Some Results on the Pinching Problem

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We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

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Abstract of the Dissertation

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Wallach has discovered that in dimension 7, there are infinitely many simply connected compact manifolds $M(p,q)$ with strictly positive curvature. These spaces are in fact homogeneous (but not normal), obtained as quotients of $SU(3)$ by the various imbeddings of the circle S^1. For a suitable choice of Wallach's metrics, which appears to be optimal, we can show that the pinching of $M(1,1)$ is $\delta=16/29\cdot 37$. Using this result, we prove that for any $0<\varkappa<\delta$, there are infinitely many topologically distinct $M(p,q)$ with pinching$\geq\varkappa$. On the other hand, the pinching of all $M(p,q)$ is not uniformly bounded away from zero. It follows from our result that there is no finiteness theorem for the topological types of simply connected positively curved compact manifolds with given pinching, in odd dimension. In particular, the

injectivity radius of such manifolds cannot be bounded away from zero uniformly. All this is in contrast to fundamental results of Cheeger, Klingenberg, Weinstein, and others in _even_ demensions.

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Introduction

Global geometric properties of a complete riemannian manifold M often interact strongly with the underlying topological structure of M. The classical Gauss-Bonnet Theorem would probably be the simplest and most famous example, saying that if M is a compact oriented surface with curvature function K, then the integral of K over M is equal to $2\pi\chi(M)$, where $\chi(M)$ denotes the Euler characteristic of M. We only consider connected compact riemannian manifolds with strictly positive curvature. There is an important geometric quantity, the <u>pinching</u> of M, defined to be the quotient of the minimum and the maximum of all sectional curvatures on M, that plays an important role in the relationship between differential geometric and topological properties of these manifolds.

There are many results on pinching problems which seem as equally exciting as the Gauss-Bonnet Theorem. By definition, the value of the pinching will lie somewhere between 0 and 1. If the pinching of a simply connected manifold M is large, say between 1/4 and 1, then by the now classical Sphere Theorem, M must be homeomorphic to a symmetric space of rank one. Fairly recently, Cheeger[5], and Weinstein[14] proved that given any positive number $0<\delta\leq 1$, the set of all <u>even</u> dimensional riemannian manifolds with pinching $\geq\delta$ contains only finitely many different topological types (and also only finitely many distinct differentiable structures).

It has been an outstanding problem for quite a while to decide whether or not the same finiteness results are true for <u>odd</u> dimensional simply connected riemannian manifolds. (It is well known that in any odd dimension $n\geq 3$, there are already infinitely many topological distinct manifolds of <u>constant</u> curvature, all covered by the standard sphere S^n.) Various geometric questions are linked to this problem, notably a-priori estimates from below for the injectivity radius of the exponential map. In section 5 we shall prove that the above conjecture is false, at least in dimension 7.*

Another very interesting question then arises: What is the smallest number α such that there are only finitely many different topological types for all simply connected manifolds with pinching greater than α? (The number α might depend on the dimension.) This problem is still unsolved.

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*Our result should be contrasted with a remarkable recent finiteness result, announced by M. Gromov, according to which in particular, the Betti numbers of all n-dimensional, δ-pinched manifolds can be bounded in terms of δ and n alone.

But one knows from the Sphere Theorem that $\alpha \leq 1/4$. On the other hand, we shall prove that in dimension 7, the number α cannot be less than $16/29 \cdot 37$.

The above results have some other important geometric applications. For example, the injectivity radius of the exponential map, or equivalently, the length of the shortest periodic geodesic, is known to have an a-priori lower positive bound for all compact even dimensional manifolds with normalized positive curvature ≤ 1 (Synge, Klingenberg). It had been attempted for many years to prove a corresponding result for odd dimensions, in the simply connected case. We will see that such a uniform lower bound does not exist in dimension 7, not even for fixed pinching less than $16/29 \cdot 37$. But on the other hand, by the Sphere Theorem, there is a lower bound if the pinching is at least $1/4$.

One major difficulty to study pinching problems is the lack of enough examples. Topologically, all known examples are so far locally homogeneous spaces. Comparatively few (simply connected) homogeneous spaces admit strictly positive sectional curvature. Berger [1] has classified all normal homogeneous spaces with strictly positive curvature. He proved that all such spaces are symmetric spaces of rank one, with two exceptions in dimension 7 and in dimension 13. The symmetric spaces of rank one are spheres, complex projective spaces, quaternionic projective spaces, and the Cayley plane; see [3]. Non-normal homogeneous metrics with positive curvature have been analyzed completely only very recently by Wallach [12], and Bérard Bergery [3]. It turned out that there are just a few more examples than the above, except for an infinite string of new simply connected spaces in dimension 7, discovered by Wallach [13]. These examples are the starting point for our work. Wallach's crucial technique was to deform the metric on a suitable subspace of the tangent space of a normal homogeneous space (which always has curvature ≥ 0). This procedure usually introduces some negative curvatures, but it may also lead to strictly positively curved spaces, in some instances.

The paper is organized as follows: In Section 1, we briefly review and compile the facts about Lie groups, homogeneous spaces, and invariant metrics that will be used later. Section 2 and 3 contain a sketch of the necessary curvature computations. In Section 4, we discuss Wallach's examples from our point of view. All the results mentioned above then will be proved in Section 5.

1. Basic Facts.

A <u>Lie group</u> is a group G which is also an analytic manifold such that the mapping $(x,y) \to xy^{-1}$ of the product manifold $G \times G$ into G is analytic. There is a special kind of algebra - the Lie algebra - canonically associated with every Lie group.

A <u>Lie algebra</u> is a vector space $\mathfrak{g}$ together with a mapping $[,]$ from $\mathfrak{g} \times \mathfrak{g}$ into $\mathfrak{g}$ such that

$$[a_1V_1+a_2V_2,U] = a_1[V_1,U] + a_2[V_2,U],$$
$$[V,U] = -[U,V],$$
$$[V_1,[V_2,V_3]] + [V_3,[V_1,V_2]] + [V_2,[V_3,V_1]] = 0.$$

We recall how a Lie algebra $\mathfrak{g}$ corresponds to a Lie group G.

$L_a(R_b)$ is a left (right) <u>translation</u> on G if $L_a(R_b)$ is the transformation on G such that $L_a(x) = ax$ $(R_b(x) = xb)$, for all x in G, where a,b are fixed elements in G. The vector field X is <u>left invariant</u> (right invariant) if

$$(dL_g)X_{g'} = X_{gg'} \quad (\quad (dR_g)X_{g'} = X_{g'g} \quad),$$

where dL_g is the differential (Jacobian) of L_g.

The left (right) invariant vector field X is uniquely determined by the value X_e of X at the identity element e in G. We denote by $\mathfrak{g}$ the set of all left invariant vector fields. $\mathfrak{g}$ will form a Lie algebra if we define [U,V] to be UV–VU, $\mathfrak{g}$ is called the <u>Lie algebra of G</u>. It may be identified with the tangent space of G at e.

There is a natural homomorphism Ad, called the <u>adjoint representation</u>, from G to the general linear group $GL(\mathfrak{g})$ of $\mathfrak{g}$, defined as follows:

$$Ad_g(X) = dR_g \circ dL_{g^{-1}}(X).$$

We define $ad: \mathfrak{g} \to gl(\mathfrak{g})$ to be the differential of the adjoint representation Ad at the point e (the identity of G). ad is a Lie algebra homomorphism from $\mathfrak{g}$ into the Lie algebra $gl(\mathfrak{g})$ of all linear transformations on $\mathfrak{g}$. One has

$$ad_X(Y) = [X,Y] = XY-YX.$$

We can always put a riemannian metric on the Lie group G to make it a riemannian manifold. Now we discuss special metrics on G, the so-called invariant metrics. The metric $\langle\, ,\, \rangle$ defined on a Lie group G is called <u>left (right) invariant</u> if $\langle X,Y \rangle$ is constant for all left (right) invariant vector fields X and Y on G, i.e. all left (right) translations are isometries. By the above, it is always possible to put a left (right) invariant metric on any Lie group, just by choosing an inner product for the

Lie algebra. If the metric is invariant under both left and right translations, then this metric is called biinvariant. A left invariant metric is biinvariant if and only if it is Ad_G-invariant on $\mathfrak{g}$. For a compact group G, one can always choose a more or less canonical biinvariant metric, essentially given by the Killing form. The bilinear form

$$B(X,Y) = Tr(ad_X \circ ad_Y) \text{ on } \mathfrak{g},$$

where Tr is the trace of a vector space endomorphism, is called the Killing form of G. B is invariant under Ad_G. A connected Lie group G is called semi-simple if the Killing form of G is nondegenerate. The following facts are well known:

(1) Let G be a semi-simple Lie group, then G is compact if and only if the Killing form of G is strictly negative definite.

(2) If G is a compact Lie group, then its Lie algebra is the direct sum $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g},\mathfrak{g}]$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$, The commutator $[\mathfrak{g},\mathfrak{g}]$ is semi-simple, and the restriction $B|[\mathfrak{g},\mathfrak{g}]$ is strictly negative definite. Now we can put a biinvariant metric $\langle\ ,\ \rangle$ on the compact Lie group G: We choose $-B$ for $\langle\ ,\ \rangle$ on $[\mathfrak{g},\mathfrak{g}]$, where B is the Killing form, we give the center $\mathfrak{z}$ some euclidean metric, and we make $\mathfrak{z}$ and $[\mathfrak{g},\mathfrak{g}]$ orthogonal.

Let G be a Lie group and H a closed subgroup. Denote by G/H the set of all left cosets gH. G/H carries a unique structure of an analytic manifold such that the natural projection $G \to G/H$ becomes an analytic principal fibration. There is a natural smooth action of G on G/H induced by left translations on G, $g_1(gH) = g_1 gH$. This is a transitive action, and one calls G/H a homogeneous space.

Now we will discuss invariant metrics on homogeneous spaces, i.e. metrics on G/H for which G acts by isometries. It is not always the case that invariant metrics exist on homogeneous spaces. However, if for example, H is a compact subgroup of G, then one can find an invariant metric for G/H. For the purpose of this thesis, we always assume that G is compact.

Let G be a compact Lie group, H a closed subgroup of G. The biinvariant metrics for G, as described above, induce invariant metrics on G/H. If $\langle\ ,\ \rangle$ denotes a biinvariant metric on G, $\mathfrak{h}$ the Lie algebra of H, then the Lie algebra $\mathfrak{g}$ of G admits an orthogonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, where $\mathfrak{p} = \mathfrak{h}^{\perp}$, and $[\mathfrak{p}, \mathfrak{h}] \subset \mathfrak{p}$, i.e. $Ad_H(\mathfrak{p}) \subset \mathfrak{p}$. The restriction of the inner product $\langle\ ,\ \rangle$ to the orthogonal complements of the tangent spaces to the fibers gH, therefore projects to an invariant metric on G/H. We call the metric obtained in this way a <u>normal metric</u>. Furthermore, if $\mathfrak{p}$ can be written as a direct sum of $\mathfrak{p}_i$, i.e. $\mathfrak{p} = \bigoplus_{i=1}^{m} \mathfrak{p}_i$, such that $Ad_H(\mathfrak{p}_i) \subset \mathfrak{p}_i$ for each $i = 1, \cdots, m$, then we can deform the normal metric on $\mathfrak{p}$ to a new metric $\overline{\langle\, ,\, \rangle}$ on $\mathfrak{p}$ as follows,

$$\overline{\langle X, Y \rangle} = \sum_{i=1}^{m} a_i \langle X_i, Y_i \rangle ,$$

where $a_i > 0$, $X = \sum_{i=1}^{m} X_i$, $Y = \sum_{i=1}^{m} Y_i$, and $X_i, Y_i \in \mathfrak{p}_i$, $i = 1, \cdots, m$.

This new metric on G/H is G-invariant, but not normal in general, since biinvariant metrics are essentially unique.

2. The Curvature of Homogeneous Spaces.

We will discuss the curvature of a Lie group first, Once we know the curvature for a Lie group, we can use O'Neill's formula to compute the curvature for homogeneous spaces.

Let G be a Lie group, $\mathfrak{g}$ the Lie algebra of G, $\langle\ ,\ \rangle$ a left invariant metric on G. We use A^* to denote the adjoint of a linear transformation A on $\mathfrak{g}$ with respect to the given metric $\langle\ ,\ \rangle$, i.e., $\langle AX,Y\rangle = \langle X,A^*Y\rangle$. For example, $\langle [X,Y],Z\rangle = \langle Y,(ad_X)^*Z\rangle$. First we need the Levi-Civita covariant derivative ∇. It suffices to work with left invariant vector fields on G.

<u>Proposition 2.1</u>.

$$(1)\quad \nabla_X Y = 1/2\{[X,Y] - (ad_X)^*Y - (ad_Y)^*X\}$$

Proof:

Since X,Y,Z are left invariant, we have:

$$X\langle Y,Z\rangle = \langle \nabla_X Y,Z\rangle + \langle Y,\nabla_X Z\rangle = 0,$$
$$Y\langle X,Z\rangle = \langle \nabla_Y X,Z\rangle + \langle X,\nabla_Y Z\rangle = 0,$$
$$Z\langle X,Y\rangle = \langle \nabla_Z X,Y\rangle + \langle X,\nabla_Z Y\rangle = 0.$$

From the above equations, we can derive the following formula:

$$(2)\quad 2\langle \nabla_X Y,Z\rangle = \langle [X,Y],Z\rangle - \langle Y,[X,Z]\rangle - \langle X,[Y,Z]\rangle.$$

This yields (1).

We now compute the riemannian curvature tensor R for G, in terms of first covariant derivatives. Again, it suffices to consider only left invariant vector fields X,Y,Z,W on G.

Proposition 2.2.

$$(3)\ \langle R(X,Y)Z,W\rangle = \langle \nabla_X Z, \nabla_Y W\rangle - \langle \nabla_Y Z, \nabla_X W\rangle - \langle \nabla_{[X,Y]}Z, W\rangle.$$

Proof:

By (2), $\langle \nabla_X Y, Z\rangle$ is a constant function on G, hence $\nabla_X Y$ is a left invariant vector field on G. It follow that

$$X\langle \nabla_Y Z, W\rangle = \langle \nabla_X \nabla_Y Z, W\rangle + \langle \nabla_Y Z, \nabla_X W\rangle = 0$$

$$Y\langle \nabla_X Z, W\rangle = \langle \nabla_Y \nabla_X Z, W\rangle + \langle \nabla_X Z, \nabla_Y W\rangle = 0.$$

Subtracting the above equations and using the definition of R,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z,$$

we obtain (3).

From the last proposition we can derive a simple algebraic formula for the sectional curvature of G, in terms of data of the Lie algebra only.

$$(4)\ \langle R(X,Y)Y,X\rangle = 1/4\|(ad_X)^*Y + (ad_Y)^*X\|^2 - \langle (ad_X)^*X, (ad_Y)^*Y\rangle - 3/4\|[X,Y]\|^2 - 1/2\langle [[X,Y],Y],X\rangle - \langle [[Y,X],X],Y\rangle.$$

Proof:

We choose Z=Y,W=X in (3). Then,

$$(3') \quad \langle R(X,Y)Y,X \rangle = \langle \nabla_X Y, \nabla_X Y \rangle - \langle \nabla_Y Y, \nabla_X X \rangle - \langle \nabla_{[X,Y]} Y, X \rangle.$$

Applying (1) to each term in (3') yields

$$\langle \nabla_X Y, \nabla_Y X \rangle = 1/4\{\|(ad_X)^* Y + (ad_Y)^* X\|^2 - \|[X,Y]\|^2\},$$
$$\langle \nabla_Y Y, \nabla_X X \rangle = \langle (ad_Y)^* Y, (ad_X)^* X \rangle,$$
$$\begin{aligned} \langle \nabla_{[X,Y]} Y, X \rangle &= 1/2\{\langle [[X,Y],Y],X \rangle - \langle (ad_{[X,Y]})^* Y, X \rangle\} \\ &\quad - \langle (ad_Y)^*[X,Y],X \rangle \\ &= 1/2\{\langle [[X,Y],Y],X \rangle - \langle [[X,Y],X],Y \rangle + \|[X,Y]\|^2\}. \end{aligned}$$

Insert the last three equations into (3') to prove (4).

<u>Corollary 2.4</u>.

If $\langle\, ,\, \rangle$ is a biinvariant metric, then ad_X is skew adjoint, and

$$(5) \quad \nabla_X Y = 1/2[X,Y],$$
$$(6) \quad \langle R(X,Y)Z,W \rangle = -1/4 \langle [X,Y],[Z,W] \rangle,$$
$$(7) \quad \langle R(X,Y)Y,X \rangle = 1/4\|[X,Y]\|^2.$$

In order to generalize the curvature formula for Lie groups to homogeneous spaces, we need the assistance of O'Neill's formula which compares the curvature of two riemannian manifolds M and N related by a riemannian submersion; see also [11], [6].

Definition 2.5.

A riemannian submersion π is a smooth map from a riemannian manifold M of dimension n+k onto a riemannian manifold N of dimension n such that $d\pi$ has rank n for all $m \in M$.

By the implicit function theorem, $\pi^{-1}(p)$ is a smooth k-dimensional submanifold of M for all $p \in N$. Let V_q be the tangent space of $\pi^{-1}(p)$ at q, $H_q=V_q^{\perp}$ the orthogonal complement of V_q in M_q. We call H,V the horizontal and vertical subspace respectively, and use h,v as subscripts to denote the horizontal and vertical components. π is a riemannian submersion if the restriction $d\pi|H_q$ is isometric for each $q \in M$. Given a vector field X on N, there is a unique vector field $\bar{X}$ on M such that $\bar{X}_q \in H_q$ and $d\pi(\bar{X}_q)=X_{\pi(q)}$, for all $q \in M$. $\bar{X}$ is sometimes called the horizontal lift of X. We will derive the formula which gives the relation between the sectional curvature with respect to a horizontal 2-plane $\measuredangle$ in M and the sectional curvature with respect to the plane $d\pi(\measuredangle)$ in N. Let X,Y be vector fields on N and $K(X,Y) = \langle R(X,Y)Y,X \rangle$. Note that if X,Y are (locally) orthonormal, then K(X,Y) is just the sectional curvature with respect to the planes spanned by X,Y.

Proposition2.6. (O'Neill's Formula)

$$(8)\quad K(X,Y) = \bar{K}(\bar{X},\bar{Y}) + 3/4\|[\bar{X},\bar{Y}]_v\|^2.$$

In particular, riemannian submersions are curvature non-decreasing.

Proof:

$\bar{X},\bar{Y}$ and X,Y are π-related, i.e. $(d\pi)\bar{X}_q = X_{\pi(q)}$, for all $q \in M$. Then $[\bar{X},\bar{Y}]$ is π-related to $[X,Y]$. Let X,Y,Z,W be vector fields on N, and T a vertical vector field on M. By the above observation, we have $< [\bar{X},\bar{Y}],\bar{Z} > = < [X,Y],Z >$, $< [\bar{X},T],\bar{Y} > = 0$. Note that T is π-related to 0.

Let $\bar{\nabla},\nabla$ be the riemannian connections on M,N, respectively. Using the definition of the covariant derivative of Levi-Civita we obtain

$$< \bar{\nabla}_{\bar{X}}\bar{Y},\bar{Z} > = 1/2\{\bar{X}< \bar{Y},\bar{Z} > + \bar{Y}< \bar{X},\bar{Z} > - \bar{Z}< \bar{X},\bar{Y} > + < [\bar{X},\bar{Z}],\bar{Y} > - < [\bar{X},\bar{Z}],\bar{Y} > - < [\bar{Y},\bar{Z}],\bar{X} > = < \nabla_X Y,Z >,$$

and $< \bar{\nabla}_{\bar{X}}\bar{Y},T > = 1/2 < [\bar{X},\bar{Y}],T >$. Therefore,

(9) $\bar{\nabla}_{\bar{X}}\bar{Y} = \overline{\nabla_X Y} + 1/2[\bar{X},\bar{Y}]_v$. Furthermore,

$$< \bar{\nabla}_T\bar{X},\bar{Y} > = < \bar{\nabla}_{\bar{X}}T,\bar{Y} > + < [T,\bar{X}],\bar{Y} > = -< \bar{\nabla}_{\bar{X}}\bar{Y},T >$$
$$= -1/2< [\bar{X},\bar{Y}],T > = -1/2< [\bar{X},\bar{Y}]_v,T >, \text{ and}$$

$\bar{X}< \bar{\nabla}_{\bar{Y}}\bar{Z},\bar{W} > = X< \nabla_Y Z,W >$. Hence,

$$< \bar{\nabla}_{\bar{X}}\bar{\nabla}_{\bar{Y}}\bar{Z},\bar{W} > = \bar{X}< \bar{\nabla}_{\bar{Y}}\bar{Z},\bar{W} > - < \bar{\nabla}_{\bar{Y}}\bar{Z},\bar{\nabla}_{\bar{X}}\bar{W} >$$
$$= X< \nabla_Y Z,W > - < \nabla_Y Z,\nabla_X W > - 1/4< [\bar{Y},\bar{Z}]_v,[\bar{X},\bar{W}]_v >$$
$$= < \nabla_X\nabla_Y Z,W > - 1/4< [\bar{Y},\bar{Z}]_v,[\bar{X},\bar{W}]_v >, \text{ and}$$

$$< \bar{\nabla}_{[\bar{X},\bar{Y}]}\bar{Z},\bar{W} > = < \nabla_{[X,Y]}Z,W > - 1/2< [\bar{Z},\bar{W}]_v,[\bar{X},\bar{Y}]_v >.$$

Combining all those results yields

(10) $$< \bar{R}(\bar{X},\bar{Y})\bar{Z},\bar{W} > = < R(X,Y)Z,W > -1/4< [\bar{Y},\bar{Z}]_v,[\bar{X},\bar{W}]_v > +1/4< [\bar{X},\bar{Z}]_v,[\bar{Y},\bar{W}]_v > + 1/2< [\bar{X},\bar{Y}]_v,[\bar{Z},\bar{W}]_v >.$$

In particular, if we choose $\bar{\bar{Z}}=\bar{Y}$, $\bar{\bar{W}}=\bar{X}$, then

$$\bar{K}(\bar{X},\bar{Y}) = < \bar{R}(\bar{X},\bar{Y})\bar{Y},\bar{X} > = <R(X,Y)Y,X > - 3/4\|[\bar{X},\bar{Y}]_V\|^2$$
$$= K(X,Y) -3/4\|[\bar{X},\bar{Y}]_V\|^2.$$

This completes the proof of (8).

<u>Remark 2.7</u>. The term $[U,V]_V$ in (9) is not a derivation, but linear in two arbitary horizontal vector fields U,V on M. Its value at any point $q \in M$ depends only on the vectors U_q, $V_q \in M_q$.

We can now easily compute the curvature of any riemannian homogeneous space $\pi\colon G \to G/H$ in terms of the Lie algebras $\mathfrak{g} \supset \mathfrak{h}$ of $G \supset H$. By assumption, G is equipped with a left invariant metric, i.e. an inner product on $\mathfrak{g}$ which is Ad_H-invariant. The metric on G induces a metric on G/H such that G acts on G/H by isometries, and π becomes a riemannian submersion. By homogeneity, it suffices to compute the curvature of G/H at one point, say $\pi(e) = eH$. We have an orthogonal splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, and the projection π identifies $\mathfrak{p}$ isometrically with the tangent space of G/H at $\pi(e)$. Let $X,Y \in \mathfrak{p}$ be two orthonormal vectors, and let K(X,Y) denote the sectional curvature of G/H at $\pi(e)$ with respect to the plane spanned by $(d\pi)X$ and $(d\pi)Y$. Then we have, Proposition 2.8.

$$(11)\quad K(X,Y) = 1/4\|(ad_X)^*Y+(ad_Y)^*X\|^2$$
$$-< (ad_X)^*X,(ad_Y)^*Y >$$
$$-3/4\|[X,Y]_{\mathfrak{p}}\|^2$$
$$-1/2< [[X,Y],Y],X > - 1/2< [[Y,X],X],Y >.$$

Here $[X,Y]_{\mathfrak{p}}$ denote the $\mathfrak{p}$ -component of $[X,Y]$.

Corollary 2.9. (Samelson, Nomizu)

If G/H is a normal homogeneous space, then (11) simplifies to

$$(12)\quad K(X,Y)= 1/4\|[X,Y]\|^2 + 3/4\|[X,Y]_{\mathfrak{h}}\|^2$$
$$= 1/4\|[X,Y]_{\mathfrak{p}}\|^2 + \|[X,Y]_{\mathfrak{h}}\|^2.$$

In particular, the curvature of a normal homogeneous space is always nonnegative.

Proof: Formula (11) and (12) are an immediate consequence of (8),(9), and Remark 2.7.

3. Deformations of the Normal Metric.

We shall now study a much more special situation. Let G be a compact connected Lie group and H a closed subgroup of G. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie algebra of $H \subset G$. The normal homogeneous space G/H is said to satisfy condition II if for the orthogonal complement $\mathfrak{p}$ of $\mathfrak{h}$ in $\mathfrak{g}$, the following holds:

(1) $\mathfrak{p} = V_1 \oplus V_2$ (orthogonal direct sum),
$Ad(H)V_i \subset V_i$ $(i=1,2)$,

(2) $[V_1,V_2] \subset V_2$,

(3) $[V_1,V_1] \subset \mathfrak{h} \oplus V_1$,

(4) $[V_2,V_2] \subset \mathfrak{h} \oplus V_1$,

(5) If $X = X_1+X_2$, $Y = Y_1+Y_2$, $X_i,Y_i \in V_i$ $(i=1,2)$, $X \wedge Y \neq 0$, and if $[X,Y] = 0$, then $[X_1,Y_1] \neq 0$.

The formulation of condition II and the following lemma were given by Wallach; see [3]. However, we will present a different and straightforward proof of this lemma using 2.(11).

<u>Lemma 3.1</u>.

Let G/H satisfy (1),(2),(3),(4) of condition II. Let $<\ ,\ >$ denote the Ad_G-invariant metric on $\mathfrak{g}$. Fix $-1<t<\infty$, and let $<\ ,\ >_t$ be the inner product on $\mathfrak{p}$ given by

$$< X_1+X_2, Y_1+Y_2>_t = (1+t)< X_1,Y_1> + < X_2,Y_2>$$

for $X_i, Y_i \in V_i$ Extend $< , >_t$ to a metric on $\mathfrak{g}$: Make $\mathfrak{h}$ and $\mathfrak{p}$ orthogonal and choose $< , >_t$ to be $< , >$ on $\mathfrak{h}$. Now $< , >_t$ is Ad_H-invariant, hence it induces a new riemannian structure for G/H. Relative to this metric, let $K(X,Y)$ denote the sectional curvature of G/H at eH with respect to the plane spanned by orthonormal vectors $X,Y \in \mathfrak{p}$. As bofore, we have identified $\mathfrak{p}$ with the tangent space of G/H at eH. Then we have:

$$\begin{aligned} K(X,Y) = {} & \|[X,Y]_{\mathfrak{h}}\|^2 + t(1-t)< [X_1,Y_1]_{\mathfrak{h}},[X,Y]_{\mathfrak{h}}> \\ (6) \qquad & + t^2\|[X_1,Y_1]_{\mathfrak{h}}\|^2 + (1-3t)/4\|[X,Y]_1\|^2 \\ & + t(1-t)< [X_1,Y_1]_1,[X,Y]_1> + t^2\|[X_1,Y_1]_1\|^2, \end{aligned}$$

where if $Z \in \mathfrak{g}$, $Z = Z_{\mathfrak{h}}+Z_1+Z_2$, $Z_{\mathfrak{h}} \in \mathfrak{h}$, $Z_i \in V_i$, $i=1,2$.

Proof: By 2.(11), we have

$$\begin{aligned} K(X,Y) = {} & 1/4\|(ad_X)^*Y+(ad_Y)^*X\|_t^2 - < (ad_X)^*X,(ad_Y)^*Y >_t \\ (7) \qquad & -1/2< [[X,Y],Y],X >_t - 1/2< [[Y,X],X],Y >_t \\ & -3/4\|[X,Y]_{\mathfrak{p}}\|_t^2. \end{aligned}$$

We have to express everything in terms of the biinvariant metric $< , >$ on $\mathfrak{g}$. Recall that

$$< X,Y>_t = (1+t)< X_1, Y_1> + < X_2,Y_2> = t< X_1,Y_1> + < X,Y >.$$

Since $< , >_t$ is Ad_H-invariant, $ad_{\mathfrak{h}}$ is skew adjoint with respect to $< , >_t$ on $\mathfrak{p}$, i.e.

$$(8)\quad < [X,Z],Y >_t + < [Y,Z],X >_t = 0,$$

for $X,Y \in \mathfrak{p}$, $Z \in \mathfrak{k}$. Choose orthonormal bases $W_1,\cdots,W_r$; $Z_1,\cdots, Z_m$; $Z_{m+1},\cdots,Z_n$ for $\mathfrak{k}$, V_1,V_2, with respect to $< , >_t$. Then, using (8), condition II, and the fact that $ad_{\mathfrak{g}}$ is skew adjoint on $\mathfrak{g}$ with respect to $< , >$, we have

$$\begin{aligned}
(ad_X)^*Y + (ad_Y)^*X &= \sum_{j=1}^{r} (< [X,W_j],Y >_t + < [Y,W_j],X >_t)W_j \\
&\quad + \sum_{i=1}^{n} (< [X,Z_i],Y >_t + < [Y,Z_i],X >_t)Z_i \\
&= t\sum_{i=1}^{n} (< [X,Z_i]_1,Y_1> + < [Y,Z_i]_1,X_1>)Z_i \\
&= t\sum_{i=1}^{m} (< [X_1,Z_i]_1,Y_1> + < [Y_1,Z_i]_1,X_1>)Z_i \\
&\quad + \sum_{i=m+1}^{n} (< [X_2,Z_i]_1,Y_1> + < [Y_2Z_i]_1,X_1>)Z_i \\
&= -t\sum_{i=m+1}^{n} (< [X_2,Y_1],Z_i> + < [Y_2,X_1],Z_i>)Z_i \\
&= t([X_1,Y_2] - [X_2,Y_1]).
\end{aligned}$$

Therefore,

$$(9)\quad \begin{aligned}
&1/4\|(ad_X)^*Y+(ad_Y)^*X\|_t^2 \\
&= t^2/4\|[X_1,Y_2] - [X_2,Y_1]\|^2 \\
&= t^2/4\|[X_1,Y_2] + [X_2,Y_1]\|^2 - t^2< [X_1,Y_2],[X_2,Y_1]> \\
&= t^2/4\|[X,Y]_2\|^2 - t^2< [X_1,Y_2],[X_2,Y_1]>.
\end{aligned}$$

Similarly,

$$(ad_X)^*X = \sum_1^r < [X,W_j],X>_t W_j + \sum_1^n < [X,Z_i],X >_t Z_i$$
$$= t \sum_1^n < [X,Z_i]_1,X_1 > Z_i$$
$$= t \sum_{m+1}^n < [X_2,Z_i],X_1 > Z_i$$
$$= -t[X_2,X_1] = t[X_1,X_2],$$

and by the same reasoning, $(ad_Y)^*Y = t[Y_1,Y_2]$.

So we have,

$$< (ad_X)^*X,(ad_Y)^*Y >_t = t^2 < [X_1,X_2],[Y_1,Y_2] >.$$

Now,

$$< [X_1,X_2],[Y_1,Y_2] > = -< X_1,[[Y_1,Y_2],X_2] >$$
$$= < X_1,[[Y_2,X_2],Y_1] > + < X_1,[[X_2,Y_1],Y_2] >$$
$$= -< [X_1,Y_1],[Y_2,X_2] > - < [X_1,Y_2],[X_2,Y_1] >$$
$$= < [X_1,Y_1],[X_2,Y_2] > - < [X_1,Y_2],[X_2,Y_1] >.$$

Therefore,

$$\begin{aligned} &-<(ad_X)^*X,(ad_Y)^*Y >_t \\ (10) \quad &= -t^2 < [X_1,Y_1],[X_2,Y_2] > + t^2 < [X_1,Y_2],[X_2,Y_1] >. \end{aligned}$$

Furthermore,

$$< [[X,Y],Y],X >_t + < [[Y,X],X],Y >_t$$
$$= 2< [[X,Y],Y],X > + t\{< [[X,Y],Y]_1,X_1> + < [[Y,X],X]_1,Y_1>\}$$
$$= -2< [X,Y],[X,Y] > + t\{< [[X,Y]_h,Y_1],X_1> + < [[Y,X]_h,X_1],Y_1>$$
$$+ < [[X,Y]_1,Y_1],X_1> + < [[Y,X]_1,X_1],Y_1> + < [[X,Y]_2,Y_2],X_1>$$
$$+ < [[Y,X]_2,X_2],Y_1>\}$$
$$= -2\|[X,Y]\|^2 - 2t< [X,Y]_h,[X_1,Y_1] > -2t< [X,Y]_1,[X_1,Y_1] >$$
$$-t\|[X,Y]_2\|^2.$$

So,

$$-1/2< [[X,Y],Y],X >_t -1/2< [[Y,X],X],Y >_t$$

(11) $$=\|[X,Y]\|^2+t< [X,Y]_h,[X_1,Y_1]_h> +t< [X,Y]_1,[X_1,Y_1]_1>$$

$$+t/2\|[X,Y]_2\|^2.$$

Finally,

(12) $$-3/4\|[X,Y]\|_t^2 = -3/4\{t\|[X,Y]_1\|^2+\|[X,Y]\|^2\}.$$

Putting together all these results (9) through (12) yields (6).

<u>Corollary 3.2</u>.

If G/H satisfies condition II , then the G-invariant metric $< , >_t$ on G/H defined above, has strictly positive curvature, for $-1<t<0$.

The argument is very easy and can be found in [13].

4. The Examples of Wallach.

In this section, we study the positively curved homogeneous spaces discovered by Wallach [13]. The main purpose is to compute explicitly the curvature of these spaces.

Let G = SU(3), the special unitary group of degree 3 Every non-trivial circle in SU(3) is of the form,

$$T(p,q) = \left\{ \begin{bmatrix} e^{2\pi i p\theta} & 0 & 0 \\ 0 & e^{2\pi i q\theta} & 0 \\ 0 & 0 & e^{-2\pi i(p+q)\theta} \end{bmatrix} \;\middle|\; \begin{matrix} \theta \in R \\ p^2+q^2 > 0 \end{matrix} \right\},$$

$p,q \in Z$, up to conjugacy in SU(3).

Consider the following subgroup $G \subset SU(3)$,

$$G = \left\{ \begin{bmatrix} g & 0 \\ 0 & \overline{\det(g)} \end{bmatrix} \;\middle|\; g \in U(2) \right\}.$$

Then $T(p,q) \subset G$.

Let $\mathfrak{g}$ be the Lie algebra of G, i.e. the algebra of all complex skew hermitian 3×3 matrices with trace 0. Let $\mathfrak{t} = \mathfrak{t}(p,q) \subset \mathfrak{g}_1 \subset \mathfrak{g}$ be the Lie subalgebras of $T(p,q) \subset G_1 \subset G$. Clearly,

$$\mathfrak{h} = \left\{ \begin{bmatrix} 2\pi i p\theta & 0 & 0 \\ 0 & 2\pi i q\theta & 0 \\ 0 & 0 & -2\pi i(p+q)\theta \end{bmatrix} \middle| \; \theta \in R \right\},$$

and

$$\mathfrak{g}_1 = \left\{ \begin{bmatrix} Z & 0 \\ 0 & -\mathrm{Tr} Z \end{bmatrix} \middle| \; \begin{array}{l} Z \text{ skew hermitian} \\ 2\times 2 \text{ matrix} \end{array} \right\}.$$

Define the (standard) Ad_G-invariant inner product on $\mathfrak{g}$, by means of the Killing form:

$$\langle X,Y \rangle = -1/2 \; \mathrm{Re}(\mathrm{Tr}\; XY).$$

Let $A^{\perp}$ denote the orthogonal complement of a subspace A in $\mathfrak{g}$. We have,

$$\mathfrak{g}_1^{\perp} = \left\{ \begin{bmatrix} 0 & z \\ -{}^t\bar{z} & 0 \end{bmatrix} \middle| \; z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad z_i \in C \right\}.$$

For the following, we refer to Section 3. Set

$$V_1 = \mathfrak{h}^{\perp} \cap \mathfrak{g}_1,$$
$$V_2 = \mathfrak{g}_1^{\perp}.$$

Then $\mathfrak{g} = \mathfrak{h} \oplus V_1 \oplus V_2$.

One can easily verify that SU(3)/T(p,q) satisfies condition II, if $pq \neq 0$. We shall denote SU(3)/T(p,q) by M(p,q).

If p,q are relatively prime integers, then M(p,q) is simply connected. The following result is due to Lashof.

Proposition 4.1.

$$H^4(M(p,q);Z) = Z/rZ, \text{ where } r = |p^2 + q^2 + pq|,$$

provided p,q are relatively prime.

Corollary 3.2 and proposition 4.1 contain Wallach's result that there are infinitely many simply connected, topologically distinct homogeneous spaces with strictly positive curvature, in dimension 7. It should be pointed out that the spaces $M(p,q)$, with respect to their normal metrics, always have some zero curvatures. We now proceed to give a more quantitative analysis of the curvature of the manifolds $M(p,q)$.

We define a bases of $\mathfrak{g}$ as follows:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_5 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \quad A_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$$

$$A_7 = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{bmatrix} \quad A_8 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}$$

The set $A = \{A_1,\cdots,A_8\}$ is not orthonormal yet. But we can easily produce an orthonormal basis $M = \{M_1,\cdots,M_8\}$, by setting $M_i=A_i$ for $1\leq i\leq 6$,

$$\sqrt{3}\Delta M_7 = (2q+p)A_7-(2p+q)A_8,$$

$$\Delta M_8 = pA_7+qA_8,$$

where $\Delta = \sqrt{p^2+q^2+pq}$.

Notice that M_8 spans $\mathfrak{h}$, the matrices M_1,M_4,M_7 span V_1, and M_2,M_3,M_5,M_6 span V_2.

We need the Lie brackets $[A_i,A_j]$ and $[M_i,M_j]$. They are compiled in the following tables.

<u>Table of Lie brackets</u> $[A_i,A_j]$, $i<j$

j=	1	2	3	4	5	6	7	8	i =
		$-A_3$	A_2	$2A_7-2A_8$	$-A_6$	A_5	$-A_4$	A_4	1
			$-A_1$	$-A_6$	$2A_7$	A_4	$-2A_5$	$-A_5$	2
				$-A_5$	A_4	$2A_8$	$-A_6$	$-2A_6$	3
					$-A_3$	$-A_2$	A_1	$-A_1$	4
						$-A_1$	$2A_2$	A_2	5
							A_3	$2A_3$	6
								0	7

Table of Lie brackets $[M_i, M_j]$, $i<j$

j=	2	3	4	5	6	7	8	i =
	$-M_3$	M_2	$\frac{(p-q)}{\Delta}M_8 + \frac{\sqrt{3}(p+q)}{\Delta}M_7$	$-M_6$	M_5	$-\frac{\sqrt{3}(p+q)}{\Delta}M_4$	$\frac{(q-p)}{\Delta}M_4$	1
		$-M_1$	$-M_6$	$\frac{(2p+q)}{\Delta}M_8 + \sqrt{3}qM_7$	M_4	$\frac{-\sqrt{3}q}{\Delta}M_5$	$-\frac{(2p+q)}{\Delta}M_5$	2
			$-M_5$	M_4	$\frac{(2q+p)}{\Delta}M_8 - \sqrt{3}p\ M_7$	$\frac{-\sqrt{3}p}{\Delta}M_6$	$-\frac{(2q+p)}{\Delta}M_6$	3
				$-M_3$	$-M_2$	$\frac{\sqrt{3}(p+q)}{\Delta}M_1$	$\frac{(p-q)}{\Delta}M_1$	4
					$-M_1$	$\frac{\sqrt{3}q}{\Delta}M_2$	$\frac{(2p+q)}{\Delta}M_2$	5
						$\frac{-\sqrt{3}p}{\Delta}M_3$	$\frac{(p+2q)}{\Delta}M_3$	6
							0	7

We now endow $M(p,q)$ with the metric $\langle\ ,\ \rangle_t$ of Section 3, $-1<t<0$, and apply 3.(6) to compute the sectional curvature $K(X,Y)$ for a plane spanned by vectors $X,Y \in \mathfrak{u}^{\perp}$, orthonormal with respect to $\langle\ ,\ \rangle_t$.

We can express X,Y as $X = \sum_{i=1}^{7} a_i M_i'$, $Y = \sum_{i=1}^{7} b_i M_i'$,

where $M_i' = M_i \quad (i=2,3,5,6)$,

$\sqrt{1+t}\, M_i' = M_i \quad (i=1,4,7)$,

$\sum_{i=1}^{7} a_i b_i = 0,\quad \sum_{i=1}^{7} a_i^2 = \sum_{i=1}^{7} b_i^2 = 1.$

$\{M_i'\}$ is orthonormal with respect to the metric $\langle\ ,\ \rangle_t$.

We choose once and for all $t=-1/2$, which seems to be optimal for all pinching estimates. Then we obtain

<u>Proposition 4.2.</u>

$$\begin{aligned} K(X,Y) = {} & 5/8\langle [X_1,Y_1]_1,[X_1,Y_1]_1 \rangle \\ & +1/16\langle [X,Y]_2,[X,Y]_2 \rangle \\ & + \langle [X,Y]_{\mathfrak{u}},[X,Y]_{\mathfrak{u}}\rangle \\ & -3/4\langle [X_1,Y_1]_{\mathfrak{u}},[X,Y]_{\mathfrak{u}} \rangle \\ & +1/4\langle [X_1,Y_1]_{\mathfrak{u}},[X_1,Y_1]_{\mathfrak{u}}\rangle, \end{aligned}$$

where $Z = Z_1+Z_2+Z_{\mathfrak{u}}$, $Z_1\in V_1$, $Z_2\in V_2$, $Z \in \mathfrak{u}$.

We can use the table on page 26 to express $K(X,Y)$ as follows:

<u>Corollary 4.3</u>.

$$K(X,Y)=1/2\{-a_2b_3+a_3b_2-a_5b_6+a_6b_5+\sqrt{3}(p+q)/\Delta(a_4b_7-a_7b_4)\}^2$$

$$+1/2\{a_2b_6-a_6b_2+a_3b_5-a_5b_3+\sqrt{3}(p+q)/\Delta(-a_1b_7+a_7b_1)\}^2$$

$$+1/2\{\sqrt{3}q/\Delta(a_2b_5-a_5b_2)-\sqrt{3}p/\Delta(a_3b_6-a_6b_3)+\sqrt{3}(p+q)/\Delta(a_1b_4-a_4b_1)\}^2$$

$$+1/8\{-a_2b_3+a_3b_2-a_5b_6+a_6b_5\}^2$$

$$+1/8\{a_2b_6-a_6b_2+a_3b_5-a_5b_3\}^2$$

$$+1/8\{\sqrt{3}q/\Delta(a_2b_5-a_5b_2)-\sqrt{3}p/\Delta(a_3b_6-a_6b_3)\}^2$$

$$+1/8\{a_1b_3-a_3b_1-a_4b_6+a_6b_4+\sqrt{3}q/\Delta(a_5b_7-a_7b_5)\}^2$$

$$+1/8\{-a_1b_2+a_2b_1-a_4b_5+a_5b_4-\sqrt{3}p/\Delta(a_6b_7-a_7b_6)\}^2$$

$$+1/8\{a_1b_6-a_6b_1-a_3b_4+a_4b_3-\sqrt{3}q/\Delta(a_2b_7-a_7b_2)\}^2$$

$$+1/8\{-a_1b_5+a_5b_1-a_2b_4+a_4b_2+\sqrt{3}p/\Delta(a_3b_7-a_7b_3)\}^2$$

$$+2(p-q)^2/\Delta^2(a_1b_4-a_4b_1)^2+5(p-q)/2\Delta^2(a_1b_4-a_4b_1)$$

$$\times\{(2p+q)(a_2b_5-a_5b_2)+(p+2q)(a_3b_6-a_6b_3)\}$$

$$+1/\Delta^2\{(2p+q)(a_2b_5-a_5b_2)+(p+2q)(a_3b_6-a_6b_3)\}^2.$$

5. Proof of the Main Theorems.

Proposition 5.1.

Suppose the pair of integers p', q' is a multiple of the pair p, q by a positive integer d (i.e. $p' = pd, q' = qd$). Then $M(p', q')$ and $M(p, q)$ have the same pinching.

Proof: The coefficients of the curvature formula in 4.3 are either constant or only depend on the ratio of p and q. This proves the proposition. In fact, $M(p, q)$ is a d-sheeted riemannian covering of $M(p', q')$.

Proposition 5.2.

Suppose p, q are relatively prime. Let $\delta(p, q)$ be the pinching of $M(p, q)$. Then there exists an infinite family of simply connected spaces $\{M(p_i, q_i)\}$ such that the $M(p_i, q_i)$ are mutually topologically distinct, and their pinching approaches $\delta(p, q)$ as $i \to \infty$.

Proof: Since p, q are relatively prime, we can find positive integers m, n such that $|mp - nq| = 1$. Then, $p_i = n + ip$, $q_i = m + iq$ are relatively prime for all i, and $p_i/q_i \to p/q$ as $i \to \infty$. From Proposition 4.1 and Corollary 4.3 (using the compactness of the set of tangential 2-planes) it follows that the sequence $M(p_i, q_i)$ is as desired.

It seems to be difficult to determine the pinching $\delta(p,q)$ of $M(p,q)$ explicitly, in general, However, we can compute $\delta=\delta(1,1)$, which we can prove to be the maximal value. Therefore, Proposition 5.2 contains as an important special case:

Corollary 5.3.

$\{M_i = M(i+1,i)\}$ is a sequence of strictly positively curved simply connected spaces such that each M_i has a distinct topological type and its pinching δ_i approaches the pinching δ of $M(1,1)$ as $i \to \infty$.

Another consequence is:

Corollary 5.4.

There exists a positive number such that, in dimension 7, we can find infinitely many simply connected topologically different compact manifolds whose pinchings are all greater than that number.

It is interesting to know, what is the supremum α of numbers satisfying 5.4? This is still an open question. However, by the Sphere Theorem, we have $\alpha \leq 1/4$. From Corollary 5.3, it follows that $\alpha \geq \delta$, and $\delta = 16/29\cdot 37$ by Proposition 5.8. We restate these results as:

Theorem 5.5.

Let α be the supremum of numbers satisfying 5.4. Then

$$16/29\cdot 37 \leq \alpha \leq 1/4.$$

Remark 5.6. The pinching of the spaces $M(p,q)$ is not bounded away from zero uniformly in p,q. In the sequence $\{M_i = M(i,1)\}$ of homogeneous spaces, the pinching approaches zero as $i \to \infty$.

Proof: Let K_i denote the sectional curvature function on $M(i,1)$. Choose $X = M_2'$, $Y = M_5'$, $X' = M_5'$, $Y' = M_7'$ (see Section 4). Then by 4.3,

$$K_i(X',Y') \geq (2i+1)^2/(i^2+i+1) \geq 3,$$
$$K_i(X,Y) = 3/\{8(i^2+i+1)\}.$$

Hence, the pinching of M_i will be less than $K_i(X,Y)/K(X',Y') \leq 1/\{8(i^2+i+1)\}$.

If we consider all compact strictly positively curved manifolds with normalized sectional curvature $0 < K \leq 1$, then one knows by a theorem of Klingenberg that the injectivity radius of the exponential map is always greater than $\pi/2$, in any <u>even</u> dimension. However,

Theorem 5.7.

The injectivity radius (and therefore the length of the shortest nontrivial periodic geodesic) on 7-dimensional simply connected compact manifolds with (normalized) positive curvature $0<\varkappa\leq K\leq 1$ is not uniformly bounded away from zero for fixed $\varkappa$, if $\varkappa<\delta=16/29\cdot 37$.

Proof: It might be possible to give a fairly simple direct geometric argument, based on Corollary 5.3. But we prefer to apply the finiteness results of Cheeger [5], and Weinstein [14]. Since K and the volume are bounded, the existence of an a-priori lower bound for the injectivity radius would imply that there are only finitely many topological types possible for the manifolds in question. This would contradict 5.5, and the theorem is proved.

At last we compute the pinching of $M(1,1)$.

<u>Proposition 5.8</u>.

The pinching of $M(1,1)$ is $\delta=16/29\cdot 37$.

It is very surprising that this is exactly the same number as the pinching of Berger's 13-dimensional exceptional normal homogeneous space, as determined by Heintze [8].

Proof: In order to calculate the pinching, we have to determine the maximum and minimum of the sectional curvature of $M(1,1)$. Since $p=q=1$, the curvature formula 4.3 reduces to

$$(1)\ K(X,Y) = 1/2\sum_{i=1}^{3}(A_i+2B_i)^2 + 1/8\sum_{i=1}^{3}A_i^2+3C^2+1/8\sum_{i=4}^{7}A_i^2\ ,$$

where

$$A_1 = (-a_2b_3+a_3b_2-a_5b_6+a_6b_5),$$
$$A_2 = (\ a_2b_6-a_6b_2+a_3b_5-a_5b_3),$$
$$A_3 = (\ a_2b_5-a_5b_2-a_3b_6+a_6b_3),$$
$$A_4 = (\ a_1b_3-a_3b_1-a_4b_6+a_6b_4+a_5b_7-a_7b_5),$$
$$A_5 = (-a_1b_2+a_2b_1-a_4b_5+a_5b_4-a_6b_7+a_7b_6),$$
$$A_6 = (\ a_1b_6-a_6b_1-a_2b_7+a_7b_2-a_3b_4+a_4b_3),$$
$$A_7 = (-a_1b_5+a_5b_1-a_2b_4+a_4b_2+a_3b_7-a_7b_3),$$
$$B_1 = (\ a_4b_7-a_7b_4),$$
$$B_2 = (-a_1b_7+a_7b_1),$$
$$B_3 = (\ a_1b_4-a_4b_1),$$
$$C\ = (\ a_2b_5-a_5b_2+a_3b_6-a_6b_3).$$

Since X,Y are orthonormal, i.e.

$$\sum_{i=1}^{7}a_ib_i = 0,\quad \sum_{i=1}^{7}a_i^2 = \sum_{i=1}^{7}b_i^2 = 1,$$

we have the following inequalities,

$$\sum_{i=1}^{3}(A_i+B_i)^2+\sum_{i=4}^{7}A_i^2 = 1 + 4\sum_{i=1}^{3}A_iB_i \geq 0,$$
$$\sum_{i=1}^{3}A_iB_i \leq 1/4.$$

First we show that the minimum of the sectional curvature is not less than 2/37.

$$K(X,Y) = 1/2\sum_{i=1}^{3}(A_i+2B_i)^2 + 1/8\sum_{i=1}^{3}A_i^2 + 1/8\sum_{i=4}^{7}A_i^2 + 3C^2$$

$$\geq 1/2\sum_{i=1}^{3}(A_i+2B_i)^2 + 1/8\sum_{i=1}^{3}A_i^2 + 2/37\sum_{i=4}^{7}A_i^2$$

$$= 2/37\sum_{i=1}^{3}(A_i+B_i)^2 + 2/37\sum_{i=4}^{7}A_i^2 + 1/2\sum_{i=1}^{3}(A_i+2B_i)^2$$

$$+1/8\sum_{i=1}^{3}A_i^2 - 2/37\sum_{i=1}^{3}(A_i+B_i)^2$$

$$= 2/37(1+4\sum_{i=1}^{3}A_iB_i) + 169/8\cdot 37\sum_{i=1}^{3}A_i^2 + 70/37\sum_{i=1}^{3}A_iB_i$$

$$+72/37\sum_{i=1}^{3}B_i^2$$

$$= 2/37 + 2/37\sum_{i=1}^{3}(13/4A_i+6B_i)^2 \geq 2/37.$$

IF we choose $X = \sqrt{24/37}\, M_2' + \sqrt{13/37}\, M_4'$,

$Y = \sqrt{24/37}\, M_3' + \sqrt{13/37}\, M_7'$,

then by formula (1),

$$K(X,Y) = 2/37.$$

Hence we know that 2/37 is the minimum of K(X,Y).

In order to find the maximum of K(X,Y), we need the following inequalities:

$$\sum_{i=1}^{3}A_i^2 = \sum_j a_j^2(B'^2-b_j^2) - 2\sum_{k\neq l}a_kb_ka_lb_l$$

(2) $$= A'^2B'^2 - (\sum_k a_kb_k)^2$$

$$\leq A'^2B'^2 \leq 1 \qquad (j,k,l = 2,3,5,6),$$

where

$$A' = \sqrt{a_2^2+a_3^2+a_5^2+a_6^2},\quad B' = \sqrt{b_2^2+b_3^2+b_5^2+b_6^2},$$

$$A = \sqrt{a_1^2+a_4^2+a_7^2},\qquad B = \sqrt{b_1^2+b_4^2+b_7^2},$$

(3) $\quad A_i \le A'B' \quad (i=2,3,5,6).$

(4) $\quad C \le A'B'.$

$$\sum_{i=1}^{3} A_iB_i \le A_1\sqrt{(a_4^2+a_7^2)(b_4^2+b_7^2)} + A_2\sqrt{(a_1^2+a_7^2)(b_1^2+b_7^2)}$$
$$+ A_3\sqrt{(a_1^2+a_4^2)(b_1^2+b_4^2)}$$

(5)
$$\le A'B'\{\sqrt{(a_4^2+a_7^2)(b_4^2+b_7^2)} + \sqrt{(a_1^2+a_7^2)(b_1^2+b_7^2)} + \sqrt{(a_1^2+a_4^2)(b_1^2+b_4^2)}$$
$$\le 1/2A'B'(a_4^2+a_7^2+b_4^2+b_7^2+a_1^2+a_7^2+b_1^2+b_7^2+a_1^2+a_4^2+b_1^2+b_4^2)$$
$$= A'B'(a_1^2+a_4^2+a_7^2+b_1^2+b_4^2+b_7^2)$$
$$= A'B'(2-A'^2-B'^2).$$

(6)
$$\sum_{i=1}^{3} B_i^2 = A^2B^2 - (\sum_j a_jb_j)^2 \quad (j=1,4,7)$$
$$\le A^2B^2 = (1-A'^2-B'^2+A'^2B'^2).$$

Using (2) through (6), we have in (1),

$$K(X,Y) = 1/2\sum_{i=1}^{3}(A_i+2B_i)^2 + 1/8\sum_{i=1}^{7}A_i^2 + 3C^2$$
$$\le 1/2\sum_{i=1}^{3}(A_i+B_i)^2 + \sum_{i=1}^{3}A_iB_i + 3/2\sum_{i=1}^{3}B_i^2 + 1/8\sum_{i=1}^{7}A_i^2 + 3C^2$$
$$\le 1/2\sum_{i=1}^{3}(A_i+B_i)^2 + 1/2\sum_{i=4}^{7}A_i^2 + \sum_{i=1}^{3}A_iB_i + 3/2\sum_{i=1}^{3}B_i^2$$
$$+ 1/8\sum_{i=1}^{3}A_i^2 + 3C^2$$
$$\le 1/2(1+4\sum_{i=1}^{3}A_iB_i) + \sum_{i=1}^{3}A_iB_i + 3/2\sum_{i=1}^{3}B_i^2 + 1/8\sum_{i=1}^{3}A_i^2 + 3C^2$$
$$= 1/2 + 3\sum_{i=1}^{3}A_iB_i + 3/2\sum_{i=1}^{3}B_i^2 + 1/8\sum_{i=1}^{3}A_i^2 + 3C^2.$$

Furthermore,

$$\sum_{i=1}^{3} A_i^2 \leq 1, \quad C^2 \leq (A'B')^2,$$

$$\sum_{i=1}^{3} A_i B_i \leq A'B'(2-A'^2-B'^2),$$

$$\sum_{i=1}^{3} B_i^2 \leq (1-A'^2-B'^2+A'^2B'^2).$$

Therefore,

(7) $K(X,Y) \leq 5/8+3/2(1-A'^2-B'^2+A'^2B'^2)+3A'B'(2-A'^2-B'^2)+3(A'^2B'^2)$,

where

$$0 \leq A', B' \leq 1.$$

Now consider the function

$$f(x,y) = 1-x^2-y^2+x^2y^2 + 2xy(2-x^2-y^2) + 2x^2y^2,$$

defined on $\Omega = \{(x,y) \mid 0 \leq x,y \leq 1\}$.

The maximum of f is assumed at the point $(\bar{x},\bar{y})$

where $\partial f/\partial x = \partial f/\partial y = 0$,

or along the boundary of Ω.

$$\partial f/\partial x = 6xy^2+4y-2x(1+2xy)-(x^2+y^2)2y,$$

$$\partial f/\partial y = 6x^2y+4x-2y(1+2xy)-(x^2+y^2)2x.$$

The solutions of $\partial f/\partial x = \partial f/\partial y = 0$ in Ω,

are $\bar{x}=\bar{y}=0$, or $\bar{x}=\bar{y}=1$. Therefore one has to look at the boundary of Ω. It turns out easily that, when $\bar{x}=\bar{y}=1$, $f(\bar{x},\bar{y})$ reaches its maximum $f(1,1)=2$.

Hence we know in (7), $K(X,Y) \leq 5/8+3/2\{f(x,y)\}$ $\leq 5/8+(3/2)2 = 29/8$.

If we choose $X=M_2'$, $Y=M_5'$, then $K(X,Y) = 29/8$.
This implies that the maximum of $K(X,Y)$ is $29/8$. So we have finally determined the pinching of $M(1,1)$ to be

$$\delta = \frac{2/37}{29/8} = \frac{16}{29 \cdot 37} .$$

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