ON SURFACES OBTAINED FROM QUATERNION ALGEBRAS OVER REAL QUADRATIC FIELDS

by

Ira H. Shavel

to

The Graduate School

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in

Mathematics

State University of New York

at

Stony Brook

August, 1976

STATE UNIVERSITY OF NEW YORK

AT STONY BROOK

THE GRADUATE SCHOOL

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Abstract of the Dissertation

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In this dissertation we investigate a class of algebraic surfaces of general type which are of the form $\Gamma \backslash H \times H$, where H is the usual upper half plane, and Γ is a discontinuous group obtained from a quaternion algebra A, with center a real quadratic number field $k = \mathbb{Q}(\sqrt{d})$.

By combining a formula of Matsushima-Shimura, and the Riemann-Roch and Gauss-Bonnet theorems, we obtain formulae for the various numerical invariants of these surfaces, i.e., the Euler characteristic, geometric genus, plurigenera and c_1^2. Using these invariants, we show that these surfaces are of general type. We also give smoothness criteria for these surfaces.

As an example, if $\Gamma = \Gamma(1)$ = the group of units of a maximal order having reduced norm 1, and if $U(1) = \Gamma(1)/\{\pm 1\} \backslash H \times H$ is smooth, then the Euler characteristic E and geometric genus

Pg are given by:

$$E = \frac{1}{12} B_{\chi,2} \prod_{P\in S(A)} (N_{k/\mathbb{Q}}P-1)$$

$$Pg = \frac{1}{4} E-1$$

where $S(A)$ is the set of primes of k at which A is ramified, and $B_{\chi,2}$ is the generalized Bernoulli number of the numerical character modulo d associated to the field $k = \mathbb{Q}(\sqrt{d})$.

We have found several surfaces with geometric genus 0. These surfaces have $c_1^2 = 8$. All previously known surfaces of general type with $Pg = 0$ had $c_1^2 \leq 3$.

To Shahin

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Acknowledgements

I would like to thank my teacher,Professor Michio Kuga, for suggesting this problem. Without his unfailing patience and support during these last three years, this work would not have been possible. I am very grateful to Professor Han Sah for reading this manuscript, and for his advice and encouragement. I am also indebted to Professor Ronald Douglas, the Chairman of my thesis committee, for all the help he has given me during this last year. Finally, I would like to thank Carole Alberghine for her fine typing of this manuscript.

CHAPTER 0

INTRODUCTION

Eicher [4], Kuga [11], Shimizu [17], Shimura [18] have studied algebraic varieties which are quotients of products of upper half planes H^n, by discrete groups Γ of holomorphic automorphisms of H^n constructed from quaternion algebras over number fields. These investigations have been primarily number theoretic. More recently, Hirzebruch [6] considered Hilbert modular surfaces as both number-theoretic and geometric objects. Hilbert modular surfaces can be viewed as a particular case of the above construction; they arise when the quaternion algebra is non-division. At the suggestion of Professor Kuga, I investigated the algebraically more complicated, but geometrically simpler division algebra case. Unlike the Hilbert modular case, the quotient surface is automatically compact. This avoids the necessity to first compactify and then resolve the resulting cusp singularity. The present investigation is mainly concerned with quaternion algebras A over real quadratic number fields.

Under certain conditions, the unit group of a maximal order of A yields a discrete subgroup Γ of $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$, such that the quotient surface $\Gamma\backslash H^2 = U$ is smooth. These surfaces are of general type. This means that U has a particular kind of embedding into projective space; it is embeddable via a pluricanonical system (see I.1).

Pluricanonical systems and surfaces of general type have been studied extensively by Bombieri [1], [2], and Kodaira [8], [9]. I have found examples of surfaces with geometric genus Pg = 0, irregularity q = 0, and $c_1^2 = 8$. It is known that for general type surfaces with Pg = q = 0, $c_1^2 \leq 9$. All other examples of Pg = 0, q = 0 surfaces of general type have $c_1^2 \leq 3$, so these surfaces are topologically new.

Geometric genus 0 surfaces of general type are of particular interest, because the canonical line bundle K over U is positive definite in the sense of Kodaira, but the associated divisor [K] is not a positive divisor. The geometric genus is the complex dimension of the space of holomorphic sections of the line bundle K. Since this is 0, there are no holomorphic sections, but there is always a meromorphic section ψ. [K] is then $(\psi)_0 - (\psi)_\infty$, the difference of the zero set of ψ and the polar set of ψ. Since ψ is not holomorphic $(\psi)_\infty$ is non-empty, and [K] is not a positive divisor, i.e., it is not linearly equivalent to a sum of curves with positive integral coefficients.

In Chapter I, we construct the surfaces which are the objects of this investigation. We also present the necessary background material from the theory of algebraic surfaces and the number theory of quaternion algebras.

In Chapter II, we compute the numerical invariants of U, and we calculate the Euler number of a particular surface from which, it turns out, we will be able to determine the invariants of all the other surfaces.

In Chapter III we give smoothness conditions, and in Chapter IV we work out some examples.

CHAPTER I

PRELIMINARIES

§1 Surfaces of General Type

We will be concerned with surfaces U which are quotients of the product H×H of the usual upper half plane with itself, by a discrete subgroup of $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ acting on H×H without fixed points, and with compact quotient. These groups will be constructed from division quaternion algebras over totally real number fields. These surfaces are projective algebraic varieties because their universal covering space H×H is complex analytically homeomorphic to a bounded domain in $\mathbb{C}^2$. (See Morrow and Kodaira [15].)

<u>Definition I.1.1</u>. A non-singular rational C on U is called an exceptional curve of the first kind, if the intersection multiplicity of C with itself C·C is -1.

Let T^* denote the holomorphic cotangent bundle of U, and K denote the canonical line bundle $\Lambda^2 T^*$ over U. Let $H^p(U,K)$ denote the p-th cohomology group of U with coefficients in the sheaf of germs of local holomorphic sections of K.

<u>Definition I.1.2</u>. The m-th plurigenus P_m of U is the complex dimension of $H^0(U,mK)$, where mK is the sum of the bundle K with itself m times.

Let c_1^2 be the square of the first Chern class evaluated on the fundamental cycle of U.

Definition I.1.3. A smooth surface U is of general type if U has no exceptional curves of the first kind, and c_1^2 and P_2 are both positive.

We shall see in II.2 that our surfaces U are of general type.

We now give a more intuitive definition of general type. Consider a basis $\{\varphi_0, \varphi_2, \ldots, \varphi_N\}$ for the group $H^0(U, mK)$. The φ_i are global holomorphic sections of the line bundle mK. Such a basis is called the m-th pluricanonical system. Consider the mapping $\Phi(z) = (\varphi_0(z), \varphi_1(z), \ldots, \varphi_N(z))$ from U to $\mathbb{C}^{N+1}$. The points $z \in U$ for which $\varphi_i(z) = 0$, for all i $0 \leq i \leq N$, are called base points of Φ. The set of base points is the union of a finite number of points and curves. Let [K] denote the canonical divisor on U and let $\mathcal{E}$ denote the union of all non-singular rational curves E for which $E \cdot [K] = 0$. The restriction of K to E induce the trivial bundle on E, and, therefore, the holomorphic sections of mK over E are constant and the map Φ sends the curve E to a point. The number of such curves is finite, in fact, less than the second Betti number of U. If $\Phi(z)$ has no base points, then it provides a biholomorphic embedding of $U-\mathcal{E}$ into some $P^N(\mathbb{C})$. In our case $\mathcal{E}$ is empty (see the proof of

Corollary II.2.1).

Definition I.1.3'. A smooth surface U is of general type if it has no exceptional curves of the first kind, and for sufficiently large m, the m-th pluricanonical system provides a biholomorphic embedding of U-$\mathcal{E}$ into projective space.

§2 The Groups $\Gamma(1)$, E^{++} and B^{++}

Definition I.2.1. A quaternion algebra A with center k is a central simple algebra of dimension 4 over a number field k.

A central simple algebra always has dimension n^2 over its center. If A is a division algebra, we call n the degree of A. We denote by $\mathfrak{O}_k$ the ring of integers of k.

Definition I.2.2. An order in $\mathfrak{O}$ in a quaternion algebra A is an $\mathfrak{O}_k$-lattice in A which is also a subring of A, that is, it is a subring of A and a finitely generated $\mathfrak{O}_k$-submodule in A such that $k\mathfrak{O} = A$. A maximal order is an order not properly contained in any other order of A.

For a prime divisor P of k we let k_P denote the P-adic completion of k, and we let A_P denote $A \otimes_k k_P$.

Definition I.2.3. We say A is split at P is A_P is isomorphic to the total matrix algebra $M_2(k_P)$, and we say A is ramified at P if A_P is isomorphic to a division algebra over k_P. For an extension K of k, we say K splits A if $A \otimes_k K$ is isomorphic to $M_2(K)$.

From now on k is a totally real number field of degree m over $\mathbb{Q}$. If $k_P = \mathbb{R}$, then A_P is isomorphic to either $M_2(\mathbb{R})$ or $\mathbb{H}$ the Hamiltonian quaternions. We say A is indefinite if it is split at at least one infinite prime, and totally indefinite if it is split at all infinite primes.

Proposition I.2.1. Let A be a quaternion algebra with center k. Let S(A) be the set of primes at which A is ramified. Then A is determined up to isomorphism by k and S(A), and the cardinality of S(A) $|S(A)|$ is even. Moreover, given any set S(A) of primes with $|S(A)|$ even, there exist up to isomorphism a unique quaternion algebra A(k,S(A)) with center k which is split at precisely the primes in S(A).

Proof: See Weil [20].

The following propositions are fundamental.

Proposition I.2.2. Let K be an extension of a number field k. For any prime divisor P of k, let $\mathfrak{P}_1, \ldots, \mathfrak{P}_g$ be the primes of k lying above P. Then we have an isomorphism of k_P-algebras:

$$K \otimes_k k_P \cong \bigoplus_{i=1}^{g} K_{\mathfrak{P}_i}.$$

Proof: See Weil [20].

<u>Proposition I.2.3.</u> K splits A if and only if there is a k-linear embedding of K into A.

<u>Proof:</u> See Reiner [16].

<u>Proposition I.2.4.</u> A is isomorphic to $M_2(k)$ if and only if A_P is isomorphic to $M_2(k_P)$ for all P.

<u>Proof:</u> See Reiner [16].

<u>Corollary.</u> A is division if and only if S(A) is non empty.

Let $P_{\infty_1},\ldots,P_{\infty_m}$ be the real prime divisors of k corresponding to m non-conjugate embeddings $\varphi_1,\ldots,\varphi_m$ of k into $\mathbb{R}$. Let A be split at $P_{\infty_1},\ldots,P_{\infty_n}$ and ramified at $P_{\infty_{n+1}},\ldots,P_{\infty_m}$.

<u>Proposition I.2.5.</u> $\nu(\mathfrak{O}^x) = \{a \in \mathfrak{O}_k^x | \varphi_{\infty_i}(a) > 0 \text{ for } n+1 \le i \le m\}$.

<u>Proof:</u> See Reiner [16].

From now on A will be assumed to be a division algebra. Let ν denote the reduced norm map from A to k. For any ring R, R^x will denote the units of R.

<u>Definition I.2.4.</u> For a fixed maximal order $\mathfrak{O}$ of A, $\Gamma(1)$ is the group of units in $\mathfrak{O}$ with reduced norm 1, i.e. $\Gamma(1) = \{\gamma \in \mathfrak{O}^x | \nu(\gamma) = 1\}$.

Considering $\mathbb{R}$ as a k-algebra via φ_i we have $A \otimes_k \mathbb{R}_{\varphi_i} \cong M_2(\mathbb{R})$ for $1 \le i \le n$, and $A \otimes_k \mathbb{R}_{\varphi_i} \cong \mathbb{H}$ for $n+1 \le i \le m$. Fix such a set of isomorphisms λ_i extending the φ_i. Consider $A \otimes_{\mathbb{Q}} \mathbb{R}$. We have:

$$A \to A_{\mathbb{R}} = A \otimes_{\mathbb{Q}} \mathbb{R} \cong A \otimes_k (k \otimes_{\mathbb{Q}} \mathbb{R}) \cong A \otimes_k \left(\bigoplus_{i=1}^{m} \mathbb{R}_{\varphi_i}\right) \overset{\lambda=(\lambda_1,\ldots,\lambda_m)}{\cong}$$

$$\underbrace{M_2(\mathbb{R}) \oplus \ldots \oplus M_2(\mathbb{R})}_{n\text{-factors}} \oplus \underbrace{\mathbb{H} \oplus \ldots \oplus \mathbb{H}}_{m-n \text{ factors}}$$

For the invertible elements of A we have

$$A^{x} \xrightarrow{\lambda} GL_2(\mathbb{R}) \times \ldots \times GL_2(\mathbb{R}) \times \mathbb{H}^{x} \times \ldots \times \mathbb{H}^{x}.$$

Let A^{x++} denote those elements of A^{x} with totally positive reduced norm. Then

(1) $$A^{x++} \xrightarrow{\lambda} GL_2^{+}(\mathbb{R}) \times \ldots \times GL_2^{+}(\mathbb{R}) \times \mathbb{H}^{x} \times \ldots \times \mathbb{H}^{x}.$$

The reduced norm and the determinant are compatible, that is, the following diagram commutes (for $1 \le i \le n$):

$$\begin{array}{ccccc}
A & \xrightarrow{\text{inclusion}} & A \otimes \mathbb{R}_{\varphi_i} & \xrightarrow{\lambda_i} & M_2(\mathbb{R}) \\
\downarrow \nu & & \downarrow \nu & & \downarrow \det \\
k & \longrightarrow & \mathbb{R} & = & \mathbb{R}
\end{array}$$

Thus,

$$\Gamma(1) \xrightarrow{\lambda} SL_2(\mathbb{R}) \times \ldots \times SL_2(\mathbb{R}) \times \mathbb{H}_1^{x} \times \ldots \times \mathbb{H}_1^{x},$$

where $\mathbb{H}_1^{x}$ are the Hamiltonian quaternions with reduced norm 1. Let $\tilde{\lambda}$ denote the composition of λ with projection to the first n factors.

$$\Gamma(1) \xrightarrow{\tilde{\lambda}} SL_2(\mathbb{R}) \times \ldots \times SL_2(\mathbb{R}) = SL_2(\mathbb{R})^{n}.$$

This map is injective, and the image is a discrete subgroup of $SL_2(\mathbb{R})^n$. Identify $\Gamma(1)$ with its image in $SL_2(\mathbb{R}^n)$. Let the map j denote "reduction modulo center". The center of $\Gamma(1)$ is $\{\pm 1\}$, and so $j(\Gamma(1)) = \Gamma(1)/\{\pm 1\}$ is a discrete subgroup of $PSL_2(\mathbb{R})^n$, and $j(\Gamma(1))$ acts on $H \times \ldots \times H = H^n$ via fractional linear transformation. It is well known that since A is division, the quotient surface is compact (see Shimura [19]).

Let $A^{\times ++}$ be those elements of $A^{\times}$ with totally positive reduced norm. The center of $A^{\times ++}$ is $k^{\times}$.

For the remainder of this section, k denotes a real quadratic field.

<u>Definition I.2.5</u>. E^{++} is the subgroup of $\mathcal{O}^{\times}$ consisting of those elements of $\mathcal{O}^{\times}$ having totally positive reduced norm, i.e. $E^{++} = A^{\times ++} \cap \mathcal{O}^{\times}$.

We now compute $j(E^{++})/j(\Gamma(1))$. The center of E^{++} is $\mathcal{O}_k^{\times}$. Consider the subgroup $\mathcal{O}_k\Gamma(1)$ of E^{++}. $j(\mathcal{O}_k^{\times}\Gamma(1)) = j(\Gamma(1))$. Therefore,

$$j(E^{++})/j(\Gamma(1)) = j(E^{++})/j(\mathcal{O}_k^{\times}\Gamma(1)) = E^{++}/\mathcal{O}_k^{\times} \Big/ \mathcal{O}_k^{\times}\Gamma(1)/\mathcal{O}_k^{\times}$$

$$\cong E^{++}/\mathcal{O}_k^{\times}\Gamma(1).$$

Consider the exact sequences

$$1 \to \Gamma(1) \to E^{++} \xrightarrow{\nu} \mathcal{O}_k^{\times ++} \to 1$$

and

$$1 \to \Gamma(1) \to \mathcal{O}_k^x\Gamma(1) \overset{\nu}{\to} (\mathcal{O}_k^x)^2 \to 1$$

where $\mathcal{O}_k^{x++}$ are the totally positive units of $\mathcal{O}_k$. For the first sequence, the surjectivity of ν follows from Proposition I.2.4.

From the sequences we have $E^{++}/\Gamma(1) \cong \mathcal{O}_k^{x++}$ and $\mathcal{O}_k\Gamma(1)/\Gamma(1) \cong (\mathcal{O}_k^x)^2$. Finally

$$j(E^{++})/j(\Gamma(1)) \cong E^{++}/\Gamma(1) \Big/ \mathcal{O}_k^x\Gamma(1)/\Gamma(1) \cong \mathcal{O}_k^{x++}/(\mathcal{O}_k^x)^2$$

$\mathcal{O}_k^x = \{\pm 1\} \times \{\varepsilon_k^n | n \in \mathbb{Z}\}$ where ε_k is a fundamental unit of k. If ε_k is totally positive then $|\mathcal{O}_k^x : (\mathcal{O}_k^x)^2| = 2$, and $|\mathcal{O}_k^x : (\mathcal{O}_k^x)^2| = 1$ otherwise. Thus, if ε_k is totally positive then $j(E^{++})/j(\Gamma(1)) \cong \mathbb{Z}/2\mathbb{Z}$, otherwise $j(E^{++}) = j(\Gamma(1))$.

<u>Definition I.2.6</u>. B^{++} is the intersection of the normalizer of $\Gamma(1)$ in A^x with A^{x++}, i.e., $B^{++} = N_{\Gamma(1)}$, where N denotes normalizer in A^{x++}.

The center of B^{++} is k^x, because for $x \in k^x$, $\nu(x) = x^2$.

<u>Proposition I.2.6</u>. $\mathcal{O}_k\Gamma(1) = \mathcal{O}$.

<u>Proof</u>: See Kudla [10].

<u>Corollary I.2.1</u>. $B^{++} = N_{\mathcal{O}}$.

<u>Proof</u>: Suppose β normalizes $\mathcal{O}$, then $\beta\Gamma(1)\beta^{-1} \subseteq \mathcal{O}$, and for $\gamma \in \Gamma(1)$, $\beta\gamma^{-1}\beta^{-1}$ is the inverse (in $\mathcal{O}$) of $\beta\gamma\beta^{-1}$. Therefore,

$\beta\Gamma(1)\beta^{-1} \subseteq \mathfrak{O}^x$. $\nu(\beta\gamma\beta^{-1}) = \nu(\beta)\nu(\gamma)\nu(\beta^{-1}) = \nu(\gamma) = 1$. Thus, β normalizes $\Gamma(1)$. Conversely, suppose β normalizes $\Gamma(1)$. Then $\beta\mathfrak{O}\beta^{-1} = \beta\mathfrak{O}_k\Gamma(1)\beta^{-1} = \mathfrak{O}_k(\beta\Gamma(1)\beta^{-1}) = \mathfrak{O}_k\Gamma(1) = \mathfrak{O}$. Thus, β also normalizes $\mathfrak{O}$.

Take $\alpha \in B^{++}$, then by the Corollary $\alpha\mathfrak{O} = \mathfrak{O}\alpha$, i.e., α generates a 2-sided principal $\mathfrak{O}$-ideal. The 2-sided $\mathfrak{O}$-ideals form an abelian group generated by the 2-sided maximal $\mathfrak{O}$-ideals. Corresponding to each P of $\mathfrak{O}_k$, there is a unique maximal 2-sided $\mathfrak{O}$-ideal $\mathfrak{P}$. If $P \notin S(A)$, then $P\mathfrak{O} = \mathfrak{P}$. If $P \in S(A)$, then $\mathfrak{P}^2 = P\mathfrak{O}$. Every 2-sided $\mathfrak{O}$-ideal is uniquely a product of the $\mathfrak{P}$'s. Thus $\alpha\mathfrak{O} = \mathfrak{O}\alpha = \mathfrak{P}_1^{\nu_1} \ldots \mathfrak{P}_r^{\nu_r} \mathfrak{Q}_1^{\mu_1} \ldots \mathfrak{Q}_s^{\mu_s}$ where $\nu_i, \mu_i \in \mathbb{Z}$, and the $\mathfrak{P}_i$ correspond to the $P_i \in S(A)$, and the $\mathfrak{Q}_i$ correspond to the $Q_i \notin S(A)$.

The reduced norm of a 2-sided maximal $\mathfrak{O}$-ideal is a maximal $\mathfrak{O}_k$-ideal, that is

Proposition I.2.7. $\nu(\mathfrak{P}_i) = P_i$.

Proof: See Reiner [16].

Now suppose the class number of k is 1, then the class number of A is also 1. For all P, choose a generator π for P, and for $\mathfrak{P}$ corresponding to $P \in S(A)$, choose a generator Π. Since for $P \in S(A)$, $P\mathfrak{O} = \mathfrak{P}^2$, $\Pi^2 = \pi\varepsilon$ where $\varepsilon \in \mathfrak{O}^x$. So $\alpha\mathfrak{O} = \mathfrak{P}_1^{\nu_1} \ldots \mathfrak{P}_r^{\nu_r} \mathfrak{Q}_1^{\mu_1} \ldots \mathfrak{Q}_s^{\mu_s} = \Pi_1^{\nu_1} \ldots \Pi_r^{\nu_r} \pi_1^{\mu_1} \ldots \pi_s^{\mu_s}\mathfrak{O}$. Thus

$$\alpha = \Pi_1^{\nu_1} \ldots \Pi_r^{\nu_r} \pi_1^{\mu_1} \ldots \pi_s^{\mu_s} \lambda_0 \varepsilon;$$

where $\lambda_0 \in k^x$ and $\varepsilon \in \mathfrak{O}^x$. Since π_i and $\Pi_i^2 \in k^x$, $\alpha = \Pi_1^{\nu_1} \cdots \Pi_r^{\nu_e} \lambda\varepsilon$ where $\lambda \in k^x$ and $\nu_i \in \{0,1\}$. Thus

$$B^{++} = \{\Pi_{i_1} \cdots \Pi_{i_r} \lambda\varepsilon \mid \lambda \in k^x,\ \varepsilon \in \mathfrak{O}^x \text{ and } \nu(\Pi_{i_1} \cdots \Pi_{i_r} \varepsilon)\text{'is totally positive}\}.$$

Let k^{++} denote the totally positive elements of k^x. Similarly let k^{+-}, k^{-+} and k^{--} denote the elements of k^x whose image via the first embedding of k into R is positive and negative image via the second, negative image the first embedding and positive via the second, and negative image via both embeddings, respectively. Let ε_k be a fundamental unit greater than 0. Choose η and ξ of $\mathfrak{O}^x$ such that $\nu(\eta) = -1$ and $\nu(\xi) = \varepsilon_k$. If $\nu(\Pi) \in k^{--}$ replace Π by $\Pi\eta$, and then $\nu(\Pi\eta) \in k^{++}$. If $\nu(\Pi) \in k^{+-}$ (respectively $\in k^{-+}$) and $\varepsilon_k \in k^{+-}$, replace Π by $\Pi\varepsilon$ (respectively $\Pi\eta\varepsilon$) and then $\nu(\Pi\varepsilon)$ (respectively $\nu(\Pi\eta\varepsilon)$) is in k^{++}. If $\varepsilon_k \in k^{++}$ and some $\nu(\Pi) \in k^{+-}$, we cannot replace Π by an element having totally positive reduced norm. Since $\Pi^2 \in k^x$, we have

$$j(B^{++})/j(\Gamma(1)) \cong \begin{cases} (\mathbf{Z}/2\mathbf{Z})^{|S(A)|} & \text{if } \varepsilon_k \notin k^{++} \\ (\mathbf{Z}/2\mathbf{Z})^{|S(A)|} & \text{if } \varepsilon_k \in k^{++} \text{ and some } \nu(\Pi) \notin k^{++} \text{ or } k^{--}. \\ (\mathbf{Z}/2\mathbf{Z})^{|S(A)|+1} & \text{if } \varepsilon_k \in k^{++} \text{ and all } \nu(\Pi) \in k^{++} \text{ or } k^{--}. \end{cases}$$

CHAPTER II

THE NUMERICAL INVARIANTS

§1 The Geometric Genus and the Euler Number

Throughout this section Γ will denote a discrete subgroup of $SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \times \ldots \times SL_2(\mathbb{R}) = SL_2(\mathbb{R})^n$ such that $j(\Gamma) = \Gamma/\text{center of } \Gamma$ acts on $H \times H \times \ldots \times H = H^n$ without fixed points and with compact quotient. $H \times H$ will be denoted by X, and for $\Gamma \subseteq SL_2(\mathbb{R})^2$ the quotient surface $j(\Gamma)\backslash X$ will be denoted by $U(\Gamma)$. Let U denote an aribitrary compact complex manifold. The r-th Betti number b^r of U is the complex dimension of $H^r(U,C)$ and the Euler number $E(U)$ of U is $\sum_{r=0}^{2n}(-1)^r b^r$ where $n = \dim_{\mathbb{C}}(U)$.

In this section we find simple relationships between the Euler number, geometric genus and arithmetic genus of $U(\Gamma)$ in the case where Γ is commensurable with $\Gamma(1)$. We begin with some definitions.

<u>Definition II.1.1.</u> The sheaf of germs of holomorphic p-forms on U is denoted by Ω^p.

Observe that $\Omega^0 = \mathcal{O}$ = the sheaf of germs of holomorphic functions on U.

<u>Definition II.1.2.</u> The complex dimension of $H^q(U,\Omega^p)$ is denoted by $h^{(p,q)}$.

Definition II.1.3. The geometric genus Pg of U is $h^{(0,n)}$.

Definition II.1.4. The irregularity q of U is $h^{(0,1)}$.

Definition II.1.5. The arithmetic genus Pa of U is $\sum_{j=0}^{n}(-1)^j h^{(0,j)}$.

Proposition II.1.1 (Serre Duality). Let U be a compact complex manifold of dimension n. Then $H^q(U,\Omega^p) \cong H^{n-q}(U,\Omega^{n-p})$.

Proof: See Morrow and Kodaira [15].

Corollary II.1.1. For $U(\Gamma)$, $h^{(0,1)} = h^{(2,1)}$, $h^{(1,0)} = h^{(1,2)}$ and $h^{(2,0)} = h^{(0,2)}$.

Proof: This follows immediately with n = 2.

Definition II.1.6. The space of C^∞-differential r-forms on U is denoted by $A^r(U)$, and the space of r-forms of bidegree (p,q) is denoted by $A^{(p,q)}(U)$.

Proposition II.1.2. $A^r(U) = \bigoplus_{p+q=r} A^{(p,q)}(U)$ where $\oplus$ denotes direct sum.

Proof: This is immediate from the definitions.

A differential form can be represented locally in terms of local coordinates. Let $\omega \in A^{(p,q)}(U)$ be given locally by

$$\omega_{\alpha\beta} = f_{\alpha_1\ldots\alpha_p\beta_1\ldots\beta_q} dz_{\alpha_1}\wedge\ldots\wedge dz_{\alpha_p}\wedge d\bar{z}_{\beta_1}\wedge\ldots\wedge d\bar{z}_{\beta_q}.$$

Definition II.1.7. The operators $\partial : A^{(p,q)}(U) \to A^{(p+1,q)}(U)$ and $\bar{\partial} : A^{(p,q)}(U) \to A^{(p,q+1)}(U)$ are defined by

$$\partial\omega_{\alpha\beta} = \sum_{i=1}^{n} \frac{\partial(f_{\alpha_1\ldots\alpha_p\beta_1\ldots\beta_q})}{\partial z_{\alpha_i}} dz_{\alpha_i}\wedge dz_{\alpha_1}\wedge\ldots\wedge dz_{\alpha_p}\wedge$$

$$d\bar{z}_{\beta_1}\wedge\ldots\wedge d\bar{z}_{\beta_q}$$

and

$$\bar{\partial}\omega_{\alpha\beta} = \sum_{i=1}^{n} \frac{\partial f_{\alpha_1\ldots\alpha_p\beta_1\ldots\beta_q}}{\partial\bar{z}_{\beta_i}} dz_{\beta_i}\wedge dz_{\alpha_1}\wedge\ldots\wedge dz_{\alpha_p}\wedge d\bar{z}_{\beta_1}\wedge\ldots\wedge d\bar{z}_{\beta_q}.$$

Clearly $\partial+\bar{\partial} = d$ is the ordinary exterior differential operator.

Let $ds^2 = 2\sum_{\alpha,\beta=1}^{n} g_{\alpha\beta}dz_\alpha\cdot d\bar{z}_\beta$, ${}^t(\overline{g_{\alpha\beta}}) = (g_{\alpha\beta})$ be a Hermitian metric on U.

Definition II.1.8. A compact complex manifold U is called Kähler if there exists a Hermitian metric ds^2 on U such that the associated 2-form $\Omega = \sqrt{-1}\sum_{\alpha,\beta=1}^{n} g_{\alpha\beta}dz_\alpha\wedge d\bar{z}_\beta$ is closed, i.e., $d\Omega = 0$. Such a metric is called a Kähler metric.

Proposition II.1.3. A projective algebraic manifold is Kähler.

Proof: See Morrow and Kodaira [15].

Thus $U(\Gamma)$ is Kähler and we can apply the following to $U(\Gamma)$:

Proposition II.1.4. If U is a Kähler manifold, then

a) $H^q(U,\Omega^p) \cong H^p(U,\Omega^q)$.

b) $H^r(U,\mathbb{C}) \cong \bigoplus_{p+q=r} H^p(U,\Omega^q)$.

<u>Proof</u>: See Morrow and Kodaira [15].

<u>Corollary II.1.2</u>. For a Kähler manifold U

a) $h^{(p,q)} = h^{(q,p)}$

b) $b^r = \sum_{p+q=r} h^{(p,q)}$

c) $Pg = h^{(n,0)}$ = dimension of the space of all holomorphic n-forms on U.

d) $q = h^{(1,0)}$ = dimension of the space of all holomorphic 1-forms on U.

e) $Pa = \sum_{j=0}^{n} (-1)^j h^{(j,0)}$.

For a complex 2-dimensional Kähler manifold we have $q = h^{(0,1)} = h^{(1,0)} = h^{(1,2)} = h^{(2,1)}$ and $Pg = h^{(0,2)} = h^{(2,0)}$.

Let U be a complex manifold with universal covering space X, $G = \pi_1(U)$ be the group of covering transformation and $\pi : X \to U = G\backslash X$ the natural covering map. For $g \in G$ and $\eta \in A^r(X)$ let $g^*\eta$ be $\eta \circ g$ and for $\omega \in A^r(U)$ let $\pi^*\omega = \omega \circ \pi$ be the pullback of ω via π. Since π is surjective π^* is injective, and we can identify an r-form $\omega \in A^r(U)$ with its image $\pi^*\omega$ in $A^r(X)$. With this identification we have

<u>Proposition II.1.5</u>. $A^r(U) = \{\eta \in A^r(X) | g^*\eta = \eta \text{ for all } g \in G\}$.

<u>Proof</u>: $\pi^*\omega = g^*(\pi^*\omega)$ because $\omega \circ \pi = \omega \circ \pi \circ g$. Thus the image of $A^r(U)$ in $A^r(X)$ are Γ-invariant forms on X.

Conversely, let η be a Γ-invariant form on X. For $p \in U$ choose a neighborhood V of p small enough so that $\pi^{-1}(V)$ is a disjoint union of open sets V_i, $i = 1,2,\ldots,n$ of X. π restricted to V_i is a diffeomorphism of V_i onto V. Let g_{ij} be the covering transformation taking V_j to V_i. Then restricting $\pi \circ g_{ij}^{-1} = \pi$ to V_j and taking inverses we have

$$(\pi|_{V_j})^{-1} = g_{ij} \circ (\pi|_{V_i})^{-1}$$

and so

$$(\pi|_{V_j})^{-1*}(\eta) = (\pi|_{V_i})^{-1}(g_{ij}^*(\eta))$$

Since η is Γ-invariant, this is equal to $(\pi|_{V_i})^{-1*}(\eta)$. Thus the induced r-form on V is independent of the choice of V_i and we have a unique r-form η_V defined on V.

Now suppose W is another neighborhood of p which is again small enough so that $\pi^{-1}(W)$ is a disjoint union of W_i, $i = 1,2,\ldots,n$. Let V_i and W_i have the common point p_i contained in the fiber over p. Then

$$\eta_V = (\pi|_{V_i})^{-1*}(\eta) = \eta \circ (\pi|_{V_i})^{-1}$$

and

$$\eta_W = (\pi|_{W_i})^{-1*}(\eta) = \eta \circ (\pi|_{W_i})^{-1}.$$

From this, it follows that $\eta_V|_{V \cap W} = \eta_W|_{V \cap W}$. Thus we can define a global r-form $(\pi^{-1})^*\eta$ on U and the proposition follows.

In the case where $j(\Gamma)$ acts properly discontinuously and without fixed points on X, X is the universal cover of $U(\Gamma)$ and $\pi_1(U(\Gamma))$ is isomorphic to $j(\Gamma)$. The proposition then allows us to apply a theorem of Matsushima and Shimura [14] to $U(\Gamma)$, because the forms $A^r(X,\Gamma,\rho)$ defined in [14] with ρ the trivial representation are nothing but Γ-invariant forms on X, i.e. $A^r(U)$. We use their theorem to compute the irregularity q of $U(\Gamma)$ and $h^{(1,1)}$.

Definition II.1.9. Let G_i, $i = 1,2,\ldots,n$ be copies of a connected, non-compact, simple Lie group, and let $G = G_1 \times G_2 \times \ldots \times G_n$. We call a discrete subgroup Γ of G irreducible if the projection of Γ to any partial factor of G different from G itself is not discrete in the partial factor.

If Γ is irreducible and Γ_1 is commensurable with Γ then Γ_1 is also irreducible.

In the case where G_i is either $SL_2(\mathbb{R})$ or $PSL_2(\mathbb{R})$ the following criterion is useful.

Lemma II.1.1 (Shimizu). Γ is irreducible in $G = PSL_2(\mathbb{R})^n$ if and only if Γ contains no element $\gamma = (\gamma^{(1)},\gamma^{(2)},\ldots,\gamma^{(n)})$ such that $\gamma^{(i)} = 1$ and $\gamma^{(j)} \neq 1$ for some j.

Proof: See Shimizu [17], Corollary, page 40.

Lemma II.1.2. A discrete subgroup Γ $SL_2(\mathbb{R})^n$ is irreducible in $SL_2(\mathbb{R})^n$ if and only if $j(\Gamma)$ is irreducible in $PSL_2(\mathbb{R})^n$.

Proof: Let G_o be a topological group, F a finite normal subgroup and let j denote the natural projection of G_o onto G_o/F. Let Γ_o be a subgroup of G_o. Then Γ_o is discrete in G_o if and only if $j(\Gamma_o)$ is discrete in G_o/F. Applying this to the case where G_o is a partial product of $SL_2(\mathbb{R})^n$ and F is its center, the lemma follows at once.

Lemma II.1.3. Let Γ be commensurable with $\Gamma(1)$. Then Γ is irreducible in $SL_2(\mathbb{R})^2$.

Proof: By Lemma II.1.2, it suffices to show that $j(\Gamma(1))$ is irreducible in $PSL_2(\mathbb{R})^2$. Consider $\mathbb{R}$ as a k-algebra in two ways via $\varphi_{\infty 1}$ and $\varphi_{\infty 2}$, that is via two non-conjugate embeddings $\varphi_{\infty 1}$ and $\varphi_{\infty 2}$ of k into $\mathbb{R}$. Extend these embeddings to isomorphisms $A \otimes_k \mathbb{R} \overset{\lambda_i}{\cong} M_2(\mathbb{R})$, $i = 1,2$. The λ_i, $i = 1,2$ restricted to $\Gamma(1)$ are injective maps of $\Gamma(1)$ into $SL_2(\mathbb{R})$. Thus if $\lambda_1(\gamma) = 1$. Then $\gamma = 1$ and $\lambda_2(\gamma) = 1$. So for $\gamma = (\gamma^{(1)}, \gamma^{(2)})$ if $\gamma^{(1)} = 1$ then $\gamma^{(2)} = 1$ and the same holds for $j(\Gamma(1))$. Thus using Shimizu's criterion $j(\Gamma(1))$ is irreducible in $PSL_2(\mathbb{R})^2$ and the lemma follows.

Proposition II.1.6. Let Γ be a discrete irreducible subgroup of $SL_2(\mathbb{R})^n$ such that $j(\Gamma)$ acts on H^n without fixed points. Then for $j(\Gamma)\backslash H^n$

a) $h^{(p,q)} = 0$ for $p \neq q$ and $p + q \neq n$.

b) $h^{(n-q,q)} = \binom{n}{q}[\delta_{n-q,q} + h^{(n,0)}]$

where $\delta_{i,j}$ is the Kronecker delta symbol.

Proof: See Matsushima and Shimura [14].

Corollary II.1.3. Let Γ be commensurable with $\Gamma(1)$ and $j(\Gamma)$ act on X without fixed points. Then $q = h^{(1,0)} = h^{(0,1)} = h^{(2,1)} = h^{(1,2)} = 0$ and $h^{(1.1)} = 2Pg + 2$.

Proposition II.1.7. Let Γ be commensurable with $\Gamma(1)$ and act on X without fixed points. Then for $U(\Gamma)$, $b^0 = b^4 = 1$.

Proof: U is a connected compact 4-dimensional real manifold. Since it admits a complex structure it is orientable. Therefore, $b^0 = b^4 = 0$.

Theorem II.1.1. Let Γ be commensurable with $\Gamma(1)$ and act on X without fixed points. Then $E(U(\Gamma)) = 4(Pg + 1) = 4Pa$.

Proof: $b^1 = b^3 = 0$, and $b^2 = 4Pg + 2$ by Corollary II.1.3. Thus $E(U(\Gamma)) = 4(Pg + 1)$. $Pa = Pg + 1$ follows from the definition of Pa.

§2 The Plurigenera and c_1^2.

In this section we calculate c_1^2 and the m-th plurigenus of $U(\Gamma) = j(\Gamma)\backslash X$ where Γ is commensurable with $\Gamma(1)$, and as a Corollary we show that $U(\Gamma)$ is of general type.

Let F be a complex analytic line bundle over U a compact complex manifold, and let c(F) denote the Chern class of F. Let T^* denote the holomorphic cotangent bundle of U and K denote the canonical line bundle $\Lambda^2 T^*$ over U. Let c_i denote the i-th Chern classes of U. $c_1 = -c(K)$ and for an n-dimensional complex manifold U ${}_n$ is the Euler number of U. The p-th cohomology group of U with coefficients in the sheaf of germs of local holomorphic sections of F will be denoted by $H^p(U,F)$. The m-th plurigenus Pm of U is $\dim_{\mathbb{C}} H^0(U,mK)$.

The Riemann-Roch-Hirzebruch Theorem can be formulated as follows:

<u>Theorem II.2.1.</u>

$$h^0(U,F) - h^1(U,F) + h^2(U,F) = \frac{1}{2}(c(F)^2 + c(F)\cdot c_1) + \frac{1}{12}(c_1^2 + c_2)$$

where $h^p(U,F)$ is the complex dimension of $H^p(U,F)$.

<u>Proof</u>: See Hirzebruch [5].

We now compute c_1^2 of $U(\Gamma)$.

$$(1) \quad h^0(U,K) - h^1(U,K) + h^2(U,K) = \frac{1}{2}(c(K)^2 + c(K)\cdot c_1) + \frac{1}{12}(c_1^2 + c_2).$$

Since $c(K) = -c_1$ and $c_2 = E(U)$, then (1) becomes

$$(2) \quad h^0(U,K) - h^1(U,K) + h^2(U,K) = \frac{1}{12}(c_1^2 + E(U)).$$

$H^p(U,K) = H^{2-p}(U,\Omega^0)$ therefore (2) becomes

(3) $$Pg - q + 1 = \frac{1}{12}(c_1^2 + E(U))$$

and finally we have

(4) $$c_1^2 = 12Pa - E(U)$$

which for $U(\Gamma)$ becomes

(5) $$c_1^2 = 8Pa.$$

To determine Pm we apply the Riemann-Roch-Hirzebruch Theorem to the line bundle mK. The sum of the line bundle K with itself m-times. We have:

$$h^0(U(\Gamma),mK) - h^1(U(\Gamma),mK) + h^2(U(\Gamma),mK)$$
$$= \frac{1}{2}(c(mK)^2 + c(mK)\cdot c_1) + \frac{1}{12}(c_1^2 + c_2)$$
$$= \frac{1}{2}(m^2c(K)^2 + mc(K)\cdot c_1) + Pa$$
$$= \frac{1}{2}(m^2-m)c_1^2 + Pa$$
$$= 4(m^2-m)Pa + Pa$$
$$= Pa(2m-1)^2.$$

For the quotient V of a bounded domain in $\mathbb{C}^2$ by a discontinuous group of automorphisms acting without fixed points $h^1(V,mK) = h^2(V,mK) = 0$ for $m \geq 2$. (See Hirzebruch [5].) Thus we have

Theorem II.2.2. For $U(\Gamma)$ we have

(1) $c_1^2 = 8Pa$

(2) $Pm = Pa(2m-1)^2$.

Corollary II.2.1. $U(\Gamma)$ is of general type.

Proof: By the Theorem c_1^2 and P_2 are both positive. We must show that $U(\Gamma)$ has no exceptional curves of the first kind. Suppose D is an exceptional curve of the first kind. Then there is a rational map f of $P'(\mathbb{C})$ into $U(\Gamma)$ with image D. This map lifts to a holomorphic map $\tilde{f}$ of $P'(\mathbb{C})$ into $X = H \times H$ a bounded domain in $\mathbb{C}^2$. By Liouville's Theorem $\tilde{f}$ must be constant, but then D must be a point. This contradiction completes the proof.

§3 Computation of the Euler number of $U(\Gamma(1))$

In this section we determine the Euler number of $U(1) = j(\Gamma(1))\backslash X$. This will also enable us to determine the Euler numbers of surfaces obtained from certain subgroups of $\Gamma(1)$ and certain subgroups of the normalizer of $\Gamma(1)$ in A^x.

Recall that the measure

$$dz = \frac{dx_1 \wedge dy_1}{y_1^2} \wedge \frac{dx_2 \wedge dy_2}{y_2^2} \wedge \ldots \wedge \frac{dx_n \wedge dy_n}{y_n^2}$$

is $(PSL_2\mathbb{R})^n$-invariant on H^n.

Definition II.3.1. For measurable set $E \subseteq H^n$ the volume of E Vol(E) is the number $\int_E dz$.

Let Γ be a discrete subgroup of $PSL_2(\mathbb{R})^n$ acting on H^n without fixed points and with compact quotient. Then the Gauss-Bonnet Theorem gives the Euler number $E(\Gamma\backslash H^n)$ of $\Gamma\backslash H^n$.

$$(1) \qquad E(\Gamma\backslash H^n) = \left(\frac{-1}{2\pi}\right)^n \mathrm{Vol}(F)$$

where F is a fundamental domain for the action Γ on H^n. In the case where $X = H \times X$ this gives

Proposition II.3.1. If $\Gamma \subseteq PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ is discrete acting on X without fixed points and with compact quotient, then

$$(2) \qquad E(\Gamma\backslash X) = \frac{1}{4\pi^2}\,\mathrm{Vol}(F).$$

Let k be a totally real algebraic number field of degree m over $\mathbb{Q}$. Let $P_{\infty_1}, P_{\infty_2}, \ldots, P_{\infty_m}$ be the m real places of k corresponding to the m non-conjugate embeddings $\varphi_1, \varphi_2, \ldots, \varphi_m$ of $k/\mathbb{Q}$ into $\mathbb{R}$. Let A be a quaternion algebra with center k which is unramified at the first n infinite places and ramified at the next m-n infinite places. Recall (Chapter I) that this gives an isomorphism

$$A \otimes_{\mathbb{Q}} \mathbb{R} \cong \underbrace{M_2(\mathbb{R}) \oplus \ldots \oplus M_2(\mathbb{R})}_{n\text{-copies}} \oplus \underbrace{\mathbb{H} \oplus \ldots \oplus \mathbb{H}}_{m-n \text{ copies}}$$

and that via this isomorphism

$$(3) \qquad \Gamma(1) \hookrightarrow \underbrace{SL_2(\mathbb{R}) \times \dots \times SL_2(\mathbb{R})}_{\text{n-copies}} \times \underbrace{\mathbb{H}_1^{\times} \times \dots \times \mathbb{H}_1^{\times}}_{\text{m-n copies}}$$

where $\mathbb{H}_1^{\times}$ denotes the multiplicative group of elements of $\mathbb{H}$ with norm 1. Consider the projection of (3) onto the first n-factors. We then have injective maps

$$\Gamma(1) \hookrightarrow \underbrace{SL_2(\mathbb{R}) \times \dots \times SL_2(\mathbb{R})}_{\text{n-copies}}$$

$$j(\Gamma(1)) \hookrightarrow \underbrace{PSL_2(\mathbb{R}) \times \dots \times PSL_2(\mathbb{R})}_{\text{n-copies}}$$

The images are discrete and the quotient $j(\Gamma)\backslash H^n$ has finite volume. Moreover, if A is a division algebra, then $j(\Gamma(1))\backslash H^n$ is compact.

Shimizu [17] gives a formula for the volume of a fundamental domain F for the action of $j(\Gamma(1))$ on H^n:

$$(4) \qquad \mathrm{Vol}(F) = \frac{2^{n-m+1} d^{3/2} h(k) \zeta_k(2)}{\pi^{2m-n} [\mathcal{O}_k^{\times} : \mathcal{O}_k^{\times'}] h(A)} \prod_{P \in S'(A)} (N_{k/\mathbb{Q}} P - 1)$$

where $A = A(k, S(A))$, $S'(A)$ is the subset of all finite primes in $S(A)$, d is the discriminant of $k/\mathbb{Q}$, $\zeta_k(2)$ is the value of the Dedekind zeta function $\zeta_k(S)$ at 2, h(k) is the class number of k, h(A) is the class number of a maximal order of A, $\mathcal{O}_k$ is the group of units of k, and $\mathcal{O}_k^{\times'}$ are those units ε of k such that $\varphi_{\infty_i}(\varepsilon) > 0$ for $i = n+1, n+2, \dots, m$.

The following relationship between the class numbers of k and A greatly simplifies (4).

Lemma II.3.1. $h(A) = h(k) \frac{2^{m-n}}{[\mathfrak{O}_k^{x} : \mathfrak{O}_k^{x'}]}$.

Proof: The ideal class group of $\mathfrak{O}$ is isomorphic to the ray class group of k and $P_{\infty_{n+1}} P_{\infty_{n+2}} \cdots P_{\infty_m}$ (see Reiner [16]). Thus $h(A) = h_0(k)$ = the order of the ray class group mod $P_{\infty_{n+1}} P_{\infty_{n+2}} \cdots P_{\infty_m}$. The order of the ray class group mod $P_{\infty_{n+1}} P_{\infty_{n+2}} \cdots P_{\infty_m}$ is given by

$$\frac{h(k)2^{m-n}}{[\mathfrak{O}_k^{x} : \mathfrak{O}_k^{x'}]}$$

where $\mathfrak{O}_k^{x'}$ are those elements a of $\mathfrak{O}_k^{x}$ for which $\varphi_{\infty_j}(a) > 0$ for $n+1 \leq j \leq m$ (see Lang [12]). Thus

$$h(A) = \frac{h(k)2^{m-n}}{[\mathfrak{O}_k^{x} : \mathfrak{O}_k^{x'}]} .$$

Equation (4) now becomes

$$(5) \qquad \mathrm{Vol}(F) = \frac{2d^{3/2}\zeta_k(2)}{\pi^{2m-n}2^{2m-2n}} \prod_{P \in S'(A)} (N_{k/\mathbb{Q}}P-1)$$

We now specialize to the case $k = \mathbb{Q}(\sqrt{m})$, m a square free positive integer, and d is the discrimant of k.

Definition II.3.2. The generalized Bernoulli numbers $B_{\chi,\ell}$, $\ell = 0,1,2,\ldots$ and the coefficients of the Maclaurin expansion

$$\frac{\sum_{n=1}^{d-1} \chi(n)te^{nt}}{e^{dt}-1}$$

where χ is the numerical character modulo d associated to the field $k = \mathbb{Q}(\sqrt{m})$. That is

$$\sum_{\ell=0}^{\infty} \frac{B_{\chi,\ell}}{\ell!} t^{\ell} = \frac{\sum_{n=1}^{d-1} \chi(n) t e^{nt}}{e^{dt}-1}.$$

The character $\chi(r)$ is given by

$$(6) \qquad \chi(r) = \begin{cases} (\frac{r}{m}) & \text{if } m \equiv 1 \pmod 4 \\ (\frac{r}{m})(-1)^{\frac{r-1}{2}} & \text{if } m \equiv 3 \pmod 4 \\ (\frac{r}{m'})(-1)^{\frac{r^2-1}{8} + \frac{r-1}{2}} & \text{if } m = 2m' \\ 0 & \text{if } r \mid d \end{cases}$$

where $(-)$ is the Kronecker symbol (see Borevich-Shafarevich [3]).

<u>Lemma II.3.2</u>. $\zeta_k(s) = \zeta(s)\, L(s,\chi)$ where ζ is the Riemann zeta function and $L(s,\chi)$ is the L-function with character χ mod d associated to k.

<u>Proof</u>: See Hecke [7] for the lemma in this form, or Weil [20] for the corresponding result for any abelian extension.

<u>Proposition II.3.2</u>. Let $k = \mathbb{Q}(\sqrt{m})$, $m > 0$ and let d be the discriminant of k. Then the value of the L-function $L(2n,\chi)$ for $n \geq 1$ is given by

$$(7) \qquad L(2n,\chi) = \frac{\tau(\chi)}{2}\left(\frac{2\pi}{d}\right)^{2n} \frac{B_{\chi,2n}}{(2n)!}$$

where $\tau(\chi)$ is the Gauss sum $\sum_{r=1}^{d-1} \chi(r)e^{2\pi ir/d}$.

Proof: See Leopoldt [13].

Corollary II.3.1. Let $k = \mathbb{Q}(\sqrt{m})$, $m > 0$ and let d be the discriminant of k. Then the value of the zeta function $\zeta_k(s)$ at 2 is

$$\zeta_k(2) = \frac{\pi^4}{6d^2} \tau(\chi)B_{\chi,2}. \tag{8}$$

Proof: $\zeta_k(2) = \zeta(2)\, L(2,\chi)$ by Lemma II.2.2. Evaluating (7) at $n = 1$ and noting that $\zeta(2) = \pi^2/6$ gives the desired result.

Lemma II.3.3. $|\tau(\chi)| = \sqrt{d}$.

Proof: See Lang [12].

Going back to (5) we have

$$\text{Vol}(F) = \frac{\pi^2}{3\sqrt{d}} \tau(\chi)B_{\chi,2}$$

and using (2) we have

$$E(\Gamma(1)) = \frac{B_{\chi,2}}{12\sqrt{d}} \tau(\chi) \prod_{P\in S(A)} (N_{k/\mathbb{Q}}P - 1)$$

Since the Euler characteristic is positive this becomes

$$E(\Gamma(1)) = \frac{|B_{\chi,2}|}{12\sqrt{d}} |\tau(\chi)| \prod_{P\in S(A)} (N_{k/\mathbb{Q}}P - 1)$$

$$= \frac{|B_{\chi,2}|}{12} \prod_{P\in S(A)} (N_{k/\mathbb{Q}}P - 1)$$

Theorem II.3.1. For $j(\Gamma(1))$ acting on X without fixed points and with compact quotient we have

$$(9) \qquad E(\Gamma(1)) = \frac{|B_{\chi,2}|}{12} \prod_{P \in S(A)} (N_{k/\mathbb{Q}}P - 1)$$

Proposition II.3.3. $B_{\chi,2} = \frac{1}{d} \sum_{r=1}^{d-1} r^2\chi(r)$.

Proof: By definition

$$\sum_{\ell=0}^{\infty} \frac{B_{\chi,\ell}}{\ell!} t^\ell = \frac{\sum_{r=1}^{d-1} \chi(r)te^{rt}}{e^{dt}-1} .$$

Expanding the right hand side we have

$$\sum_{\ell=0}^{\infty} \frac{B_{\chi,\ell}}{\ell!} t^\ell = \frac{\sum_{r=1}^{d-1} \chi(r)t(1+rt+(rt)^2/2!+ \ldots)}{dt+(dt)^2/2!+(dt)^3/3!+\ldots}$$

$$= \frac{\sum_{r=1}^{d-1} \chi(r)(1+rt+(rt)^2/2!+ \ldots)}{d(1+dt/2!+(dt)^2/3!+\ldots)}$$

$$= \frac{\sum_{r=1}^{d-1} \chi(r)+\sum_{r=1}^{d-1} rt\chi(r)+\sum_{r=1}^{d-1} (r^2t^2/2!)\chi(r)+ \ldots}{d(1+dt/2!+(dt)^2/3!+ \ldots)}$$

Since χ is not the trivial character $\sum_{r=1}^{d-1} \chi(r) = 0$. Reindexing the sum we have

$$\sum_{r=1}^{d-1} t\chi(r) = \sum_{r=1}^{d-1} (d-r)t\chi(d-r)$$

$$= dt\sum_{r=1}^{d-1} \chi(d-r) - t\sum_{r=1}^{d-1} r\chi(d-r)$$

$$= -t\sum_{r=1}^{d-1} r\chi(d-r) \qquad (\text{since } \sum_{r=1}^{d-1}\chi(d-r) = 0)$$

$$= -t\sum_{r=1}^{d-1} r\chi(-r)$$

$$= -t\sum_{r=1}^{d-1} r\chi(-1)\chi(r)$$

By (6) $\chi(-1) = 1$. Therefore

$$t\sum_{r=1}^{d-1} r\chi(r) = -t\sum_{r=1}^{d-1} r\chi(r)$$

and hence

$$rt\sum_{r=1}^{d-1} \chi(r) = 0.$$

Thus

$$\sum_{\ell=0}^{\infty} \frac{B_{\chi,\ell}}{\ell!} t^{\ell} = \frac{\frac{1}{2!}\sum_{r=1}^{d-1}(rt)^2\chi(r) + \frac{1}{3!}\sum_{r=1}^{d-1}(rt)^3\chi(r) + \ldots}{d(1+dt/2! + dt/3! + \ldots)}$$

Comparing coefficients we have

$$B_{\chi,2} = \frac{1}{d}\sum_{r=1}^{d-1} r^2\chi(r).$$

With the aid of a computer, James Maiorana has determined $B_{\chi,2}$ for real quadratic fields with discriminant less than 750. For the purpose of finding all surfaces with small geometric genus this is more than sufficient. Table II.2.1 give all d such that $B_{\chi,2}$ is less than 200.

Table II.3.1

$B_{\chi,2}$ for $d < 750$ and $B_{\chi,2} \leq 200$

d	$B_{\chi,2}$	d	$B_{\chi,2}$
5	0.8	88	92
8	2	89	104
12	4	92	80
13	4	93	72
17	8	97	136
21	8	101	76
24	12	104	100
28	16	105	144
29	12	109	108
33	24	113	144
37	20	120	136
40	28	124	160
41	32	129	200
44	28	133	136
53	28	136	184
56	40	137	192
57	56	140	152
60	48	141	144
61	44	149	140
65	64	152	164
69	48	157	172
73	88	165	176
76	76	173	156
77	48	197	196
85	72		

The following inequality is useful:

<u>Lemma II.3.4</u>. $\frac{d^{3/2}}{30} < |B_{\chi,2}| < \frac{d^{3/2}}{6}$

<u>Proof</u>: $L(2,\chi) < \zeta(2) = \frac{\pi^2}{6}$. Therefore, by (7)

$$\frac{\tau(\chi)}{d^2} \pi^2 B_{\chi,2} < \frac{\pi^2}{6}$$

$$|B_{\chi,2}| < \frac{d^{3/2}}{6}$$

$$\zeta(2) + L(2,\chi) > 2$$

therefore

$$2-\zeta(2) < L(2,\chi)$$

$$2 - \frac{\pi^2}{6} < \frac{\tau(\chi)}{d^2} \pi^2 B_{\chi,2}$$

$$d^{3/2}(\frac{2}{\pi^2} - \frac{1}{6}) < (B_{\chi,2})$$

$$d^{3/2} \frac{1}{30} < (B_{\chi,2}).$$

CHAPTER III

SMOOTHNESS CONDITIONS

§1 A Smoothness Condition for $U(\Gamma(1))$

First we recall some notation. Let A^{x++} be those elements of A^x with totally positive reduced norm, and let B^{++} denote the intersection of A^{x++} with the normalizer of $\Gamma(1)$ in A^x. In this chapter we give conditions for a subgroup Γ of $j(B^{++})$ to yield a quotient space which is a smooth algebraic surface. Such a surface is smooth if and only if $j(\Gamma)$ acts on X without fixed points.

For $\gamma \in A^x - k^x$, $k(\gamma)$ is a maximal subfield of A, therefore $k(\gamma)$ is a quadratic extension of k. Moreover, if $\gamma \in A^{x++}$, then $j(\gamma)$ acts on X and $j(\gamma)$ is the identity automorphism if and only if $\gamma \in k^x$.

<u>Proposition III.1.1</u>. Let K be a totally imaginary quadratic extension of k, and let φ be a k-linear isomorphism of k into A (an embedding of k in A). Then for $a \in K^x - k^x$ $\varphi(a) = \gamma$ is an element of A^{x++}, and $j(\gamma)$ has a unique fixed point on X which is the same for all $a \in K^x - k^x$. Conversely, if $j(\gamma) \in j(A^{x++})$, $j(\gamma) \neq 1$, has a fixed point on X, then $k(\gamma)$ is isomorphic to a totally imaginary quadratic extension of k.

<u>Proof</u>: See Shimura [19].

Suppose $\gamma \in A^{x++}$ and $j(\gamma)$ is an element of a discontinuous group Γ. If $j(\gamma)$ has a fixed point on X, then $j(\gamma)^r = 1$ for some $r > 0$ and so $\gamma^r \in k^x$.

Let K be a Galois extension of k of degree n. Then all primes $\mathfrak{P}_i$ of K lying above P of k have the same ramification index $e(K/k,P)$ and residue class degree $f(K/k,P)$. Furthermore, for an arbitrary extension K of k

$$\sum_{\mathfrak{P}_i | P} e(K/k,\mathfrak{P}_i)f(K/k,\mathfrak{P}_i) = n.$$

Thus, in the Galois case

$$e(K/k,P)f(K/k,P)g(K/k,P) = n$$

where $g(K/k,P)$ is the number of primes of K lying above P.

We need a lemma about embedded subfields of division algebras over local fields.

Lemma III.1.1. Let D be a division algebra over a local field F with degree r. Then a finite extension L of F splits D if and only if r divides $|L : F|$.

Proof: See Reiner [16].

Proposition III.1.2. A quadratic extension K of k is embeddable in A if and only if $K \otimes_k k_P$ is embeddable in $A \otimes_k k_P$ for all P.

Proof: If K is embeddable in A then $K \otimes_k k_P$ is embeddable in $A_P = A \otimes_k k_P$ for all P. Now consider the converse. By Proposition I.2.3 K is embeddable in A if and only if K splits A, that is $A \otimes_k K \cong M_2(K)$. Let B denote $A \otimes_k K$, and let $\mathfrak{P}$ lie above P. By Proposition I.2.4 $B \cong M_2(K)$ if and only if $B_{\mathfrak{P}} = B \otimes_k k_{\mathfrak{P}} \cong M_2(K_{\mathfrak{P}})$ for all $\mathfrak{P}$.

For a central simple algebra C over a local field F, let inv(C) denote the Hasse invariant of C. inv(C) is an element of $\mathbb{Q}/\mathbb{Z}$, and in the quaternion case, inv(C) (as an element of $\mathbb{Q}/\mathbb{Z}$) is 1/2 if C is a division algebra and 0 otherwise. In addition $inv(B_P) = |K_P : k_P| \cdot inv(A_P)$. Consider the following three cases:

i) $P \notin S(A)$, then $inv(A_P) = 0$ which implies that $inv(B_{\mathfrak{P}}) = 0$ and so $B_{\mathfrak{P}}$ is isomorphic to $M_2(K_{\mathfrak{P}})$.

ii) $P \in S(A)$ and P does not split in K/k, that is $|K_{\mathfrak{P}} : k_P| = 2$. In this case $inv(B_{\mathfrak{P}}) = 0$ and so $B_{\mathfrak{P}}$ is isomorphic to $M_2(K_{\mathfrak{P}})$.

iii) $P \in S(A)$ and P splits in K/k, that is $|K_{\mathfrak{P}} : k_P| = 1$. $inv(B_{\mathfrak{P}}) = inv(A_P) = 1/2$ and so $B_{\mathfrak{P}}$ is a division algebra.

$K \otimes_k k_P \cong k_P \oplus k_P$ (see Proposition I.2.2) and thus has zero divisors. Therefore $K \otimes_k k_P$ is not embeddable in $B_{\mathfrak{P}}$ which is contrary to hypothesis and so does not occur.

Thus for all P, $B_{\mathfrak{P}}$ is isomorphic to $M_2(K_{\mathfrak{P}})$ and so K is embeddable in A.

<u>Proposition III.1.3</u>. A quadratic extension K of k is embeddable in $A = A(k,S(A))$ if and only if for all finite $P \in S(A)$, $g(K/k,P) = 1$, and for all infinite $P \in S(A)$ there exists a unique extension of P to K.

<u>Proof</u>: By the last Proposition, K is embeddable in A if and only if $K \otimes_k k_P$ is embeddable in $A \otimes_k k_P$ for all P.

For P finite or infinite, Proposition I.2.2 gives:

$$K \otimes_k k_P \cong \begin{cases} k_P \oplus k_P & \text{if there are two primes in K lying above P.} \\ K_{\mathfrak{P}} & \text{if there is only one prime in K lying above P.} \end{cases}$$

First consider $P \notin S(A)$. In this case $A \otimes_k k_P \cong M_2(k_P)$. If two distinct primes lie above P then $K \otimes_k k_P \cong k_P \oplus k_P$, and $k_P \oplus k_P$ is embeddable in $M_2(k_P)$ by $(a,b) \to \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ where $a,b \in k_P$. If one prime $\mathfrak{P}$ lies above P then $K \otimes_k k_P \cong K_{\mathfrak{P}}$ is a quadratic extension of k_P. Let $\{e_1,e_2\}$ be a basis for $K_{\mathfrak{P}}/k_P$. Then the map $\gamma \to M(\gamma) \in M_2(k_P)$ given by the regular representation

$$\gamma(e_1,e_2) = (e_1,e_2)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is an embedding of K over k into A. So for $P \notin S(A)$ we have shown that $K \otimes_k k_P$ over k_P is always embeddable in $A \otimes_k k_P$. Thus, K over k embeddable in A if and only if $K \otimes_k k_P$ over k_P

is embeddable in $A \otimes_k k_P$ for all $P \in S(A)$.

For $P \in S(A)$, $A \otimes_k k_P = D_P$ is a division algebra and thus has no zero divisors. By Lemma III.1.4 with $r = 2$, any quadratic extension of k_P splits D_P. In particular, if there exists only one prime $\mathfrak{P}$ lying above P, $K_{\mathfrak{P}}$ splits D_P and by Proposition I.2.3 $K_{\mathfrak{P}}$ is embeddable in D_P. In the case where there are two primes lying above P, $K \otimes_k k_P \cong k_P \oplus k_P$ has zero divisors and cannot be embedded in D_P. Thus we finally have:

> K is embeddable in A if and only if for all finite $P \in S(A)$, $g(K/k,P) = 1$ and for all infinite $P \in S(A)$, there exists a unique extension of P to K.

We now apply the above discussion to $\Gamma(1)$. The center of $\Gamma(1)$ is $\{\pm 1\}$. Thus, for $\gamma \in \Gamma(1)$ $j(\gamma)$ has finite order (i.e. has a fixed point on X) if and only if $\gamma^r = \pm 1$ for some $r > 0$, and so we have

<u>Proposition III.1.4</u>. Assume $h(k)$ = class number of $k = 1$. Then $j(\Gamma(1))$ has a fixed point on X if and only if there exists $N > 2$ such that $\mathbb{Q}(e^{2\pi i/N})$ is embeddable in A.

<u>Proof</u>: If $j(\Gamma(1))$ has a fixed point on X, then $\gamma^r = \pm 1$ for some $\gamma \in \Gamma(1)$, $\gamma \neq \pm 1$, and so there exists $\gamma \in \Gamma(1)$ and $N > 2$ such that $\gamma^N = 1$ with N minimal. Then $\mathbb{Q}(\gamma) \cong \mathbb{Q}(e^{2\pi i/N})$ is embeddable in A. To show the necessity, recall that $h(k) = 1$

implies that $h(A) = 1$ (see Lemma II.2.1). Thus all maximal orders of A are conjugate. If $\mathbb{Q}(e^{2\pi i/N})$ is embeddable in A, then there exists $\gamma \in \mathfrak{O}$, for some maximal order $\mathfrak{O}'$, such that $\gamma^N = 1$. There exists $\beta \in A^x$ such that $\beta\mathfrak{O}'\beta^{-1} = \mathfrak{O}$. Thus $\beta\gamma\beta^{-1} \in \mathfrak{O}$ also has order N. Since $\nu((\beta\gamma\beta^{-1})^N) = \nu(\beta\gamma^N\beta^{-1}) = \nu(\gamma^N) = \nu(\gamma)^N = 1$, and $\nu(\gamma)$ is an element of a real quadratic field, $\nu(\gamma) = \pm 1$, and hence either γ or γ^2 is in $\Gamma(1)$. This completes the proof.

Remark. Without the restriction $h(k) = 1$, the condition is still sufficient.

Now let $k = \mathbb{Q}(\sqrt{d})$, $d > 0$, and let A be totally indefinite. If $\mathbb{Q}(e^{2\pi i/N})$ is embeddable in A, then $|\mathbb{Q}(e^{2\pi I/N}) : \mathbb{Q}|$ divides 4. Then $\varphi(N)$ divides 4, where $\varphi(N)$ is the order of $(\mathbb{Z}/N\mathbb{Z})^x$. An easy calculation shows that N is one of the integers: 2, 3, 4, 5, 6, 8, 10, or 12.

Let ζ_N denote the primitive N-th root of unity $e^{2\pi i/N}$. Considering the eight cases above, we have:

$N = 2$, $\varphi(2) = 1$, $\mathbb{Q}(\zeta_2) = \mathbb{Q}$

$N = 3$, $\varphi(3) = 2$, $\mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3})$

$N = 4$, $\varphi(4) = 2$, $\mathbb{Q}(\zeta_4) = \mathbb{Q}(\sqrt{-1}) = \mathbb{Q}(i)$

$N = 5$, $\varphi(5) = 4$, $\mathbb{Q}(\zeta_5) > \mathbb{Q}(\sqrt{5})$. Since $Gal(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ is cyclic of degree 4 and therefore has a unique subgroup of index 2, $\mathbb{Q}(\zeta_5)$ contains a unique quadratic subfield whose discriminant divides 5^3.

$N = 6$, $\varphi(6) = 2$, $\mathbb{Q}(\zeta_6) = \mathbb{Q}(\sqrt{-3})$.

$N = 8$, $\varphi(8) = 4$, $\mathbb{Q}(\zeta_8) = \mathbb{Q}(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = \mathbb{Q}(\sqrt{2}, i) \supset \mathbb{Q}(\sqrt{8})$.

$N = 10$, $\varphi(10) = 4$, $\mathbb{Q}(\zeta_{10}) = \mathbb{Q}(\zeta 5) \supset \mathbb{Q}(\sqrt{5})$.

$N = 12$, $\varphi(12) = 4$, $\mathbb{Q}(\zeta_{12}) = \mathbb{Q}(\sqrt{3} + i) \supset \mathbb{Q}(\sqrt{3})$.

Consider the cases $N = 5, 8, 10$ and 12. For these cases $|k(\zeta_N) : k| = 2$, since $k(\zeta_N)$ is embeddable in A and $k(\zeta_N) \subsetneq R$ and therefore $k(\zeta_N) \neq k$. $k(\sqrt{N}) \subsetneq k(\zeta_N)$. Therefore $k(\sqrt{N}) = k$, thus we have

<u>Lemma III.1.2</u>. Let $N = 5, 8, 10$ or 12. If $\Gamma(1)$ contains an element of order N, then $k = \mathbb{Q}(\sqrt{N})$ if $N = 5, 8, 12$ and $k = \mathbb{Q}(\sqrt{40})$ if $N = 10$.

<u>Proposition III.1.5</u>. Assume $h(k) = 1$. For $d \neq 5$ or 10, $j(\Gamma(1))$ acts on X without fixed points if and only if there exists $P \in S(A)$ such that $g(k(\sqrt{-3})/k, P) = 2$ and there exists $Q \in S(A)$ such that $g(k(\sqrt{-1})/k, Q) = 2$ (P and Q may coincide).

<u>Proof:</u> By Lemma III.1.5 the only possibilities are elements of orders 3, 4, 6, 8 or 12.

If there is an element γ of order 3, then $-\gamma$ is of order 6. On the other hand, if γ is of order 6, then γ^2 is of order 3. Thus $\Gamma(1)$ contains an element of order 6 if and only if it contains an element of order 3. By Proposition III.1.4

this is so if and only if $\mathbb{Q}(\zeta_3)$ is embeddable in A. $\mathbb{Q}(\zeta_3)$ contains $\sqrt{-3}$ and $|k(\sqrt{-3}) : k| = 2$. Thus, $\mathbb{Q}(\zeta_3)$ is embeddable in A if and only if $k(\sqrt{-3})$ is. By Proposition III.1.3, this is the case if and only if for all $P \in S(A)$, $g(k(\sqrt{-3})/k,P) = 1$.

$\Gamma(1)$ contains an element of order 4 if and only if $\mathbb{Q}(\zeta_4)$ is embeddable in A. $\mathbb{Q}(\zeta_4)$ contains i and $|k(i) : k| = 2$. Thus $\mathbb{Q}(\zeta_4)$ is embeddable in A if and only if k(i) is, and again by Proposition III.1.3 this is the case if and only if for all $P \in S(A)$ $g(k(i)/k,P) = 1$.

If $\Gamma(1)$ contains an element of order 8 (then $d = 8$) it contains an element of order 4 (its square). $\Gamma(1)$ contains an element of order 8 if and only if $\mathbb{Q}(\zeta_8)$ is embeddable in A. $\mathbb{Q}(\zeta_8)$ contains i and $|k(i) : k| = 2$. Thus $\Gamma(1)$ contains an element of order 8 if and only if k(i) is embeddable in A. But this is exactly the condition for $\Gamma(1)$ to contain an element of order 4.

Finally, if $\Gamma(1)$ contains an element of order 12 (then $d = 12$) it also contains elements of orders 3 and 4 (γ^4 and γ^3 respectively). Thus, $k(\sqrt{-3})$ and k(i) are embeddable in A, but since $k = \mathbb{Q}(\sqrt{12})$, $k(\sqrt{-3}) = k(i)$. Thus, $\Gamma(1)$ contains an element of order 12 if and only if $\Gamma(1)$ contains an element of order 4 and this is so if and only if $g(k(i)/k,P) = 1$ for all $P \in S(A)$.

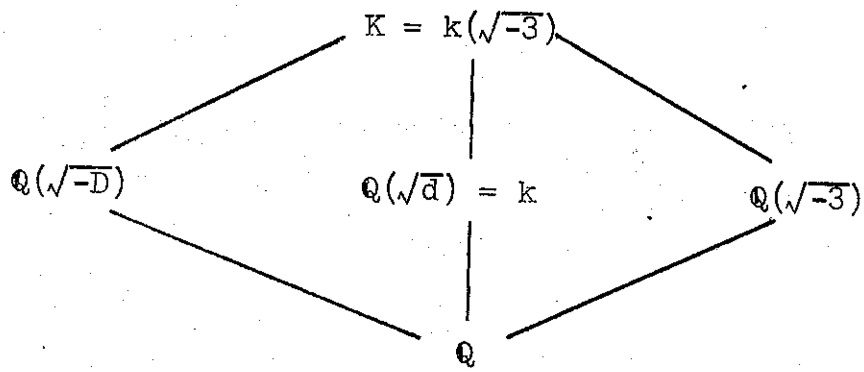
Thus, for $d \neq 5$ or 10, $\Gamma(1)$ contains no element of finite order ($\neq \pm 1$) if and only if there exists $P \in S(A)$ such that

$g(k(\sqrt{-3})/k,P) = 2$, and there exists $Q \in S(A)$ such that $g(k(i)/k,Q) = 2$.

Proposition III.1.6. For $d = 5$ or 10, $\Gamma(1)$ contains an element of order 5 or 10 if and only if for all $P \in S(A)$ $g(k(\zeta_5)/k,P) = 1$.

Proof: This follows immediately from Proposition III.1.4.

Lemma III.1.2. $Gal(k(\sqrt{-3})/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and we have the following diagram of fields:

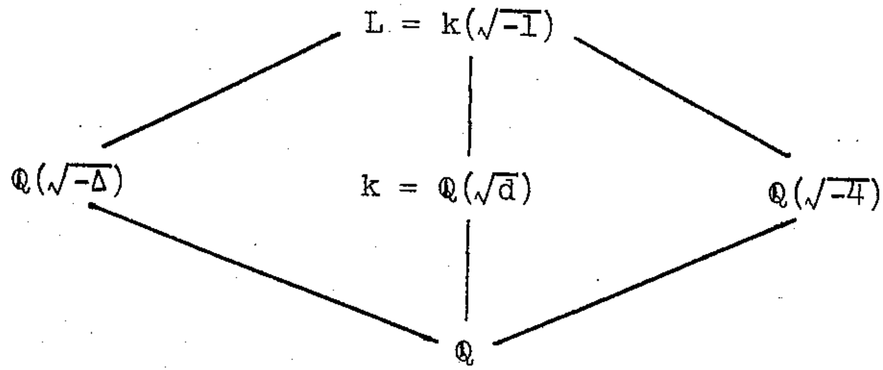


where $D = 3d$, if $3 \nmid d$

$D = d/3$ if $3 | d$.

Proof: $k(\sqrt{-3}) = \mathbb{Q}(\sqrt{d},\sqrt{-3})$ is a biquadratic extension of $\mathbb{Q}$, so $Gal(k(\sqrt{-3})/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The subgroup diagram is an easy computation.

Lemma III.1.3. $Gal(k(\sqrt{-1})/k) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and we have the following diagram of fields:



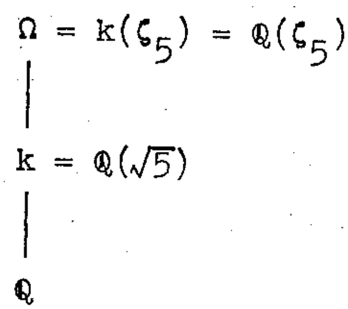
where $\Delta = 4d$ if $d \equiv 1 \pmod 4$

$\Delta = m$ if $d = 4m$ and $m \equiv 3 \pmod 4$

$\Delta = 4m$ if $d = 4m$ and $m \equiv 2 \pmod 3$, i.e. $\Delta = d$.

Proof: See Lemma III.1.2.

Lemma III.1.4. For $k = \mathbb{Q}(\sqrt{5})$ we have the following diagram of fields:



Proof: $\mathbb{Q}(\sqrt{5})$ is the unique quadratic subfield of $\mathbb{Q}(\zeta_5)$.

Using the diagrams in Lemmas III.1.2 and III.1.3, we wish to determine what relationship the three quadratic extensions of $\mathbb{Q}$ must satisfy in order for $g(K/k,P) = 2$ (or $g(L/k,P) = 2$). There are only three possibilities for a prime in a quadratic extension, that is, it either splits, ramifies, or remains prime. We denote these three possibilities by g, e and f respectively.

For a quadratic extension $\mathbb{Q}(\sqrt{\delta})$ of $\mathbb{Q}$, the Kronecker symbol $(\frac{\delta}{p})$ determines the splitting of (p). Using the multiplicative property of (-), that is $(\frac{a}{p})(\frac{b}{p}) = (\frac{ab}{p})$, we compile the following tables which give all the possibilities for the three quadratic extensions $\mathbb{Q}$ for the diagram in Lemma III.1.2.

Table III.1.1

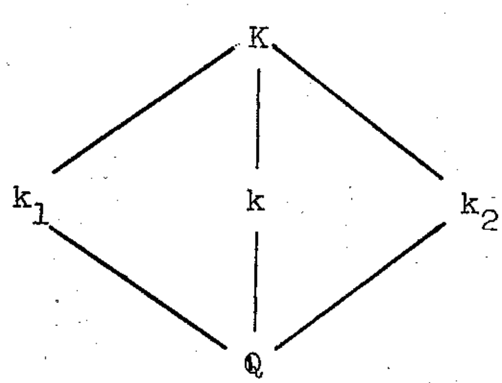
$\mathbb{Q}(\sqrt{d})$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-D})$	Remarks
g	g	g	
g	f	f	
f	g	f	
f	f	g	
e	g	e	
e	f	e	
g	e	e	Can occur only for $3 \nmid d$ and (p) = (3)
f	e	e	Can occur only for $3 \nmid d$ and (p) = (3)
e	e	g	Can occur only for $3 \mid d$ and (p) = (3)
e	e	f	Can occur only for $3 \mid d$ and (p) = (3)
e	e	e	Can occur only for $3 \mid d$ and (p) = (3)

For the three quadratic extensions in Lemma III.1.3, we have the following table:

Table III.1.2

$\mathbb{Q}(\sqrt{d})$	$\mathbb{Q}(\sqrt{-4})$	$\mathbb{Q}(\sqrt{-\Delta})$	Remarks
g	g	g	
g	f	g	
f	g	f	
f	f	g	
e	g	e	
e	f	e	
g	e	e	Can occur only for $(p) = (2)$ and $d \equiv 1 \pmod 4$
f	e	e	Can occur only for $(p) = (2)$ and $d \equiv 1 \pmod 4$
e	e	g	Can occur only for $(p) = (2)$ and $d = 4m$, $m \equiv 3 \pmod 4$
e	e	f	Can occur only for $(p) = (2)$ and $d = 4m$, $m \equiv 3 \pmod 4$
e	e	e	Can occur only for $(p) = 2$ and $d = 4m$, $m \equiv 2 \pmod 4$

Representing the subgroup diagrams for both K and L by

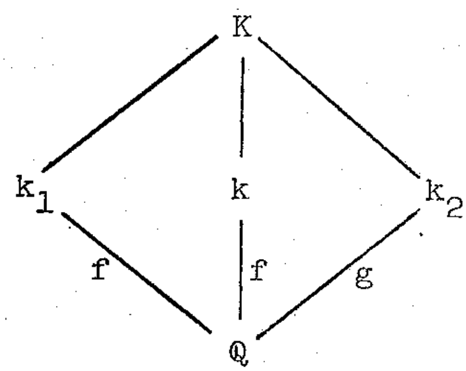


we have the following

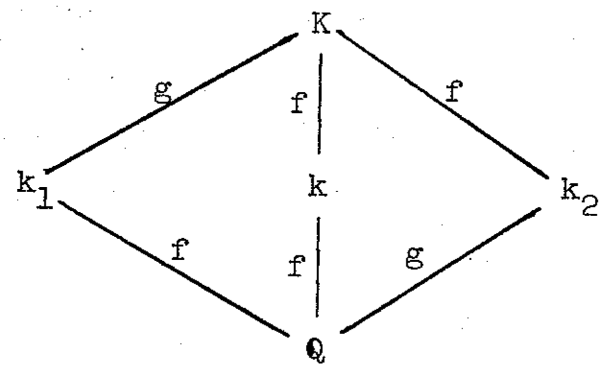
Table III.1.3

	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
$k_2/\mathbb{Q}$	g	g	f	f	g	f	e	e	e	e	e
$k/\mathbb{Q}$	g	f	g	f	e	e	f	g	e	e	e
$k_1/\mathbb{Q}$	g	f	f	g	e	e	e	e	f	g	e

Consider the possibility described by column II for some prime (p)

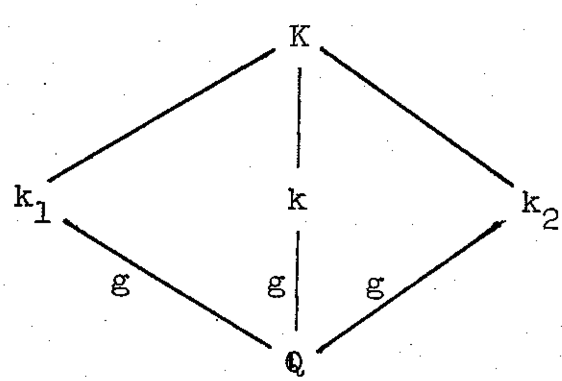


All of the extensions are abelian. Choose primes P_1, P, P_2 and $\mathfrak{P}$ in k_1, k, k_2 and K respectively lying above (p). $\mathfrak{P}$ lies above P_1, P, and P_2. The integers e, f and g are multiplicative in towers, that is $e(K/Q,(p)) = e(K/k,P)e(k/\mathbb{Q},(p))$ with analogous relations for f and g. Moreover, $e(K/\mathbb{Q},(p))$ $f(K/\mathbb{Q},(p))$ and $g(K/\mathbb{Q},(p))$ are independent of the intermediate extension. Thus, for this example $g(K/\mathbb{Q},(p)) = 2$, $f(K/\mathbb{Q},(p)) = 2$ and $g(K/k,P) = 2$.



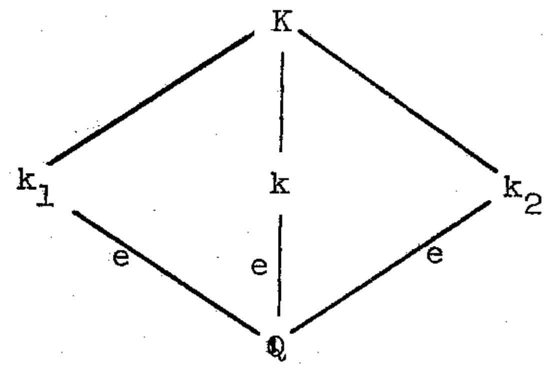
Similar arguments allow us to determine e, f and g for all the extensions in the diagrams for the situations described by columns III through X.

To complete the table (columns I and XI), we consider the decomposition and inertia group of the extension $K/\mathbb{Q}$. Let G be the galois group of $K/\mathbb{Q}$, $G_{\mathfrak{P}}$ decomposition group of $\mathfrak{P}$ and $T_{\mathfrak{P}}$ the inertia group of $\mathfrak{P}$. $G \supseteq G_{\mathfrak{P}} \supseteq T_{\mathfrak{P}}$. Let $K^{G_{\mathfrak{P}}}$ and $K^{T_{\mathfrak{P}}}$ be the fixed fields of $G_{\mathfrak{P}}$ and $T_{\mathfrak{P}}$, respectively. $K^{G_{\mathfrak{P}}}$ is the largest subfield of K in which (p) splits completely. Considering the situation of column I

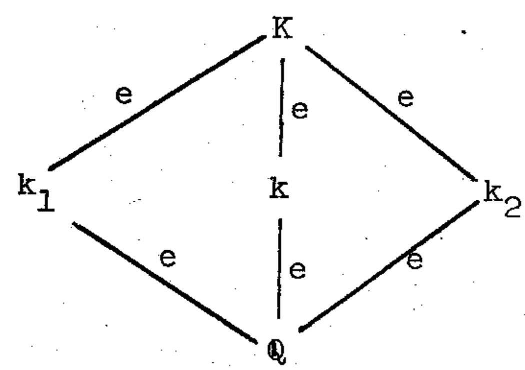


$K^{G_{\mathfrak{P}}}$ must be K since it must contain k_1, k and k_2. Thus, $g(K/k,P) = 2$.

The last case to be considered is the following:



K is a totally ramified extension of $K^{T_\mathfrak{P}}$ (relative to $\mathfrak{P}$, P_1, P and P_2) and $K^{T_\mathfrak{P}}$ is an unramified extension of $K^{G_\mathfrak{P}}$. The only possibility is that $K^{T_\mathfrak{P}} = K^{G_\mathfrak{P}} = \mathbb{Q}$, and therefore, all extensions are ramified, that is



Finally we have

Table III.1.4

	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
$k_2/\mathbb{Q}$	g	g	f	f	g	f	e	e	e	e	e
$k/\mathbb{Q}$	g	f	g	f	e	e	f	g	e	e	e
$k_1/\mathbb{Q}$	g	f	f	g	e	e	e	e	f	g	e
K/k	g	g	f	g	g	f	e	e	f	g	e

Consider the situation in Lemma III.1.4, that is

$$\begin{array}{l} \Omega = \mathbb{Q}(\zeta_5) \\ \quad | \\ k = \mathbb{Q}(\sqrt{5}) \\ \quad | \\ \mathbb{Q} \end{array}$$

$g(\Omega/k,P)$ can be 2 only if $g(\Omega/Q,(p)) = 4$ for $P|(p)$. It is known that (p) splits completely in $\mathbb{Q}(\zeta_5)/\mathbb{Q}$ if and only if $p \equiv 1 \pmod 5$. Thus $g(\Omega/k,P) \equiv 2$ if and only if $p \equiv 1 \pmod 5$.

Summarizing we have

<u>Theorem III.1.1</u>. Assume $h(k) = 1$. Let $A = A(k,S(A))$, $k = \mathbb{Q}(\sqrt{d})$, $d > 0$. $j(\Gamma(1))$ acts on X without fixed points if and only if all of the following hold:

1) $(\frac{-3}{p}) = 1$ or $(\frac{-D}{p}) = 1$ for some $P \in S(A)$, where $p\mathbb{Z} = P \cap \mathbb{Z}$ and $-D$ is the discriminant of $\mathbb{Q}(\sqrt{-3d})$.

2) $(\frac{-1}{p}) = 1$ or $(\frac{-\Delta}{p}) = 1$ for some $P \in S(A)$ where $p\mathbb{Z} = P \cap \mathbb{Z}$ and $-\Delta$ is the discriminant of the field $\mathbb{Q}(\sqrt{-d})$.

3) If $d = 5$, there exists $P \in S(A)$ such that $p\mathbb{Z} = P \cap \mathbb{Z}$ and $p \equiv 1 \pmod 5$.

§2 The Groups B^{++} and E^{++}

Throughout this section the class number of k is assumed to be 1, and $\mathfrak{O}$ is a fixed maximal order of A. Recall that E^{++} is $\mathfrak{O}^{\times} \cap A^{\times ++}$ and $j(\Gamma(1))$ is an index 2 subgroup of $j(E^{++})$ if a fundamental unit of k is totally positive, and $j(\Gamma(1))$ coincides with $j(E^{++})$ otherwise.

Theorem III.2.1. Assume ε_k is totally positive. Then $j(E^{++})$ acts on X without fixed points if and only if both of the following hold:

1) $j(\Gamma(1))$ has no fixed points on X.

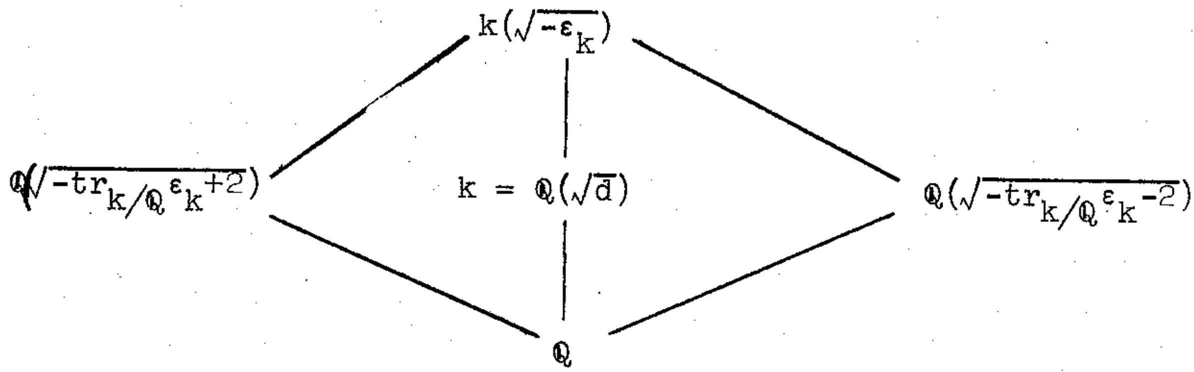
2) There exists $P \in S(A)$ such that $\underset{\sim}{P}$ splits in $k(\sqrt{-\varepsilon_k})/k$.

Proof: E^{++} contains $\Gamma(1)$, and the center of E^{++} is $\mathfrak{O}_k^{\times}$. Suppose $j(\gamma)$, $1 \neq j(\gamma) \in j(E^{++})$ has a fixed point on X. By Proposition III.1.2, $k(\gamma)$ is a totally imaginary quadratic extension of k, and there exists a positive integer r with $j(\gamma)^{+} \in k^{\times}$. Choose r to be minimal. Since $\gamma \in E^{++}$ and ε_k is totally positive, $\nu(\gamma) = \varepsilon_k^m$. In addition, we may assume m is non-negative. (If this is not the case, replace ε_k by $\frac{1}{\varepsilon_k}$). We can find $\gamma_1 \in E^{++}$ such that $j(\gamma_1) = j(\gamma)$ and $\nu(\gamma)$ is either 1 or ε_k. To see this suppose $m = 2\ell$ (respectively $2\ell+1$). Let $\gamma_1 = \gamma\varepsilon_k^{-\ell}$. Then $\nu(\gamma_1) = \nu(\gamma)\nu(\varepsilon_k^{2\ell}\varepsilon_k^{-2\ell}) = 1$ (respectively $= \varepsilon_k^{2\ell+1}\varepsilon_k^{-2\ell} = \varepsilon_k$). Since γ and γ_1 differ by a multiple of elements of $k^{\times}$, $k(\gamma) = k(\gamma_1)$. If m is even,

$\gamma \in \Gamma(1)$ and $j(\Gamma(1))$ has a fixed point on X. If m is odd, $\gamma_1^r \in k^x$ (again r is minimal). Actually, $\gamma_1^r \in \mathfrak{G}_k^x$ and so $\gamma_1^r = \pm \varepsilon_k^t$. Taking reduced norms of both sides gives: $\varepsilon_k^r = \nu(\gamma_1^r) = \varepsilon_k^{2t}$. Therefore, $r = 2t$. Let $\gamma_2 = \gamma_1^t$. $j(\gamma_2) = j(\gamma)$. By the minimality of r, $\gamma_2 \notin k$ and $k(\gamma_2) = k(\gamma)$ is a totally imaginary quadratic extension of k. Since $\gamma_2^2 = \pm \varepsilon_k^t \in k^x$ and $\sqrt{-\varepsilon_k^t}$ is totally imaginary, $k(\gamma_2)$ must be isomorphic to $k(\sqrt{-\varepsilon_k^t})$. If t is even, then $k(\sqrt{-\varepsilon_k^t}) = k(\sqrt{-1})$ is embeddable in A and $j(\Gamma(1))$ has a fixed point on X. If t is odd, say $t = 2s+1$, then $k(\sqrt{-\varepsilon_k^t}) = k(\sqrt{-\varepsilon_k})$ is embeddable in A. Thus, if $j(\gamma)$ has a fixed point on X implies either $j(\Gamma(1))$ has a fixed point on X or $k(\sqrt{-\varepsilon_k})$ is embeddable in A.

Conversely, suppose either $j(\Gamma(1))$ has a fixed point on X or $k(\sqrt{-\varepsilon_k})$ is embeddable in A. In the first case, $j(E^{++})$ has a fixed point on X because $j(\Gamma(1)) \subset j(E^{++})$. Suppose φ is an embedding of $k(\sqrt{-\varepsilon_k})$ in A and let γ_1 denote the image of $\sqrt{-\varepsilon_k}$ in A. $\sqrt{-\varepsilon_k}$ is integral over $\mathfrak{G}_k$ and is therefore contained in some maximal order $\mathfrak{G}'$ of A. Since the class number of A is 1, the fixed maximal order $\mathfrak{G}$ is conjugate to $\mathfrak{G}'$, i.e. $\mathfrak{G} = a\mathfrak{G}'a^{-1}$ for some $a \in A^x$. $\gamma = a\gamma_1 a^{-1} = a\varphi(\sqrt{-\varepsilon_k})a^{-1}$ is a unit of $\mathfrak{G}$, and since it is the image of a totally imaginary element, $\gamma \in A^{x++} \cap \mathfrak{G}^x = E^{++}$ and $j(\gamma)$ has a fixed point on X. This completes the proof.

<u>Lemma III.2.1</u>. $k(\sqrt{-\varepsilon_k})$ is an abelian extension of $\mathbb{Q}$ with Galois group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Moreover we have the following diagram of subfields:



<u>Proof</u>: Let α denote $\sqrt{-\varepsilon_k}$ and β denote $\sqrt{-\varepsilon_k'}$ where prime is Galois conjugation in k and the square roots are chosen such that Im $\alpha > 0$ and Im$(\beta) > 0$. $(\alpha\beta)^2 = 1$, therefore $\alpha\beta = \pm 1$, but α and β are purely imaginary, and Im α and Im β are both positive. Therefore $\alpha\beta = -1$ and $\alpha = \frac{-1}{\beta}$. Let ξ denote $\alpha+\beta$ and η denote $\alpha-\beta$. Then we have

$$(2) \qquad \xi^2 = (\alpha+\beta)^2 = -\varepsilon_k - \varepsilon_k' + 2\alpha\beta = -tr_{k/\mathbb{Q}}\varepsilon_k - 2 \in k(\sqrt{-\varepsilon_k})$$

$$(3) \qquad \eta^2 = (\alpha-\beta)^2 = -\varepsilon_k - \varepsilon_k' - 2\alpha\beta = -tr_{k/\mathbb{Q}}\varepsilon_k + 2 \in k(\sqrt{-\varepsilon_k}).$$

Adjoining ξ and η to $\mathbb{Q}$ give two distinct intermediate quadratic extensions of $\mathbb{Q}$ neither of which is k. The composition of $\mathbb{Q}(\sqrt{-tr_{k/\mathbb{Q}}\varepsilon_k+2})$ and $\mathbb{Q}(\sqrt{-tr_{k/\mathbb{Q}}\varepsilon_k-2})$ coincides with $k(\sqrt{-\varepsilon_k})$. Therefore, $k(\sqrt{-\varepsilon_k})$ is a biquadratic extension of $\mathbb{Q}$ and the lemma follows.

Let $\mathfrak{P}_i$ be a maximal 2-sided $\mathfrak{O}$-ideal. To study B^{++}, for $P_i \in S(A)$ we fix a generator $\pi_i > 0$ of P_i and a generator Π_i for the ideal $\mathfrak{P}_i$ such that $\mathfrak{P}_i^2 = P_i\mathfrak{O}$. By Proposition I.2.7 $\nu(\mathfrak{P}_i) = P_i$, and therefore

(1) $$\nu(\Pi_i) = \pi_i \varepsilon_i$$

where $\varepsilon_i \in \mathfrak{O}_k^x$. Recall that a typical element γ of B^{++} is of the form $\Pi_{i_1} \dots \Pi_{i_\ell} \varepsilon\lambda$ where $\varepsilon \in \mathfrak{O}^x$, $\lambda \in k^x$ and $\nu(\gamma)$ is a totally positive element of k^x.

<u>Theorem III.2.2</u>. Let $\varepsilon_k > 0$ be a fundamental unit of k. Then $j(B^{++})$ acts on X without fixed points if and only if all of the following hold:

1) $j(E^{++})$ acts on X without fixed points.

2) For all totally positive $\pi_{i_1}\pi_{i_2} \dots \pi_{i_\ell}$, there exists $P \in S(A)$ such that P splits in the extension $k(\sqrt{-\pi_{i_1} \dots \pi_{i_\ell}})$ of k.

3) For all totally positive $\pi_{i_1}\pi_{i_2} \dots \pi_{i_\ell}\varepsilon_k$, there exists $P \in S(A)$ such that P splits in the extension $k(\sqrt{-\pi_{i_1} \dots \pi_{i_\ell}\varepsilon_k})$ of k.

<u>Proof</u>: Suppose $\gamma \in B^{++}$, $j(\gamma) \neq 1$ has a fixed point on X. γ is of the form $\Pi_{i_1}\Pi_{i_2} \dots \Pi_{i_\ell}\varepsilon\lambda$ where $\varepsilon \in \mathfrak{O}^x$; $\lambda \in k^x$; Π_i generates $\mathfrak{P}_i$ and $\nu(\Pi_{i_1} \dots \Pi_{i_\ell}\varepsilon\lambda)$ is a totally positive element of k. We can replace γ by $\gamma_1 = \Pi_{i_1} \dots \Pi_{i_\ell}\varepsilon$ since $j(\gamma) = j(\gamma_1)$. By Proposition III.1.1, $k(\gamma_1)$ is a totally

imaginary quadratic extension of k. Let r be the least positive integer such that $\gamma_1^r \in k^x$. Consider the 2-sided $\mathfrak{O}$-ideals $\gamma_1\mathfrak{O} = \mathfrak{P}_{i_1}\mathfrak{P}_{i_2} \cdots \mathfrak{P}_{i_\ell}$ and $\gamma_1^r\mathfrak{O} = \mathfrak{P}_{i_1}^r \cdots \mathfrak{P}_{i_\ell}^r$. Since γ_1^r is integral over $\mathbb{Z}$ and $\gamma_1^r \in \mathfrak{O}$,

$$\gamma_1^r\mathfrak{O} = (\gamma_1^r\mathfrak{O}_k)\mathfrak{O} = P_1^{\nu_1} \cdots P_m^{\nu_m} Q_1^{\mu_1} \cdots Q_n^{\mu_n}\mathfrak{O}$$
$$= \mathfrak{P}_1^{2\nu_1} \cdots \mathfrak{P}_m^{2\nu_m} \mathfrak{Q}_1^{\mu_1} \cdots \mathfrak{Q}_n^{\mu_n}$$

where $P_i \in S(A)$ and $P_i\mathfrak{O} = \mathfrak{P}_i^2$, and $Q_i \notin S(A)$ and $Q_i\mathfrak{O} = \mathfrak{Q}_i$. Since a 2-sided ideal has a unique expression as a product of maximal 2-sided ideals

$$\gamma_1^r\mathfrak{O} = \mathfrak{P}_{i_1}^r \cdots \mathfrak{P}_{i_\ell}^r = \mathfrak{P}_1^{2\nu_1} \cdots \mathfrak{P}_m^{2\nu_m} \mathfrak{Q}_1^{\mu_1} \cdots \mathfrak{Q}_n^{\mu_n}$$

implies $r = 2\nu_j$, $1 \le j \le m$ and $\mu_k = 0$, $1 \le k \le n$. Therefore, r is even, say $r = 2s$. Put $\gamma_2 = \gamma_1^s$. $\gamma_2 \notin k^x$ because of the minimality of r. Since $k \subsetneq k(\gamma_2) \subset k(\gamma_1)$ and $|k(\gamma_1) : k| = 2$ we must have $k(\gamma_2) = k(\gamma_1)$ and therefore $j(\gamma_1)$ and $j(\gamma_2)$ have the same fixed point. (See Proposition III.1.1.)

If s is even, say $s = 2t$, then $\gamma_2 = a\varepsilon^s$ where $a = (\Pi_{i_1}^2)^t(\Pi_{i_2}^2)^t \cdots (\Pi_{i_\ell}^2)^t \in k^x$. Therefore $j(\gamma_2) = j(\varepsilon^s)$ and $\nu(\varepsilon^s)$ is totally positive. So in this case if $j(B^{++})$ has a fixed point then $j(E^{++})$ has a fixed point.

If s is odd, say, $s = 2t+1$, then $\gamma_2 = \Pi_{i_1}^{2t+1} \cdots \Pi_{i_\ell}^{2t+1} \varepsilon^{2t+1} = \Pi_{i_1} \cdots \Pi_{i_\ell} \varepsilon^{2t+1}a$ where $a = (\Pi_{i_1}^2)^t \cdots (\Pi_{i_\ell}^2)^t \in k^x$. Thus, $j(\gamma_2) = j(\Pi_{i_1} \cdots \Pi_{i_\ell} \varepsilon^{2t+1})$.

$\nu(\gamma_2)^2 = \nu(\gamma_2^2) = \gamma_2^4$ since $\gamma_2^2 \in k^x$. Therefore, $\gamma_2^2 = \pm\nu(\gamma_2) = \nu(\Pi_{i_1}) \ldots \nu(\Pi_{i_\ell})a^2\nu(\varepsilon^{2t+1}) = \pi_{i_1} \ldots \pi_{i_\ell}a^2\varepsilon_0$ where $\varepsilon_0 \in \mathfrak{O}_k^x$. Since $\varepsilon_0 = \varepsilon_k^q$ (the π_i and ε_k are positive), $k(\gamma_2) \cong k(\sqrt{\pm\pi_{i_1} \ldots \pi_{i_\ell}(\varepsilon_k)})$, where ε_k appears if q is odd and does not appear if q is even. $\pi_{i_1} \ldots \pi_{i_\ell}(\varepsilon_k)$ is totally positive and $k(\gamma_2)$ is a totally imaginary extension of k. Therefore we must choose the minus sign, i.e. $k(\gamma_2) \cong k(\sqrt{-\pi_{i_1} \ldots \pi_{i_\ell}(\varepsilon_k)})$. So in this case, if $j(\gamma)$ has a fixed point, then $k(\gamma) \cong k(\sqrt{-\pi_{i_1} \ldots \pi_{i_\ell}(\varepsilon_k)})$ is embeddable in A. By Proposition III.1.3, if $j(\gamma)$ has no fixed point, then there is a $P \in S(A)$ such that P splits in the extension $k(\sqrt{-\pi_{i_1} \ldots \pi_{i_\ell}(\varepsilon_k)})$ of k.

Conversely, suppose that either $j(E^{++})$ has a fixed point, or $k(\sqrt{-\pi_{i_1} \ldots \pi_{i_\ell}(\varepsilon_k)})$ is embeddable in A where $\pi_{i_1} \ldots \pi_{i_\ell}(\varepsilon_k)$ is totally positive. In the first case $j(B^{++})$ has a fixed point since $j(E^{++}) \subset j(B^{++})$. In the second case let φ be an embedding of $k(\sqrt{-\pi_{i_1} \ldots \pi_{i_\ell}(\varepsilon_k)})$ into A, and let $\varphi(\sqrt{-\pi_{i_1} \ldots \pi_{i_\ell}(\varepsilon_k)}) = \gamma$. γ is totally positive since it is the image of a totally imaginary element. Consider the ideals $\gamma\mathfrak{O}$ and $\gamma^2\mathfrak{O}$. $\varphi(-\pi_{i_1}\ldots\pi_{i_\ell}(\varepsilon_k)) =$ $= -\pi_{i_1}\ldots\pi_{i_\ell}(\varepsilon_k)$ since φ is k-linear. Therefore

$$(\gamma\mathfrak{O})(\gamma\mathfrak{O}) = \gamma^2\mathfrak{O} = \pi_{i_1} \ldots \pi_{i_\ell}(-\varepsilon_k)\mathfrak{O}$$

$$= \pi_{i_1} \ldots \pi_{i_\ell}\mathfrak{O} = \mathfrak{P}_{i_1}^2 \ldots \mathfrak{P}_{i_\ell}^2 .$$

Thus $\gamma\mathfrak{G} = \mathfrak{P}_{i_1} \dots \mathfrak{P}_{i_\ell}$ is a 2-sided ideal. Therefore, $\gamma\mathfrak{G} = \mathfrak{G}\gamma$ and γ normalizes $\mathfrak{G}$. By Corollary I.2.1, γ also normalizes $\Gamma(1)$. Thus γ is in B^{++} and has a fixed point on X. This completes the proof.

There is no simple way of determining whether a prime P of k splits in $k(\sqrt{-\pi_{i_1} \cdots \pi_{i_r}(\varepsilon_k)})/k$. The difficulty is that these extensions are not necessarily Galois. This will become apparent in Chapter IV.

For groups Γ with $j(E^{++}) \subseteq \Gamma \subseteq j(B^{++})$ the theorem can be used to determine if Γ has elements of finite order. This will be done in Chapter IV.

CHAPTER IV

EXAMPLES

Let A be a totally indefinite quaternion algebra over a real quadratic field $k = \mathbb{Q}(\sqrt{d})$, and let $\mathcal{O}$ be a fixed maximal order of A. Assume further that k has class number 1.

In Chapter III we have given conditions for the smoothness of $U(\Gamma) = j(\Gamma)\backslash X$, where $\Gamma(1) \subseteq \Gamma \subseteq B^{++}$. The conditions for $U(\Gamma(1))$ and $U(E^{++})$ are given in theorems III.1.1 and III.2.1. In practice, these conditions are easy to verify.

Let Γ be a subgroup of B^{++} properly containing E^{++}. The condition given in Theorem III.2.2 can easily be extended to Γ. Choose a complete set of coset representatives for $j(\Gamma)/j(E^{++})$. These representatives are products $\Pi_1 \cdots \Pi_r(\varepsilon)$ where $\nu(\Pi_1 \cdots \Pi_r(\varepsilon))$ are totally positive. (For notation see I.2 and III.2) The condition for Γ to give a smooth surface is the same as that given in Theorem III.2.2, except that it is only necessary to consider those $k(\sqrt{-\pi_1 \cdots \pi_r(\varepsilon_k)})$ that arise from the coset representatives.

<u>Lemma IV.1.1</u>. $P_i = \pi_i \mathcal{O}_k$ ramifies in $k(\sqrt{-\pi_1 \cdots \pi_r(\varepsilon_k)})/k$ for $1 \le i \le r$.

<u>Proof</u>: Let $\alpha = \sqrt{-\pi_1 \cdots \pi_r(\varepsilon_k)}$. Consider k_{P_i}, the P_i-adic completion of k. There exists a unique extension $\mathfrak{P}_i$ of the P_i-adic valuation on k_{P_i} to $k_{P_i}(\alpha) = K_{\mathfrak{P}_i}$ given by

$$|x|_{\mathfrak{B}_i} = \sqrt{|N_{K_{\mathfrak{B}_i}/k_{P_i}}(x)|_{P_i}}.$$

Consider $|\alpha|_{\mathfrak{B}_i} = \sqrt{(\alpha\bar{\alpha})_{P_i}}$ where $^-$ denotes Galois conjugation.

$$|\alpha|_{\mathfrak{B}_i} = \sqrt{(\alpha\bar{\alpha})_{P_i}} = \sqrt{(\alpha)^2_{P_i}} = \sqrt{|\pi_1|_{P_i}\cdots|\pi_i|_{P_i}\cdots|\pi_r|_{P_i}|(-\varepsilon_k)|_{P_i}}$$
$$= \sqrt{|\pi_i|_{P_i}} = \frac{1}{\sqrt{NP_i}}$$

since $|\pi_j|_{P_i} = 1$ for $i \neq j$.

Let $B_{k_{P_i}}$ and $B_{K_{\mathfrak{B}_i}}$ denote the value groups of k_{P_i} and $K_{\mathfrak{B}_i}$ respectively. $B_{k_{P_i}}$ is a subgroup of $B_{K_{\mathfrak{B}_i}}$, and since $B_{k_{P_i}}$ is generated by NP_i and $B_{K_{\mathfrak{B}_i}}$ is generated by $\sqrt{NP_i}$, the index of $B_{k_{P_i}}$ in $B_{K_{\mathfrak{B}_i}}$ is 2. Thus, $e(K/k,P_i) = 2$ and P_i ramifies.

<u>Corollary IV.1.1</u>. If ε_k is not totally positive, then $U(B^{++})$ is never smooth.

<u>Proof</u>: By the lemma, all $P \in S(A)$ ramify in $k(\sqrt{-\pi_1\cdots\pi_s(\varepsilon_k)})$ where $S(A) = \{\pi_1\mathfrak{O}_k,\pi_2\mathfrak{O}_k,\ldots,\pi_s\mathfrak{O}_k\}$, since $\pi_1\cdots\pi_s(\varepsilon)$ is

a coset representative for $j(B^{++})/j(\Gamma(1))$, $U(B^{++})$ cannot be smooth.

Lemma IV.1.2. Let K be an algebraic number field of degree n over $\mathbb{Q}$, with minimal polynomial $f(x) \in \mathbb{Z}[x]$. Then the discriminant $d(f)$ of f divides the discriminant d of K over $\mathbb{Q}$ and the quotient is the square of an integer, i.e., $d(f) = m^2 d$. Moreover, if p does not divide m the number of distinct irreducible factors of $f(x)$ in $\mathbb{Z}/p\mathbb{Z}[x]$ is the same as the number of primes lying above (p).

Proof: See Borevich-Shafarevich [3].

If $U(\Gamma)$ is smooth, then $E(1) = E(\Gamma)\cdot|j(\Gamma)/j(\Gamma(1)))|$ because $U(\Gamma(1))$ is a $|j(\Gamma)/j(\Gamma(1))|$-fold covering of $U(\Gamma)$.

For simplicity we denote $A(\mathbb{Q}(\sqrt{d}), \{P_1,\ldots,P_r\})$ by $A(d;P_1,P_2,\ldots,P_r)$, and $B_{\chi,2}$ by B_d.
group Γ between E^{++} and B^{++} by a complete set of coset representatives for $j(\Gamma)/j(E^{++})$. For an element $a \in k^x$, $a >> 0$ denotes that a is totally positive.

For simplicity we write $A(d;P_1,\ldots,P_r)$ instead of $A(Q(\sqrt{d}),\{P_1,\ldots,P_r\})$, and B_d in place of $B_{\chi,2}$. For $a \in k^x$, $a >> 0$ denotes that a is totally positive, and for notational purposes, we identify a group Γ between E^{++} and B^{++} by a complete set of coset representatives for $j(\Gamma)/j(E^{++})$.

Example 1. $A(12;P_2,P_{13}$ where P_2 is a prime of $\mathbb{Q}(\sqrt{12})$ lying above (2), and P_{13} is a prime of $\mathbb{Q}(\sqrt{12})$ lying above (13).

$B_{12} = 4$, and therefore, $E(\Gamma(1)) = E(1) = \frac{4}{12}(NP_2-1)(NP_{13}-1)$. Since (2) ramifies, $NP_2 = 2$. Thus, $E(\Gamma(1)) = \frac{1}{3}(2-1)(13-1) = 4$ and U(1) is a candidate for a Pg = 0 surface.

Let us check the smoothness conditions.

$$(\frac{-1}{13}) = (-1)^{\frac{13-1}{2}} = 1$$

and

$$(\frac{-3}{13}) = (\frac{-1}{13})(\frac{3}{13}) = (\frac{13}{3})(-1)^{\frac{13-1}{2}\cdot\frac{3-1}{2}} = (\frac{1}{3}) = 1.$$

Therefore, $A(12;P_2,P_{13})$ gives a smooth Pg = 0 surface U(1).

Example 2. $A(8;P_2,P_5)$. $B_8 = 2$. $NP_2 = 2$ and $NP_5 = 25$. $E(1) = \frac{2}{12}(2-1)(25-1) = 4$.

Let us now check the smoothness condition. $(\frac{-1}{5}) = 1$ and $(\frac{-24}{5}) = 1$, therefore $A(8;P_2,P_5)$ yields a smooth Pg = 0 surface U(1).

Similarly

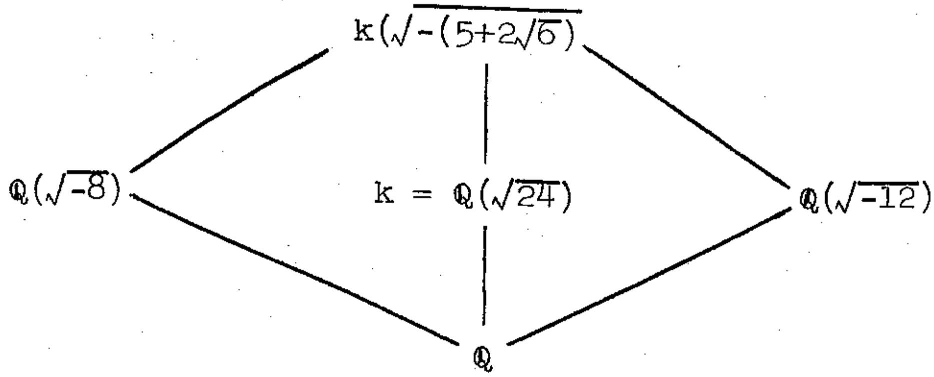
Example 3. $A(5;P_3,P_{31})$ gives a smooth Pg = 0 surface U(1).

There are no other Pg = 0 U(1) surfaces with k a real quadratic field.

Now let us consider $U(E^{++})$ surfaces.

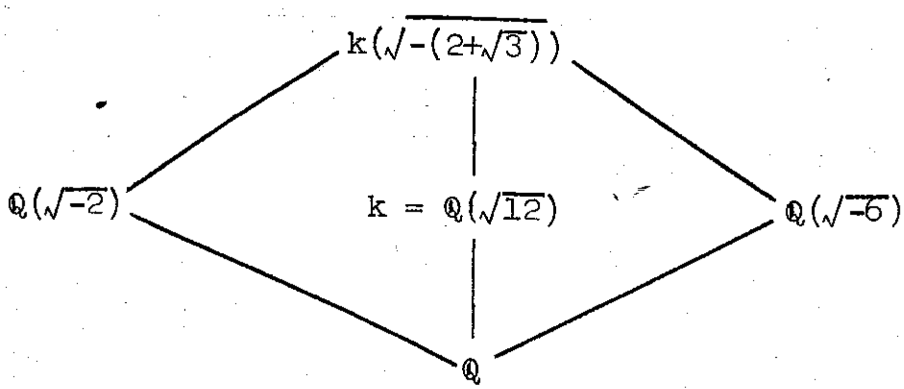
Example 4. $A(24;P_3,P_5)$ gives a smooth Pg = 1 surface, and $\varepsilon_k = 5+2\sqrt{6} >> 0$. We must investigate the splitting of P_3 and P_5 in $k(\sqrt{-\varepsilon_k})/k$.

By Lemma III.2.1 we have the following diagram:



(3) splits in $\mathbb{Q}(\sqrt{-8})/\mathbb{Q}$, and ramifies in $\mathbb{Q}(\sqrt{-8})/\mathbb{Q}$ and in $\mathbb{Q}(\sqrt{24})/\mathbb{Q}$. By the same reasoning as in the proof of Theorem III.1.1 (see Table III.1.4), P_3 splits in $k(\sqrt{-(5+2\sqrt{6})})/k$ and $U(E^{++})$ is smooth. $E(U(E^{++})) = \frac{1}{2}E(1) = 4$, and therefore, $U(E^{++})$ is a $Pg = 0$ surface.

Example 5. $A(12;P_2,P_5)$. $\varepsilon_k = 2 + \sqrt{3} >> 0$. $U(1)$ is smooth and has $Pg = 1$. Consider $U(E^{++})$.



(5) splits in $\mathbb{Q}(\sqrt{-6})/\mathbb{Q}$, and remains prime in $\mathbb{Q}(\sqrt{-2})/\mathbb{Q}$ and in $\mathbb{Q}(\sqrt{12})/\mathbb{Q}$. Therefore, P_5 splits in $k(\sqrt{-(2+\sqrt{3})})/k$ and $U(E^{++})$ is smooth with $Pg = 0$.

Similarly, we have

Example 6. $A(12;P_3,P_{13})$ yields a smooth $Pg = 0$ surface $U(E^{++})$,

and

Example 7. $A(21;P_3,P_5)$ yields a smooth $Pg = 0$ surface $U(E^{++})$.

These four examples are all of the smooth $Pg = 0$ $U(E^{++})$ surfaces.

We now give an example where the group Γ properly contains E^{++} and yields a smooth $Pg = 0$ surface.

Example 8. $A(8;P_3,P_7)$. $B_8 = 2$. $NP_3 = 9$ and $NP_7 = 7$. $\varepsilon_k = 1 + \sqrt{2}$ and ε_k is not totally positive. $E(1) = \frac{2}{12}(9-1)(7-1) = 8$ and $U(1)$ is smooth. Since ε_k is not totally positive, $U(1) = U(E^{++})$. To find a $Pg = 0$ surface, we look for Γ such that $|j(\Gamma)/j(\Gamma(1))| = 2$. $\pi_3 = 3$, and $\pi_7 = 3 + \sqrt{2} \gg 0$.

A) Consider Γ such that $j(\Gamma)/j(\Gamma(1)) = \{\Pi_3\}$. We must investigate whether either P_3 or P_7 splits in $k(\sqrt{-3})/k$. But, this is exactly one of the conditions for $U(1)$ to be smooth. Therefore, $j(\Gamma)\backslash X$ is smooth and has Euler characteristic 4, i.e. $Pg = 0$.

B) Consider Γ such that $j(\Gamma)/j(\Gamma(1)) = \{\Pi_7\}$. $k(\sqrt{-\pi_7}) = k(\sqrt{-3-\sqrt{2}}) = \mathbb{Q}(\sqrt{-3-\sqrt{2}}) = \mathbb{Q}(\alpha)$. $\alpha^2 = -3-\sqrt{2}$, $\alpha^2+3 = -\sqrt{2}$, $\alpha^4+6\alpha^2+9 = 2$, and therefore the minimal polynomial for $k(\sqrt{-\pi_7})/\mathbb{Q}$ is $f(x) = x^4 + 6x^2 + 7$. The roots of this polynomial are $\alpha_1 = i\sqrt{3+\sqrt{2}}$, $\alpha_2 = -i\sqrt{3+\sqrt{2}}$, $\alpha_3 = i\sqrt{3-\sqrt{2}}$ and $\alpha_4 = -i\sqrt{3-\sqrt{2}}$.

$d(f) = (\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4)(\alpha_2-\alpha_3)(\alpha_2-\alpha_4)(\alpha_3-\alpha_4) = 2^{10}7$.

By Lemma IV.1.2, since $\tilde{f}^3$, the reduction of f modulo 3, is x^4-2, P_3 does not split in $k(\sqrt{-\pi_7})/\mathbb{Q}$. Now, by Lemma IV.1.1 $P_7 = \pi_7\mathcal{O}_k$ ramifies in $K(\sqrt{-\pi_7})/k$. Then neither P_3 nor P_7 split in $k(\sqrt{-\pi_7})/k$ and $j(\Gamma)$ does not yield a smooth surface.

C) Consider Γ such that $j(\Gamma)/j(\Gamma(1)) = \{\Pi_3\Pi_7\}$. By Lemma IV.1.1 both P_3 and P_7 ramify in $k(\sqrt{-\pi_3\pi_7})/k$ and so $j(\Gamma)$ does not yield a smooth surface.

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