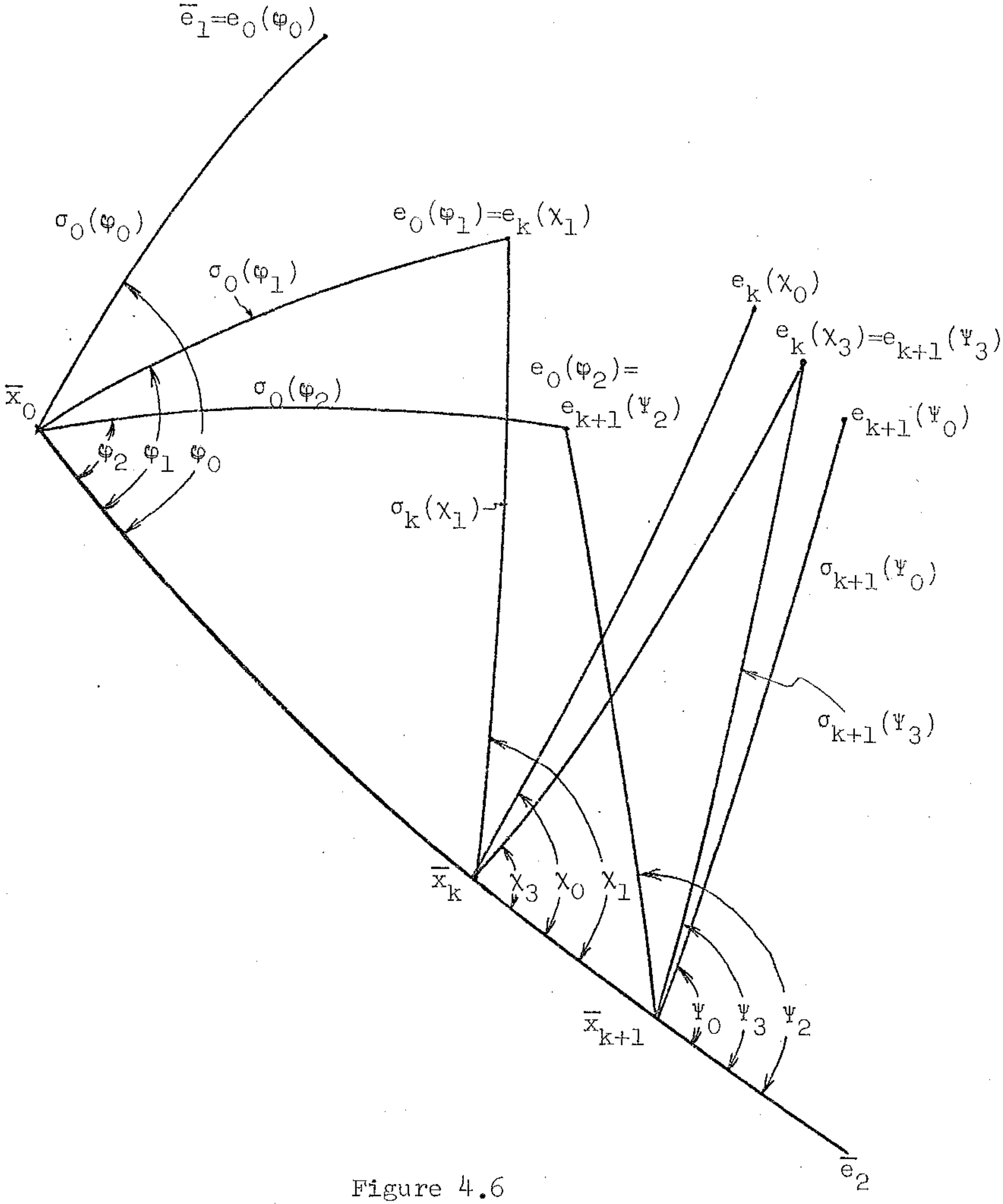
$$e_{k+1}(\Psi) := \sigma_{k+1}(\Psi)(\lambda_{k+1}).$$

We consider $\overline{\gamma}_2$, and thus $\overline{\tau}_{i,j}$ for $0 \leq i < j \leq n$, to be fixed; so that increasing or decreasing angles φ, χ, or Ψ will be by moving the σ_i. Let $(-\sigma_0(\varphi_1), \overline{\tau}_{0,k}, \sigma_k(\chi_1))$ be the triangle in $\overline{M}$ which corresponds to $(-\sigma_0, \tau_{0,k}, \sigma_k)$. Observe that $\varphi_1 < \varphi_0$ and $\chi_1 > \chi_0$ by the induction hypothesis. It may help to refer to figure 4.6 on page 54.

Note that in 4.2.2 $(-\sigma_0(\varphi_1), \tau_{0,k}, \sigma_k(\chi_1))$ is contained in $\overline{M}_+$, while in 4.2.1 we have no such assurance. This is one of the reasons why in 4.2.1 it is convenient to work in $\overline{M}$*.

Since 4.2.1 will be used as a lemma for 4.2.2, it is necessary to first finish 4.2.1, and then come back to 4.2.2. Sections (7) and (8) refer only to 4.2.1.

(7) We will first show that as φ is decreased from φ_0



ē1=e0(φ0)
σ0(φ0)
e0(φ1)=ek(χ1)
σ0(φ1)
ek(χ0)
ek(χ3)=ek+1(Ψ3)
e0(φ2)=
ek+1(Ψ2)
σ0(φ2)
x̄0
ek+1(Ψ0)
φ2 φ1 φ0
σk(χ1)
σk+1(Ψ0)
σk+1(Ψ3)
x̄k
χ3 χ0 χ1
x̄k+1
Ψ0 Ψ3 Ψ2
ē2

Figure 4.6

to φ_1, the distance from $\sigma_0(\varphi)$ to $\overline{x}_{k+1}{}^*$ is decreased. Specifically, since $\sigma_0(\varphi_0) \subset \overline{M}_+{}^* \cup \overline{p}^*$, 3.5.7 implies that $d/d\varphi\ d(e_0(\varphi),\overline{x}_{k+1}{}^*)\ |_{\varphi=\varphi_0} \geq 0$. This remains true, as φ is decreased, so long as $\sigma_0(\varphi) \subset \left(\overline{M}_+{}^* \cup T_{d(\overline{x}_{k+1}{}^*,\overline{p}^*)+\delta_1}(\overline{x}_{k+1}{}^*)\right)$ and $\sigma_0(\varphi)$ is either in $\overline{M}_+{}^*$ or below $\overline{x}_{k+1}{}^*$. The latter condition clearly remains in force as φ is decreased. To see that the former does also, suppose that $e_0(\overline{\varphi}) \in \mathrm{Cl}\left(T_{d(\overline{x}_{k+1}{}^*,\overline{p}^*)}(\overline{x}_{k+1}{}^*)\right)$ for some $\overline{\varphi}$. But since the hypotheses of 3.5.7 are satisfied, we see that $d(e_0(\varphi),\overline{x}_{k+1}{}^*)$ continue to decrease, which forces $e_0(\varphi)$ to remain in $T_{d(\overline{x}_{k+1}{}^*,\overline{p}^*)+\delta_1}(\overline{x}_{k+1}{}^*)$. This of course applies as well to each point on $\sigma_0(\varphi)$, so that

$$d(e_0(\varphi_1),\overline{x}_{k+1}{}^*) \leq d(e_0(\varphi_0),\overline{x}_{k+1}{}^*) = d(\overline{e}_1{}^*,\overline{x}_{k+1}{}^*) .$$

This same argument applied to $\sigma_k(\chi)$ implies that

$$d(e_0(\varphi_1),\overline{x}_{k+1}{}^*) = d(e_k(\chi_1),\overline{x}_{k+1}{}^*) \geq d(e_k(\chi_0),\overline{x}_{k+1}{}^*) .$$

Now just as in part (4) we conclude that

$$d(e_k(\chi_0),\overline{x}_{k+1}{}^*) > L[\sigma_{k+1}] = d(e_1,x_{k+1}) .$$

Combining all inequalities yields

$$d(\overline{e}_1{}^*,\overline{x}_{k+1}{}^*) \geq d(e_1,x_{k+1}) ,$$

which is (A).

(8) We will now show that (B) holds for $(\sigma_0,\tau_{0,k+1},\sigma_{k+1})$.

The techniques of part (5) show that (B) is true for the triangle $(-\sigma_k, \tau_{k,k+1}, \sigma_{k+1})$. Let $(\sigma_k(\chi_3), \tau_{k,k+1}{}^*, \sigma_{k+1}(\Psi_3))$ denote the corresponding triangle in $\overline{M}^*$. Note that $\Psi_3 > \Psi_0$, and that increasing Ψ_3 will bring $e_{k+1}(\Psi)$ still closer to $\overline{x}_k{}^*$.

Decrease φ_1 to φ_2, so that $d(e_0(\varphi_2), \overline{x}_{k+1}{}^*) = L[\sigma_{k+1}]$. Clearly 3.5.7 continues to apply during this movement. Let Ψ_2 be the angle determined by the (unique) geodesic $\sigma_{k+1}(\Psi_2) \in \mathrm{Geo}(\overline{x}_{k+1}{}^*, e_0(\varphi_2))$.

As noted above, $\varphi_2 < \varphi_1 < \varphi_0$. To see that $\Psi_2 > \Psi_0$, observe that $d(e_0(\varphi_2), \overline{x}_k{}^*) < L[\sigma_k]$, and thus (working in $\overline{M}$ if needed) Ψ_3 must be <u>increased</u> to bring $e_{k+1}(\Psi)$ nearer to $\overline{x}_k{}^*$; i.e., so that $d(e_{k+1}(\Psi_2), \overline{x}_k{}^*) < L[\sigma_k]$.

Hence B is true for $(\sigma_0, \tau_{0,k+1}, \sigma_{k+1})$, and the proof of 4.2.1 is complete.

We now repeat steps (7) and (8) for theorem 4.2.2, using 4.2.1 as needed.

(9) Since $(-\sigma_0(\varphi_1), \overline{\tau}_{0,k}, \sigma_k(\chi_1)) \subset \overline{M}_+$ by the induction hypothesis, it is clear that $d(\overline{e}_1, \overline{x}_{k+1}) \geq d(e_k(\chi_1), x_{k+1})$. Since $\sigma_k(\chi_0) \subset \overline{M}_+$, it is clear that $d(e_k(\chi_1), \overline{x}_{k+1}) \geq d(e_k(\chi_0), \overline{x}_{k+1})$. Theorem 4.2.1, applied to the hinge $(\tau_{0,k}, \sigma_k, \beta_k)$, implies that $d(\sigma_k(t), p) \leq d(\overline{\sigma}_k(t), \overline{p})$. Thus 4.1.3 implies that $d(e_k(\chi_0), \overline{x}_{k+1}) \geq d(e_1, x_{k+1})$. Hence

$d(\bar{e}_1, \bar{x}_{k+1}) \geq d(e_1, x_{k+1})$.

(10) We again apply 4.2.2, this time to the hinge $(\tau_{0,k+1}, \sigma_{k+1}, \beta_{k+1})$, and conclude that $d(\sigma_{k+1}(t), p) \leq d(\bar{\sigma}_{k+1}(t), \bar{p})$. Thus 4.1.3 implies that $d(e_{k+1}(\Psi_0), \bar{x}_k) \geq L[\sigma_k]$. (B) now follows for the triangle $(-\sigma_0, \tau_{0,k+1}, \sigma_{k+1})$ in a manner analogous to that in part (8). Uniqueness is discussed in (11).

(11) Loose ends:

First, the uniqueness of the triangle in (B) of 4.2.2 is clear since the triangle is contained in $\bar{M}_+$ and has one vertex at p.

For the assumption in part (3), note that any nice surface of revolution is a deformation of $\mathbb{R}^2$ through nice surfaces. Suppose that $\{S^t\}$ is such a deformation with $S^0 = \mathbb{R}^2$ and $S^1 = \bar{M}$, and let γ_i^t and σ_i^t denote the geodesics on S^t resulting from the construction in part 2. Let B^t denote the region of S_+^t which is bounded by $\mu_{e_1}^t$ and $\mu_{e_2}^t$.

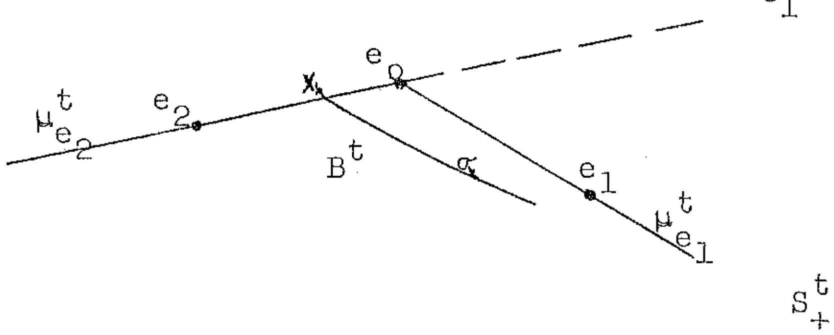


Figure 4.7

Steps (4)-(10) show that if σ_i^t is in S_+^t, it is actually restricted to B^t. Since the σ_i^t are continuous with respect to t, since no σ_i^t passes through p, and since the σ_i^0 are clearly in S_+^0, it follows that the σ_i^1 are in $S_+^1 = \overline{M}_+$. Here we are also using the fact that $\alpha_0 < \pi$.

For the assumption in part (1) concerning δ-correspondence, construct $\overline{M}$ corresponding to M. Since the theorem holds if the metric of $\overline{M}$ is multiplied by any constant greater than 1, it must, by continuity, hold on $\overline{M}$ itself.

If for any x, $\langle \gamma_x', \gamma_2' \rangle(x) = -1$ (see step (1)), then $\alpha_0 = \pi$ and the theorem holds by continuity.

Again referring to the assumptions in step (1), suppose that $\overline{t} \in [0, b_2]$ is the smallest number such that $\langle \gamma_y', \gamma_2' \rangle(y) = +1$, where $y = \gamma_2(\overline{t})$. Then, by continuity, the theorem holds for the hinge $(\gamma_1, \gamma_2|[0,\overline{t}], \alpha_0)$. Since $\gamma_2|[\overline{t}, b_2]$ is contained in the minimal geodesic from y to e_1, the theorem clearly holds for $(\gamma_1, \gamma_2, \alpha_0)$. ∎ (whew!)

5. MANIFOLDS DIFFEOMORPHIC TO EUCLIDEAN SPACE

Finally, we are in a position to prove the main result of this thesis. We begin by recalling several facts from the paper of Cheeger and Gromoll [1972].

M^n will continue to denote a complete open non-negatively curved manifold.

5.1 The Soul of a Manifold

5.1.1 Definition: A set $C \subset M$ is called totally convex if for any $x, y \in C$, $\mathrm{Geo}(x,y) \subset \mathrm{Geo}(C)$.

5.1.2 Theorem (Cheeger and Gromoll): M contains a compact totally geodesic submanifold S without boundary which is totally convex, $0 \leq \dim S < \dim M$.

Proof: We merely outline part of the construction of the set S. For further details see Cheeger and Gromoll [1972], or Cheeger and Ebin [1975].

Let $x \in M$ and let $\gamma : [0,\infty) \to M$ be any geodesic ray beginning at x; i.e., γ is a globally minimal geodesic. Let $B_x(\gamma) := \{y \in S \mid d(\gamma(t),y)<t,\ t \in [0,\infty)\}$, and set $H_x(\gamma) := M \setminus B_x(\gamma)$. We call $H_x(\gamma)$ a complementary half space.

Define $C_x := \cap(H_x(\gamma))$ as γ ranges over all rays beginning at x. It is not difficult to show that C_x is compact. S is now constructed as a certain subset of C_x. ∎

5.1.3 Definition: The set $S \subset M$ constructed in 5.1.2 is called a soul of M.

5.1.4 Theorem (Cheeger and Gromoll): Let S be a soul of M. Then M is diffeomorphic to the normal bundle $\nu(S)$ of S in M. Furthermore, if $K(x) > 0$ for each $x \in S$, then S is a point and hence M is diffeomorphic to $\mathbb{R}^n$.

Proof: See Cheeger and Gromoll [1972], and either Poor [1974] or Šarafutdinov [1974]. ∎

5.2 Compact Half Spaces and Shriveled Souls

Theorem: Let M^n be a complete, open, nonnegatively curved manifold. Suppose $p \in M$ and γ is a ray starting at p.

5.2.1 If there is an $r_1 \in \mathbb{R}$ such that for each $x \in T_{r_1}(p)$, $K(x) > (\pi/3r_1)^2$, then $H_x(\gamma)$ is compact.

5.2.2 If there is an $r_2 \in \mathbb{R}$ such that for each $x \in T_{r_2}(p)$, $K(x) > (\pi/\lambda r_2)^2$, where $\lambda \cong 2.46057$, then M is diffeomorphic to $\mathbb{R}^n$.

Proof: This is an easy application of 4.2.2 to the construction outlined in 5.1.2. We first need to construct appropriate comparison surfaces.

Let C_r^H denote a cone with spherical cap of curvature H and radius r. That is, C_r^H is the surface of revolution generated by the curve $f : \mathbb{R}_+ \to \mathbb{R}^2$, f defined by

$$f(t) = \begin{cases} \left(\dfrac{\sin(t)}{\sqrt{H}} , \dfrac{1-\cos(t)}{\sqrt{H}}\right) & 0 \le t \le \theta \\ \left(\dfrac{\sin(\theta)}{\sqrt{H}} + (t-r)\sin(\theta) , \dfrac{1-\cos(\theta)}{\sqrt{H}} + (t-r)\cos(\theta)\right) & t > \theta \end{cases}$$

where $\theta := r\sqrt{H}$. See figure 5.1(a).

If $x \in C_r^H$ and $y \in \overline{\mu}_x$, let g denote a minimal geodesic from x to y. Let P denote the parallel through the point $(H^{-1/2}\cdot\sin(\theta) , H^{-1/2}\cdot(1-\cos(\theta)) \in f \subset S$, and let $p := d(x,P)$, $q := d(y,P)$. Let s and φ be as indicated in figure 5.1(b).

We will now find conditions on θ which allow us to choose $x \in C_r^H$ so that $L[g] \le p+r$ in the cases

(1) g minimal between x and $\overline{\mu}_x$, and

(2) $q = 0$.

<u>Case</u> (1) $L[g] = (p+s)\sin(\varphi)$

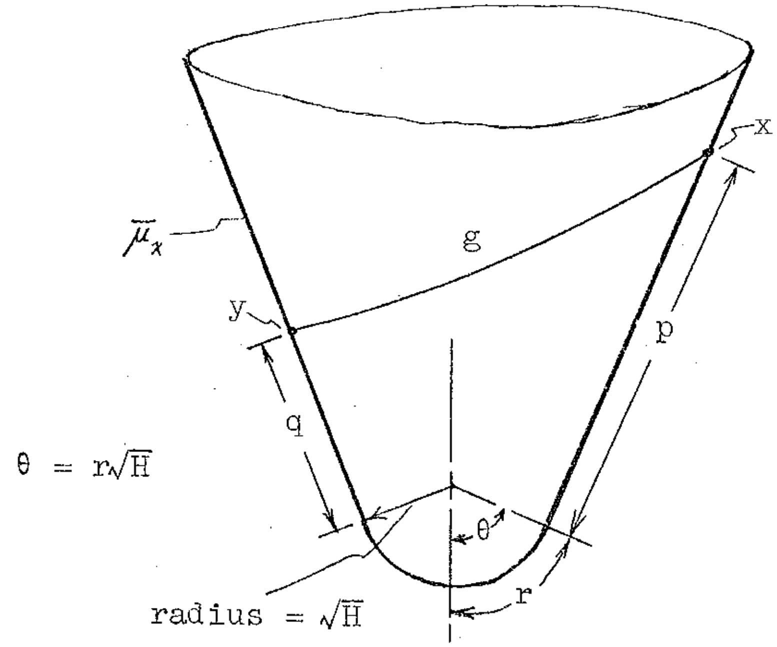
We need conditions so that

$$(p+s)\sin(\varphi) < (p+r) \text{ , or equivalently,}$$

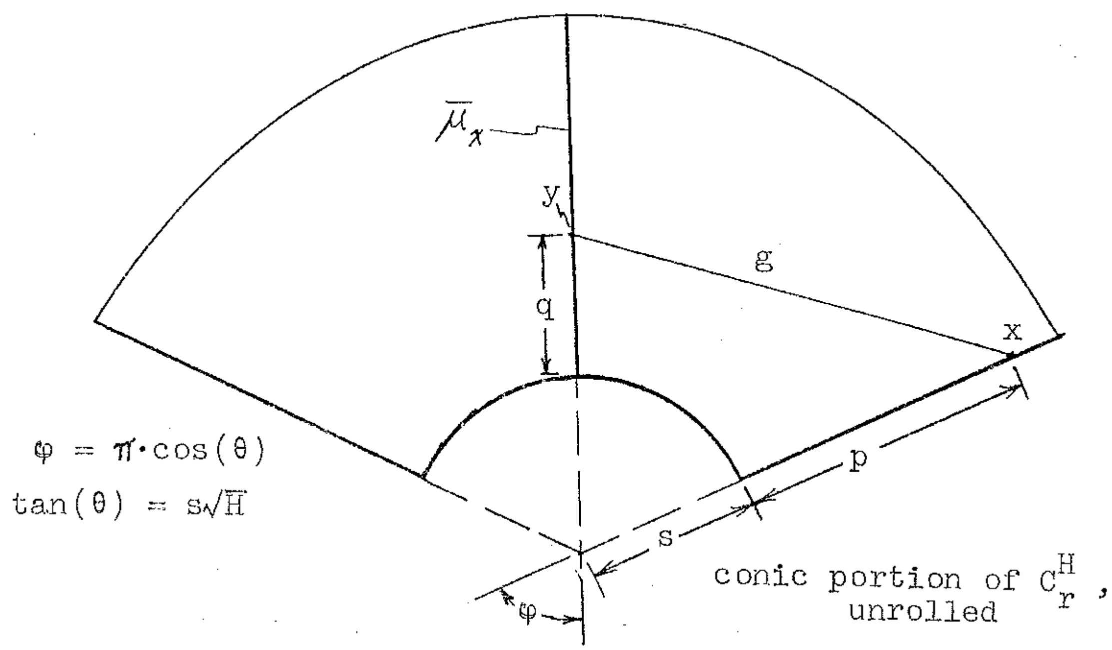
$$p(1-\sin(\varphi)) + (r-s\cdot\sin(\varphi)) > 0.$$

Hence if $\sin(\varphi) \neq 1$, we may choose p large enough to insure that the inequality holds. Thus we need $\varphi < \pi/2$, or $\pi/3 < \theta \le \pi/2$; which is to say

$$\frac{\pi}{3\sqrt{H}} < r \le \frac{\pi}{2\sqrt{H}} .$$



5.1(a)



5.1(b)

Figure 5.1

Case (2) $L[g]^2 = s^2 + (p+s)^2 - 2s(p+s)\cdot\cos(\varphi)$

We need conditions so that

$$s^2 + (p+s)^2 - 2s(p+s)\cdot\cos(\varphi) < (p+r)^2 ,$$

i.e.,
$$s^2 + p^2 + 2ps + s^2 - 2ps\cdot\cos(\varphi) - 2s^2\cdot\cos(\varphi) < p^2 + 2pr + r^2 ,$$

i.e.,
$$2p[s\cdot\cos(\varphi) + r - s] + [r^2 + 2s^2\cdot\cos(\varphi) - 2s^2] > 0 .$$

Hence if $s\cdot\cos(\varphi) + r - s > 0$, we may choose p large enough so that the inequality holds.

Thus the condition is

$$\frac{\tan(\theta)}{\sqrt{H}} \cdot \cos(\varphi) + \frac{\theta}{\sqrt{H}} - \frac{\tan(\theta)}{\sqrt{H}} > 0 ,$$

i.e.,
$$f(\theta) := [\tan(\theta)]\cdot[\cos(\pi\cos(\theta))] + \theta - \tan(\theta) > 0.$$

This is true for $\pi/\lambda < \theta \leq \pi/2$, where $\lambda \cong 2.46057$. Thus we need

$$\frac{\pi}{\lambda\sqrt{H}} < r \leq \frac{\pi}{2\sqrt{H}} .$$

Despite the fact that C_r^H is not smooth, we may use it as a comparison surface by virtue of the previously mentioned approximation theorems. (cf. 2.7)

For 5.2.1, choose $H_1 \in \mathbb{R}$ such that

$$(\pi/3r_1)^2 \leq H_1 \leq K(x) \quad \forall\, x \in T_{r_1}(p) ,$$

and for 5.2.2 choose $H_2 \in \mathbb{R}$ such that

$$(\pi/\lambda r_2)^2 \leq H_2 \leq K(x) \quad \forall\, x \in T_{r_2}(p).$$

Let η be any minimal geodesic in M which starts at p.

In 5.2.1, by comparison with $C^{H_1}_{r_1}$, there is some $t_o > 0$ such that $\eta(t) \in B_p(\gamma)$ for $t > t_o$; and thus $H_p(\gamma) \subset T_{t_o}(p)$.

In 5.2.2, by comparison with $C^{H_2}_{r_2}$, $\eta(t) \in B_p(\gamma)$ for $t > r_2$. Thus $S \subset H_p(\gamma) \subset T_{r_2}(p)$. But $K(x) > 0$ for all x in $T_{r_2}(p)$. Hence by 5.1.4, M is diffeomorphic to $\mathbb{R}^n$. ∎

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