DIAMETERS OF COMPACT RIEMANN SURFACES AND AUTOMORPHIC FORMS

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Fwu-Ranq Chang

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We, the dissertation committee for the above candidate for the Ph. D. degree, hereby recommend acceptance of the dissertation.

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Abstract of the Dissertation

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Let G be a finitely generated Fuchsian group of the first kind without parabolic elements so that $S = U/G$ is a compact Riemann surface furnished with the Poincaré metric. Let d be the diameter of S and let m be the length of the shortest closed geodesic on S. We have the following inequalities:

$$\sinh(m/4)d \leq (1/2)\text{area}(U/G) \leq m\sinh(d).$$

We also realize the diameter of S in some Poincaré polygon as the distance from the origin to the farthest vertex.

Now let G be a finitely generated Fuchsian group of the second kind with no parabolic elements. Then $S = U/G$ is a compact Riemann surface of genus g with n disjoint closed discs removed. Let $\{T_1, T_2, \ldots, T_n\}$ be the hyperbolic transformations corresponding to the closed discs, where

each T_i is conjugate to $z \to \lambda_i z$, $\lambda_i > 1$. Under projection, the axes of the T_i's bound m annuli; their complement S^* is relatively compact. Let d^* be the diameter of S^* measured in the Poincaré metric. Then for any $q > 1$ and positive integer p, the norm of the inclusion map

$$L : A_q^p(U,G) \to A_q^\infty(U,G)$$

has an upper bound given by

$$\max_{i=1,2,\ldots,n} \{ c_1 (3 + \frac{\log 3}{\log \lambda_i})^{1/p},\ c_2 \frac{(1+\tanh(d^*+b))^{q+1/p}}{(1-\tanh(d^*+b))^{1/p}} \},$$

where c_1, c_2 and b are constants. We then prove the inclusion of $A_q^p(U,G)$ in $A_q^\infty(U,G)$ for any finitely generated Fuchsian group G. When $p = 1$, this is the well-known $A_q(U,G) \subset B_q(U,G)$ conjecture.

To my parents

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CHAPTER 1

Introduction

Poincaré's first essay on automorphic function, published in 1880, began a new era in function theory. He not only generalized the concepts of circular, elliptic, hyperbolic and certain other functions of elementary analysis, but he also provided a powerful tool for studying Fuchsian groups. Later he extended the concept to automorphic forms of weight $(-2q)$ via the introduction of Poincaré theta series. With the proof of the uniformization theorem, given by Koebe in 1912, the classical theory of automorphic functions came to a close.

In the early 1920's, Hecke defined the Eisenstein series such that the resulting automorphic form φ is a cusp form, i.e., φ vanishes at all parabolic cusps. Meanwhile Petersson, Hecke's student, was working on the correspondence between Riemann surface theory and automorphic function theory, and in 1939 he introduced the important Petersson scalar product which makes the set of cusp forms a Hilbert space. In 1960, Bers obtained the Banach spaces $A_q^p(U,G)$, $1 \leq p \leq \infty$, by applying the L^p-norm to the set of holomorphic automorphic forms of weight $(-2q)$ with respect to the group

G. Then he proved in 1965 ([3]) that $A_q^1(U,G) = A_q^\infty(U,G)$ provided that the group G is finitely generated of the first kind. They are, however, different when G is of the second kind. As a matter of fact that $A_q^1(U,G)$ is always separable, whereas $A_q^\infty(U,G)$ is not. It is natural to conjecture that $A_q^1(U,G) \subset A_q^\infty(U,G)$ for all Fuchsian groups G. Several authors had made contributions to this problem. In 1968, Drasin and Earle [4], among others, proved the conjecture for finitely generated Fuchsian groups of the second kind and, using the closed graph theorem, they also proved the continuity of the inclusion. In 1973-74, Lehner [13,14] proved the conjecture for certain infinitely generated Fuchsian groups with additional restrictions.

The major difficulty of this conjecture is that we did not have enough information about the behavior of automorphic forms near the holes. To attack this problem, it is important to investigate the norm of the inclusion when G is generated by a single hyperbolic transformation $z \to \lambda z$, $\lambda > 1$. As we shall see in this paper, the norm of the inclusion map, in this special case, is bounded both from above and from below by constant multiples of the reciprocal of the circumference of the hole, i.e., $1/\log\lambda$. Hence the norm will blow up when the size of the hole shrinks.

This suggests that the conjecture must be false and a Riemann surface with infinitely many holes decreasing in size should give us a counterexample. In May of 1974, Pommerenke [19] announced this result. In his counterexample, he first built a Fuchsian group G containing infinitely many such hyperbolic generators $\{z \to \lambda_i z, \lambda_i > 1, i = 1,2,\ldots\}$ with the property that $\lambda_i \to 1$ as $i \to \infty$. Then he used the technique of Bloch functions to show that there exists an integrable automorphic form which is not essentially bounded.

In estimating the norm of the inclusion, two geometric objects came to our attention: the circumferences of the holes and the "diameter" of Nielsen convex region. Mumford [18] discovered an inequality:

> Let d be the diameter of a compact Riemann surface, and let m be the length of a shortest simple closed geodesic on S, then we have
>
> $$md \leq 2\text{area}(S).$$

It is of interest to find out whether there is a lower bound for these two geometric terms.

In this dissertation, we first derive a lower bound and sharpen Mumford's upper bound. More precisely, we have

$$\sinh(m/4)d \leq (1/2)\mathrm{area}(S) \leq m\sinh(d).$$

We then discuss the general inclusion map $A_q^p(U,G) \subset A_q^\infty(U,G)$ for positive integer p and finitely generated Fuchsian group G. Our second result is an explicit bound for the norm of the inclusion in terms of the maximum of the "diameter" of the Nielsen convex region and the circumferences of the holes on S. As a direct consequence of this theorem, we present a new and simple proof for the $A_q^1(U,G) \subset A_q^\infty(U,G)$ conjecture when G is finitely generated.

In Chapter 2, we summarize some known facts about Fuchsian groups and Riemann surfaces that will be needed subsequently, and we fix our notations. In Section 1, we classify the Möbius transformations and define Fuchsian groups. In Section 2, we discuss Riemann surfaces and the corresponding fundamental domains. In Section 3, we introduce hyperbolic geometry and Dirichlet regions.

Chapter 3 is devoted to the development of the inequalities and a discussion on the geodesics of S.

Chapter 4 is on automorphic forms. Some well-known results in the spaces of automorphic forms are introduced in Section 1. In Section 2, we discuss the $A_q^p(U,G) \subset A_q^\infty(U,G)$ conjecture and state the main theorems. In Section 3, we prove the result for the two special cases: G is cyclic

(G is generated by a single transformation $z \to \lambda z$, $\lambda > 1$) and U/G is compact. Both cases are important in themselves. Moreover, they yield the decisive facts for the main theorems. In the last section, we present the proofs of the main theorems.

CHAPTER 2

Uniformizations

2.1. <u>Möbius transformations</u>.

We denote the extended complex plane $\mathbb{C} \cup \{\infty\}$ by $\hat{\mathbb{C}}$. We shall study the group $PSL(2;\mathbb{C})$ whose elements are 2x2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with complex entries and the normalizations $ad-bc=1$, where we identify A with $-A$. Each matrix A is called a <u>Möbius transformation</u>. Sometimes we also write this matrix as a rational function mapping $\hat{\mathbb{C}}$ onto itself;

$$A(z) = \frac{az+b}{cz+d}, \; ad-bc=1.$$

This mapping is one-to-one and directly conformal, it preserves circles (including straight lines) and it preserves the cross-ratio, i.e.,

$$(Az_1, Az_2, Az_3, Az_4) = (z_1, z_2, z_3, z_4),$$

where

$$(z_1, z_2, z_3, z_4) = \frac{z_1-z_3}{z_1-z_4}\,\frac{z_2-z_4}{z_2-z_3}\,.$$

One sees at once that every non-trivial element of $PSL(2;\mathbb{C})$ has at least one and at most two fixed points. This leads to the well-known classification of elements of $PSL(2;\mathbb{C})$ by the number of fixed points. If A has exactly one fixed point, then A is called <u>parabolic</u>, and A is conjugate to the translation $z \to z+1$. One can also obtain, by conjugation, a general form for a parabolic transformation A with fixed point $z_o \neq \infty$ as

$$\frac{1}{Az - z_o} = \frac{1}{z - z_o} + p, \quad p \neq 0.$$

In the case that A has two fixed points x_1, x_2, we can find some B in PSL(2;$\mathbb{C}$) with $Bx_1 = 0$ and $Bx_2 = \infty$. Thus BAB^{-1} has the form

$$BAB^{-1}(z) = \varkappa z, \quad z \neq 0.$$

Since we could have chosen B to take x_1 to ∞, and x_2 to 0, we may always assume $|\varkappa| \geq 1$. If $|\varkappa| = 1$, $\varkappa \neq 1$, then BAB^{-1} is a rotation, and A is called <u>elliptic</u>. If $|\varkappa| > 1$, then A is called <u>loxodromic</u>; in the special case that $\varkappa$ is real, $\varkappa > 1$, we say that A is <u>hyperbolic</u>.

We may also classify them by the well-defined

$$\mathrm{trace}^2 A = \mathrm{tr}^2 A = (a+d)^2, \text{ if } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If $A \neq$ id., then A is

elliptic	if and only if	$\mathrm{tr}^2 A \in [0,4)$,
parabolic	if and only if	$\mathrm{tr}^2 A = 4$,
hyperbolic	if and only if	$\mathrm{tr}^2 A > 4$,
loxodromic	otherwise.	

Let G be a subgroup of PSL(2;$\mathbb{C}$), then for any $z \in \hat{\mathbb{C}}$, we denote by G_z the stabilizer (or the stability subgroup) of z; that is,

$$G_z = \{ A \in G : Az = z \}.$$

We shall say that G is discontinuous at z, if

(i) G_z is finite,

(ii) there is a neighborhood V of z such that

$A(V) = V$, for all $A \in G_z$,

$A(V) \cap V = \emptyset$, for all $A \in G - G_z$.

The set of points in $\hat{\mathbb{C}}$ at which G acts discontinuously is called the regular set or the set of discontinuity, and is denoted by $\Omega(G)$, or simply Ω if there is no danger of confusion. Since Ω is open, we may call Ω the region of discontinuity. Each point in Ω is called an ordinary point. Obviously, Ω is invariant under G, i.e., $A(\Omega) = \Omega$, for all $A \in G$. A connected component of Ω which is invariant under G is called an invariant component. The group is discontinuous if $\Omega \neq \emptyset$. A discontinuous subgroup of PSL(2;$\mathbb{C}$) is called Kleinian. If G is discontinuous and $A \in$ PSL(2;$\mathbb{C}$), then AGA^{-1} is also discontinuous; we shall regard these groups as being the same. It is a basic result that a discontinuous group is countable (Ford [5]).

A point z in $\hat{\mathbb{C}}$ is called a limit point of G if there is a point x in Ω, and there is a sequence $\{A_n\}$ of distinct elements of G, so that $A_n(x) \to z$. The set of limit points, called the limit set, is denoted by $\Lambda(G)$, or simply Λ if there is no danger of confusion. It is known that $\Lambda \cap \Omega = \emptyset$ and $\Lambda \cup \Omega = \hat{\mathbb{C}}$. We recall that a set is perfect if it is closed and dense in itself; i.e., every point of the set is a limit point of other points of the set. We also recall that a perfect subset of Euclidean space can not be countable. It can be shown that if Λ contains more than two points, then Λ is a nowhere dense, perfect set. Hence Λ is uncountable. For the case Λ contains at most two points, G is called elementary. The elementary groups are completely classified. The reader may find the details in Ford [5].

A non-elementary Kleinian group G is called Fuchsian if all its loxodromic elements are hyperbolic. Then G leaves a disc or a half plane fixed ([11]), whose boundary is called the principal circle. One can achieve, by conjugation, that this be the unit disc U or the upper half plane H. Since a Möbius transformation A which maps the unit disc U onto itself has the form

$$A(z) = \frac{az + \bar{c}}{cz + \bar{a}}, \quad a\bar{a} - c\bar{c} = 1,$$

we may regard a Fuchsian group G as a subgroup of

$$SU(1,1) = \{ A \in PSL(2;\mathbb{C}) : A = \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix} \}.$$

In the case of the upper half plane H, all elements $A \in G$ have real coefficients. Thus we may also regard a Fuchsian group G as a discontinuous subgroup of $PSL(2;\mathbb{R})$.

We call a Fuchsian group <u>of the first kind</u>, if every point of the principal circle is a limit point; <u>of the second kind</u>, otherwise. The points of Λ are either fixed points of hyperbolic elements of G or accumulation points of such points ([11]). There are two invariant components for a Fuchsian group of the first kind; whereas there is only one for a Fuchsian group of the second kind.

2.2 <u>Riemann surfaces</u>.

A <u>Riemann surface</u> if a connected Hausdroff space S together with a <u>conformal structure</u> $\{U_\alpha, z_\alpha\}_{\alpha\in A}$. In this context $\{U_\alpha\}_{\alpha\in A}$ constitute an open cover of S and the mapping $z_\alpha : U_\alpha \to \mathbb{C}$ is a homeomorphism onto an open subset of the complex plane $\mathbb{C}$ such that the <u>transition functions</u>

$$f_{\alpha\beta} = z_\alpha \circ z_\beta^{-1} : z_\beta(U_\alpha \cap U_\beta) \to z_\alpha(U_\alpha \cap U_\beta)$$

are conformal (i.e., holomorphic with non-vanishing derivatives). Each z_α together with its domain U_α is called a <u>local parameter</u>, and U_α is called a <u>parametric disc</u>.

Classically, a compact Riemann surface is called <u>closed</u>; while a non-compact one is called <u>open</u>.

Let S be a Riemann surface, and let $\tilde{S}$ be its universal covering space. Again $\tilde{S}$ is a Riemann surface. Then we have the (cf. [21])

<u>Uniformization Theorem</u>. Any simply connected Riemann surface is conformally equivalent to one of the following:

(1) the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$,

(2) the complex plane $\mathbb{C}$,

(3) the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ or its conformal equivalent, the upper half plane $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

Let $\pi : \tilde{S} \to S$ be a holomorphic universal covering map, and let G be the covering group of π; that is, $G = \{$ conformal self-maps g of $\tilde{S} : \pi g = \pi\}$. Then G is discontinuous and fixed point free. Furthermore, $G \cong \pi_1(S)$, the fundamental group of S, and S is conformally equivalent to the orbit space, $\tilde{S}/G$, of $\tilde{S}$ under the action of G.

The uniformization theorem implies that any compact Riemann surface of genus zero is conformally equivalent to the sphere. Thus a Riemann surface with $\hat{\mathbb{C}}$ as its universal covering space is itself $\hat{\mathbb{C}}$. It can be shown that the only Riemann surfaces with $\mathbb{C}$ as their universal covering spaces are conformally equivalent to the once-punctured sphere, the doubly-punctured sphere, or a compact Riemann surface of genus 1. In all other cases, $\tilde{S} = U$ (or, equivalently, $\tilde{S} = H$). This is the case in which we are primarily interested.

Let G be a Fuchsian group. The projection $\pi : H \to H/G$ is locally one-to-one, except at points $z \in H$ with non-trivial stabilizer G_z. In this case G_z is a cyclic group of order k, the projection is k-to-one near z, and the image of z under the projection is called a <u>ramification point of order k</u>. We also call a point $z \in R$ a <u>ramification point of order ∞</u>, or simply a <u>puncture</u>, if z is a fixed point of some parabolic element in G. If G is finitely generated, then $S = H/G$ is a closed Riemann surface of genus g with n ramification points and m disjoint conformal discs removed (cf. Ahlfors [1]). These ramification points correspond to the images of the fixed points of the elliptic and parabolic elements of G under the projection. Let the conjugacy classes of these elliptic and parabolic elements be $C_1, C_2, \ldots, C_n$ and order them so that their corresponding orders ν_j satisfy $2 \le \nu_1 \le \nu_2 \le \ldots \le \nu_n \le \infty$. Each removed conformal disc, called a <u>hole</u>, is the image of a conjugacy class of <u>intervals of discontinuity</u>, by which we mean the intervals on $\mathbb{R}$ whose points are all ordinary points of G. G is said to be of <u>type</u> $(g;m;n)$ and of <u>signature</u> $(g;m;n;\nu_1,\ldots,\nu_n)$: We also say that G is of <u>finite type</u> if G is of type $(g;n)$, i.e., $m = 0$, there is no hole.

Conversely, let S be a Riemann surface of genus g, and let $\{p_1, p_2, .., p_n\}$ be a finite set of points, where there is a number ν_i, $2 \le \nu_i \le \infty$, assigned to each p_i. We assume

$$2g - 2 + \sum_{i=1}^{n} [1 - 1/\nu_i] > 0.$$

A classical theorem in Fuchsian group theory, due to Koebe and Wong, states that there exists a finitely generated Fuchsian group G of the first kind so that U/G has n ramification points q_k, each is of order ν_k, $k = 1,2,..,n$, and $U/G - \{q_1, q_2, .., q_n\}$ is conformally homeomorphic to $U/G - \{p_1, .., p_n\}$, i.e., G is of signature $(g;o;n;\nu_1,..,\nu_n)$ (Wong [22]). Under the above circumstances, we say that G uniformizes $\{S;p_1,..,p_n;\nu_1,..,\nu_n\}$. Thus the notation for, and the terms, type and signature are applied to Riemann surfaces as well.

A fundamental domain D for a given Fuchsian group G is an open connected subset of U such that

(1) $A(D) \cap D = \emptyset$, for all $A \in G$, $A \neq 1$,

(2) $\bigcup_{A \in G} \overline{A(D)} = U$.

It is known that for every Fuchsian group there are infinitely many fundamental domains. We shall construct a special one in the next section, and hence prove the existence of such domains.

2.3. Poincaré metric and Dirichlet regions

Hyperbolic geometry is obtained from plane Euclidean geometry by replacing the parallel postulate by the following axiom: Through a given point not on a given line in a plane, there passes more than one line which does not meet the given line. There are two models for this (Lobatchevski's) non-euclidean geometry: the unit disc model U and the upper half plane H. They are equivalent.

Let Q be the boundary of U, that is, the unit circle. A point in the hyperbolic plane is represented by a point in U, and a line is represented by the arc of a circle orthogonal to Q which lies in U, including the diameters of Q. Also, the angle measure is the same as Euclidean angle measure. We define the Poincaré metric $ds = \lambda(z)|dz|$, $z \in U$ by

$$\lambda(z) = 2\,(1 - |z|^2)^{-1}.$$

Since the Möbius transformations which map U onto itself are of the form

$$A = \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix}, \quad a\bar{a}-c\bar{c}=1,$$

we have

$$\lambda(Az)|A'(z)| = \lambda(z), \quad z \in U.$$

The Poincaré metric thus defined may be used to define a new metric on U, other than the Euclidean metric on U, that gives rise to the standard topology. If ℓ is a rectifiable curve in U, we define the <u>length</u> of ℓ by

$$|\ell| = \int_{\ell} \lambda(z)|dz|.$$

We then define the <u>distance</u> between any pair of points z_1 and z_2 as the length of the arc of a "line" passing through z_1 and z_2 with center on Q. Thus the distance between any two points z_1 and z_2 is the infimum of the lengths of curves joining z_1 to z_2. Since we can always transform z_1 to the origin of U by some transformation A of U, we can derive the distance formula for a pair of points z_1 and z_2 as

$$d(z_1,z_2) = \log \frac{1 + |(z_2-z_1)/(1-\bar{z}_1 z_2)|}{1 - |(z_2-z_1)/(1-\bar{z}_1 z_2)|}.$$

Since the cross-ratio is invariant under Möbius transformations, the same is true for the distance d. Hence the conformal self-maps of U are rigid motions for the hyperbolic plane. We also define the area of a Lebesgue measurable set ω in U, denoted as area(ω), to be the Lebesgue integral

$$(1/2) \iint_{\omega} \lambda(z)^2 |dz \wedge d\bar{z}| = \iint_{\omega} \lambda(z)^2 \, dxdy.$$

Using polar coordinates, if $z = \rho e^{i\theta}$, then the area can be expressed as

$$\iint_{\omega} \frac{\rho d\rho d\theta}{(1-\rho^2)^2}.$$

This leads us to the following

Lemma. The disc $B(0;R)$ centered at the origin with euclidean radius R is of area $\pi R^2/(1-R^2)$.

Proof.

$$\text{area}(B(0;R)) = \int_0^R \int_0^{2\pi} \frac{\rho d\rho d\theta}{(1-\rho^2)^2} = \frac{\pi R^2}{1-R^2}.$$

Now let us turn to the second model: the upper half plane H. The hyperbolic lines are represented by the arcs of circles orthogonal to the real axis $\mathbb{R}$, including the straight lines parallel to the imaginary axis, and the Euclidean angle is retained. Also note that the group of conformal self-maps is known to be $PSL(2;\mathbb{R})$. We define the Poincaré metric $ds = \lambda(z)|dz|$, $z = x + iy \in H$ by

$$\lambda(z) = 2(z-\bar{z})^{-1} = y^{-1}.$$

Similarly, we have the length of a rectifiable curve ℓ as

$$|\ell| = \int_{\ell} \lambda(z)|dz| = \int_{\ell} |dz|/y.$$

and the area of a Lebesgue measurable set ω as

$$\text{area}(\omega) = \iint_{\omega} y^{-2}\, dxdy.$$

A remark should be made is that the Poincaré metric thus defined gives a constant Gaussian curvature $K \equiv -1$. Recall that K is defined by

$$K = -\frac{\Delta(\log \lambda)}{\lambda^2},$$

where Δ is the Laplace operator, i.e., $\Delta u = u_{xx}+u_{yy}$. Another remark is that the Poincaré metric is the unique complete conformal Riemannian metric of curvature $\equiv -1$.

We shall now construct a fundamental region by means of hyperbolic geometry. The hyperbolic concepts will be denoted by prefixing H: H-line, H-distance, etc.

Let G be a finitely generated Fuchsian group. Let $z_o \in U$ be a non-fixed point of G. We enumerate the elements of G by $\{I=A_o, A_1, A_2, \ldots\}$. The images $z_i = A_i z_o$ are all distinct. Denote by B_i the perpendicular bisector of the H-segment $z_o z_1$. The line B_i divides U into two regions; the one which contains z_o we call L_i. $w \in L_i$ if and only if $d(w, z_o) < d(w, z_i)$, where $d(.,.)$ denotes H-distance. Define

$$D = \text{Int}\left(\bigcap_{i=1}^{\infty} L_i\right).$$

D is called the Dirichlet region (or the Poincaré polygon, in some literature) of G with center z_o. We may characterize D by saying that D consists of those points of U that are strictly nearer z_o than to any translate of z_o, i.e.,

$$D = \{ z : d(z,z_o) < d(z,z_j),\ j>0 \}.$$

By construction, D is open, connected and convex.

Let us now examine the boundary points of D. Suppose z lies on a single bisector B_i. One notices that there is no other bisectors in some neighborhood of z. It follows that there is an arc of B_i all of whose points are boundary points of D. Let s be the largest such arc (it may be the whole B_i) and call it a side of D. The endpoints of a side, are points lying on two or more sides, are called vertices of D. If a side terminates in a point p of Q, there may be another side of D, issuing from p, but there connot be more than one, because of the convexity of D. Suppose there is no side of D in the neighborhood of p except s. Then there is a maximal arc f of Q beginning at p which forms part of the boundary of D; f is called a free side. The only limit point that could possibly lie on the boundary of D are parabolic fixed points. Thus the boundary of D consists of sides, free sides and parabolic fixed points.

The Dirichlet region thus constructed is a fundamental domain ([11]). Moreover,

Theorem 1 (Heins [6]). Let G be a finitely generated Fuchsian group. Then any convex fundamental domain is finite sided. In particular, any Dirichlet region is of finite sides.

Conversely, if a Fuchsian group G has a fundamental domain with a finite number of sides, then G is finitely generated (Ford [5]). As a matter of fact, G can be read off through D.

Given a Fuchsian group G with signature $(g;m;n;\nu_1,\ldots,\nu_n)$, there exists a fundamental domain D bounded by $4g+2m+2n$ Jordan curves in U and m boundary arcs on Q with the following properties (Keen [9]). Suppose the sides of D are suitably labelled in order:

$$a_1,b_1,a_1',b_1',\ldots,a_g,b_g,a_g',b_g',c_1,c_1',\ldots,c_n,c_n',$$

$$d_1,e_1,d_1',\ldots,d_m,e_m,d_m',$$

there exist hyperbolic elements A_i, B_i and D_j in G, $i = 1,2,\ldots,g$, $j = 1,2,\ldots,m$, such that $A_i(a_i) = -a_i'$, $B_i(b_i) = -b_i'$ and $D_j(d_j) = -d_j'$, and elliptic elements $C_k \in G$ of order ν_k (parabolic if $\nu_k = \infty$), $k = 1,2,\ldots,n$, such that $C_k(c_k) = -c_k$.

These elements satisfy the relation

$$(*) \qquad \prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1} \prod_{j=1}^{n} C_j \prod_{k=1}^{m} D_k = 1.$$

By a <u>standard set of generators</u> we mean the set $\{A_1, B_1, \ldots A_g, B_g, C_1, \ldots, C_n, D_1, \ldots, D_m\}$ with the relation (*) and the relations: $C_k^{\nu_k} = 1$, and it generates G.

We remark that since conformal mappings are rigid motions of the hyperbolic plane, we can always transform the center z_o of D to the origin of U without changing any metric property. The advantage is that the distance from any point z on S to z_o can be realized as the H-length of the ray emanating from the origin to z. We shall in general assume this to be done.

The following is a standard theorem in Fuchsian group theory. For proof the reader is referred to Kra [10].

<u>Theorem 2</u>. Let G be a Fuchsian group acting on U and let D be a Dirichlet region as defined above, then the following conditions are equivalent:

(a) $\text{area}(D) < \infty$,

(b) $\text{area}(\tilde{D}) < \infty$, where $\tilde{D}$ is any fundamental domain for G in U, with $\text{meas}(\partial\tilde{D}) = 0$,

(c) D has a finite number of sides and no free sides,

(d) G is finitely generated of the first kind,

(e) G is of finite type.

Since area(D) is invariant for all fundamental domains of G, we may define the hyperbolic area of a Riemann surface $S = U/G$ as

$$\text{area}(U/G) = \text{area}(S) = \text{area}(D),$$

for any fundamental domain D of G. By Theorem 2, area(S) is finite if and only if G is of finite type.

There are two important consequences of the well-known Gauss-Bonnet theorem which we shall need later. The reader may refer to Kra [10] for the proofs.

<u>Theorem 3</u>. Let G be a finitely generated Fuchsian group with signature $(g;n;\nu_1,..,\nu_n)$. Then

$$\text{area}(U/G) = 2\pi\,\{2g - 2 + \sum_{j=1}^{n}(1 - 1/\nu_j)\}.$$

<u>Corollary</u>. For a Riemann surface S without ramification points, we have

$$\text{area}(S) = 4\pi(g-1).$$

<u>Theorem 4.</u> The hyperbolic area of a hyperbolic triangle in H with angles θ_1, θ_2 and θ_3 is finite and equal to

$$\pi - (\ \theta_1 + \theta_2 + \theta_3\ ).$$

CHAPTER 3

The Diameters of Compact Riemann Surfaces

In this chapter, we consider a finitely generated Fuchsian group of the first kind containing no parabolic elements for which the orbit space $S = U/G$ is a compact Riemann surface of genus g. This in general includes groups with elliptic elements, unless the contrary is specifically stated.

3.1. Geometric Preliminaries.

Here we summarize some basic definitions and results. For a detailed exposition, please refer to Hicks [7].

Recall that the Poincaré metric on H, $ds = \lambda(z)|dz| = (\operatorname{Im} z)^{-1}|dz|$, is the unique Riemannian metric with constant curvature -1. The Poincaré metric on S is that induced from the Poincaré metric on H by the group G, and is still written as $\lambda(z)$.

Let ℓ be a rectifiable curve on S. We define the length of ℓ, written as $|\ell|$, to be

$$|\ell| = \int_{\ell} \lambda(z)|dz| .$$

Then the distance between any two points x and y on S is defined by

$$d(x,y) = \inf \{ \ |\ell| \ : \ell \text{ is a rectifiable curve joining } x \text{ to } y\}.$$

It is well-known in Riemann surface theory that there is a curve ℓ realizing that distance. Such a curve is called a minimal geodesic. It can be shown that it is in fact a geodesic in the differential geometry sense. However, it is not necessarily unique. We shall prove later in this section that there are only finitely many of them. Another remark that should be made is that there may be other non-minimal geodesics from x to y.

The distance function $d : S \times S \longrightarrow R^{+} \cup \{0\}$, is clearly continuous. If S is compact, d is bounded from above. It is then natural to define the diameter of S, written as diam(S),

$$\mathrm{diam}(S) = \sup \{ \ d(x,y) \ : (x,y) \in S \times S \ \}.$$

This is well-defined. By compactness, there exists a pair of points that realize this diameter.

On S, there are many closed curves which are not homotopically trivial. In each free homotopy class of curves through a given point, there is a unique minimal geodesic. A simple closed minimal geodesic on S which gives the shortest length will play an important role in our theory.

We conclude this section by the following result.

Theorem 1. Given any two points x and y on S, there are at most finitely many minimal geodesics joining x to y.

Proof. Let $\tilde{x}$ be a preimage of x of the projection map $\pi : U \longrightarrow U/G = S$. Fix $\tilde{x}$. Then every minimal geodesic from x to y will be lifted uniquely to a minimal geodesic from $\tilde{x}$ to some $\tilde{y}$ in U, where $\tilde{y}$ is a preimage of y with $d(x,y) = d(\tilde{x},\tilde{y}) = c$. The hyperbolic disc $B(\tilde{x},c)$ is relatively compact in U. Since in any compact subset of U there are only finitely many points equivalent to y ([11]), there are only finitely many G-equivalent points of y in $\overline{B(\tilde{x};c)}$. This proves the finiteness.

Remark. Let $S = U/G$ be a compact Riemann surface with ramification points $\{ p_1, p_2, \ldots, p_n \}$. Let D_x be a Dirichlet region with center at $x = 0$, where $x \neq p_i$, $i = 1, 2, \ldots, n$. Denote the radius of D_x by $R(x)$, which is defined to be the distance between the point $x = 0$ and the farthest vertex of D_x. By definition,

$$R(x) \leq \operatorname{diam}(U/G),$$

for all $x \in U/G - \{p_1, \ldots, p_n\}$. It follows that

$$\sup \{ R(x) : x \in U/G - \{p_1, \ldots, p_n\} \} \leq \operatorname{diam}(U/G).$$

This inequality becomes equality provided that the diameter of $S = U/G$ is not realized by a minimal geodesic δ whose endpoints are both ramified. Let x_o be the non-fixed endpoint of δ. Form the Dirichlet region D_{x_o}. Then the other endpoint must be the farthest vertex of D_{x_o}, by the definition of the diameter. This distance is exactly $R(x_o)$. Thus,

$$\mathrm{diam}(U/G) = \sup\{R(x)\colon x \in U/G-\{p_1,..,p_n\}\}.$$

In other words, the diameter of S can be realized as the length of the hyperbolic line segment emanating from the origin of U to the farthest vertex in some Dirichlet region. In the case that the diameter of S is realized by a minimal geodesic whose two endpoints are both fixed points of G, the result is still known.

3.2. The inequalities.

In [18], Mumford proved a general compactness theorem and obtained, as an application, that for any $g \geq 2$ and constant $c > 0$, the set of compact Riemann surface of genus g all of whose closed geodesics, measured in the Poincaré metric, have length $\geq c$, is itself compact. Later Bers [2] proved that the theorem remains valid for Fuchsian groups containing elliptic and parabolic elements. Along the line of Mumford's proof, he derived the following inequality: Let S be a compact Riemann surface of genus g without ramification points. Let d be the diameter of S and let m be the length of a shortest closed (non-trivial) geodesic on S. Then we have

$$md \leq 2\mathrm{area}(S).$$

We shall sharpen this inequality and derive another one. Our main result is

Theorem 2. Let G be a finitely generated Fuchsian group of the first kind without parabolic elements. Then S = H/G is a compact Riemann surface of genus g together with the Poincaré metric. Let d be the diameter of S and let m be the length of a shortest simple closed geodesic on S. We have the following inequalities:

$$\sinh(m/4)d \leq (1/2)\mathrm{area}(U/G) \leq m\sinh(d).$$

As a direct consequence of Theorem 2 and Mumford's compactness theorem, we have

Corollary 1. For any genus $g \geq 2$ and constant $c > 0$, the set of all compact Riemann surfaces of genus g with diameter, measured in the Poincaré metric, having length $\leq c$, is itself compact.

Corollary 2. For any compact Riemann surface of genus $g \geq 2$ without ramification points, we have

$$\sinh(m/4)d \leq 2\pi(g-1) \leq m\sinh(d).$$

We start by establishing the following lemmas. Let x and y be any two points on S with $d(x,y) = r$. Let τ be a minimal geodesic on S realizing the distance between x and y. Construct a belt B around τ of radius s as the union of all geodesics on S perpendicular to τ, and each is of length s; in addition, we also assume that no two such geodesics will meet. We shall give B a geometric interpretation on the upper half plane H.

We may assume that τ is lifted to the imaginary axis on H so that τ is the segment $[i, ie^r]$. Then the geodesics perpendicular to τ are those circular arcs on H with euclidean center at the origin. We then observe that

Lemma 1. The ray $\theta = \theta_o$ in H is of distance $\delta = \log|\csc\theta_o + \cot\theta_o|$ to the imaginary axis I. Conversely, if a point $z = \rho e^{i\theta}$ in H has distance δ to I, then $\arg z = \theta_o = \csc^{-1}(\cosh(\delta))$.

Proof. Let $z = \rho e^{i\theta_o}$ be a point on the ray $\theta = \theta_o$. Then

$$d(z,I) = \int_{\theta_o}^{\pi/2} \frac{\rho d\theta}{\rho\sin\theta} = \int_{\theta_o}^{\pi/2} \csc\theta \, d\theta = \log|\csc\theta_o + \cot\theta_o|.$$

Conversely, if $\delta = \log|\csc\theta_o + \cot\theta_o|$, then a simple calculation shows that $\theta_o = \csc^{-1}(\cosh(\delta))$.

Thus B is a region in H bounded by the curves $\rho = 1$, $\rho = e^r$, $\theta = \csc^{-1}(\cosh(s))$ and $\theta = \pi - \csc^{-1}(\cosh(s))$.

Lemma 2. A region Ω in H bounded by the curves $\rho = 1$, $\rho = e^a$, $\theta = b$, $\theta = \pi - b$ is of area $2a\cot(b)$.

Proof.

$$\text{area}(\Omega) = \int_1^{e^a}\int_b^{\pi-b} \frac{\rho d\rho d\theta}{\rho^2\sin^2\theta} = \int_1^{e^a}\frac{d\rho}{\rho}\int_b^{\pi-b}\frac{d\theta}{\sin^2\theta}$$

$$= 2\,a\cot(b).$$

Lemma 3. The area of B is $2r\sinh(s)$.

Proof. This is directly from Lemma 2 and

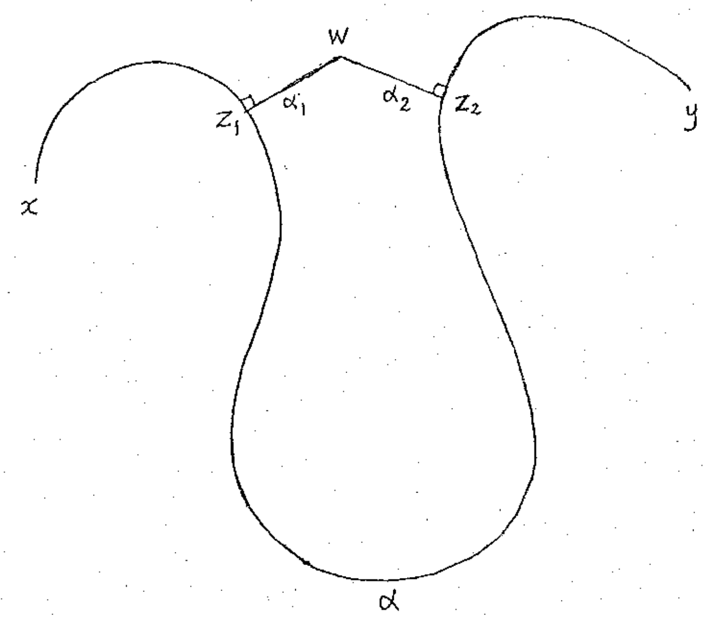
$$\cot(\csc^{-1}(\cosh(s))) = \sqrt{\csc^2(\csc^{-1}(\cosh(s))) - 1}$$

$$= \sinh(s).$$

Proof of Theorem 2.

Let α be a minimal geodesic realizing the diameter d of $S = H/G$. Let m be the length of a shortest simple closed geodesic on S and construct a belt B around α of radius $m/4$. Then there are two possibilities: either no two of these geodesics meet, or else some pair α_1, α_2 meet. We shall prove the second one is impossible.

Suppose α_1 and α_2 meet. Let z_1, z_2 and w be the points indicated in the figure and let e be the distance from z_1 to



z_2 along α. Then we can go from x to y by going from x to z_1 on α, following α_1, then α_2 and going from z_2 to y along α. This has length $\leq (d-e)+m/2$. Since α is the shortest path from x to y,

$$d \leq (d-e) + m/2,$$

i.e., $e \leq m/2$. But then α_1, α_2 and part of α between z_1 and z_2 is a closed path τ of length at most m. τ is certainly not homotopic to zero, for otherwise τ would be lifted to a triangle in the upper half plane H with two interior right angles, which is impossible. Moreover, τ has corners and so it is not itself a geodesic. Therefore there is a closed geodesic freely homotopic to τ of length $< m$. This contradicts the definition of m and thus only the first possiblity is correct. This also shows that the whole belt B is in S.

Now we can apply Lemma 3 to measure the area of the belt B, which is then $2d\sinh(m/4)$. Since the whole belt B is in S, the area of B is bounded by the area of S. This proves the first inequality.

For the second inequality, let β be a shortest simple closed geodesic on S with length m. Let p be a point on β which is not a fixed point of any elliptic element of G at all. We may assume that the point $ie^{m/2}$ of H is a preimage of p of the projection map $\pi : H \to H/G$. Form the Dirichlet region D of G in H with center at $p = ie^{m/2}$. We may assume, without loss of generality, that β is on the imaginary axis I of H. More precisely, β is the segment $[i, ie^m]$ with two endpoints identified by the group G. Construct the belt B around β of radius d. We claim that D is contained in B.

Let A be a Möbius transformation in G so that $A(i) = ie^m$. We first observe that i and ie^m are lying on some sides s_1 and s_2 of D respectively. By the construction of D, s_1 is an arc of the curve $\rho = 1$, and s_2 is an arc of the curve $\rho = e^m$, with $A(s_1) = s_2$. In other words, D is contained in the strip bounded by the curves $\rho = 1$ and $\rho = e^m$. Since every point q on S is at most of distance d to the point $p = ie^{m/2}$, q is at most of distance d to β. Thus D is contained in B. Hence we have

$$\text{area}(D) \leq \text{area}(B).$$

Again by Lemma 3, area(B) = 2msinh(d) gives the result.

<u>Remark</u>. There is a natural isomorphism between the set of conjugacy classes in G and the set of all closed geodesics in $S = H/G$. This map is defined by taking an element $A \neq 1$ in G to the shortest geodesic from x to Ax, for all $x \in S$. It is easy to check that it is in fact an isomorphism. Let the conjugacy class of A correspond to β, then A is conjugate to a hyperbolic transformation

$$\begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}, \ \lambda > 1.$$

Since the square of the trace is invariant under conjugation, we have

$$|\mathrm{Tr}\ A| = \sqrt{\lambda} + 1/\sqrt{\lambda}\ .$$

Moreover, the length of β, which is also invariant under conjugation, is log λ. Then a simple calculation shows

$$|\operatorname{Tr} A| = 2 \cosh\left(\frac{\text{length of } \beta}{2}\right).$$

Thus, for a shortest simple closed geodesic β of length m, we have

$$\cosh(m/2) = \left|\frac{\operatorname{Tr} A}{2}\right|.$$

Hence we can rewrite Theorem 2 as

Corollary 3. Let G be a finitely generated Fuchsian group of the first kind without paraboloc elements. Then S = H/G is a compact Riemann surface of genus g together with the Poincaré metric. Let d be the diameter of S and let m be the length of a shortest simple closed geodesic on S. If the conjugacy class of A corresponds to the shortest simple closed geodesic, then

$$\frac{1}{2} \sinh\left(\frac{1}{2} \cosh^{-1}\left(\left|\frac{\operatorname{Tr} A}{2}\right|\right) d \le \frac{1}{4} \operatorname{area}(H/G)$$

$$\le \cosh^{-1}\left(\left|\frac{\operatorname{Tr} A}{2}\right|\right) \sinh(d).$$

CHAPTER 4

Automorphic Forms

4.1 Spaces of Automorphic Forms

Let G be a Fuchsian group acting on the unit disc U and let D be a fundamental domain for G with area $(\partial D) = 0$. By a holomorphic automorphic form of weight $(-2q)$ for G in U we mean a holomorphic function $f(z)$, $z \in U$, satisfying the functional equation

$$f(A(z))A'(z)^q = f(z), \text{ for all } A \in G.$$

For any real numbers $p \geq 1$ and $q > 1$, the holomorphic automorphic forms with

$$\|f\|_{q,p,G} = \{ \iint_{U/G} \lambda(z)^{2-qp} |f(z)|^p |dz \wedge d\bar{z}| \}^{1/p} < \infty,$$

where $\lambda(z) = 2(1-|z|^2)^{-1}$ is the Poincaré metric on U, form a Banach space $A_q^p(U,G)$ of p-integrable forms.

The holomorphic automorphic forms with

$$\|f\|_{q,\infty,G} = \sup_{z \in U} \{\lambda(z)^{-q} |f(z)|\} < \infty,$$

form a Banach space $A_q^\infty(U,G)$ of bounded forms. The integral is independent of the choice of the fundamental domain D. The supremum is not changed if we replace U by D. The spaces $A_q^1(U,G)$ and $A_q^\infty(U,G)$ are of particular interest in the theory of Fuchsian groups and Kleinian groups; in some literature they are also denoted as $A_q(U,G)$ and $B_q(U,G)$ respectively. The Petersson scalar product

$$(f,g)_{q,p,G} = \iint_{U/G} \lambda(z)^{2-pq} f(z)\overline{g(z)}\, dz\wedge d\bar{z},$$

exists whenever $f \in A_q^p(U,G)$ and $g \in A_q^{p'}(U,G)$ for $1 \leq p < \infty$, with $1/p + 1/p' = 1$.

The above definitions also apply to $G = \{1\}$, the trivial group. If $f \in A_q^p(U,1)$, the Poincaré theta series

$$\Theta f(z) = \sum_{A\in G} f(A(z))A'(z)^q,$$

converges uniformly and absolutely on compact subsets of U.

We state three basic results on automorphic forms, for a more detailed exposition, the reader is referred to Kra [10]:

Theorem A. Θ is a continuous linear mapping of $A_q^1(U,1)$ onto $A_q^1(U,G)$ of norm ≤ 1. Furthermore, for every $f \in A_q^p(U,G)$, there is a $g \in A_q^p(U,1)$ such that

$$f = \Theta g,$$

and

$$\|g\|_{q,p} \leq C_q \|f\|_{q,p,G},$$

where $C_q = (2q-1)/(q-1)$.

Theorem B. (Bers [3]). For $1 \leq p < \infty$ with $1/p + 1/p' = 1$, the Petersson scalar product establishes an anti-linear isomorphism between $A_q^{p'}(U,G)$ and the dual space to $A_q^p(U,G)$.

Furthermore, if $f \in A_q^{p'}(U,G)$ and the linear functional L on $A_q^p(U,G)$ correspond to each other under this isomorphism, then

$$C_q^{-1}\|f\|_{q,p',G} \leq \|L\| \leq \|f\|_{q,p',G},$$

where $\|L\|$ is the norm of the linear function L.

The situation becomes much simpler if G is of finite type. Then $A_q(U,G) = B_q(U,G)$, is finite dimensional, and is called the space of <u>cusp forms</u>. If G has signature $(g;0;n;\ \nu_1,\ldots,\nu_n)$, then

$$d_q = \dim A_q(U,G) = (2q-1)(g-1) + \sum_{j=1}^{n} [q-q/\nu_j],$$

where $[x]$ is the greatest integer function, and it is agreed that $[q-q/\infty] = q-1$. This formula is a well-known corollary of the Riemann-Roch theorem.

<u>Theorem C</u>. If T is a möbius transformation defined on U (or H, the upper half plane), then the mapping

$$T_q^* : A_q^p(T(U),TGT^{-1}) \to A_q^p(U,G)$$

defined by

$$T_q^*f(z) = f(T(z))T'(z)^q, \quad z \in U,$$

is an isometric isomorphism, and

$$(T_q^*f, T_q^*g)_{q,G} = (f,g)_{q,TGT^{-1}},$$

for all $f \in A_q^p(T(U),TGT^{-1})$, $g \in A_q^{p'}(T(U),TGT^{-1})$ with $1/p + 1/p' = 1$.

We end this section with the following standard theorem:

Theorem 1. Let G be a Fuchsian group. If for some p, $1 \le p < \infty$,

$$A_q^p(U,G) \subset A_q^\infty(U,G),$$

then

$$A_q^p(U,G) \subset A_q^{p'}(U,G),$$

for all p' satisfying $1 \le p \le p' \le \infty$.

Proof. Let $f \in A_q^p(U,G)$, then

$$\begin{aligned}\|f\|_{q,p',G}^{p'} &= \iint_{U/G} \lambda(z)^{2-p'q}|f(z)|^{p'}|dz\wedge d\bar z| \\ &= \iint_{U/G} (\lambda(z)^{-q}|f(z)|)^{p'-p}(\lambda(z)^{2-pq}|f(z)|^p)|dz\wedge d\bar z| \\ &\le \|f\|_{q,\infty,G}^{p'-p}\ \|f\|_{q,p,G}^{p}\end{aligned}$$

Hence $\|f\|_{q,p',G} \le \|f\|_{q,\infty,G}^{1-p/p'}\ \|f\|_{q,p,G}^{p/p'} < \infty$.

4.2 $A_q(U,G) \subset B_q(U,G)$ conjecture.

It was conjectured some years ago that

$$(*) \qquad A_q(U,G) \subset B_q(U,G),$$

for all Fuchsian group G and that the inclusion map is continuous. The situation is trivial when G is of finite type. Several authors, Drasin and Earle [4], Metzger and Rao [15],

[16], Knopp [8] and Lehner [12], have proved (*) for Fuchsian group G of the second kind. A variety of tools has been used in these proofs, in particular, Abel's theorem on Riemann surfaces and a reproducing formula. Drasin and Earle also proved, using the closed graph theorem, that the inclusion map is bounded as soon as there exists an inclusion. Recently Lehner [13,14] proved (*) for infinitely generated Fuchsian groups under additional restrictions. Pommerenke [19] gave a counterexample for G containing infinitely many hyperbolic generators, using the technique of Bloch functions.

Instead, we shall deal with the more general case: the inclusion of $A_q^p(U,G)$ in $A_q^\infty(U,G)$ for any $q > 1$ and positive integer p. Also, we associate to each finitely generated Fuchsian group G with a standard set of generators some intrinsic geometric objects, measured in the Poincaré metric, of the surface $S = U/G$ serving as an upper bound for the norm of the inclusion map

$$A_q^p(U,G) \subset A_q^\infty(U,G).$$

We start with the cutting of a fundamental domain of a finitely generated Fuchsian group G of the second kind. As we mentioned in Chapter 2, U/G is a closed Riemann surface with n holes removed and some punctures under parabolic elements in G. Let $\{T_1,T_2,\ldots,T_n\}$ be the hyperbolic transformations corresponding to the holes, where each T_i is conjugate to

$$z \mapsto \lambda_i z,\ \lambda_i > 1,\ z \in H.$$

Then the <u>axis</u> of T_i is defined as the conformal image of the imaginary axis I on H under the previous conjugation.

Assume ∞ is a parabolic fixed point of G, and T generates the stabilizer G_∞. Let H_1 denote the set $\{z \in H : \mathrm{Im}\, z > 1\}$. Then it is known that the action of G on H_1 is the same as the action of the cyclic subgroup generated by T on H_1 (Shimizu [20]). Thus the line $\ell = \{z \in H : \mathrm{Im}\ z = 1\}$ encircles a punctured disc

$$V_\infty = H_1 \cap \{z \in H : 0 < \mathrm{Re}\ z < 1\}$$

as a natural neighborhood of ∞.

The Dirichlet region D of G in the unit disc U consists of a finite number of sides and a finite number of free sides. There is exactly one side of D terminating in the left endpoint of a free side and exactly one terminating at the right endpoint; these two sides are identified by some hyperbolic element T_i of G. The open quadrilateral bounded by these two sides, the free side and the axis of T_i is called a <u>funnel</u>. If G has parabolic elements, D will have a finite number of cusps p lying on $Q = \{z \in \mathbb{C} : |z| = 1\}$ which are fixed points of parabolic elements. The region formed by the two sides of D meeting at p and the conformal image of ℓ, which is the conformal image of V_∞, is called a <u>cusp sector</u>. The complement of these funnels and cusp sectors is relatively compact in U. We call it the <u>compact component</u> of D and denote it as D*. Then D consists of a compact region D*, a finite number (possibly zero) of cusp sectors, and a finite number of funnels. The decomposition of D obtained this way is completely determined by a given standard set of generators $\mathcal{S}$ of a group G. By the <u>reduced diameter</u> d* of a group G with respect to a standard set of generator $\mathcal{S}$, we mean the diameter of D*, measured in the Poincaré metric. The axis of T_i in a funnel is of length $\log \lambda_i$. Both terms, the reduced diameter and $\log \lambda_i$, are conformally invariant and hence are given by the group G.

Under the projection, S = U/G is a compact Riemann surface of genus g with n disjoint closed discs removed and possibly a finite number of punctures. The axes of the T_i's bound n annuli, and the conformal images of ℓ (which are circles tangent internally at cusps) bound a finite number of punctured discs. Their complement S* in S, which corresponds to D*, is relatively compact having d* as the reduced diameter. Our main result is

<u>Theorem 2</u>. Let G be a finitely generated Fuchsian group of the second kind containing no parabolic elements. Let D be a Dirichlet region of G. Let $\{T_1,\ldots,T_n\}$ be the hyperbolic elements corresponding to the free sides, where each T_i is conjugate to $z \to \lambda_i z$, $\lambda_i > 1$, $z \in H$. Let S* be the compact region of S = U/G with reduced diameter d*. Then for any

$q > 1$ and positive integer p, there is an inclusion map

$$A_q^p(U,G) \hookrightarrow A_q^\infty(U,G)$$

whose norm has an upper bound given by

$$\max_{i=1,2,\ldots,n} \{c_1(3 + \frac{\log 3}{\log \lambda_i})^{1/p},\ c_2 \frac{(1 + \tanh(d^*+b))^{q+1/p}}{(1 - \tanh(d^*+b))^{1/p}}\}$$

where c_1, c_2 and b are constants.

Theorem 3. Let G be a finitely generated Fuchsian group. Then for $q > 1$ and positive integer p, we have

$$A_q^p(U,G) \subset A_q^\infty(U,G)$$

And the inclusion map is continuous.

Remarks. 1. We obtain a new, elementary proof for the $A_q(U,G) \subset B_q(U,G)$ conjecture, whenever G is finitely generated, by setting $p = 1$ in the proof of Theorem 3.

2. We find an explicit upper bound for the norm of the inclusion map $A_q(U,G) \subset B_q(U,G)$, whenever G is finitely generated of the second kind containing no parabolic elements, by setting $p = 1$ in Theorem 2.

3. There are two special cases, both are important in themselves. First, $S = H/G$ is compact. The reduced diameter d^* is the diameter d of S and the $\log \lambda_i$ terms disappear from the upper bound. More precisely, it is

$$c_2 \frac{(1 + \tanh d)^{q+1/p}}{(1 - \tanh d)^{1/p}} .$$

Second, if G is cyclic and is generated by a single hyperbolic transformation

$$z \mapsto \lambda z, \ \lambda > 1, \ z \in H.$$

The norm then is bounded by

$$c_1\left(3 + \frac{\log 3}{\log \lambda}\right)^{1/p}.$$

As a corollary to Theorem 1 and Theorem 3, we have

<u>Corollary 1.</u> For $q > 1$, $p \geq 1$, $p \in \mathbb{Z}$, we have

$$A_q^1(U,G) \subset A_q^p(U,G) \subset A_q^\infty(U,G),$$

whenever G is finitely generated. Furthermore, the first inclusion map satisfies the following

$$\|f\|_{q,p,G} \leq \|L\|^{1-1/p}\|f\|_{q,1,G},$$

where $L : A_q(U,G) \subset B_q(U,G)$.

For the reverse inclusions, we have (cf. Kra [10])

<u>Proposition.</u> For any real numbers $q \geq 1$ and $p \geq 1$, we have

$$A_q^\infty(U,G) \subset A_q^p(U,G) \subset A_q^1(U,G),$$

whenever U/G is of finite type. Furthermore, in this case, the inclusion maps are all continuous.

<u>Corollary 2.</u> For $p \in \mathbb{Z}$, $p \geq 1$, $q \geq 1$, we have

$$A_q^1(U,G) = A_q^p(U,G) = A_q^\infty(U,G)$$

whenever U/G is of finite type.

This is the space of cusp forms.

4.3 Two special cases.

In order to prove Theorem 2, we need to study the behavior of the norm of the inclusion map $L : A_q^p(U,G) \to A_q^\infty(U,G)$ in two special cases.

Theorem 4. Let $G_\lambda = \langle T_\lambda : z \mapsto \lambda z, \lambda > 1\rangle$ be the cyclic group generated by a single hyperbolic transformation T_λ. Then for any $q > 1$ and integral $p > 0$, the inclusion

$$L_\lambda : A_q^p(U,G_\lambda) \to A_q^\infty(U,G_\lambda)$$

holds and is continuous. Furthermore, $\|L_\lambda\|$ satisfies the following inequalities

$$k_1(\log \lambda)^{-1/p} \le \|L_\lambda\| \le k_2\left(3 + \frac{\log 3}{\log \lambda}\right)^{1/p},$$

where k_1 and k_2 are constants, $\|L_\lambda\|$ is the norm of the map L_λ.

Proof. In this proof, we shall use the upper half plane H rather than the unit disc U. Let $D_\lambda = \{z \in H : 1 < |z| < \lambda\}$ be a fundamental domain for G_λ. Let $f \not\equiv 0$ be in $A_q^p(U,G)$. Note that f is holomorphic in H, then so is f^p for positive integer p. Let $\zeta = \xi + i\eta \in H$. Recall the Poincaré metric on

H is $\lambda(\zeta) = \eta$. We define

$$\Delta(\zeta) = \{z \in H : |z-\zeta| < \eta/2\},$$
$$M(\zeta) = \eta^q |f(\zeta)|,$$

and

$$B_\lambda = \bigcup_{\zeta \in D_\lambda} \Delta(\zeta).$$

By the mean value property, we have

$$f^p(\zeta) = \frac{1}{\pi(\eta/2)^2} \iint_{\Delta(\zeta)} f^p(z)dxdy, \quad z = x+iy.$$

In $\Delta(\zeta)$, $(2/3)y < \eta < 2y$ implies

$$C_{pq}(y/\eta)^{pq-2} \geq 1, \text{ for all } p \geq 1,\ q > 1,$$

where

$$C_{pq} = \begin{cases} 2^{pq-2}, & \text{if } pq \geq 2. \\ (2/3)^{pq-2}, & \text{if } pq < 2. \end{cases}$$

Then we have

$$\begin{aligned} |f(\zeta)|^p &\leq 4\pi^{-1}\eta^{-2} \iint_{\Delta(\zeta)} |f(z)|^p dxdy \\ &\leq 4\pi^{-1}\eta^{-2} \iint_{\Delta(\zeta)} C_{pq}(y/\eta)^{pq-2}|f(z)|^p dxdy \\ &\leq 4\pi^{-1}\eta^{-pq}C_{pq} \iint_{B_\lambda} y^{pq-2} \ |f(z)|^p dxdy \end{aligned}$$

Hence

$$M(\zeta)^p = \eta^{pq}|f(\zeta)|^p \leq 2\pi^{-1}C_{pq} \iint_{B_\lambda} y^{pq-2}|f(z)|^p |dz\wedge d\bar{z}|.$$

We know that B_λ can be covered by finitely many translates of D_λ; we can estimate that number as follows. In B_λ, the farthest and the nearest points to the origin are $3\lambda/2$ and $1/2$ respectively. We also know that all the translates of D_λ are of the form

$$\{z \in H \mid \lambda^{k-1} < |z| < \lambda^k,\ k \neq 1,\ k \in \mathbb{Z}\}.$$

It suffices to find integers m and n so that

$$\lambda^m \geq 3\lambda/2 \text{ and } \lambda^{-n} \leq 1/2,$$

for then the sum m+n is the number desired. A simple calculation shows that

$$m = [(\log \tfrac{3\lambda}{2})/\log \lambda] + 1,$$
$$n = [\log 2/\log \lambda] + 1,$$

where [ ] is the greatest integer function. Choose

$$h_1(x) = 3 + \log 3/\log x.$$

Clearly, $h_1(\lambda) \geq m+n$. Then

$$M(\zeta)^p \leq 2\pi^{-1} C_{pq} h_1(\lambda) \iint_{D_\lambda} y^{pq-2} |f(z)|^p |dz \wedge d\bar{z}|$$
$$= C_o h_1(\lambda) \|f\|^p_{q,p,G_\lambda},$$

where C_o is a constant. Therefore,

$$\|f\|_{q,\infty,G} = \sup_{\zeta \in H} M(\zeta) \leq k_2 h_1^{1/p}(\lambda) \|f\|_{q,p,G_\lambda}.$$

By definition,

$$\|L\| = \sup_{f \neq 0} \frac{\|f\|_{q,\infty,G_\lambda}}{\|f\|_{q,p,G_\lambda}} \leq k_2 h_1^{1/p}(\lambda).$$

This gives the second inequality of the theorem, and thus proves the continuity of L_λ by finding an explicit upper bound.

For the first inequality, we choose a function

$$h(z) = z^{-q}.$$

$h(z)$ is analytic in H and is an automorphic form of weight $(-2q)$. Then

$$\|h\|_{q,\infty,G_\lambda} = \sup_{z \in H} (y)^q |h(z)| = 1,$$

and

$$\|h\|^p_{q,p,G_\lambda} = \int_0^\pi \int_1^\lambda y^{pq-2} |h(z)|^p \, r dr d\theta$$

$$= \int_0^\pi \int_1^\lambda (r \sin \theta)^{pq-2} \, r^{-pq} \, r dr d\theta$$

$$= \int_0^\pi \int_1^\lambda \sin^{pq-2}\theta \; r^{-1} dr d\theta = \log \lambda \, K_{pq},$$

where

$$K_{pq} = \int_0^\pi \sin^{pq-2}\theta d\theta.$$

We claim that this integral converges for $pq > 1$, and depends only on p and q. For $pq \geq 2$, it is obvious, since $\sin \theta$ is always bounded from above by θ. It remains to show that K_{pq} is convergent for $1 < pq < 2$.

For convenience, write $t = pq - 2$. Then $-1 < t < 0$. Note that

$$\int_{\infty}^{\pi} \sin^t\theta \, d\theta = \int_{0}^{\pi/2} \sin^t\theta \, d\theta + \int_{\pi/2}^{\pi} \sin^t\theta \, d\theta.$$

By changing of variables, $s = \pi - \theta$, in the second integral

$$\int_{\pi/2}^{\pi} \sin^t\theta \, d\theta = \int_{\pi/2}^{0} \sin^t(\pi-s)d(\pi-s) = \int_{0}^{\pi/2} \sin^t s \, ds,$$

we have

$$\int_{0}^{\pi} \sin^t\theta \, d\theta = 2\int_{0}^{\pi/2} \sin^t\theta \, d\theta.$$

Also, the elementary fact that $\lim\limits_{\theta\to 0} (\sin\theta/\theta) = 1$ implies $(\sin\theta/\theta)^t$ is bounded from above, say by $M > 0$, in $(0,\pi/2]$. Hence

$$\int_{0}^{\pi} \sin^t\theta \, d\theta = 2\int_{0}^{\pi/2} \sin^t\theta \, d\theta = 2 \lim_{\varepsilon\to 0} \int_{\varepsilon}^{\pi/2} (\sin\theta/\theta)^t \theta^t \, d\theta$$

$$= 2M \lim_{\varepsilon\to 0} \int_{\varepsilon}^{\pi/2} \theta^t d\theta = 2M \int_{0}^{\pi/2} \theta^t d\theta < \infty.$$

<u>Corollary</u>. Let G_λ be the cyclic group generated by $z \mapsto \lambda z$, $\lambda > 1$. Then, for $pq > 1$, $p \geq 1$ and $p \in \mathbb{Z}$, the inclusion map

$$L_\lambda : A_q^p(U,G_\lambda) \to A_q^\infty(U,G_\lambda)$$

satisfies

1) $\|L_\lambda\| \to \infty$, as $\lambda \to 1$,

2) $\|L_\lambda\| \to 0$, as $\lambda \to \infty$.

<u>Remark</u>. This corollary suggests that to obtain a counter-example for the $A_q \subset B_q$ conjecture we need to have infinitely many hyperbolic generators $z \mapsto \lambda_n z$ in G with $\lambda_n \to 1$. This is

exactly Pommerenke's approach.

Next, we consider those groups G of the first kind which contain no parabolic elements. Then $S = U/G$ is a compact Riemann surface of genus g, and

$$\text{area}(U/G) = (1/2) \iint_{U/G} \lambda(z)^2 |dz \wedge d\bar{z}| < \infty.$$

If G contains no elliptic elements, then by Gauss-Bonnet theorem,

$$\text{area}(S) = 4\pi(g-1), \quad g \geq 2.$$

The following is a well-known result.

Lemma 1. (Marden [17]) There is a universal number $r > 0$ so that given any Fuchsian group G there exists a Möbius transformation I such that the Dirichlet region D of TGT^{-1} (in the unit disc U) contains the hyperbolic disc $B(0;r)$.

In the unit disc U, $M(\zeta) = \lambda(\zeta)^q |f(\zeta)| = 2^q (1-|\zeta|^2)^q |f(\zeta)|$ is invariant under G. And we have

Lemma 2. Let D be a Dirichlet region of a finitely generated Fuchsian group G with center at the origin of U. Let D* be the compact component of D, then D* is contained in a hyperbolic disc $B(0;R)$ with $0 < R < \infty$. Then for any $\zeta \in D^*$ and for any $f \in A^p_q(U,G)$, we have

$$M(\zeta)^p \leq K \frac{(1 + \tanh(R))^{pq+1}}{(1 - \tanh(R))} \|f\|^p_{q,p,G},$$

where K is a constant.

Proof. Let $0 \neq f \in A^p_q(U,G)$ and $\zeta \in U$. Then f^p is holomorphic in U. Define

$$\Delta(\zeta) = \{z \in U : |z-\zeta| < (1-|\zeta|)/2\},$$

and

$$C'_{pq} = \begin{cases} 2^{pq-2} & , \text{ if } pq \geq 2, \\ (1/3)^{pq-2}, & \text{if } pq < 2. \end{cases}$$

Then for all z in $\Delta(\zeta)$

$$|\zeta| - \frac{1-|\zeta|}{2} \leq |z| \leq |\zeta| + \frac{1-|\zeta|}{2},$$

we have

$$\frac{3}{2}(1-|\zeta|) \geq 1-|z| \geq \frac{1-|\zeta|}{2}.$$

Since $2 \geq 1 + |z| \geq 1$ in U, we obtain

$$3(1-|\zeta|) \geq 1-|z|^2 \geq \frac{1}{2}(1-|\zeta|).$$

Thus

$$(1-|\zeta|)^{pq-2} \leq C'_{pq}(1-|z|^2)^{pq-2}.$$

By the mean value property, we have

$$f^p(\zeta) = \frac{1}{\pi a^2} \iint_{\Delta(\zeta)} f^p(z)dxdy, \quad a = \frac{1-|\zeta|}{2}, \quad z = x+iy.$$

Then, for all $\zeta \in D^*$,

$$\begin{aligned} M(\zeta)^p &= 2^{pq}(1-|\zeta|^2)^{pq}|f(\zeta)|^p \\ &\leq 2^{pq+2}\pi^{-1}(1+|\zeta|)^{pq}(1-|\zeta|)^{pq-2}\iint_{\Delta(\zeta)}|f(z)|^p dxdy \\ &= 2^4\pi^{-1}(1+|\zeta|)^{pq}\iint_{\Delta(\zeta)}(2(1-|\zeta|))^{pq-2}|f(z)|^p dxdy \\ &\leq 16\pi^{-1}(1+|\zeta|)^{pq}\iint_{\Delta(\zeta)} C'_{pq}(2(1-|z|^2))^{pq-2}|f(z)|^p dxdy \\ &= 8\pi^{-1}(1+|\zeta|)^{pq}\, C'_{pq}\iint_{\Delta(\zeta)}(2(1-|z|^2))^{pq-2}|f(z)|^p|dz\wedge d\bar{z}|. \end{aligned}$$

The region on which the mean value property applied is $\bigcup_{\zeta\in D^*} \Delta(\zeta)$, which can be covered by a disc centered at the origin with euclidean radius $x = (1 + \tanh R)/2$. In other words, $\bigcup_{\zeta\in D^*} \Delta(\zeta)$ is contained in some hyperbolic disc $B(0;s)$ with center at the origin and

$$s = (1/2)\log \frac{1+x}{1-x} = (1/2)\log \frac{3+\tanh R}{1-\tanh R}.$$

As formulated in Chapter 2, the hyperbolic area of $B(0;s)$ is given by

$$\text{area}(B(0;s)) = \int_0^{2\pi}\int_0^x \frac{rdrd\theta}{(1-r^2)^2} = \frac{\pi(1+\tanh R)^2}{(3+\tanh R)(1-\tanh R)}.$$

Lemma 1 tells us that the compact component D^* of D contains a disc $B(0;r)$ with the universal number r as its radius. Every translate of D^* will certainly contain a region of the same area as that of $B(0;r)$. Since the hyperbolic area of $B(0;s)$ is finite, the number of translates of D^* needed to cover $B(0;s)$ will not exceed

$$1 + \left[\frac{\text{area}(B(0;s))}{\text{area}(B(0;r))}\right] \leq \frac{2\ \text{area}(B(0;s))}{\text{area}(B(0;r))},$$

where [ ] is the greatest integer function. Hence

$$M(\zeta)^p \leq 8\pi^{-1}C'_{pq}(1+|\zeta|)^{pq} \iint_{\Delta(\zeta)} (2(1-|z|^2))^{pq-2}|f(z)|^p|dz\wedge d\bar{z}|$$

$$\leq 8\pi^{-1}C'_{pq}(1+|\zeta|)^{pq} \iint_{B(0;s)} (2(1-|z|^2))^{pq-2}|f(z)|^p|dz\wedge d\bar{z}|$$

$$\leq 8\pi^{-1}C'_{pq}(1+|\zeta|)^{pq}\left(\frac{2\ \text{area}(B(0;s))}{\text{area}(B(0;r))}\right) \iint_{D^*}(2(1-|z|^2))^{pq-2} |f(z)|^p|dz\wedge d\bar{z}|$$

$$\leq 8\pi^{-1}C'_{pq}(1+|\zeta|)^{pq}\left(\frac{2\ \text{area}(B(0;s))}{\text{area}(B(0;r))}\right) \iint_{D}(2(1-|z|^2))^{pq-2} |f(z)|^p|dz\wedge d\bar{z}|$$

$$\leq K\ \frac{(1+\tanh R)^{pq+1}}{1-\tanh R}\ \|f\|^p_{q,p,G},$$

where $K = 16\ C'_{pq}(\text{area}(B(0;r)))^{-1}$. This proves Lemma 2.

As a direct consequence of Lemma 2, we have

<u>Theorem 5</u>. Let G be a finitely generated Fuchsian group of the first kind containing no parabolic elements. Then the inclusion map

$$L : A^p_q(U,G) \to A^\infty_q(U,G)$$

is continuous and satisfies the following inequalities:

$$K_1 \leq \|L\| \leq K_2\ \frac{(1+\tanh d)^{q+1/p}}{(1-\tanh d)}$$

where K_1 and K_2 are constants, d is the diameter of $S = U/G$.

Proof. Choose a point p in S, which is not a fixed point of any element in G. We may assume that the origin in U is a preimage of p of the projection map $\pi : U \to U/G$. Form the Dirichlet region D of G with respect to the origin in U. By the definiton of d,

$$D \subseteq B(0;d).$$

Then the second inequality follows from Lemma 2 with R replaced by d, and hence the continuity of L also follows.

For the first inequality, let $f \in A_q^\infty(U,G)$. Then

$$\begin{aligned}
\|f\|^p_{q,p,G} &= \iint_{U/G} \lambda(z)^{2-pq} |f(z)|^p |dz \wedge d\bar{z}| \\
&= \iint_{U/G} \lambda^2(z) |\lambda^{-q}(z) f(z)|^p |dz\wedge d\bar{z}| \\
&\le \|f\|^p_{q,\infty,G} \iint_{U/G} \lambda^2(z) |dz \wedge d\bar{z}| \\
&= 2 \text{ area}(U/G) \|f\|^p_{q,\infty,G}.
\end{aligned}$$

Thus

$$\|f\|_{q,p,G} \le C\|f\|_{q,\infty,G},$$

where $C = (2 \text{ area}(U/G))^{1/p}$ is a constant. And $K = C^{-1}$.

4.4 The Proofs.

We now proceed with the proof of Theorem 2. Let G be a finitely generated Fuchsian group of the second kind. Let D be a Dirichlet region of G with center at a non-fixed point

$x = 0$. We shall modify the decomposition of D described in section 2 to serve our purpose. We extend the compact component D^* to

$$D_1 = \bigcup_{\zeta \in D^*} \{z \in D : d(z,\zeta) < b/2\},$$

where $d(.,.)$ is the hyperbolic distance in U determined by the Poincaré metric, and

$$\begin{aligned} b &= \log(\csc \tan^{-1}2 + \cot \tan^{-1}2) \\ &= \log((\sqrt{5} + 1)/2). \end{aligned}$$

The number $b/2$ has a geometric meaning. It is the distance of the curve $\{z \in H : \arg z = \tan^{-1}2\}$ to the imaginary axis on H. More precisely, we obtain D_1 by extending the "axes" over into the funnels by a distance of $b/2$. The compact region D_1 thus obtained has the reduced diameter at most d^*+b. And the funnels obtained herewith are the conformal images of

$$F_i = \{z \in H : 1 < |z| < \lambda_i, 0 < \arg z < \tan^{-1}2\},$$

under the Möbius transformations A_i, $i = 1,2,\ldots,n$.

Suppose f assumes its essential supremum in one of the funnels, say $A_k(F_k)$. As mentioned in section 1 (Theorem C), the conformal mappings are isometric isomorphisms between the spaces of automorphic forms. Hence we can evaluate the essential supremum of the corresponding automorphic form of f in F_k. Since there is no danger of confusion, we can still write that automorphic form as f. Then f is holomorphic in H, and so is f^p for positive integer p. Let $\zeta \in F_k$. By the mean value property of f^p,

$$f^p(\zeta) = \frac{1}{\pi(\eta/2)^2} \iint_{\Delta(\zeta)} f^p(z)\, dxdy,$$

where $\zeta = \xi+i\eta$, $z = x+iy$ and $\Delta(\zeta) = \{z \in H : |z-\zeta| < \eta/2\}$.

The region the mean value property applied is

$$B_k = \bigcup_{\zeta \in F_k} \Delta(\zeta)$$

which lies completely in $\{z \in H : 0 < \arg z < \pi/2\}$; and D, in this case, lies completely in the strip $\{z \in H : 1 < |z| < \lambda_k\}$. Define

$$R_k = \{z \in D : 0 < \arg z < \pi/2\}.$$

Note that R_k is a funnel of S cutting along the axis. Using the estimation technique in Theorem 6, $h_1(\lambda_k) = (3+\log 3/\log \lambda_k)$ copies of R_k will certainly cover B_k. Hence

$$\begin{aligned} M(\zeta)^p &\le \eta^{pq}|f(\zeta)|^p \le \eta^{pq}\pi^{-1}(\eta/2)^{-2} \iint_{\Delta(\zeta)} |f(z)|^p dxdy \\ &\le 2\pi^{-1}\eta^{pq-2} \iint_{\Delta(\zeta)} C_{pq}(y/\eta)^{pq-2}|f(z)|^p|dz\wedge d\bar{z}| \\ &= 2\pi^{-1}C_{pq} \iint_{\Delta(\zeta)} y^{pq-2}|f(z)|^p|dz\wedge d\bar{z}| \\ &\le 2\pi^{-1}C_{pq} \iint_{B_k} y^{pq-2}|f(z)|^p|dz\wedge d\bar{z}| \\ &\le 2\pi^{-1}C_{pq}h_1(\lambda_k) \iint_{R_k} y^{pq-2}|f(z)|^p|dz\wedge d\bar{z}| \end{aligned}$$

$$\leq 2\pi^{-1} C_{pq} h_1(\lambda_k) \iint_D y^{pq-2} |f(z)|^p |dz \wedge d\bar{z}|$$

$$= 2\pi^{-1} C_{pq} h_1(\lambda_k) \, \|f\|^p_{q,p,G}.$$

Hence we obtain

$$M(\zeta) \leq c_1 (3+\log 3/\log \lambda_k)^{1/p} \, \|f\|_{q,p,G},$$

where

$$c_1 = (2\pi^{-1} C_{pq})^{1/p}.$$

Now suppose f assumes its essential supremum in the compact component D_1 of D. Since the reduced diameter of G is at most d*+b, we have

$$D_1 \subseteq B(0; d^*+b).$$

We can apply Lemma 2, with d*+b in place of R,

$$M(\zeta)^p \leq K \frac{(1+\tanh(d^*+b))^{pq+1}}{(1-\tanh(d^*+b))} \|f\|^p_{q,p,G}.$$

Thus

$$M(\zeta) \leq c_2 \frac{(1+\tanh(d^*+b))^{q+1/p}}{(1-\tanh(d^*+b))^{1/p}} \|f\|_{q,p,G}.$$

Since f must assume its essential supremum either in the compact component D_1 or in one of the funnels $\{A_1(F_1), \ldots, A_n(F_n)\}$, i.e.

$$\|f\|_{q,\infty,G} = \max_{i=1,2,\ldots,n} \{\sup_{\zeta \in D_1} M(\zeta), \sup_{\zeta \in A_i(F_i)} M(\zeta)\}.$$

Thus the theorem is proved.

To prove Theorem 3, we shall need the following well-known result. For the convenience of the reader we insert a proof.

Lemma 3. Let f be in $A_q^p(U,G)$ with respect to a Fuchsian group G. Then $f(z) \to 0$ as $z \to p$ within a cusp sector at p. Moreover, $M(\zeta)$ is bounded within a cusp sector at p.

Proof. We may assume, without loss of generality, that $i\infty$ is a cusp and the stabilizer $G_{i\infty}$ is generated by $T : z \mapsto z+1$ acting on H. Let D be a fundamental domain for G. Set R to be the strip

$$\{\zeta \in H : |\mathrm{Re}\ \zeta| < 1/2\}.$$

Clearly, $D \subseteq R$.

Let f be holomorphic in D, so is f^p. Since $f \in A_q^p(U,G)$, $f(\zeta+1) = f(\zeta)$, we have $f^p(\zeta+1) = f^p(\zeta)$. It follows that f^p has a Fourier expansion in H

$$f^p(\zeta) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n\zeta}.$$

Shimizu's theorem guarantees us that for $\zeta = u+iv$, $v > 1$, the Fourier coefficients

$$a_n = \int_{-1/2}^{1/2} f^p(\zeta)e^{-2\pi i n\zeta}du.$$

Then

$$|a_n| \leq \int_{-1/2}^{1/2} |f(\zeta)|^p e^{2\pi nv}du,$$

and

$$|a_n| \int_{h_o}^{h} v^{pq-2}dv \le \int_{h_o}^{h}\int_{-1/2}^{1/2} |f(\zeta)|^p v^{pq-2} e^{2\pi nv} du dv,$$

where $h > h_o > 1$. Hence

$$a_n \cdot K_o \le e^{2\pi nh} \int_{h_o}^{h}\int_{-1/2}^{1/2} |f(\zeta)|^p v^{pq-2} du dv$$

$$\le (1/2)e^{2\pi nh}\|f\|^p_{q,p,G},$$

where

$$K_o = \int_{h_o}^{h} v^{pq-2}dv = \begin{cases} (h^{pq-2} - h_o^{pq-2})/(pq-1), & \text{if } pq > 1, \\ \log(h/h_o) & , \text{if } pq = 1. \end{cases}$$

Now let $h \to \infty$ we deduce that $a_n = 0$ for $n \le 0$. Therefore

$$f^p(\zeta) = \sum_{n=1}^{\infty} a_n e^{2\pi i n\zeta}.$$

Hence $f(\zeta) \to 0$ as $\zeta \to i\infty$. Moreover,

$$M(\zeta)^p = v^{pq}|f(\zeta)|^p \le \sum_{n=1}^{\infty} |a_n| v^{pq} e^{-2\pi nv},$$

implies that $M(\zeta) \to 0$ as $v \to \infty$.

Proof of Theorem 3. Let $z \in D$. If z lies in the compact region D_1, we obviously have

(*) $$M(z) = \lambda(z)^q |f(z)| = C < \infty,$$

for some constant $C > 0$.

Next, let z lie in a funnel. From Theorem 4,

$$M(z)^p = C_o h_1(\lambda) \|f\|^p_{q,p,G},$$

so (*) holds in this case as well. Finally, let z lie in a cusp sector. By Lemma 3, $M(z) \to 0$ as $z \to p$ within a cusp sector at p, (*) is also valid. Hence

$$\|f\|_{q,\infty,G} = \sup_{z \in U} M(z) < \infty,$$

and the theorem is proved.

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