Conformal Structures on 3-Manifolds

A Dissertation presented

by

Mark Charles Wolff

to

The Graduate School

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in

Mathematics

State University of New York

at

Stony Brook

August, 1975

STATE UNIVERSITY OF NEW YORK

AT STONY BROOK

THE GRADUATE SCHOOL

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The dissertation is accepted by the Graduate School

Herbert Weisinger, Dean
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July, 1975

Abstract of the Dissertation

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In this paper the existence of a strong type of conformal structure on 3-dimensional manifolds is examined. To this end the group $\overline{G^n}$ of all conformal self maps of S^n, the standard Euclidean n-sphere, is examined in section 2. We present various transformation groups isomorphic to $\overline{G^n}$. We list a number of the most important properties of conformal transformations and of the group $\overline{G^n}$.

Beginning with section 3 we concentrate on G^3, the orientation preserving half of $\overline{G^3}$. For most purposes we utilize a model of G^3 which operates on $\overline{R^3}$, the one point compactification of Euclidean 3-space. In section 3 we classify the elements of G^3 according to their fixed point sets. Using this classification

we show that every element is conjugate in G^3 to a transformation in one of 4 "normal forms."

In section 4 we determine the set of circles left invariant by a transformation in normal form. This refines our understanding of these four types of transformations.

In section 5 we give geometric conditions for two transformations of G^3 to commute. The primary tools for doing this are the results in sections 3 and 4.

The remaining two sections of this paper are devoted to demonstrating the existence of a closed, orientable, 3-dimensional manifold which does not admit a conformal structure. For our purposes a conformal structure for a manifold M is an order pair (G,D) where G is a discrete subgroup of G^3 acting on $\overline{R}^3$ and D is an open subset of $\overline{R}^3$. D must be an invariant, connected component of the set of proper discontinuity of G, and D modulo the action of G must be homeomorphic to M. Thus a necessary condition for (G,D) to be a conformal structure for M is the existence of a homomorphism h from the fundamental group of M to G^3 such that G equals the image of h.

In section 6 we construct a particular

3-dimensional closed, orientable manifold M_0 and compute its fundamental group. In section 7, using the relations established in section 5, we are able to show that "very few" homomorphisms of the required type exist. Finally we examine those that do exist and show that they cannot yield a conformal structure for M_0.

To my parents

and

P.G.K.

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Acknowledgements

I wish to thank my thesis advisor Professor Bernard Maskit. Professor Maskit suggested the problem I have worked on and gave me many valuable suggestions. I would also like to thank Professors David Frank, Irwin Kra and John Thorpe for their mathematical assistance, and for their personal encouragement and patience.

Section 1

Introduction

1.1 In this paper we examine the group G^3 of orientation preserving conformal maps of S^3 onto itself. We then use this information to prove the existence of a 3-dimensional closed, orientable manifold which does not admit a strong type of conformal structure as explained below.

The paper is divided into 7 sections. Each section is further divided into subsections. Theorems and definitions are referred to by the subsection in which they appear, so that definition 2.3 is the definition given in the third subsection of section 2.

1.2 The material in section 2 is primarily background material. The reader is expected only to be familiar with the concept of a differentiable manifold and basic related structures. Many of the definitions in section 2 can be found in an introductory text on Differential Geometry such as Hicks [7]. Most of the theorems, propositions and corollaries in section 2 can be found in the literature in some form. They come from a wide variety of sources (both Ford's

Automorphic Functions [4] and Lehner's *A Short Course in Automorphic Functions* [8] provide much of this material in the case of dimension 2.) For this reason we either give short proofs whenever possible or list references when this is not possible.

In particular we carefully define the group $\overline{G^n}$ of all conformal transformations of the standard Riemannian unit n-sphere S^n to itself. We denote the orientation preserving half of $\overline{G^n}$ by G^n. Four other models of $\overline{G^n}$ are also presented. Each model is a group of conformal transformations of a Riemannian manifold onto itself. The most important model for our purposes is obtained by putting a Riemannian metric on $\overline{R^n} = R^n \cup \{\infty\}$, the one point compactif of R^n. The metric is chosen so that stereographic-projection from S^n to $\overline{R^n}$ is conformal.

One of our major tools for understanding the $\overline{R^n}$-model of $\overline{G^n}$ is Liouville's Theorem. It states that $\overline{G^n}$ is generated by translations, dilation, rotations and inversions of $\overline{R^n}$. We prove four important corollaries to the theorem which describe the structure of $\overline{G^n}$ and the geometry of a conformal transformation on $\overline{R^n}$.

The remaining three models of $\overline{G^n}$ which we describe all act on a manifold homeomorphic to D^{n+1}, open unit ball in R^{n+1}. In these models $\overline{G^n}$ is in fact the full group of isometries of the Riemannian manifold. For the model acting on $\underline{H^{n+1}} = \{(x_1,\ldots,x_{n+2}) | \sum_{i=2}^{n+2} x_i^2 = x_1^2 - 1\}$, this implies that G^n is isomorphic to a subgroup of index 2 in $SO(n+1,1)$. Finally we define the notion of a discrete subgroup of $\overline{G^n}$.

1.3 Beginning with section 3 we restrict our attention to the group G^3. In this section we prove two theorems about the $\overline{R^3}$-model of G^3. The first theorem states that every element of G^3 has 0, 1 or 2 fixed points or an entire circle of fixed points. The second theorem gives normal forms for the elements of G^3 according to the cardinality of their fixed point set. For example we show that a transformation with a circle of fixed points is conjugate in G^3 to a rotation.

There are two important results which we use in the proofs of these theorems. One is a corollary to Liouville's Theorem which shows that an element of G^n which fixes an $(n-1)$-dimensional hypersphere must be the identity. This allows us to determine the action of a transformation in G^3 by restricting our attention to any invariant 2-sphere.

The other result is a lemma in section 3. Given any two 2-dimensional hyperspheres S_1 and S_2 in $\overline{R^3}$ the lemma establishes the existence of an element f in G^3 shich maps S_1 onto S_2 and allows us to preassign the value of f at any 3 points on S_1. This gives us great flexibility in conjugating a given transformation into another one which is more easily understood. We end section 3 by defining a transformation f of G^3 to be parabolic if it has one fixed point, loxodromic if it has two fixed points, elliptic if it has a circle of fixed points and bielliptic if it has no fixed points.

1.4 One of the important properties of conformal maps of $\overline{R^n}$ which we establish in section 2 is that they preserve the family of k-dimensional hyperspheres in $\overline{R^n}$ for all positive integers $k \leq n$. We will use the term circle to denote both orindary Euclidean circles and extended lines thru ∞. This property then implies that conformal maps carry circles to circles. In section 4 we examine which circles in $\overline{R^3}$ are sent to themselves by elements of G^3. Given a transformation f in G^3 the number and relative location of any f-invariant circles is the same for any conjugate of f. Thus it suffices to determine which circles are invariant for transformations in one of our normal forms. Section 4 ends with a theorem which describes the set of invariant circle for parabolic, loxodromic, elliptic and bielliptic transformations of $\overline{R}^3$ in normal form. These results are particularly important in the bielliptic case, for they give us a much clearer description of the action of a bielliptic transformation than we get in section 3.

1.5 In section 5 we determine conditions for two elements in G^3 to commute. Since this property is not effected by conjugation in G^3 it suffices to give conditions for a transformation to commute with a given transformation in normal form. Many of our results are obtained by repeated applications of the following simple observation: If S is a f-invariant set then fg = gf implies that fg(S) = gf(S) = g(S), i.e., g(S) must also be f-invariant. For our purposes the two most important examples of f-invariant sets are the fixed points of f which we examined in section 3 and the f-invariant circles which we examined in section 4. The importance to this paper of understanding when two elements in G^3 commute is clarified in subsection 1.7.

1.6 In section 6 we construct a closed, orientable 3-dimensional manifold M_0. In section 7 we will show that M_0 does not admit a conformal structure. The manifold M_0 is a torus bundle over a circle. Using the exact homotopy sequence of a fiber bundle we compute $\Pi_1(M_0)$. The group $\Pi_1(M_0)$ has a presentation consisting of 3 generators, a, b and c and the 3 relations $[a,b]$ = identity, $[c,b]$ = identity and $[a,c] = b$.

1.7 In section 7 we define the type of conformal structure that we wish to consider and show that M_0, as constructed, in section 6, does not admit such a structure. We will define a conformal structure for a manifold M to be a pair (G,D) where G is a discrete subgroup of G^3 which is invariant and freely discontinuous on the set D in $\overline{R^3}$ and such that D/G is homeomorphic to M_0.

The first lemma we prove in section 7 states a necessary condition for (G,D) to be a conformal structure for a manifold M. Necessarily there exists a homomorphism from $\Pi_1(M)$ to G^3 with image G and such that D is a maximal connected G-invariant component of the set of discontinuity of G. This lemma clarifies the importance of knowing when two elements of G^3 commute.

Broadly, we prove that M_0 has no conformal structure by first showing that there does not exist a 1-1 homomorphism from $\Pi_1(M_0)$ into G^3. This implies that D cannot be simply connected. Next we show that if (G,D) is a conformal structure for M_0 then the limit set of G (= $\{x_0 \in \overline{R^3} \mid \lim_{k\to\infty} g_k(x) = x_0\}$ where $\{g_k\}_{k=1}^{\infty}$ is any collection of distinct elements of G and x is any element of $\overline{R^3}$) cannot be finite. If it were finite D would be simply connected or G/D would not be compact. The proof of the theorem is completed in two steps. First we show that any homomorphism $h: \Pi_1(M_0) \to G^3$ must carry b to an elliptic element of finite order. Finally we prove that h(b) of finite order implies that the limit set of G is finite.

Section 2

Definitions and Background

2.1 In this section we define what we will mean by a conformal transformation on R^k and on various submanifolds of R^k. This will enable us to define $\tilde{G}^n$, the group of all orientation preserving transformations of S^n onto itself as well as a number of other transformation groups isomorphic to $\tilde{G}^n$. Let X be a smooth n-dimensional manifold. For each $x \in X$ let $T(X,x)$ be the tangent space to X at x and let $I(X,x)$ be the set of all inner products on $T(X,x)$.

Definition: A *Riemannian metric on X* is a map, $\langle\ ,\ \rangle$, from X to the union over all $x \in X$ of $I(X,x)$ such that for each x in X $\langle\ ,\ \rangle(x)$ is an element of $I(X,x)$ and for any two smooth vector fields V_1 and V_2 on X, $\langle V_1, V_2\rangle(x)$ is a smooth function from X to R, the set of real numbers. The pair $(X, \langle\ ,\ \rangle)$ is called a *Riemannian manifold*. It will be necessary to distinguish between different Riemannian metrics on the same manifold X. This will either be done by using different shaped brackets, $\big($e.g., $\langle\ ,\ \rangle(x)$ and $[\ ,\ ](x)\big)$ or by assigning different letters to the various metrics and writing $M(\ ,\ )(x)$ or $P(\ ,\ )(x)$.

2.2 Definition: Two Riemannian metrics, $\langle\ ,\ \rangle$ and $[\ ,\ ]$, are said to be conformally equivalent if there exists a C^∞ positive real-valued function F defined on X such that $\langle v_1,v_2\rangle(x) = F(x)\cdot[v_1,v_2](x)$ for all $x \in X$ and for all v_1,v_2 in $T(X,x)$.

2.3 Definition: Given a smooth manifold X and a smooth map f from X to a Riemannian manifold $(Y,\langle\ ,\ \rangle)$ the pullback of $\langle\ ,\ \rangle$ by f, written $f^*\langle\ ,\ \rangle$, (or simply $^*\langle\ ,\ \rangle$ when it is clear which function is involved) is defined by the equation $f^*\langle v_1,v_2\rangle(x) = \langle f_*(v_1),f_*(v_2)\rangle\big(f(x)\big)$ where f_* is the differential of f.

An immediate consequence of the definitions of $\langle\ ,\ \rangle$ and f_* is that $f^*\langle\ ,\ \rangle$ is a Riemannian metric on X.

2.4 Definition: A smooth map f from a Riemannian manifold $(X,\langle\ ,\ \rangle)$ to a Riemannian manifold $(Y,[\ ,\ ])$ is said to be a conformal map if $\langle\ ,\ \rangle$ and $f^*[\ ,\ ]$ are conformally equivalent i.e., if there exists a smooth positive real-valued function F defined on X such that for all $x \in X$ and all v_1,v_2 in

$T(X,x), \langle v_1,v_2 \rangle = F(x)\cdot f^*[v_1,v_2](x)$
$= F(x)\cdot[f_*v_1,f_*v_2]\big(f(x)\big)$. If $F(x)$ is identically equal to 1 and f is a diffeomorphism then f is said to be an isometry, and $(X,\langle\ ,\ \rangle$ and $(Y,[\ ,\ ])$ are said to be isometric.

2.5 Let X be a submanifold of n-dimensional Euclidean space, R^n $(n \geq 1)$. Let f be any smooth real valued function on X. The tangent space $T(X,x)$ is spanned by the vectors $e_1(x),\ldots,e_n(x)$ where $e_i(x)(f) = (\partial f/\partial x_i)(x)$.

Definition: Given $v_1 = \sum_{i=1}^{n} \alpha_i e_i(x)$ and $v_2 = \sum_{i=1}^{n} \beta_i e_i(x)$ in $T(X,x)$ then the standard Riemannian metric (or the Euclidean metric) on X is given by $\langle v_1,v_2\rangle(x) = \sum_{i=1}^{n} \alpha_i\cdot\beta_i$.

2.6 Let S^n be the unit-sphere in R^{n+1}. We will denote the group of all conformal homeomorphisms of $(S^n$, standard Riemannian metric) onto itself by $\overline{G^n}$ $(n \geq 2)$. The subgroup of all orientation preserving elements will be denoted by G^n.

2.7 Let $\overline{R^n} = R^n \cup \{\infty\}$ be the one-point compactification of R^n. Let $\overline{R^n}$ have the differentiable structure compatible with the coordinate charts determined by the natural inclusion, inc, of R^n into $\overline{R}^n$ and the inversion map, i, from R^n into $\overline{R}^n$ defined as follows:

$$i(x_1,\ldots,x_n) = \begin{cases} \left(1/\sum_{i=1}^{n} x_i^2\right)\cdot(x_1,\ldots,x_n); & (x_1,\ldots,x_n) \neq (0,\ldots,0) \\ \infty & (x_1,\ldots,x_n) = (0,\ldots,0). \end{cases}$$

These coordinate charts give us a basis for $T(\overline{R}^n,x)$ for all $x \in R^n$. For $x \in \overline{R}^n - \{\infty\}$ let $\gamma_j(x) = \mathrm{inc}_*\big(e(x)\big)$ and let $\delta_j\big(i(x)\big) = i_*\big(e(x)\big)$ for $j = 1,\ldots,n$, then $\{\gamma_1(x),\ldots,\gamma_n(x)\}$ is a basis for $T(\overline{R}^n,x)$, $x \neq \infty$ and $\{\delta_1(x),\ldots,\delta_n(x)\}$ is a basis for $T\big(\overline{R}^n,i(x)\big)$, $x \neq 0$.

2.8 In this subsection we define a Riemannian metric $E(\ ,\ )$ on $\overline{R}^n$. In subsection 2.16 we will present an orientation-preserving conformal diffeomorphism from $\big(\overline{R}^n,E(\ ,\ )\big)$ to (S^n, standard Riemannian metric). Such a map implies that the group of conformal (orientation preserving) homeomorphisms of $\big(\overline{R}^n,E(\ ,\ )\big)$ onto itself is isomorphic to $\overline{G}^n$ (respectively G^n). We will label both of these groups $\overline{G}^n$ (respectively G^n) and refer to them as the "S^n-model" or "$\overline{R}^n$-model" when we wish to distinguish which space a transformation is acting on.

For $y \in R^n$ let $\|y\|$ be the standard Euclidean norm of y, i.e., $\|(y_1,\dots,y_n)\| = (y_1^2 + \dots + y_n^2)^{\frac{1}{2}}$. Let y be a C^∞ function from $[0,\infty]$ to $[0,1]$ such that $g(t) = 1$ for $t \leq \frac{1}{2}$ and $g(t) = 0$ for $t \geq 2$. Define g_1, g_2 from $\overline{R^n}$ to $[0,1]$ by:

$$g_1(x) = g(\|x\|)/g(\|x\|) + g(\|i(x)\|)$$

and $$g_2(x) = g(\|i(x)\|/g(\|x\|) + g(\|i(x)\|).$$

$\{g_1, g_2\}$ is clearly a partition of unity for $\overline{R^n}$ with respect to the cover $S_1 = \{x \in \overline{R^n} : \|x\| < 2\}$ and $S_2 = \{x \in \overline{R^n} : \|x\| > \frac{1}{2}\}$. (By a slight abuse of notation we let $\|\infty\| = \infty$ and have $\infty \in S_2$.) Note that $g_1\big(i(x)\big) = g_2(x)$ and $g_2\big(i(x)\big) = g_1(x)$. For $v_k = \sum_{j=1}^{n} \alpha_{k,j}\gamma_j(x)$ $(k = 1,2)$ let $[v_1, v_2](x) = \sum_{j=1}^{n} \alpha_{1,j} \cdot \alpha_{2,j}$. For $w_k = \sum_{j=1}^{n} \beta_{k,j}\delta_j(x)$ $k = 1,2)$ let $\langle w_1, w_2\rangle(x) = \sum_{j=1}^{n} \beta_{1,j}\beta_{2,j}$. Finally we define our Riemannian metric E(,) by the equation $E(u_1,u_2)(x) = g_1(x)\cdot[u_1,u_2](x) + g_2(x)\langle u_1,u_2\rangle(x)$ for $u_1, u_2 \in T(\overline{R^n}, x)$. Unless it is explicitly stated otherwise $\overline{R^n}$ will always be assumed to carry this metric.

2.9 Examples of elements in the $\overline{R^n}$ model of $\overline{G^n}$: the conformality of these transformations is proven below.

1. Translation: $T(x) = \begin{cases} x + x_0 & x \in \overline{R^n} - \{\infty\} \\ \infty & x = \infty \end{cases}$

   for some fixed x_0 in $\overline{R^n} - \{\infty\}$.

2. Rotation: $R(x_1,\ldots,x_n)$

   $$= \Big(\sum_{j=1}^{n} r_{i,j}\cdot x_r,\ldots,\sum_{j=1}^{n} r_{n,j}x_j\Big)$$

   and $R(\infty) = \infty$

   where $\sum_{i=1}^{n} r_{i,j}\cdot r_{i,k} = \delta_{j,k}$ for $i = 1,\ldots,n$.

3. Dilation: $D(x) = t_0\cdot x,\ x \in R^n - \{\infty\},\ 0 < t_0 < \infty$

   $D(\infty) = \infty$

4. Inversion: $I_n(x) = x/\|x\|^2,\ x \neq (0,\ldots,0),\infty$

   $I_n(0,\ldots,0) = \infty$ and

   $I_n(\infty) = (0,\ldots,0)$.

We remark that inversion is just the extension of the inversion map defined in subsection 2.7. (We will suppress the subscript from I_n whenever the dimension, n is clear or arbitrary.)

2.10 In this subsection we verify the conformality of the above transformation on $(\overline{R^n},E)$. We note first that I_n is in fact an isometry of $\big(\overline{R^n},E(\ ,\ )\big)$. For v_1,v_2 in $T(\overline{R^n},x)$

$E(v_1,v_2)(x) = g_1(x)\cdot[v_1,v_2](x) + g_2(x)\langle v_1,v_2\rangle(x)$.

$I_*\big(\gamma_j(x)\big) = \delta_j\big(I(x)\big)$ and $I_*\big(\delta_j(x)\big) = \gamma_j\big(I(x)\big)$

therefore $E(v_1,v_2)(x) = g_1(x)\langle I_*(v_1),I_*(v_2)\rangle\big(I(x)\big)$

$$+ g_2(x)\cdot[I_*(v_1),I_*(v_2)]\big(I(x)\big).$$

We also have $g_1(x) = g_2\big(I(x)\big)$ and $g_2(x) = g_1\big(I(x)\big)$, therefore $E(v_1,v_2)(x) = g_2\big(I(x)\langle I_*(v_1),I_*(v_2)\rangle\big(I(x)\big)$

$$+ g_1\big(I(x)[I_*(v_1),I_*(v_2)]\big(I(x)\big)$$

$$= E\big(I_*(v_1),I_*(v_2)\big)\big(I(x)\big).$$

We next observe that restricted to $\overline{R^n} - \{\infty\}$ $E(\ ,\ )$ is conformally equivalent to the standard Riemannian metric on R^n (which we denote by $\langle\ ,\ \rangle(x)$), i.e., $inc:(R^n,\langle\ ,\ \rangle) \to \big(\overline{R^n} - \{\infty\},E(\ ,\ )\big)$ is conformal. We compute:

$$E(v_1,v_2) = g_1(x)[v_1,v_2](x) + g_2(x)\langle v_1,v_2\rangle(x)$$

$$= g_1(x)\langle(inc^{-1})_*(v_1),(inc^{-1})_*(v_2)\rangle(x)$$

$$+ g_2(x)/\|x\|^4\cdot\langle(inc^{-1})_*(v_1),(inc^{-1})_*(v_2)\rangle(x)$$

$$= \big(g_1(x) + \big(g_2(x)/\|x\|^4\big)$$

$$\langle(inc^{-1})_*(v_1),(inc^{-1})_*(v_2)\rangle(x).$$

Since ∞ is a fixed point for T, R and D the conformality of these maps on $\big(\overline{R^n} - \{\infty\},E(\ ,\ )\big)$ is equivalent to their conformality on $(R^n,\langle\ ,\ \rangle)$. T and R are isometries on $(R^n,\langle\ ,\ \rangle)$ since with respect to $\{e_1(x),\ldots,e_n(x)\}$ the matrix for T_* is the identity matrix and the matrix for R_* is R so that $\langle R_*(e_i(x)), R_*(e_j(x))\rangle = \delta_{i,j} = \langle e_i(x), e_j(x)\rangle$. With respect to this same basis D_* has the matrix t times the identity matrix therefore $\langle D_*(e_i(x)), D_*(e_j(x))\rangle = t^2 \cdot \delta_{i,j}$. To complete the verification of the conformality of T, R and D it is only necessary to consider what happens at infinity. Since $I^{-1}RI(x) = R(x)$ for all x in $\overline{R^n}$

$$E(R_*(\delta_i(\infty)), R_*(\delta_j(\infty)))(\infty)$$
$$= (I_*R_*(\gamma_i(0)), I_*R_*(\gamma_j(0)))(0)$$
$$= E\left(\sum_{k=1}^{n} r_{k,i} \cdot \delta_i(\infty), \sum_{k=1}^{n} r_{k,j}\delta_j(\infty)\right)$$
$$= \delta_{i,j} = E(\delta_i(\infty), \delta_j(\infty)).$$

Similarly

$$E(D_*(\delta_i(\infty)), D_*(\delta_j(\infty)))(\infty)$$
$$= E(I_*(DI)_*(\gamma_i(0)), I_*(DI)_*(\gamma_j(0)))(0)$$

(since I is an isometry)

$$= E(1/t \cdot \gamma_i(0), 1/t\gamma_j(0))(0)$$
$$1/t^2\delta_{i,j} = 1/t^2 E(\delta_i(\infty), \delta_j(\infty)).$$

To show that T is conformal it suffices to consider the case where $x_0 = (1,0,\ldots,0)$ as any other translation can be conjugated into this one by using rotations and dilations. As before we have

$$E(T_*(\delta_i(\infty)), T_*(\delta_j(\infty)))(\infty)$$
$$= E((I^{-1}TI)_*(\gamma_i(0)), (I^{-1}TI)_*(\gamma_j(0)))$$

A computation yields

$$I^{-1}TI(x_1,\ldots,x_n) = 1/(\|x\|^2 + 2x_1 + 1)$$
$$\cdot(x_1 + \|x\|^2, x_2, \ldots, x_n)$$

At $x = (x_1,\ldots,x_n)$ the matrix for $(I^{-1}TI)_*$ with respect to the basis $\{\gamma_1(x),\ldots,\gamma_n(x)\}$ is

$$(\|x\|^2 + 2x_1 + 1)^{-2}$$

$$\cdot \begin{pmatrix} 1 + 2x_1 + 2x_1^2 - \|x\|^2 & -2x_2(x_1+1) \ldots & -2x_n(x_1+1) \\ -2x_2(x_1+1) & \|x\|^2 - 2x_2^2 + 2x_1 + 1 \ldots & -2x_2x_n \\ \vdots & \vdots \quad \ddots & \vdots \\ -2x_n(x_1+1) & -2x_nx_2 \ldots & \|x\|^2 - 2x_n^2 + 2x_1 + 1 \end{pmatrix}$$

At $x = (0,\ldots,0)$ this becomes the identity matrix. Therefore $E(I^{-1}TI)_*(\gamma_i(0)), (I^{-1}TI)_*(\gamma_j(0))(0)$
$= E(\gamma_i(0), \gamma_j(0)) = \delta_{i,j} = E(\delta_i(0), \delta_j(0))$.
This completes the verification of the conformality of I, T, R and D.

2.11 The importance of these examples of conformal maps of $(\overline{R^n}, E(\ ,\ ))$ is made clear by the following theorem due to Liouville (Blashke [1]).

Theorem: (Liouville) The $\overline{R}^n$-model of $\overline{G}^n$ is generated by translations, rotations, dilations and inversions. We have four useful corollaries to this theorem.

2.12 Corollary: $\overline{G}^n = G^n \cup IG^n$ and therefore G^n is a normal subgroup of index 2 in $\overline{G}^n$.

Proof: I is the only generator which reverses orientation and $I = I^{-1}$.

We generalize the usual notion of a "k-dimensional sphere" in $\overline{R}^n$ to include extended k-dimensional affine linear subspaces $P^k \cup \{\infty\}$ in $\overline{R}^n$ $(k = 1,\ldots,n-1)$. With this definition of k-dimensional sphere (or simply k-sphere) in $\overline{R}^n$ we get our second corollary to Liouville's theorem.

2.13 Corollary: Every element of $\overline{G}^n$ preserves the family of k-spheres in $\overline{R}^n$ $(k = 1,\ldots,n-1)$.

Proof: For translations and rotations this follows immediately from the fact that they are Euclidean isometries with a fixed point at ∞. For dilations and inversion it suffices to prove to corollary for (n-1) dimensional spheres as the lower dimensional spheres can be written as the intersection of such spheres. We first consider the action of a dilation D on $\overline{R}^n$. Let S be a Euclidean (n-1)-dimensional sphere of radius r centered at the point x_0 of the form $S = \{v \in R^n \mid \|v - x_0\| = r$. $D(S)$ is then of the form $D(S) = \{k \cdot V \in R^n \mid \|v - x_0\| = r\}$ (for some $k > 0$) which equals $\{kv \in R^n \mid \|kv - kx_0\| = kr\}$--the (n-1) dimensional sphere of radius $k \cdot r$ centered at $k \cdot x_0$. Next consider a (n-1) sphere P which contains ∞. P is of the form $P = \{v \in R^n \mid \langle v,m \rangle = p\} \cup \{\infty\}$ where $p \in R$, $\langle \ , \ \rangle$ is standard euclidean inner product on R^n and m is a fixed vector in R^n of length 1. Transforming P by a dilation D we get $D(P) = \{k \cdot v \in R^n \mid \langle v,m \rangle = p\} \cup \{\infty\}$ (for some $k > 0$) which equals $\{k \cdot v \in R \mid \langle kv,m \rangle = k \cdot p\} \cup \{\infty\}$. This is also a (n-1) sphere thru ∞ and the corollary is established for dilations.

Note that for translations, rotations and dilations ∞ is a fixed point and therefore "finite" (n-1)-sphere (i.e., spheres not containing $\{\infty\}$) are carried to themselves by these transformations. Similarly extended spheres (i.e., spheres thru ∞) are carried to themselves. However the inversion I_n interchanges 0 and ∞. This explains the necessity for enlarging our notation of a sphere to include "extended" spheres. It also necessitates separate considerations for spheres which contain 0 in the proof of this corollary.

Let $S = \{v \in R^n \mid \|v - x_0\| = r\}$ as above. If

$0 \notin S$ then a straightforward calculation shows that $I_n(S)$ is a (n-1)-sphere with center $x_0' = (1/\|x_0\|^2 - r^2)\cdot x_0$ and radius $(\|x_0\|^2 - 1)/(\|x_0\|^2 - r^2)$. S will contain 0 if and only if $\|x_0\| = r$. In this case $S = \{v \mid \|v\|^2 = 2\langle v,x\rangle\}$. For all $v \in S - \{0\}$ we obtain $\langle I_n(v), x_0/\|x_0\|\rangle = \langle v/\|v\|^2, x_0/\|x_0\|\rangle$ $= \left(1/(\|v\|^2\cdot\|x_0\|)\right)\cdot\langle v,x\rangle = 1/(2\|x_0\|)$. $I_n(S)$ must therefore be the extended sphere $\{v \in R^n \mid \langle v, x_0/\|x_0\|\rangle = 1/(2\|x_0\|)\} \cup \{\infty\}$.

Finally we consider the action of I_n on the"extended" sphere $P = \{v \in R^n \mid \langle v,n\rangle = p\} \cup \{\infty\}$ as above. If $p = 0$ then clearly $I_n(P) = P$ as $\langle v/\|v\|, n\rangle = (1/\|v\|)\cdot\langle v,n\rangle = 0$ for all $v \in P - \{0,\infty\}$. If $p \neq 0$ then an easy computation shows that $I_n(P)$ is a (n-1) sphere with center n/2p and radius 1/4p. qed

2.14 <u>Corollary</u>: Given any two k-spheres Σ_1 and Σ_2 in R^n $(k = 1,\ldots,n-1)$ there exists g in G^n such that g sends Σ_1 to Σ_2.

<u>Proof</u>: It suffices to assume that Σ_2 is an extended k-plane thru the origin. If Σ_1 is also such a plane then g can be taken to be a rotation R. If Σ_1 is an extended plane which does not contain the origin let g = RT where T is a translation carrying some point of Σ_2 to the origin. If ∞ is not an element of Σ_1

let T_2 be a translation sending a point p in Σ_1 to the origin. IT_2 will send Σ_1 to an extended plane and we are reduced to the previous case, $RTIT_2$ however, reverses orientation. Therefore we let $g = IRTIT_2$ (note that I maps Σ_2 onto itself). qed

2.15 <u>Corollary</u>: Let f be in $\overline{G^n}$ and let Σ^{n-1} be an (n-1)-sphere in $\overline{R^n}$ such that f restricted to Σ^{n-1} is the identity. Then

i. $f \in G^n$ implies that f is the identity on all of $\overline{R^n}$ and

ii. $f \cdot I \in G^n$ implies that there exists $g \in G^n$ such that $gfg^{-1} = I$.

<u>Proof</u>:

i. Σ^{n-1} divides $\overline{R^n}$ into two disjoint open, connected regions R_1 and R_2. Since f is orientation preserving, in case i. it maps each of these regions onto itself. Let p be any point in one of these regions, say R_1. Let S and S' be 1-spheres which intersect Σ^{n-1} orthogonally and such that $S \cap S' = \{p, q\}$ where $q \in R_2$. We will show that S and S' must each get mapped to themselves implying that $\{f(p), f(q)\} = \{p, q\}$. Since f sends R_1 to itself this means that $f(p) = p$ and proves our assertion. By corollary 2.13, f(S) must be a 1-sphere. By the assumption of this corollary $S \cap \Sigma^{n-1} = f(S) \cap \Sigma^{n-1}$. By the conformality of f $f(S)$ must be orthogonal to Σ^{n-1}. There can only be one 1-sphere intersecting Σ^{n-1} orthogonally at $S \cap \Sigma^{n-1}$; thus $f(S)$ must equal S. Similarly $f(S') = S'$. Note that proof of the equality $f(S) = S$ does not depend on S being a 1-sphere and would work for any k-sphere ($k = 1,2,\ldots,n-1$) orthogonal to Σ^{n-1}.

ii. By corollary 2.14 we can find $g \in G^n$ such that g maps Σ^{n-1} to S^{n-1}. $gfg^{-1}I$ is the identity on S^{n-1} therefore by part i above $gfg^{-1}I$ = identity or $gfg^{-1} = I^{-1} = I$. qed

2.16 An important consequence of corollary 2.14 is the existence of a conformal diffeomorphism from $\left(\overline{R^n}, E(\ ,\ )\right)$ to (S^n, standard Riemannian metric). Consider $\overline{R^n}$ and S^n as n-spheres in $\overline{R^{n+1}}$. By corollary 2.14 there exists a conformal map of $\left(\overline{R^{n+1}}, E(\ ,\ )\right)$ taking $\left(\overline{R^n}, E(\ ,\ )\right)$ to $\left(S^n, E(\ ,\ )\right)$. As we observed in subsection 2.10 on $\overline{R^{n+1}} - \{\infty\}$ $E(\ ,\ )$ is conformally equivalent to the standard Riemannian metric. Therefore $\left(S^n, E(\ ,\ )\right)$ is conformally equivalent to (S^n, standard Riemannian metric) and the map of corollary 2.14 restricted to $\overline{R^n}$ (as contained in

$\overline{R^{n+1}}$) establishes conformal equivalence between $\left(\overline{R^n}, E(\ ,\ )\right)$ and $(S^n$, Standard Riemannian metric).

The remainder of section 2 is devoted to presenting three models of $\overline{G^n}$. In each case $\overline{G^n}$ will be described as a group of isometries acting on a space diffeomorphic to

$H^{n+1} = \{(x_1,\ldots,x_{n+1}) \in R^{n+1} : x_{n+1} > 0\}$.

2.17 In this subsection we extend the action of $\overline{G^n}$ on $\overline{R^n}$ to $\overline{R^{n+1}}$ in a natural way. Using Liouville's theorem we must only define this extension e_n for translations, rotations, dilations, and inversion. Let T, R, D and I_n be defined as in subsection 2.9. We set

i. $e_n(T)(x_1,\ldots,x_n,x_{n+1})$

$= \left(T(x_1,\ldots,x_n),x_{n+1}\right)$,

$e_n(T)(\infty) = \infty$;

ii. $e_n(R)(x_1,\ldots,x_n,x_{n+1})$

$= \left(R(x_1,\ldots,x_n),x_{n+1}\right)$,

$e_n(R)(\infty) = \infty$;

iii. $e_n(D)(x_1,\ldots,x_n,x_{n+1})$

$= t(x_1,\ldots,x_n,x_{n+1})$,

where $D(x_1,\ldots,x_n) = t(x_1,\ldots,x_n)$;

iv. $e_n(I_n) = I_{n+1}$.

Note that $e_n(\overline{G^n})$ $\big($respectively $e_n(G^n)\big)$ is a subgroup of $\overline{G^{n+1}}$ (respectively G^{n+1}) which keeps invariant $\overline{R^n}$ and H^{n+1}.

In fact $e_n(\overline{G^n})$ is the maximal subgroup of $\overline{G^{n+1}}$ with this property. If f in $\overline{G^{n+1}}$ preserves $\overline{R^n}$ then let f_0 be f restricted to $\overline{R^n}$ and let $g = e_n(f_0)$. $g^{-1}f$ is the identity on $\overline{R^n}$ and is orientation preserving on $\overline{R^{n+1}}$ therefore by corollary 2.15 $g^{-1}f$ is the identity, i.e., $f = g \in e_n(\overline{G^n})$. Using corollary 2.15 it is also clear that e_n is an injective homomorphism from $\overline{G^n}$ (respectively G^n) into $\overline{G^{n+1}}$ (respectively G^{n+1}). We will refer to $e_n(\overline{G^n})$ $\big($respectively $e(G^n)\big)$ as the H^{n+1}-model of $\overline{G^n}$ (respectively G^n).

2.18 In this subsection we define a Riemannian metric $P(\ ,\ )$ on H^{n+1} and show that the H^{n+1}-model of $\overline{G^n}$ is the full group of isometries on $\big(H^{n+1}, P(\ ,\ )\big)$.

<u>Definition</u>: Let H^{n+1} have the standard differentiable structure of an open subset of R^{n+1}. Let $\{e_1(x),\dots,e_{n+1}(x)\}$ be a basis for $T(H^{n+1},x)$ as defined in 2.5. $P\big(e_i(x),e_j(x)\big)(x) = (1/x_{n+1})^2\cdot\delta_{i,j}$ where $x = (x_1,\dots,x_{n+1})$. $P(\ ,\ )$ is usually referred to as the <u>Poincare metric for H^{n+1}</u>.

Proposition: The H^{n+1}-model of $\overline{G^n}$ is the full group of isometries of $\left(H^{n+1}, P(\ ,\ )\right)$.

The proof of this proposition is quite lengthy. For this reason we do it in two subsections, 2.18 and 2.19. In this subsection (2.18) we show that $e_n(\overline{G^n})$ is a group of isometries of $\left(H^{n+1}, P(\ ,\ )\right)$. In 2.19 we show conversely that every isometry of $\left(H^{n+1}, P(\ ,\ )\right)$ is in $e_n(\overline{G^n})$.

To show that $e_n(\overline{G^n})$ is a group of isometries of $\left(H^{n+1}, P(\ ,\ )\right)$ it suffices to check that $e_n(T)$, $e_n(R)$, $e_n(D)$ and $e_n(I)$ of subsection 2.17 are isometries. Note that $P(\ ,\ )$ is conformal to $\langle\ ,\ \rangle$ the standard Riemannian metric defined on all of R^{n+1}. We have already observed in subsection 2.10 that translations and rotations are isometries with respect to $\langle\ ,\ \rangle$ therefore for all $v_1, v_2 \in T(H^{n+1}, x)$, where $x = (x_1, \ldots, x_{n+1})$, we have:

$$\begin{aligned} &P(v_1, v_2)(x_1, \ldots, x_{n+1}) \\ &\quad = (1/x_{n+1})^2 \langle v_1, v_2\rangle(x) \\ &\quad = (1/x_{n+1})^2 \langle (e_nT)_*(v_1), (e_nT)_*v_2\rangle\big((e_nT)(x)\big) \\ &\quad = P\big((e_nT)_*(v_1), (e_nT)_*(v_2)\big)\big((e_nT)(x)\big), \end{aligned}$$

and

$P(v_1, v_2)(x_1, \ldots, x_{n+1})$

$= (1/x_{n+1})^2 \langle v_1, v_2\rangle(x)$

$= (1/x_{n+1})^2 \langle (e_nR)_*(v_1)(e_nR)_*(v_2)\rangle\big((e_nR)(x)\big)$

$= P\big((e_nR)_*(v_1), (e_nR)_*(v_2)\big)\big(e_nR(x)\big).$

Similarly for $e_n(D)$ we have

$P(v_1, v_2)(x_1, \ldots, x_{n+1})$

$= (1/x_{n+1})^2 \langle v_1, v_2\rangle(x)$

$= (1/x_{n+1})^2 \cdot (1/t)^2 \langle (e_nD)_*(v_1), (e_nD)_*(v_2)\rangle\big(e_nD(x)\big)$

$= (1/t\cdot x_{n+1})^2 \langle (e_nD)_*(v_1), (e_nD)_*(v_2)\rangle\big(e_nD(x)\big)$

$= P\big((e_nD)_*v_1, (e_nD)_*v_2\big)\big(e_nD(x)\big).$

Finally $e_n(I_n) = I_{n+1}$. A simple computation shows that $\langle (I_{n+1})_*(v_1), (I_{n+1})_*(v_2)\rangle(x)$ $= (1/\|x\|^4)\langle v_1, v_2\rangle(x)$. Therefore we have

$P(v_1, v_2)(x_1, \ldots, x_{n+1})$

$= (1/x_{n+1})^2 \langle v_1, v_2\rangle(x)$

$= (\|x\|^2/x_{n+1})^2 \cdot \langle (I_{n+1})_*v_1, (I_{n+1})_*v_2\rangle\big(I_{n+1}(X)\big)$

$= 1/(x_{n+1}/\|x\|^2)^2 \cdot \langle (I_{n+1})_*v_1, (I_{n+1})_*v_2\rangle\big(I_{n+1}(x)\big)$

$= P\big((I_{n+1})_*v_1, (I_{n+1})_*v_2\big)\big(I_{n+1}(x)\big).$

This proves that $e_n(\overline{G^n})$ is a group of isometries of $\big(H^{n+1}, P(\ ,\ )\big)$.

2.19 In this subsection we show that any isometry g of $\left(H^{n+1}, P(\ ,\ )\right)$ is in $e_n(\overline{G^n})$. We will use the following theorem (See Helgason [5] lemma 11.2, p. 62).

<u>Theorem</u>: Given a connected Riemannian manifold M and an isometry $f: M \to M$ such that for some m in M $f(m) = m$ and $f_*: T(M,m) \to T(M,m)$ is the identity then f is the identity on M.

For any isometry g of $\left(H^{n+1}, P(\ ,\ )\right)$ we will construct an element h of $e_n(\overline{G^n})$ such that $f = hg$ satisfies this theorem. Thus $f = hg =$ identity or $g = h^{-1} \in e_n(\overline{G^n})$.

Assume first that g is orientation preserving. Let $g(0,\ldots,0,1) \in (r_1,\ldots,r_{n+1})$. Since $r_{n+1} > 0$ we can define the dilation h_1 by the equation $h_1(x_1,\ldots,x_{n+1}) = (1/r_{n+1})\cdot(x_1,\ldots,x_{n+1})$ and the translation h_2 by the equation $h_2(x_1,\ldots,x_{n+1}) = \left(x_1 - (r_1/r_{n+1}),\ldots,x_n - (r_1/r_{n+1}), x_{n+1}\right)$. $(h_2h_1g)(0,\ldots,0,1) = (0,\ldots,0,1)$ therefore $(h_2h_1g)_*$ is a rotation of $T\left(H^{n+1},(0,\ldots,0,1)\right)$. In order to simplify notation let $x_0 = (0,\ldots,0,1)$ and $(h_2h_1g)_* = \tau$. If $\tau\left(\epsilon_{n+1})(x_0)\right) = \epsilon_{n+1}(x_0)$ then τ has a matrix representation relative to the basis $\{e_1(x_0),\ldots,\epsilon_{n+1}(x_0)\}$ of the form

$$\begin{pmatrix} & & & 0 \\ & R & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$ where R is a rotation of the subspace of $T(H^{n+1}, x_0)$ generated by $\{e_1(x_0), \ldots, e_n(x_0)\}$. Let R_0 be the rotation of R^n with matrix representation R relative to the standard basis of R^n. Let $h_3 = e_n(R_0^{-1})$. Let $h = h_3 h_2 h_1$ and let $f = hg$. f clearly has a fixed point at x_0 and is the identity on $T(H^{n+1}, x_0)$. Thus $g = h^{-1} \in e_n(G^n)$. In fact $h^{-1} \in e_n(G^n)$.

We next assume that $\tau(e_{n+1}(x_0)) \neq e_{n+1}(x_0)$. To complete the proof in this case we construct an element k of $e_n(G^n)$ such that $(kh_2h_1g)_*(e_{n+1}(x_0)) = e_{n+1}(x_0)$ reducing this case to the previous one. Let l_1 and l_2 be parametrically defined lines in H^{n+1} such that $l_1(0) = l_2(0) = x_2$, $(l_1)_*(0) = e_{n+1}(x_0)$ and $(l_2)_*(0) = \tau(e_{n+1}(x_0))$. If $e_{n+1}(x_0)$ and $(e_{n+1}(x_0))$ are linearly independent then l_1 and l_2 will determine a 2-plane P in H^{n+1} passing thru x_0. If $e_{n+1}(x_0)$ and $(e_{n+1}(x_0))$ are linearly dependent redefine l_2 by the equation $l_2(t) = (0, \ldots, 0, t, 1)$. l_1 and l_2 will again determine a plane P as above. Without loss of generality we can assume $P = \{(0, \ldots, 0, x, y) \in H^{n+1}\}$. We will now construct k so that $k(x_0) = x_0$, $k \in e_n(G^n)$, $(k)_*$ is a rotation of the subspace of $T(H^{n+1}, x_0)$ determined by P and is the identity on the complement of this subspace and $k_*\big(\tau e_{n+1}(x_0)\big) = e_{n+1}(x_0)$. Let $r:\overline{R^n} \to \overline{R^n}$ be defined as before by the equation $r(x_1,\ldots,x_n) = (x_1,\ldots,x_{n-1},-x_n)$ and $r(\infty) = \infty$. Let $(s_1,\ldots,s_{n-1},s_n) = (1,\ldots,1,-1)$, then with respect to the standard bases for $T(\overline{R^n},x)$ and $T\big(\overline{R^n},r(x)\big)$, $(x \neq \infty)$, (r_*) has the matrix $\underline{(r_*)}_{i,j} = s_i \cdot \delta_{i,j}$. Clearly r is an isometry on $\overline{R^n} - \{\infty\}$. The following equalities demonstrate that r is an isometry at ∞ as well; $E(r_*\big(\delta_i(\infty)\big), r_*\big(\delta_j(\infty)\big))(\infty)$

$= E(I_* r_*\big(\gamma_i(0)\big), I_* r_*\big(\gamma_j(0)\big))(0)$ (since $r = I^{-1}rI$)

$= E\big(s_i \cdot \delta_i(\infty), s_j \delta_j(\infty)\big)(\infty) = \delta_{ij}$

$= E\big(\delta_i(\infty), \delta_j(\infty)\big)(\infty)$. A similar argument shows that the transformation $r':\overline{R^{n+1}} \to \overline{R^{n+1}}$ defined by $r'(x_1,\ldots,x_n,x_{n+1}) = (x_1,\ldots,x_{n-1},-x_n,x_{n+1})$ and $r'(\infty) = \infty$ is an isometry of $\big(\overline{R^{n+1}}, E(\ ,\ )\big)$. r' restricted to $\overline{R^n}$ equals r therefore r' must equal $e_n(r)$. We define two more maps. T_a, a translation in G^n, and D_a, a dilation in G^n, which we use to define k. Given $a \in R$ let $T_a(x_1,\ldots,x_n) = (x_1,\ldots,x_{n-1},x_n - a)$ and $D_a(x_1,\ldots,x_n) = \big(1/(1+a^2)^{\frac{1}{2}}\big)\cdot(x_1,\ldots,x_n)$. Let $k_a = e_n(rT_a^{-1}D_a^{-1}I_nD_aT_a)$. An algebraic computation gives us

$$k_a(x_1,\dots,x_{n+1})$$
$$= \Big((1+a)^2/\big(\sum_{i=1}^{n+1} x_i^2 - 2x_n a + a^2\big)(x_1,\dots,x_{n-1},a-x_n,x_{n+1})$$
$$+ (0,\dots,0,-a,0).$$

Clearly $k_a(x_0) = k_a(0,\dots,0,1) = x_0$. k_a is the composition of two isometries, $r' = e_n(r)$ and $e_n(T_a^{-1}D_a^{-1}I_nD_aT_a)$, and therefore is itself an isometry. (Note that $e_n(T_a^{-1}D_a^{-1}I_nD_aT_a)$ is an isometry as it is conjugate to the isometry $e_n(I_n) = I_{n+1}$.) The proof of this part of our proposition is completed by showing that for an appropriate choice of constant $a_0 \in R$ $(k)_* = (k_{a_0})_*$ has the desired action on $T(\overline{R^{n+1}},x_0)$. We compute $(k_a)_* = \big(e_n(rT_a^{-1}D_a^{-1}I_nD_aT_a)\big)_*$ $= e_n(r)_*e_n(D_aT_a)_*^{-1}e_n(I_n)_*e_n(D_aT_a)_*$ or equivalently $\big(e_n(r)\big)_*^{-1}(k_a)_* = e_n(D_aT_a)_*^{-1}e_n(I_n)_*e_n(D_aT_a)_*$. In matrix form $\big(\big(e_n(D_aT_a)\big)_*(x)\big)_{i,j} = \big(1/(a^2+1)^{\frac{1}{2}}\big)\cdot\delta_{1,j}$ for all x in $\overline{R^{n+1}} - \{\infty\}$. Thus we get $\big(e_n(r)\big)_*^{-1}\cdot(k_a)_*(x_0) = e_n(I_n)_*\big(D_aT_a(x_0)\big)$ or equivalently $(k_a)_*(x_0) = e_n(r)_*\cdot e_n(I_n)_*\big(D_aT_a(x_0)\big)$. $e_n(r)_*$ and $e_n(I_n)_*$ are easily computed and we get $\big((k_a)_*(x_0)\big)_{i,j} = \delta_{i,j}$ for $(i,j) \notin \{(n-1,n-1),(n-1,n),(n,n-1),(n,n)\}$.

$((k_a)_*(x_0))_{n-1,n-1} = ((k_a)_*(x_0))_{n,n} = (a^2-1)/(a^2+1)$ and $((k_a)_*(x_0))_{n,n-1} = -((k_a)_*(x_0))_{n-1,n}$ $= 2a/(a^2+1)$. For all $a \in R$ $-1 \leq (a^2-1)/(a^2+1) \leq 1$ and $((a^2-1)/(a^2+1))^2 + (2a/(a^2+1))^2 = 1$, i.e., restricted to the hyperplane P of $T(H^{n+1}, x_0)$ $(k_a)_*(x_0)$ is a rotation thru the angle arccos $((a^2-1)/(a^2+1))$. Thus for an appropriate choice of $a_0 \in R$ $(k_{a_0})_*(\tau(\epsilon_{n+1}(x_0))) = (k)_*(\tau(\epsilon_{n+1}(x_0))) = \epsilon_{n+1}(x_0)$ and we have reduced this case to the one in which $\tau(\epsilon_{n+1}(x_0)) = \epsilon_{n+1}(x_0)$. Finally if g is not orientation preserving then $e_n(I_n)g$ is an orientation preserving isometry of $(H^{n+1}, P(\ ,\ ))$ therefore $(e_n(I_n)g) \in e_n(\overline{G}^n)$ and $g \in e_n(\overline{G}^n)$.

2.20 For the sake of completeness we describe another model of $\overline{G}^n$ in this subsection which is almost identical to the H^{n+1}-model. Let $D^{n+1} = \{x \in R^{n+1} \mid \|x\| < 1\}$. Our new model will be a transformation group on D^{n+1}. We define conformal transformations T_1, T_2, and r on $\overline{R^{n+1}}$ such that $f = rT_2I_{n+1}T_1$ maps D^{n+1} to H^{n+1}. f is then used to pull back the metric P on H^{n+1} to D^{n+1}. Let

$T_1(x_1,\ldots,x_{n+1}) = (x_1,\ldots,x_n,x_{n+1}-1)$,
$T_2(x_1,\ldots,x_{n+1})=(x_1,\ldots,x_n,x_{n+1}+\frac{1}{2})$, $T_i(\infty) = \infty$,
$i = 1,2$ and $r(x_1,\ldots,x_{n+1}) = (x_1,\ldots,x_n,-x_{n+1})$.
Then $f = rT_2I_{n+1}T_1$ is the composition of conformal maps of $\left(\overline{R^{n+1}},E(\ ,\ )\right)$. A simple computation shows that

$$f(x_1,\ldots,x_{n+1}) = \left(1/2(\|x\|^2 - 2x_{n+1} + 1)\right)\cdot\left(2x_1,\ldots,2x_n,(1 - \|x\|^2)\right)$$

and $f(\infty) = (0,\ldots,0,-\frac{1}{2})$. Note that f is one to one, carries S^n to the n-dimensional extended hyperplane $x_{n+1} = 0$ and $f(0,\ldots,0) = (0,\ldots,0,\frac{1}{2})$. Therefore f maps D^{n+1} onto H^{n+1}. We use f to pull back the Poincare metric $P(\ ,\ )$ on H^{n+1} to get $P^*(\ ,\ )$ a metric on D^{n+1}. We will refer to P^* as the Poincare metric on D^{n+1}. A tedious but straight forward computation shows that for $v_1,v_2 \in T(D^{n+1},x)$

$$P^*(v_1,v_2)(x) = \left(1/(1 - \|x\|^2)\right)\cdot\langle v_1,v_2\rangle(x).$$

Since P^* is the pullback by f of P, f is an isometry of $\left(D^{n+1},P^*(\ ,\ )\right)$ onto $\left(H^{n+1},P(\ ,\ )\right)$. Conjugation by f therefore is an isomorphism of the full group of isometries of $\left(H^{n+1},P(\ ,\ )\right)$ to the full group of isometries of $\left(D^{n+1},P^*(\ ,\ )\right)$. Moreover, since f is conformal on $\left(\overline{R^{n+1}},E(\ ,\ )\right)$, the group of isometries of $\left(D^{n+1}, P^*(\ ,\ )\right)$ is exactly the subgroup of $\overline{G^{n+1}}$ which preserves D^{n+1}.

2.21 Finally we describe $\overline{G^n}$ as a matrix group. Let $\underline{H}^{n+1} = \{(x_1,\ldots,x_{n+2}) \in R^{n+2} : \sum_{i=2}^{n+2} x_i^2 = x_1^2 - 1 \text{ and } x_1 > 0\}$. For $x \in \underline{H}^{n+1}$ let $v_1 = \sum_{i=1}^{n+2} \alpha_i e_i(x)$ and $v_2 = \sum_{i=1}^{n+2} \beta_i e_i(x)$ be in the subspace of $T(R^{n+2},x)$ tangent to $\underline{H}^{n+1}$. Define a Riemannian metric $L(\ ,\ )$ on $\underline{H}^{n+1}$ by the equation
$L(v_1,v_2)(x) = (\frac{1}{4})\cdot\left(\sum_{i=2}^{n+2} \alpha_i\beta_i - \alpha_1\beta_1\right)$. The full group of isometries of $\left(\underline{H}^{n+1}, L(\ ,\ )\right)$ is known to be a simple subgroup of index 2 in $SO(n+1,1)$ (the group of $(n+2)\times(n+2)$ real matrices which preserve the form $x_1^2 - \sum_{i=2}^{n+2} x_i^2$ and have determinant $+1$). (See Artin [1], 196) The computations below show that the map $u: \left(\underline{H}^n, L(\ ,\ )\right) \to \left(D^n, P^*(\ ,\ )\right)$ defined by $u(x_1,\ldots,x_{n+1}) = \left(1/(x_1+1)\right)\cdot(x_2,\ldots,x_{n+1})$ is an isometry.

For $x = (x_1,\ldots,x_{n+1}) \in \underline{H}^n$ a vector $\sum_{i=1}^{n+1} \alpha_i e_i(x)$ in $T(\underline{H}^n,x)$ is characterized by the equation $\sum_{i=2}^{n+1} \alpha_i x_i = \alpha_1 x_1$. We compute the entries in the tangent matrix.

$\left(u_*(x)\right)_{i,1} = -x_{i+1}/(x_1+1)^2$ for $i = 1,\ldots,n$.

$\left(u_*(x)\right)_{i,j+1} = \delta_{i,j}\cdot\left(1/(x_1+1)\right)$ for $i,j = 1,\ldots,n$.

$P^*\left(u_*(v_1),u_*(v_2)\right)\left(u(x)\right)$

$= \left(1 - \|u(x)\|^2\right)^{-2}\langle u_*(v_1),u_*(v_2)\rangle$

$= (x_1+1)^2/4\cdot\langle u_*(v_1),u_*(v_2)\rangle$. If

$v_k = \sum_{i=1}^{n+1} \alpha_{i,k}e_i(x)$ for $k = 1,2$ then

$\langle u_*(v_1),u_*(v_2)\rangle$

$= \left(1/(x_1+1)\right)^4\left(\left(\sum_{j=2}^{n+2} \alpha_{j,1}\cdot\alpha_{j,2}(x_1+1)^2\right)\right.$

$- (x_1+1)\cdot(\alpha_{1,2} \sum_{j=2}^{n+1} \alpha_{j,1}x_2 + \alpha_{1,1} \sum_{j=2}^{n+1} \alpha_{j,2}x_2)$

$\left.+ \left(\alpha_{1,1}\alpha_{1,2} \sum_{j=2}^{n+2} x_j^2\right)\right)$.

Substituting $\sum_{j=2}^{n+1} \alpha_{j,k}x_j = \alpha_{i,k}x_i$ for $k = 1,2$ and $\sum_{j=2}^{n+1} x_j^2 = x_1^2 - 1$ into this expression and simplifying we get

$\langle u_*(v_1),u_*(v_2)\rangle$

$= \left(1/(x_1+1)\right)^2\cdot\left(\sum_{j=2}^{n+1} \alpha_{j,1}\cdot\alpha_{j,2} - \alpha_{1,1}\cdot\alpha_{1,2}\right)$.

Finally substituting this into our equation for $P^*\left(u_*(v_1),u_*(v_2)\right)$ we get

$P^*\big(u_*(v_1),u_*(v_2)\big)\big(u(x)\big)$

$$= (\tfrac{1}{4})\Big(\sum_{j=2}^{n+1} \alpha_{j,1}\alpha_{j,2} - \alpha_{1,1}\alpha_{1,2}\Big) = L(v_1,v_2)(x)$$

proving that $u:\big(\underline{H}^n, L(\ ,\ )\big) \to \big(D^n, P^*(\ ,\ )\big)$ is an isometry.

2.22 We conclude section 2 with a definition and a list of notations which are utilized in succeeding sections.

Definition: Let $\overline{G^n}$ have the compact-open topology (i.e., a subbasis for the topology on $\overline{G^n}$ consists of all sets of the form $\{f \in \overline{G^n} : f(k) \subseteq U$ where k is compact and U is open$\}$). A subgroup G of $\overline{G^n}$ is said to be discrete if the subspace topology on G is the discrete topology.

Notations:

1) id_X will always be the identity transformation on the topological space X.

2) I_x will denote the inversion of $\overline{R^n}$ in the $(n-1)$-dimensional sphere of radius one centered at x, (i.e., if $T(y) = y + x$ then $I_x = TI_nT^{-1}$).

3) $FP(f)$ will denote the fixed point set of the transformation f.

4) Axis (R) will be used occasionally instead of $FP(R)$ when R is a rotation.

Section 3

Normal Forms for Transformations in G^3

3.1 In section 3 we classify the elements of G^3 according to their set of fixed points. In theorem 3.2 we show that every element of G^3 has either no fixed points, one fixed point, two fixed points, or a circle of fixed points.

Examples of the last three types are easily found. A translation T of $\overline{R^3}$ or a translation T followed by a rotation R with axis (R) = $\{v \in R^3 | v = \lambda \cdot T(0)$ for $\lambda \in R\} \cup \{\infty\}$ has a single fixed point at infinity. A dilation D or a dilation followed by a rotation R with $\{0,\infty\} \subseteq$ axis (R) has fixed points $\{0,\infty\}$. A rotation R has a circle of fixed points, namely axis (R).

Throughout section 3, unless stated otherwise, $T \in G^3$ will always denote a translation of R^3, $D \in G^3$ will always be a dilation of $\overline{R^3}$ and R will always be a rotation of $\overline{R^3}$, with $\infty \in$ axis R.

Examples of elements with no fixed point are more difficult to describe directly. The existence of such transformations can be seen as follows. Let $\hat{f}$ be a rotation of $\overline{R^4}$ with the following matrix representation relative to the standard basis for R^4

$$\begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & c & -d \\ 0 & 0 & d & c \end{pmatrix}$$

where a, b, c and d $\in$ R and $a^2 + b^2 = 1 = c^2 + d^2$. $\hat{f}$ is clearly an element of $e_3(G^3)$, say $\hat{f} = e_3(f)$. $\hat{f}$ has no fixed points on S^3 therefore f must be fixed point free. In theorem 3.7 we show that up to conjugation these are the only elements of G^3. For example if $\in G^3$ has a circle of fixed points, then we can find h $\in G^3$ such that hfh^{-1} is a rotation R.

3.2 Theorem: Every non-trivial element of G^3 has either no fixed points, one fixed point, two fixed points or a circle of fixed points.

We first prove the following useful transitivity lemma for G^3. In particular this lemma will let us conjugate a given element f of G^3 so that the conjugated element hfh^{-1} has a "convenient" fixed point set. For example if we know that f has a unique fixed point, we can choose h so that $FP(hfh^{-1}) = \{\infty\}$. This will facilitate the proofs of theorem 3.2 and theorem 3.7.

Lemma: Given two 2-spheres S_1 and S_2 contained in $\overline{R^3}$ and three pairs of points (x_i, y_i) with $x_i \in S_1$ and $y_i \in S_2$, $i = 1,2,3$, there exists a unique element h of G^3 such that $h(S_1) = S_2$ and $h(x_i) = y_i$, $i = 1,2,3$.

Proof: This lemma is based on a similar transitivity for G^2 acting on $\overline{R^2}$; given three pairs of points (v_i, w_i) $i = 1,2,3$ in $\overline{R^2}$ there exists a unique k in G^2 such that $k(v_i) = w_i$, $i = 1,2,3$. (See Ford [4], p. 7.) It suffices to assume that $S_2 = \{(x,y,z) \in R^3 : z = 0\} \cup \{\infty\}$. If $\infty \in S_1$ choose (x_0, y_0, z_0) in $S_1 - \{\infty\}$ and let $T(x,y,z) = (x-x_0, y-y_0, z-z_0)$. Note that $\{0,\infty\} \subseteq T(S_1)$ and therefore $T(S_1)$ is an extended hyperplane thru 0. Let R be a rotation carrying $T(S_1)$ to S_2. By the note above there exists a unique element g of G^2 acting on S^2 which satisfies $g\big(RT(x_i)\big) = y_i$, $i = 1,2,3$. Let $l = e_2(g)$ where e_2 is the canonical isomorphism of G^2 into G^3 defined in subsection 2.17. Then $h = lRT$ satisfies the conditions of the lemma. If $\infty \notin S_1$ choose $x_0 \in S_1$ and let $I_{\overline{x}_0}$ be an inversion of $\overline{R^3}$ centered at $\overline{x}_0$ as defined in part 2 of subsection 2.22. Since $\infty \in I_{\overline{x}_0}(S_1)$, we have reduced this case to the previous one except for the fact that $1RTI_{\overline{x}_0} \in (\overline{G^3} - G^3)$. Let $r(x,y,z) = (x,y,-z)$. r also reverses orientation and restricted to S_2, r is the identity. Thus $h = rhTI_{\overline{x}_0} \in G^3$ and satisfies the conditions of the lemma. If h' is another element of G_3 satisfying the conditions of the lemma then $h'h^{-1}$ maps S_2 onto itself and fixes y_1, y_2 and y_3. Therefore by the note at the beginning of this proof $h'h^{-1}$ is the identity on S_2 and by corollary 2.15 $h'h^{-1}$ is therefore the identity on all of $\overline{R^3}$. i.e., $h' = h$. qed

In the next two subsections we consider elements of G^3 whose fixed point sets are non-empty. We will see that they must be conjugate to elements in a particular form. In subsection 3.5 we bring this information together to complete the proof of Theorem 3.2.

3.3 Lemma: Let $f \in G^3$ with $FP(f) \neq \emptyset$ then f is conjugate to a transformation of the form DRT, where $FP(D) \subseteq FP(R)$. Either D, R or T might be the trivial dilation, rotation or translation respectively.

Proof: By lemma 3.2 we can conjugate by some $h \in G^3$ so that $\infty \in FP(hfh^{-1})$. For the sake of simplicity and without loss of generality we assume that $\infty \in FP(f)$. Choose some coordinate system for R^3. Let $f^{-1}(0) = \bar{x}_0$. Let $T(x) = x - \bar{x}_0$. Tf f^{-1} fixed points at 0 and ∞. By corollary 2.13 Tf^{-1} must send any plane P_1 thru the origin to another such plane P_2. Let R be a rotation of R^3 with $0 \in FP(R)$ which sends P_2 to P_1. Then $P_1 \cup \{\infty\}$ is invariant under RTf^{-1}. R can always be chosen so that RTf^{-1} is orientation-preserving on $P_1 \cup \{\infty\}$ and has fixed points at 0 and ∞. Thus restricted to $P_1 \cup \{\infty\}$ the transformation RTf^{-1} must be a dilation followed by a rotation (See Ford [4], 18). By "absorbing" the extension of this rotation to $\overline{R^3}$ into R it can be assumed that RTf^{-1} restricted to $P_1 \cup \{\infty\}$ is a dilation with fixed points at 0 and ∞. By corollary 2.15 RTf^{-1} must be a dilation on all of $\overline{R^3}$. Equivalently $f = DRT$. qed

3.4 <u>Corollary</u>: An element f of G^3 has two or more fixed points if and only if f is conjugate to a transformation of the form DR where $FP(D) \subseteq FP(R)$. (As in lemma 3.3 D or R might be trivial.)

<u>Proof</u>: If f has two or more fixed points then by lemma 3.2 we can conjugate f by some element h in G^3 so that $\{0,\infty\} \subseteq FP(hfh^{-1})$. As above we simply assume that $\{0,\infty\} \subseteq FP(f)$. The proof of lemma 3.3 tells us

that $f = DRT$ with $T(x) = x - f^{-1}(0)$, i.e., T = identity. The converse is clear. qed

3.5 We now bring together the material of subsections 3.1 to 3.4 to prove Theorem 3.2. As we saw in subsection 3.1 there are elements of G^3 which are fixed point free, have a single fixed point, two fixed points, or a circle of fixed points. By corollary 3.4 an element f in G^3 has two or more fixed points if and only if it is conjugate to a transformation of the form DR. If D is non-trivial then for some $k > 0$, $k \neq 1$ the norm of $f(v)$ is k times the norm of v. f can therefore have only two fixed points 0 and ∞. If D is trivial then f is conjugate to a rotation and must have a circle of fixed points. This concludes the proof of Theorem 3.2.

3.6 In the proof of Theorem 3.2 we saw that information about the cardinality of the fixed point set of an element f of G^3 gave us information concerning the "type" of transformation f had to be, up to conjugation in G^3. Theorem 3.7 refines this information so that given the cardinality of $FP(f)$ we can say more precisely what type of transformation f is.

3.7 Theorem: Let f be a non-trivial element of G^3.

i) If f has exactly one fixed point then it is conjugate in G^3 to a transformation of the form RT where T is non-trivial. Moreover, if R is also non-trivial then we can find a coordinate system for $\overline{R^3}$ such that $T(0) \in$ axis (R).

ii) If f has exactly two fixed points then it is conjugate in G^3 to an element of the form DR where the dilation D is non-trivial and $FP(D) \subseteq FP(R)$.

iii) If f has a circle of fixed points then it is conjugate in G^3 to a rotation R.

iv) If f is fixed point free then f is conjugate in G^3 to transformation g of $\overline{R^3}$ which is invariant on $\Lambda = \{x,y,z) \in \overline{R^3} - \{\infty\} \mid (x,y,z) \neq (a,0,0)\}$. In cylindrical coordinates $(w, e^{i\theta})$ $g(w,e^{i\theta}) = \left(\tau(w), e^{i(\theta+\theta_0)}\right)$ for all $(w,e^{i\theta}) \in H^2 \times S^1$ where τ is a non-trivial elliptic fractional linear transformation of $\emptyset$ preserving the upper-half plane H^2 and $\theta_0 \in (0,2\pi)$. (By continuity this determines g on all of $\overline{R^3}$.)

Parts ii and iii of this theorem follow directly from corollary 3.4. In either case f is conjugate to a transformation of the form DR. If f has exactly two fixed points D must be non-trivial.

If f has a circle of fixed points then D must be trivial. Part i of this theorem is proven in subsection 3.8 and part iv is proven in the next 3 subsections.

3.8 To prove theorem 3.7 part i we assume, as in the preceding subsections that $FP(f) = \{\infty\}$. Lemma 3.3 implies that $f = DRT$, where $FP(D) \subseteq FP(R)$. As before we assume $FP(D) = \{0,\infty\}$. We can always choose a coordinate system for $\overline{R^3}$ so that T, R and D can be written as follows: $T(x,y,z) = (x + x_0, y + y_0, z + z_0)$,

$$R(x,y,z) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and $D(x,y,z) = k\cdot(x,y,z)$ for some $k > 0$. We will show that $k \neq 1$ implies that f has a finite fixed point. Let $\bar{x}_0 = (x_0, y_0, z_0)$ and $\bar{x}_1 = (x_1, y_1, z_1)$. f has a finite fixed point $\bar{x}_1$ if and only if the following sequence of equations are true;

$\bar{x}_1 = DRT(\bar{x}_1) = DR(\bar{x}_1 - \bar{x}_0) = DR(\bar{x}_1) - DR(\bar{x}_0)$.

Equivalently, let Id be the identity transformation on $\overline{R^3}$ then $\bar{x}_1 \in FP(f)$ if and only if

$DR(\bar{x}_0) = DR(\bar{x}_1) - Id(\bar{x}_1)$ which we will write as $(DR - Id)(\bar{x}_1)$. $DR - Id$ is a linear transformation of R^3. If it is non-singular we will be able to find an $\bar{x}_1$ such that $DR(\bar{x}_0) = (DR - Id)(\bar{x}_1)$ and f will have a finite fixed point. Therefore we must make sure that the determinant of the matrix representing (DR - Id) is zero. (DR - Id) is represented by the matrix

$$\begin{pmatrix} (k\cos(\theta) - 1) & -k\sin(\theta) & 0 \\ k\sin(\theta) & (k\cos(\theta) - 1) & 0 \\ 0 & 0 & (k-1) \end{pmatrix}.$$

The determinant of this matrix is

$\left(k^2 - 2k\cos(\theta) + 1\right)(k-1)$. Thus k must equal 1 or $\left((2\cos\theta \pm \left(4\cos^2(\theta) - 4\right)^{\frac{1}{2}})/2\right) = \cos\theta \pm (\cos^2\theta - 1)^{\frac{1}{2}}$. Since k is real we must have $\cos^2\theta - 1 \geq 0$, i.e., $\cos\theta$ must be ± 1. This gives us $k = \pm 1$ but k is greater than zero, therefore k must equal 1, i.e., D must be trivial.

We must still show that T is non-trivial and that in some coordinate system for R^3 $T(0) \in$ axis (R). If T is trivial then f becomes a rotation which clearly contradicts the assumption that $FP(f) = \{\infty\}$. Therefore T is non-trivial. If R were trivial, then there would be nothing further to prove; we assume from here on that R is non-trivial.

We first observe that in the coordinate system used above $T(0) = T(0,0,0) = (x_0, y_0, z_0)$ can not be of the form $(x_0, y_0, 0)$. If it were of this form then f restricted to any plane P orthogonal to axis (R) would be a translation followed by a rotation and f would then have a finite fixed point in P. To see this write f restricted to P in complex coordinates. We get $f|P(w) = (w + w_0) \cdot e^{i\theta}$ for $w_0 = x_0 + iy_0$ and $w \in \not{C}$. This has a fixed point at $w = (w_0 e^{i\theta})/(1 - e^{i\theta})$. If $T(0,0,0) = (x_0, y_0, z_0)$ with $z_0 \neq 0$ then f cannot have finite fixed points as any plane orthogonal for the z-axis is sent to another such distinct plane by f.

Finally we find a coordinate system in which $T(0,0,0) \in$ axis (R). For computational purposes it will be easier to consider f as acting on $\not{C} \times R \cup \{\infty\}$. We send (x,y,z) in R^3 to (x+iy,z) in $\not{C} \times R$. For all $(w,z) \in C \times R$ we get $f(w,z) = \left(e^{i\theta}(w + w_0), z + z_0\right)$ where $z_0 = x_0 + iy_0$. Making an affine change of coordinates in $\not{C}$ which sends 0 to $\left(w_0 e^{i\theta}/(1 - e^{i\theta})\right)$ f becomes

$$f(w,z) = (e^{i\theta}\left(w + (w - e^{i\theta})/(1 - e^{i\theta})\right) + w_0 - w_0 e^{i\theta}/(1 - e^{i\theta}), z + z_0)$$

$$= (e^{i\theta} w, z + z_0).$$

Translating this new coordinate system on $\not{C} \times R$ to R^3, we obtain f = RT. This completes the proof of theorem 3.7 part i.

3.9 In the next 3 sections we prove theorem 3.7 part iv. We begin by proving a lemma concerning elements in the D^4-model of G^3. (See subsection 2.20 for a description of this model.) We will use this lemma to show that a fixed point free transformation in the $\overline{R^3}$-model of G^3 corresponds to a particular type of rotation in the D^4-model of G^3. Finally we show that such a rotation corresponds to a transformation of the type described in theorem 3.7 part iv.

3.10 Lemma: Let f be an element of the $\overline{R^3}$-model of G^3 and let $\hat{f}$ be the corresponding element in the D^4-model. If $\hat{f}$ has a fixed point in the interior of D^4 then $\hat{f}$ is conjugate in G^3 to a rotation of D^4.

Proof: We prove this lemma by using yet another model of G^3, namely the $\underline{H}^4$-model of subsection 2.21. Let $u:\left(\underline{H}^4, L(\ ,\ )\right) \to \left(D^4, P^*(\ ,\ )\right)$ be the isometry of subsection 2.21. Recall that G^3 is isomorphic to a subgroup of index 2 of the matrix group $SO(4,1)$ in the $\underline{H}^4$-model. Let $\overline{f} = u^{-1}(\hat{f})$. In subsection 2.19 we established the transitivity of $e_3(G^3)$ on $\underline{H}^4$. This is equivalent to the transitivity of $SO(4,1)$ on $\underline{H}^4$. Thus $\overline{f}$ is conjugate in $SO(4,1)$ to a transformation $\overline{g}$ which has the point $(1,0,0,0,0)$ in its fixed point set. We will show that $u\bar{g}u^{-1}$ is a rotation of D^4.

Let F be the bilinear form associated to the quadratic form $q(x_1,\ldots,x_5) = -x_1^2 + x_2^2 + \ldots + x_5^2$ of $SO(4,1)$. Let $x_1 = (a_1,\ldots,a_5)$ and $x_2 = (b_1,\ldots,b_5)$. By the polarization identity (Hoffman & Kunze [7], p. 368)

$$F(x_1,x_2) = (\tfrac{1}{4})\Big(q(x_1+x_2) - q(x_1-x_2)\Big)$$
$$= +(\tfrac{1}{4})\Big(-(a_1-b_1)^2 + \sum_{i=2}^{5} (a_i+b_i)^2\Big)$$
$$- (\tfrac{1}{4})\Big(-(a_1-b_1)^2 + \sum_{i=1}^{5} (a_i-b_i)^2\Big).$$

$\bar{g}$ must preserves this bilinear form F. Let $e_1,\ldots,e_5$ be the standard basis on R^5. For $i \neq 1$ we get $0 = F(e_1,e_i) = F\Big(\bar{g}(e_1),\bar{g}(e_i)\Big)$

$$= F\Big(e_1,\bar{g}(e_i)\Big).$$

Letting $(t_{i,j})$ be the matrix of $\bar{g}$ relative to $e_1,\ldots,e_5$ we get $(i \neq 1)$

$$0 = F\Big(e_1,g(e_i)\Big) = (\tfrac{1}{4})\Big(-(1+t_{i,1})^2 + \sum_{j=2}^{5} t_{j,i}^2\Big)$$
$$- (\tfrac{1}{4})\Big(-(1-t_{1,i})^2 + \sum_{j=2}^{5} t_{j,i}^2\Big)$$
$$= -t_{1,i}\,.$$

A similar calculation yields $0 = t_{i,1}$ for $i \neq 1$. Therefore $(t_{i,j})$ must be of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & & & & \\ 0 & & R' & & \\ 0 & & & & \\ 0 & & & & \end{pmatrix}.$$

R' must have determinant 1 and preserve the quadratic form $x_2^2 + \ldots + x_5^2$; i.e., $\bar{g}$ is a rotation with fixed point $(1,0,\ldots,0)$. Recall that $u(x_1,\ldots,x_5) = \big(1/(x_1+1)\big)\cdot(x_2,\ldots,x_5)$. A simple algebraic computation shows that $u\bar{g}u^{-1}(x_1,\ldots,x_4) = R'(x_1,\ldots,x_4)$. qed

3.11 Corollary: If f is a conformal transformation of S^3 with no fixed points then f is conjugate in the S^3 model of G^3 to a rotation of S^3 which has the following matrix representation relative to some orthonormal basis for R^4,

$$\begin{pmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & c & -d \\ 0 & 0 & d & c \end{pmatrix} \qquad \text{where } a^2 + b^2 = 1 = c^2 + d^2 \text{ and } a \neq 1 \neq c\,.$$

Proof: Let $\hat{f}$ be the element of the D^4-model of G^3 which corresponds to f. By the Brouwer fixed point theorem $\hat{f}$ must have a fixed point in the interior of

D^4 as it has none on S^3. By lemma 3.10 above $\hat{f}$ must be conjugate to a rotation $\hat{g}$. Restricted to S^3, $\hat{g}$ must be fixed point free and therefore relative to some orthonormal basis for R^4 it is of the form indicated above.

qed

In order to complete the proof of theorem 3.7 we must still show that the elements of G^3 in the $\overline{R^3}$-model described in part iv of the theorem correspond to fixed point free rotations in the S^3 model. Define a stereographic projection Pr, of (S^3, standard Euclidean metric) to $\left(\overline{R^3}, E(\ ,\ )\right)$. We use $Pr(x_1,\ldots,x_4) = \left(1/2(1-x_4)\right)(x_1,x_2,x_3)$ when $x_4 \neq 1$ and $Pr(0,0,0,1) = \infty$. (This is essentially the conformal map from D^4 to H^4 defined in subsection 2.20.) Let $c:\Lambda \to \Lambda$ be the change of coordinates map which takes (x,y,z) to its cylindrical coordinate $\left(x,(y^2+z^2)^{\frac{1}{2}},\arctan(z/y)\right)$. Let $\overline{g}:S^3 \to S^3$ be the fixed point free rotation whose matrix is

$$\begin{pmatrix} a & 0 & 0 & -b \\ 0 & c & -d & 0 \\ 0 & d & c & 0 \\ b & 0 & 0 & a \end{pmatrix}$$

with respect to the standard basis for R^4, where $a = \cos(\theta_1)$, $b = \sin(\theta_1)$, $c = \cos(\theta_2)$,

$d = \sin(\theta_2)$, and a and c are not equal to 1. Let T be an elliptic transformation of $\mathbb{C} \cup \{\infty\}$ with fixed points $\pm i/2$ and multiplier $e^{i\theta_1}$. A simple algebraic computation shows that

$$cP_r\bar{g}P_r^{-1}c^{-1}(x,y,\psi) = \left(T(x+iy), e^{(\psi+\theta_2)i}\right),$$

i.e., $\bar{g}$ corresponds to the element described in part iv of theorem 3.7. qed

3.9 We end section 3 with some definitions based on theorem 3.7.

<u>Definition</u>: If $f \in G^3$ is of type i) then f is said to be <u>parabolic</u>. As a special case if R is known to be trivial (i.e., if f is conjugate to a translation) then f is said to be <u>purely parabolic</u>. If f is of type ii) it will be called <u>loxodromic</u>. Again as a special case if f is conjugate to a dilation then f is called <u>hyperbolic</u>. If f is of type iii) it will be called <u>elliptic</u> and <u>bielliptic</u> if it is of type iv).

Section 4

Invariant Circles for Transformations in G^3

4.1 In section 4 we try to gain a clearer geometric understanding of how the elements of G^3 act on $\overline{R^3}$. To this end we describe the invariant circles of the different types of elements of G^3. Recall that for our purposes, "circle" denotes both ordinary Euclidean circles and extended lines through infinity. First we make some preliminary definitions and remarks and then we state theorem 4.4.

4.2 Definition: Given $f \in G^3$ let $C(f)$ be the set of all f-invariant circles in $\overline{R^3}$, i.e., $C(f) = \{c \subseteq \overline{R^3} | c$ is a circle such that $f(c) = c\}$. Note that for $f,h \in G^3$, $c \in C(f)$ implies that $h(c) \in C(hfh^{-1})$. Thus in order to understand $C(f)$ it suffices to consider any conjugate of f. This motivates the next definition.

4.3 Definition: An element f of the $\overline{R^3}$-model of G^3 is said to be in normal form (with respect to a given coordinate system for $\overline{R^3}$) if f is equal to one of the standard forms of Theorem 3.7.

In the following 3 subsections we describe $C(f)$ for f in normal form when f is respectively parabolic, loxodromic and elliptic. The remaining subsections are devoted to describing $C(f)$ for f bielliptic in normal form. For the sake of future reference we bring our results together as theorem 4.18 at the end on section 4. As in section 3, T, D and R in G^3 will always denote translation, dilation and rotation respectively.

4.4 In this subsection we consider $C(f)$ for f parabolic. If f is purely parabolic, i.e., $f = T$ then $C(f)$ is the set of all extended lines whose restriction to R^3 is parallel to the line $L = \{r \cdot T(0) | r \in R\}$. If f is parabolic of the form $R \cdot T$ with R non-trivial then $C(f) = \{\text{axis } (R)\}$.

When $f = T$ then for all x_0 in R^3 the extended line determined by the three distinct points x_0, $T(x_0)$ and $T^2(x_0)$ is clearly f-invariant. Since x_0, $T(x_0)$ and $T^2(x_0)$ are always distinct for $x_0 \neq \infty$ this must be the unique f-invariant circle thru x_0. Clearly this extended line is parallel to L in $\overline{R^3} - \{\infty\}$.

Let $f = RT$ be parabolic in normal form. Recall from theorem 3.7 that axis (R) = FP(R) = $L \cup \{\infty\} = \{r \cdot T(0) | r \in R\} \cup \{\infty\}$. The circle $L \cup \{\infty\}$ is certainly T invariant by the observations of the previous paragraph and therefore it is clearly f-invariant.

In order to prove that C(RT) = axis (R) we let c_0 denote an element of C(RT) and prove that c_0 must equal axis (R). First note that for any $r > 0$ c_0 cannot be contained in $B_r = \{x \in R^3 | \|x\| \leq r\}$ since $f^k(B_r) \cap (B_r) = \emptyset$ for some natural number k. Therefore c_0 must contain ∞. Let $l_0 = c_0 - \{\infty\}$. The line l_0 cannot intersect axis (R) in single point w_0 since $f(w_0) \in$ axis (R) $- \{\infty\}$. Thus $c_0 \neq$ axis (R) implies that $l_0 \cap$ axis (R) $= \emptyset$. c_0 cannot be of the form axis (R) + w_0 where $w_0 \notin$ axis (R) since such a circle is preserved by T but not by R. Since c_0 cannot be of this form there must be a unique point z_0 in l_0 which minimizes the distance between l_0 and axis (R). To see this, project axis (R) and l_0 on any plane in R^3 orthogonal to axis (R). The line l_0 projects in a 1-1 fashion to a line and l_0 projects to a point. Take the minimizing point in the projection of l_0 and lift it back to l_0. Since c_0 contains ∞ it must also contain $z_1 = \frac{1}{2}\left(z_0 + f(z_0)\right)$ if it is to be f-invariant. However, the distance from z_1 to axis (R) is less than that of z_0 to axis (R). This can be seen by projecting onto a plane in R^3 orthogonal to axis (R) as before. Let $\bar{p}$ denote this projection. Then $\bar{p}\big(f(z_0)\big) = R\bar{p}(z_0)$ and $\bar{p}(z_1)$ is contained in the interior of the line segment connecting these points. Clearly $\bar{p}(z_1)$ is closer to $\bar{p}\big(\text{axis (R)}\big)$ than z_0 is. Thus the assumption that c_0 is f-invariant and not equal to axis (R) leads to a contradiction.

4.5 In this subsection we determine C(f) for f loxodromic in normal form, i.e., $f = D\cdot R$ with $FP(f) = \{0,\infty\}$. We show that $C(f) = \{c \in C(R) \mid \{\infty, 0\} \subseteq c\}$. (When R is trivial this implies that C(f) is the set of all circles containing $\{0,\infty\}$.)

To see that any f-invariant circle c_0 must contain the set $\{0,\infty\}$ note that for any $x_0 \in c_0$ the set $\{\|f^k(x_0)\| \mid k \text{ an integer}\}$ is neither bounded from above or below. Dilation clearly sends any line thru 0 and ∞ to itself and therefore $c_0 \in C(f) = C(DR)$ if and only if $c_0 \in C(R)$.

4.6 In this section we consider C(f) for f elliptic in normal form, i.e., f equals a rotation R.

We must consider the two cases $f^2 \neq$ identity and f^2 = identity separately.

When f^2 is not the identity C(f) contains axis (R) and the set of all Euclidean circles which lie in a (non-extended) plane P orthogonal to axis (R) and which are centered at axis (R) $\cap$ P. When f^2 is the identity then C(f) is the set of circles described above together with the set of all circles which intersect axis (R) orthogonally in 2 points.

We assume first that $f^2 = R^2$ is not the identity. Writing $\overline{R^3} - \{\infty\}$ as $\mathbb{C} \times R$ we can assume that $f(z,r) = (e^{i\theta} \cdot z, r)$ for all $(z,r) \in \mathbb{C} \times R$ and $\theta \in (0,\pi)$. Given any real number $\rho > 0$ the circle $c = \{(w,r) \in \mathbb{C} \times R : \|w\| = \rho\}$ is clearly f-invariant. When $\theta \neq \pi$ the orbit of $(\rho e^{i\theta}, r)$ under f contains at least three points and so c must be the unique f-invariant circle passing thru $(\rho e^{i\theta}, r)$. Observe that axis (R) = FP(R) and therefore axis (R) is certainly f-invariant.

We next assume that f^2 = identity or equivalently that in the representation of f in the previous paragraph, $\theta = \pi$. The circles described above are still f-invariant, but now more than one element of C(f) can pass through a given point.

Let c_0 be contained in $\overline{R^3} - \{\infty\}$ and let (z_0, r_0) be a point on c_0 such that $(z, r) \in c_0$ implies $r_0 \leq r$. Now $f(z_0, r_0) = (-z_0, r_0)$, and so if $z_0 \neq 0$ then c_0 must lie in $\{(z,r) \in \not{C} \times R : r = r_0\}$; hence c_0 is of the type described above. If $z_0 = 0$ then there exists $(z_1, r_1) \in c_0$ with $z_1 \neq 0$ and $r_1 \geq r_0$. c_0 is then determined by the set $\{(0, r_0), (z_1, r_1), (-z_1, r_1)\}$. Such a circle must be orthogonal to axis (R) and intersect it in two points. If $\infty \in c_0 \neq$ axis (R) then there exists a point (z_1, r_1) in c_0 as above and c_0 is determined by $\{(z_1, r_1), (-z_1, r_1), \infty\}$. c_0 is therefore a circle which intersects axis (R) orthogonally at zero and infinity and is clearly f-invariant.

4.7 The remainder of section 4 is devoted to describing $C(f)$ for f bielliptic and in normal form. First we must recall some facts about elliptic fractional linear transformations.

Let M be a connected, Hausdorf, C^∞ manifold with Riemannian metric $\langle\ ,\ \rangle$. Given any piecewise C^∞ curve $\sigma : [0,1] \to M$, let $T_\sigma(t)$ be the "velocity" tangent vector to σ at $\sigma(t)$, i.e., $T_\sigma(t) = \sigma_*|_t(\partial/\partial t)$. Define $|\sigma|$ by

$$|\sigma| = \int_0^1 \left(\langle T_\sigma(t), T_\sigma(t)\rangle\right)^{\frac{1}{2}} dt .$$

It can be shown that $|\sigma|$ is independent of the parametrization of σ. Given $p,q \in M$ define $d(p,q)$ by:

$d(p,q) = \inf\{|\sigma| : \sigma$ is a piecewise C^∞ curve from p to $q\}$.
$d: M \times M \to R$ defines a metric (i.e., a distance function) (Hicks [6], 69-71). Let $d(\, , \,)$ be the metric on H^2 determined by this method where $\langle \, , \, \rangle$ is the Poincare metric $P(\, , \,)$ as defined in subsection 2.18.

We have the following two facts (Lehner [8], 25):

1) Given $z_0 \in H^2$ and $r \in (0,\infty)$ then for $z_0 = x_0 + iy_0$ $S^1_r = \{z \in H^2 | d(z,z_0) = r\}$ is a circle in the ordinary Euclidean sense with center $x_0 + i\cdot[y_0\cdot(e^{2r}+1)/2e^r]$ and radius $y_0(e^{2r}-1)/2e^r$. (The Euclidean center and radius of S^1_r are easily computed once it is established that S^1_r is a Euclidean circle.)
2) If T is an elliptic fractional linear transformation acting on H^2 with fixed point z_0 then $C(T)$, the set of T-invariant circles in H^2, is $\{S^1_r\}$ $r \in (0,\infty)$. S^1_r is called the Steiner circle for T of radius r.

4.8 Let f be a bielliptic element of the $\overline{R^3}$-model of G^3. Recall from theorem 3.7 part iv that on $\Lambda = \{(x,y,z) \in R^3 | (x,y,z) \neq (a,0,0)\}$ f can be represented in cylindrical coordinates $(w,e^{i\theta}) \in H^2 \times S^1$ by $f(w,e^{i\theta}) = \left(T(w), e^{i(\theta+\theta_2)}\right)$ where T is an elliptic fractional linear transformation acting on H^2 with multiplier $e^{i\theta_1}$. Using the material of the previous subsection we can now begin to get a clearer idea of the action of f on $\overline{R^3}$. Subsequently, in subsection 4.9 we will describe C(f) under the assumption that $f^2 \neq$ identity. The case $f^2 =$ identity is left to subsection 4.17.

Fact 2) of the preceding subsection motivates the following definitions. Let $U_{\theta_0} = \{(w,\theta) \in \Lambda | \theta = \theta_0\}$ for $\theta_0 \in [-\pi,\pi]$. Define $i_\theta : H^2 \to \Lambda$ by $i_\theta(w) = (w,\theta)$, i.e., i_θ is an obvious identification of H^2 with U_θ in Λ. Given $r > 0$ let $S^1_r(T)$ be the Steiner circle for T of radius r described in subsection 4.7. Let $S^1_{r,\theta}(f) = i_\theta\left(S^1_r(T)\right)$ and $T^2_r(f) = \bigcup_{\theta\in[-\pi,\pi]}\left(S^1_{r,\theta}(f)\right)$. $T^2_r(f)$ is a right regular torus with meridian equal to $S^1_r(T)$.

<u>Proposition</u>: For all positive real numbers r, $T^2_r(f)$ is f-invariant.

<u>Proof</u>: This is clear as $f\left(S^1_{r,\theta}(f)\right)$ is $S^1_{r,\theta+\theta_2}(f)$. qed.

$T^2_r(f)$ will be called the <u>Steiner torus for f of radius r</u>. By the proposition above the action of f on $\bigcup_{r>0} T^2_r(f)$ can be understood by considering f on any Steiner torus $T^2_{r_0}(f)$. The complement in $\overline{R^3}$ of $\bigcup_{r>0} T^2_r(f)$ is two linked circles lying in orthogonal planes. One circle which we call L_f is $\overline{R^3} - \Lambda$. The other circle which we call S^1_f is $\{(z_0, e^{i\theta}) \in H^2 \times S^1 \mid z_0 = FP(T)\}$. (See diagram 1.) Note that L_f and S^1_f can be thought of an degenerate Steiner tori. S^1_f is the "limiting set" in $\overline{R^3}$ for the $T^2_r(f)$ as r tends to 0 and L_f is the "limiting set" in $\overline{R^3}$ for the $T^2_r(f)$ as r tends to ∞. By a slight abuse of notation we will write $T^2_0(f) = S^1_f$ and $T^2_\infty(f) = L_f$. We now can write $\overline{R^3}$ as the union of disjoint, f-invariant subsets, i.e., $\overline{R^3} = \bigcup_{r=0}^{\infty}\left(T^2_r(f)\right)$.

4.9 We are now ready to describe the set C(f) for f bielliptic in normal form. (The reader is reminded that we are assuming that $f^2 \neq$ identity. Although everything in the previous subsection holds for f of order 2 we will treat that case separately in subsection 4.17.)

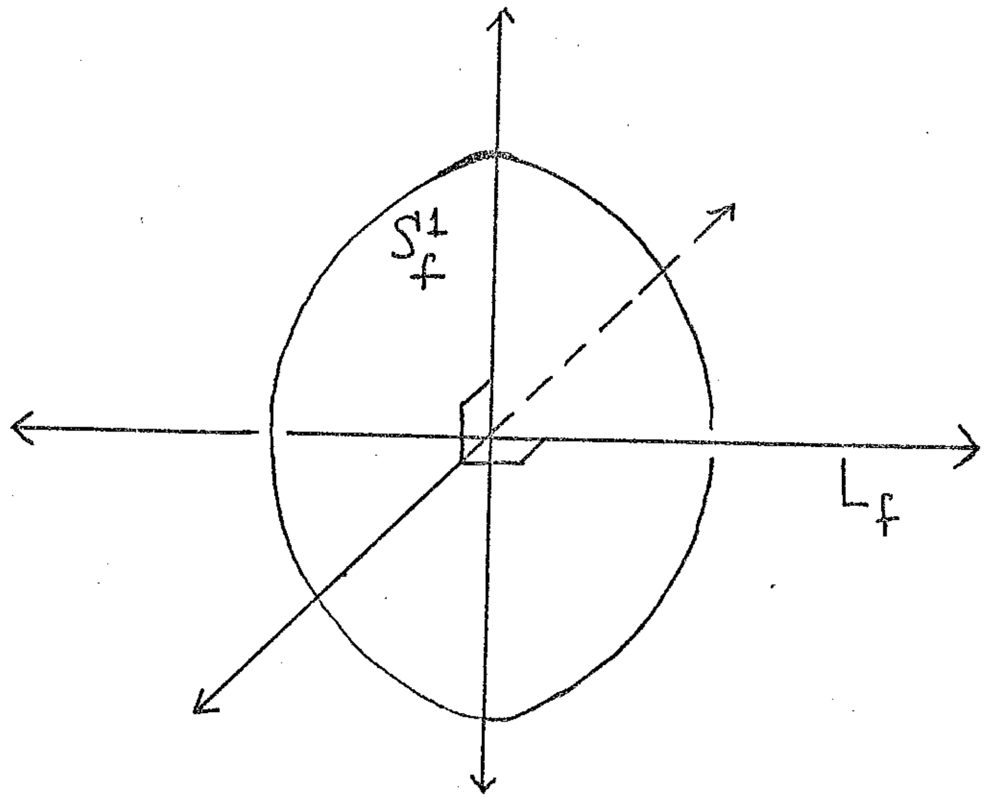


Diagram 1. The linked f-invariant circles L_f and S_f.

Certainly the circles S_f^1 and L_f are f-invariant as they each can be written as the difference of 2 f-invariant sets; $S_f^1 = \Lambda - (\bigcup_{r>0} (T_r^2(f)))$ and $L_f = \overline{R^3} - \Lambda$. Recall that $f(w, e^{i\theta}) = (T(w), e^{i(\theta+\theta_2)})$ where T is an elliptic fractional linear of H^2 with multiplier $e^{i\theta_2}$, $\theta_1 \in [-\pi, \pi]$. If $|\theta_1| \neq |\theta_2|$ we will see that $C(f) = \{S_f^1, L_f\}$. If $|\theta_1| = |\theta_2| \neq |\pi|$ then we will show that for every point $p \in T_r^2(f)$, $r > 0$, one of the oblique toroidal circles passing thru p is f-invariant. (See Coxeter [3], 132 for a description of these oblique toroidal circles.) Which of the 2 oblique toroidal circles is f-invariant will depend upon whether $\theta_1 = \theta_2$ or $\theta_1 = -\theta_2$.

In order to determine if a point p lies in an f-invariant circle we will examine the orbit of p under f, i.e., we will examine $\{f^k(p) | k \text{ is an integer}\}$. We will call this set Orb(f/p). For all $p \in \Lambda$ the cardinality of this set is the least common multiple of the orders of the transformations $r_j : S^1 \to S^1$ $i = 1,2$ where $r_j(e^{i\theta}) = e^{i(\theta+\theta_j)}$ when these orders exist and the cardinality is otherwise infinite. Since we are assuming that $|\theta_1| \neq \pi \neq |\theta_2|$ Orb(f/p) must always have at least three points and therefore Orb(f/p) would determine an f-invariant circle containing p. If $p \in T^2_r(f)$ $r > 0$ then as we have seen $\mathrm{Orb}(f/p) \subseteq T^2_r(f)$. In lemma 4.10 we show that the cardinality of Orb(f/p) being greater than 8 implies that not only must Orb(f/p) be contained in $T^2_r(f)$ but that any f-invariant circle thru p must lie entirely within $T^2_r(f)$. Thus for "most" of our normal bielliptic elements we can restrict the search for f-invariant circles thru p to circles in the Steiner torus containing p.

In subsections 4.10 to 4.15 we show that the only circles in the Steiner tori which could contain Orb(f/p) are the oblique toroidal circles and finally that they do contain Orb(f/p) if and only if $|\theta_1| = |\theta_2|$. This establishes our claim whenever the cardinality of Orb(f/p) is greater than 8. The remaining cases are dealt with in subsections 4.16 and 4.17.

For simplicity of notation we will shorten $T^2_r(f)$ to T^2_r and $S^1_{r,\theta}(f)$ to $S^1_{r,\theta}$ whenever the transformation f is clearly determined by the context.

4.10 <u>Lemma</u>: For $p \in T^2_r$ $(r \neq 0,\infty)$ if the cardinality of Orb(f/p) is greater than 8, then p is contained in an f-invariant circle if and only if $Orb(f/p)$ is contained in a circle which lies entirely in T_r^2.

Proof: Assume that $Orb(f/p)$ is contained in a circle in T_r^2. If $Orb(f/p)$ has cardinality greater than two and is contained in a circle it determines that circle. $Orb(f/p)$ is certainly f-invariant and therefore the circle it determines is f-invariant.

Assume next that p is contained in an f-invariant circle. $Orb(f/p)$ is certainly contained in T_r^2 and in the given f-invariant circle. We must show that this entire circle lies in T_r^2. Since T_r^2 $(r \neq 0,\infty)$ is a regular torus (i.e., it is the surface obtained by revolving a circle in $\overline{R^3}-\{\infty\}$ about a line which does not intersect the circle) it can be represented as the zeros of a fourth degree polynomial in 3 real variables. For $R,\rho > 0$ let
$S_{r,0}^1 = \{(x,y,0)|x^2 + \big(y-(R+\rho)\big)^2 = \rho^2\}$ then
$T_r^2 = \{(x,y,z)|x^2 + \big(\sqrt{y^2+z^2} - (R+\rho)\big)^2 = \rho^2\}$.
A circle is the zero set of a second degree polynomial in a real plane. Therefore a circle not contained in a torus could intersect the torus in at most eight points. By assumption $Orb(f/p)$ has at least 9 points therefore the given f-invariant circle must lie

entirely within T_r^2 (Walker [11], p. 111, Thm. 5.4.) qed

4.11 In order to study Orb(f/p) we can conjugate f if necessary and assume that T has fixed point at $\pm$ i, i.e.,

$T(w) = \left(w \cdot i(1 + e^{i\theta_1}) + (e^{i\theta_1} - 1)\right) / \left(w(1 - e^{i\theta_1}) + i(1 + e^{i\theta_1})\right)$.

Clearly it suffices to let p be any point on an arbitrary non-degenerate Steiner torus $T_{r_0}^2$. Let $0 < R < 1$ and let $r_0 = -\ln(R)$. By note 1) in subsection 4.7, ρ the Euclidean radius of $S_{r_0,0}^1$ is therefore equal to $(1 - R^2)/2R$. We will examine Orb(f/p) for $p = \left(i \cdot (R + 2\rho), \pi/2\right) \in T_{r_0}^2$. (See diagram 2.)

If p lies in an f-invariant circle; the circle must lie in the plane P_ψ (the subscript ψ is clarified below) determined by $\{p, f(p), f^{-1}(p)\}$. If p, f(p) and $f^{-1}(p)$ are colinear and therefore do not determine a plane then any f-invariant circle thru p would have to contain $\{\infty\}$. But $f^2 \neq$ id implies that the only f-invariant circle thru $\{\infty\}$ is $L_f = T_\infty^2$. If Orb(f/∞) has three or more points this is immediate. If Orb(f/∞) has two points then $|\theta_1| = \pi$, $T(w) = -1/w$ and $T(\infty) = 0$. Thus a f-invariant circle thru ∞ must lie in the plane $U_\theta \cup L_f \cup U_{\theta+\pi}$ for some $\theta \in [0, \pi)$, where U_θ is defined in subsection 4.8.

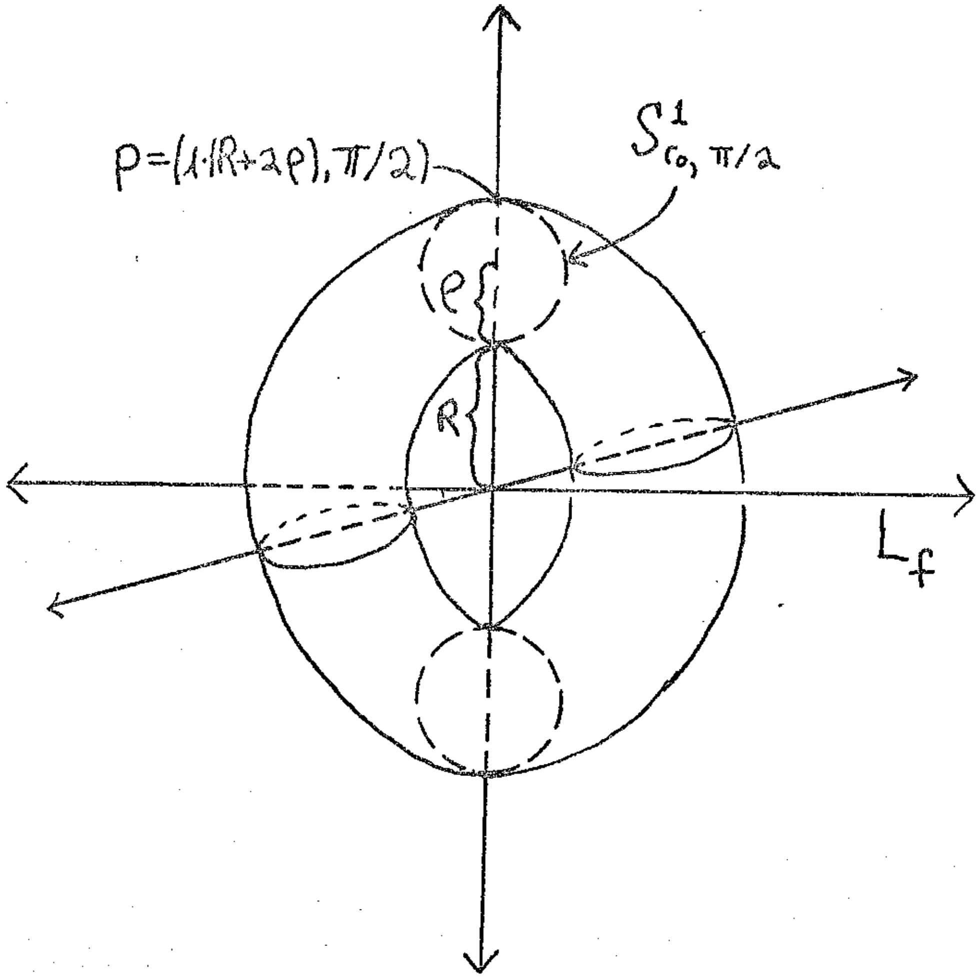


Diagram 2. The Steiner torus $T^2_{r_0}(f)$.

By assumption $f^2 \neq$ identity therefore $\theta_1 = \pi$ implies that $|\theta_1| \neq \pi$. We see that $f(U_\theta \cup L_f \cup U_{\theta+n}) = U_{\theta+\theta_2} \cup L_f \cup U_{\theta+\pi+\theta_2}$ and the only circle which is contained in both of these sets is L_f.

If $T(p) = a + bi$ then $T^{-1}(p) = a - bi$ since p is purely imaginary and T is elliptic with fixed points $\pm i$. We see that $f(p,\pi/2) = (a+bi, \pi/2+\theta_2)$ and $f^{-1}(p,\pi/2) = (a-bi, \pi/2 - \theta_2)$. From the symmetry of f(p) and $f^{-1}(p)$ we can see that P_ψ is gotten by rotating the xz plane $= \{(x,y,z) \in R^3 | y = 0\}$ about the z-axis thru some angle ψ, $\psi \in [-\pi,\pi]$, i.e., P_ψ has the linear basis $\{(0,0,1), (\cos\psi, \sin\psi, 0)\}$.

4.12. Motivated by our previous discussion and by lemma 4.10 in particular we prove the following lemma which tells us for which values of ψ the set $P_\psi \cap T_r^2$ contains a Euclidean circle. Given $p \in T_r^2$ if Orb(f/p) has cardinality greater than 8 then by lemma 4.10 the non-existence of such a Euclidean circle implies that p is not contained in a f-invariant circle.

<u>Lemma</u>: For $\psi \in [0,\pi/2]$ $P_\psi \cap T_{r_0}^2$ contains a circle if and only if $\psi = 0, \pi/2$ or $\arctan\left((2\rho R + R^2)^{\frac{1}{2}}/\rho\right)$ where $r_0 = -\ln(R)$,

$\rho = (1 - R)/2R$ and $0 < R < 1$.

Proof: Clearly, $P_0 \cap T^2_{r_0}$ and $P_{\pi/2} \cap T^2_{r_0}$ are each the union of two disjoint circles. For "small" values of ψ, $P_\psi \cap T^2_{r_0}$ is the union of two congruent disjoint, simple, closed curves $C_{1,\psi}$ and $C_{2,\psi}$ where $p = \left(i\cdot(R+2\rho),\pi/2\right) \in C_{1,\psi}$ and $-p = \left(i(R+2\rho),-\pi/2\right) \in C_{2,\psi}$. The finite plane domain bounded by $C_{1,\psi}$ is symmetric with respect to the line segment $t\cdot(0,0,R) + (1-t)(0,0,R+\rho)$ for $t \in [0,1]$. Therefore, if $C_{1,\psi}$ were a circle it would have center $(0,0,R+\rho)$ and radius ρ. The only such circle contained in $T^2_{r_0}$ is $C_{1,0} = S^1_{r_0,\pi/2}$. This argument suffices to show that $P_\psi \cap T^2_{r_0}$ contains no circle for ψ small enough to insure that $P_\psi \cap S^1_{r_0,0} = \phi$.

Claim: If $0 \leq \psi < \arctan\left((2\rho R + R^2)^{\frac{1}{2}}/\rho\right)$ then $P_\psi \cap S^1_{r_0,0} = \phi$.
If $\psi = \arctan\left((2\rho R + R^2)^{\frac{1}{2}}/\rho\right)$ then $P_\psi \cap S^1_{r_0,0}$ is a single point.

Proof of claim: This is seen by considering the planar cross section of $T^2_{r_0} \cap P_\psi$ obtained by setting $z = 0$.

$\{(x,y,z) \in T^2_{r_0} \mid z = 0\} = (S^1_{r_0,0} \cup S^1_{r_0,\pi})$.

$\{(x,y,z) \in P_\psi \mid z = 0\}$ is the line $y = \left(\tan(\psi)\right)\cdot x$,

$z = 0$. The claim is established by finding the value of $\psi \in [0,\pi/2]$ such that $\{(x,\tan(\psi)) | x \in R\}$ is tangent to $\{(x,y) | x^2 + (y-(R+\rho))^2 = \rho^2\} = S^1_{r_0,0}$. (See diagram 3.) The angle $v_1 v_2 v_3$ is $\pi/2$ and $\psi = \arctan\left((R^2 + 2R\rho)^{\frac{1}{2}}/\rho\right)$. This completes the proof of the claim.

Continuing the proof of the lemma we use a similar argument for $P_\psi \cap T^2_{r_0}$ where $\arctan\left((R^2 + 2R\rho)^{\frac{1}{2}}/\rho\right) < \psi < \pi/2$. $P_{\pi/2} \cap T^2_{r_0}$ is the disjoint union of two circles with center $(0,0,0)$ and having radius R and $R + 2\rho$ respectively. For values of ψ "close" to $\pi/2$ $P_\psi \cap T^2_{r_0}$ is the union of two disjoint simple closed curves $C_{1,\psi}$ and $C_{2,\psi}$ as above. Using the same type of arguments as we previously used we see that $C_{1,\psi}$ and $C_{2,\psi}$ cannot be circles for ψ sufficiently large to insure that $P_\psi \cap S^1_{r_0,0}$ contains two points. An argument identical to that of the claim above shows that $P_\psi \cap S^1_{r_0,0}$ contains two points for $\arctan\left((2\rho R + R^2)^{\frac{1}{2}}/\rho\right) < \psi < \pi/2$.

Finally when $\psi = \arctan\left((2R\rho + R^2)^{\frac{1}{2}}/\rho\right)$ we get the oblique toroidal circles. qed

4.13 Before continuing with the description of $C(f)$ we attempt to simplify notation by setting

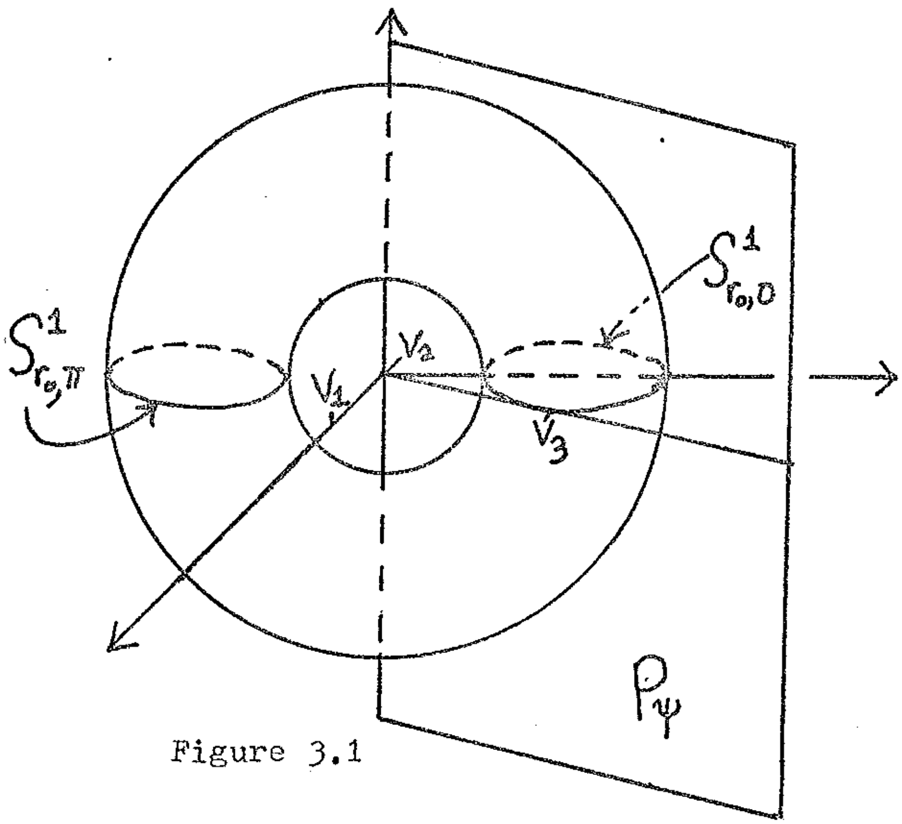


Figure 3.1

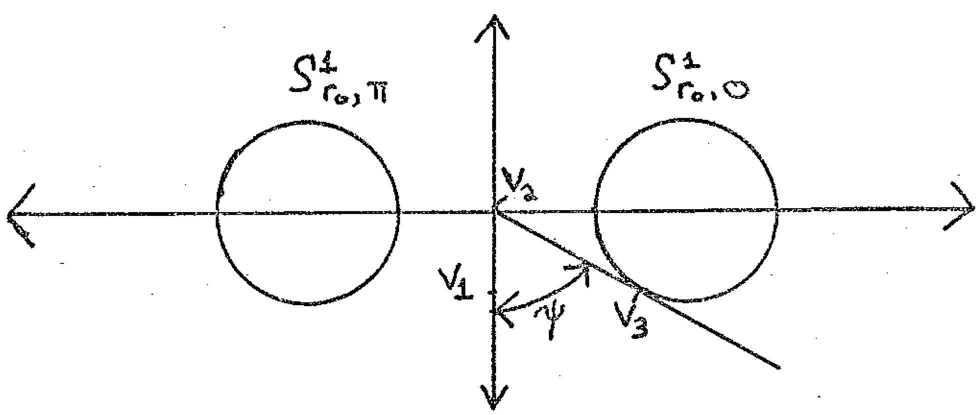


Figure 3.2

Diagram 3. Figure 3.1 shows $T^2_{r_0}$ and a portion of the plane P_ψ. Figure 3.2 is the planar cross section of $T^2_{r_0} \cap P_\psi$ obtained by set $z = 0$.

$R + 2\rho = s$. Making this substitution we get:

$$T(si) = \left(\sin(\theta_1)(1-s^2) + 2si\right) \div \left(s^2(1-\cos\theta_1) + (1+\cos\theta_1)\right).$$

When $\theta_1 = \pi$ we get $T(si) = Ri$ and therefore $R = 1/s$, $\rho = (s^2-1)/2s$ and

$$\arctan\left((2\rho R + R^2)^{\frac{1}{2}}/\rho\right) = \arctan(2s/s^2-1).$$

Let $\psi(s) = 2s/(s^2-1)$.

The description of $C(f)$ given in subsection 4.9 is completed by establishing the two statements below for the case in which the cardinality of $Orb(f/p)$ is greater than 8.

1) $f(i\cdot s,\pi/2) \in P_{\pm\psi(s)}$ if and only if $|\theta_1| = |\theta_2|$.

2) $Orb\left(f/(is,\pi/2)\right) \subseteq C_{1,\psi(s)}$ when $|\theta_1| = |\theta_2|$ for $C_{1,\psi(s)}$ as defined in lemma 4.12.

In subsection 4.16 we use special arguments to examine the cases in which the cardinality of $Orb(f/p)$ is less than or equal to 8.

4.14 In this subsection we prove statement 1) above. In rectangular coordinates $f(i\cdot s,\pi/2)$ is given by $(1/\left(s^2(1-\cos\theta_1) + (1+\cos\theta_1)\right))$ times

$$\left(\sin\theta_1(1-s^2), \sin\theta_2(2s), \cos\theta_2(2s)\right).$$

$\cos\left(\psi(s)\right) = (s^2-1)/s^2+1)$, $\sin\left(\psi(s)\right) = 2s/(s^2+1)$

and $P_{\psi(s)} = \left\{\left(\lambda_1(s^2-1)/(s^2+1), \lambda_1(2s)/(s^2+1), \lambda_2\right) \middle| \lambda_1, \lambda_2 \in R\right\}$.

We have $f(si,\pi/2) \in P_{\psi(s)}$ if and only if $\sin(\theta_1)(1-s^2)/\sin(\theta_2)(2s) = (s^2-1)/2s$ $\left(\text{for } \theta_1, \theta_2 \in (-\pi,\pi)\right)$ equivalently $\theta_1 = -\theta_2$. When $\theta_1 = \theta_2$ $f(si,\pi/2) \in P_{-\psi(s)}$. This proves statement 1).

4.15 In this subsection we prove statement 2) of subsection 4.13, i.e., we show that $\mathrm{Orb}\left(f/(is,\pi/2)\right) \subseteq C_{1,\psi(s)}$ when $|\theta_1| = |\theta_2|$. Recall that $C_{1,\psi(s)}$ is obtained by rotating the circle $y = 0$, $x^2 + \left(z - \left((s^2-1)/2s\right)\right)^2 = \left((s^2+1)/2s\right)^2$ thru the angle $\psi(s)$ about the z-axis. We will rotate $f^n(is,\pi/2)$ thru the angle $-\psi(s)$ about the z-axis and show that it is contained in our original circle.

Let $\theta_1 = \theta_2 = \theta$, then

$$f^n(is,\pi/2) = \left(1/\left(1-\cos(n\theta)\right) + \left(1+\cos(n\theta)\right)\right) \text{ times}$$
$$\left(-\sin(n\theta)\cdot(s^2-1), -\sin(n\theta)2s, \cos(n\theta)2s\right)$$

A rotation by $-\psi(s)$ about the z-axis is given by the matrix

$$M_s = \begin{pmatrix} (s^2-1)/(s^2+1) & 2s/(s^2+1) & 0 \\ -2s/(s^2+1) & (s^2-1)/(s^2+1) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with respect to the standard basis for R^3.

$$M_s\left(f^n(is,\pi/2)\right) = \left(1/\left(s^2(1-\cos\theta) + (1+\cos\theta)\right)\right) \text{ times}$$
$$\left(-\sin\theta(s^2+1), 0, 2s\cos\theta\right).$$

A long, but straightforward, algebraic computation shows that this point satisfies

$$x^2 + \left(z - \left((s^2-1)/2s\right)\right)^2 = \left((s^2+1)/2s\right)^2.$$

If $\theta_1 = -\theta_2$ then we must rotate $C_{1,\psi(s)}$ thru $+\psi(s)$ instead of $-\psi(s)$ and the result follows again.

This completes the proof except for the case of $|\theta_1| \neq |\theta_2|$ and the cardinality of Orb(f/p) less than 9 which is done in the next subsection and the final case of $|\theta_1| = |\theta_2| = \pi$ which is treated in subsection 4.17.

4.16 Thus far we have been able to show that Orb(f/p) is contained in a circle in $T^2_{r_0}$, the Steiner torus of radius r_0 ($r_0 \neq 0,\infty$) containing p, if and only if $|\theta_1| = |\theta_2|$. When the cardinality of Orb(f/p) is greater than 8 lemma 4.10 implies that p is contained in a f-invariant circle if and only if $|\theta_1| = |\theta_2|$.

We next examine what happens when the cardinality of Orb(f/p) is greater than 2 and less than 9. When $|\theta_1| = |\theta_2|$ lemma 4.12 and the two facts stated in subsection 4.13 imply that Orb(f/p) is contained in a circle and Orb(f/p) will again determine a f-invariant circle thru p. When $|\theta_1| \neq |\theta_2|$ Orb(f/p) will not be contained in $T^2_{r_0}$. However if Orb(f/p) has cardinality less than 9, this is no longer a necessary condition for p to be contained in a f-invariant circle. The cases which do arise are considered in this subsection.

If $|\theta_1| \neq |\theta_2|$ then $f^k(p) = p$ for $k \in \{2,\ldots,8\}$ if and only if the unordered pair $\{|\theta_1|,|\theta_2|\}$ is contained in the set, $\Big\{\{\pi,2\pi/3\}, \{\pi,2\pi/4\}, \{\pi,2\pi/6\}, \{\pi,2\pi/8\}, \{\pi/2,\pi/4\}, \{2\pi/3,\pi/3\}\Big\}$. When $\{|\theta_1|,|\theta_2|\} = \{\pi,2\pi/j\}$ for $j = 3$, 6 or 8 then $\{p,f^2(p),f^4(p)\}$ and $\{f(p),f^3(p),f^5(p)\}$ determine two distinct circles, either $S^1_{r_0,\pi/2}$ and $S^1_{r_0,-\pi/2}$ or $\{(p,\theta)|\theta \in [-\pi,\pi]\}$ and $\{\big(T(p),\theta\big)|\theta \in [-\pi,\pi]\}$. Therefore, Orb(f/p) cannot be contained in an f-invariant circle.

When $\{|\theta_1|,|\theta_2|\} = \{\pi/2,\pi/4\}$ or $\{2\pi/3,\pi/3\}$ then a computation shows that f(p) and $f^2(p)$ do not lie in the same plane P_ψ and therefore by lemma 4.13 Orb(f/p) cannot be contained in an f-invariant circle. For example when $(\theta_1,\theta_2) = (\pi/2,\pi/4)$ and $p = (\text{si},\pi/2)$

$f(si,\pi/2) = \big(T(si), 3\pi/4\big) = \big((1-s^2+2si)/(s^2+1), 3\pi/4\big)$ and $f^2(si,\pi/2) = \big(T^2(si),\pi\big) = (i/s,\pi)$. Converting into rectangular coordinates we get $\big(1/(s^2+1)\big)$ times $(1-s^2, -\sqrt{2}\,s, \sqrt{2}\,s)$ and $(0,-1/s,0)$ respectively. Clearly the plane P_ψ containing $f^2(si,\pi/2)$ is the plane $x = 0$ and $f(si,\pi/2)$ is not contained in this plane.

When $\{|\theta_1|,|\theta_2|\} = \{\pi, 2\pi/4\}$ $\mathrm{Orb}\big(f,(si,\pi/2)\big)$ equals $\{(si,\pi/2),(i/s,\pi),(si,3\pi/2),\big((i/s),2\pi\big)\}$ or $\{(si,\pi/2),\big(T(si),3\pi/2\big),\big(T^{-1}(si,3\pi/2\big)\}$

$$= \big(\overline{T(si)}, 3\pi/2\big),(i/s,3\pi/2)\}.$$

In either case Orb(f/p) is clearly not contained in a single circle.

4.17 Finally when $|\theta_1| = |\theta_2| = \pi$, i.e., when $f^2 = \mathrm{id}$ then $f(w,\theta) = \big(T(w),\theta+\pi\big) = (-1/w, \theta+\pi)$ for all $w \in \mathbb{C}$ with $\mathrm{im}(w) > 0$. Converting into rectangular coordinates we get $f(x,y,z) = \big(-1/(x^2+y^2+z^2)\big)(x,y,z)$. Let S^2 be the usual Euclidean unit sphere $\{(x,y,z)\,|\,x^2+y^2+z^1 = 1\}$ and let $a:\overline{R^3} \to \overline{R^3}$ be defined by $a(x,y,z) = (-x,-y,-z)$ and $a(\infty) = \infty$. (The map a will be called the antipodal map.) To complete description of C(f) in this case we show that a circle c is f-invariant if and only if $(c \cap S^2)$ is nonempty and invariant under the antipodal map.

Let c be an f-invariant circle. If there exists a point p such that $p \in c$ and $p \notin S^2$ then p and f(p) lie in different path components of $\overline{R^3} - S^2$. c must intersect S^2. f is clearly invariant on S^2 and $f|S^2 = a|S^2$ therefore $c \in C(f)$ implies that $c \cap S^2$ is a-invariant.

Assume next that $c \cap S^2 \neq \emptyset$ and is invariant under a. If $c \subseteq S^2$ then as noted above c is f-invariant if and only if it is a-invariant as $f|S^2 = a|S^2$. Assume that there exists a point $(x_0,y_0,z_0) \in \left(c \cap (\overline{R^3} - \left(S^2 \cup \{0,\infty\}\right))\right)$ and $(x_1,y_1,z_1) \in c \cap S^2$. If (x_0,y_0,z_0) and (x_1,y_1,z_1) are linearly dependent then c must be the extended line $\{\lambda(x_0,y_0,z_0)|\lambda \in R\} \cup \{\infty\}$. This "line" is clearly invariant under I_3 and a and therefore under f. If (x_0,y_0,z_0) and (x_1,y_1,z_1) are linearly independent then (0,0,0) and these two points determine the extended plane, P, containing c. $f(x_0,y_0,z_0)$ and $f(x_1,y_1,z_1)$ also lie in P. Therefore f sends P to itself. We can assume that P is the plane $z = 0$ and that $c = \{(x,y,0)|(x-b)^2 + y^2 = 1+b^2\}$ $b \neq 0$. Given

$(x_1, y_1, 0) \in c$ a simple computation shows that
$f(x_1, y_1, 0) = \left(1/(x_1^2 + y_1^2)\right)\cdot(-x_1, -y_1, 0) \in c$, i.e.,
$\left(-x_1/(x_1^2 + y_1^2) - a\right)^2 + \left(-y_1/(x_1^2 + y_1^2)\right)^2 = 1 + a^2$,
proving that c is f-invariant.

4.18 We summarize the results of section 4 in the following theorem.

Theorem: Let $f \in G^3$ be in normal form with respect to a fixed coordinate system for $\overline{R^3}$ then:

1) If f is purely parabolic, i.e., f is a translation T then C(f) is the set of all extended lines whose restriction to R^3 is parallel to the line $L = \{r \cdot T(0) | r \in R\}$.

2) If f is parabolic of the form RT with R a non-trivial rotation such that $T(0) \in FP(R)$ then $C(f) = \{\text{axis } (R)\}$.

3) If f is loxodromic of the form DR with $FP(f) = \{0, \infty\}$ where D is a dilation and R is a rotation then $C(f) = \left\{c \in C(R) | \{0, \infty\} \subseteq c\right\}$. (When R is trivial this implies that C(f) is the set of all circles containing $\{0, \infty\}$.)

4) If f is elliptic, i.e., f is a rotation R then C(f) contains axis (R) and the set of all Euclidean circles which lie in a (non-extend) plane P orthogonal

to axis (R) and which are centered at axis $(R) \cap P$. In the special case that f^2 = identity, all circles which intersect axis (R) orthogonally in 2 points are also in C(f).

5) If f is bielliptic and determined by the angles $\theta_1, \theta_2 \in [-\pi, \pi]$ then we can write $\overline{R^3}$ as the union of disjoint f-invariant tori T_r^2 and two circles $\{S_f^1, L_f\}$ which are "limiting tori" for the collection $\{T_r^2\}$ $r > 0$. S_f^1 and L_f are the only elements of C(f) unless $|\theta_1| = |\theta_2|$. If $|\theta_1| = |\theta_2| \neq \pi$ then passing thru every point p in T_r^2 $(0 < r < \infty)$ one of the two oblique toroidal circles containing p is f-invariant. The choice of this circle depends on whether $\theta_1 = \theta_2$ or $\theta_1 = -\theta_2$.

If $|\theta_1| = |\theta_2| = \pi$ or equivalently if f^2 = identity then $f(x,y,z) = \left(-1/(x^2+y^2+z^2)\right)(x,y,z)$, and C(f) is the set of all circles c such that the set $c \cap \{(x,y,z) | x^2 + y^2 + z^2 = 1\}$ is non-empty and f-invariant.

Section 5

Commutativity of Transformations in G^3

5.1 In section five we examine the problem of knowing when two elements of G^3 commute. This is done via a sequence of four propositions and a corollary. First we state and prove the following simple proposition because of its frequent use.

Given elements f and g in a group, $[f,g]$ will denote their commutation, i.e., $[f,g] = fgf^{-1}g^{-1}$. We will also write "id" to denote the identity element of G^3.

<u>Proposition</u>: Given $f,g \in G^3$ such that $[f,g] = id$ then:

1) If $S \subseteq \overline{R^3}$ is f-invariant then g(S) is also f-invariant.
2) Let S_a be a subset of $\overline{R^3}$ which is f-invariant and satisfies some property P which is preserved by all elements of G^3 (e.g., P may be the property, "is a circle"). If S is the union of all such sets S_a then S is g-invariant.

<u>Proof</u>: 1) $fg(S) = gf(S) = g(S)$

2) By 1) above $g(S_a)$ is f-invariant and by

assumption $g(S_a)$ satisfies property P. Therefore we have $g(S) \subseteq S$. Conversely $[f,g] = id$ implies $[f,g^{-1}] = id$ and we have $g^{-1}(S) \subseteq S$. Applying g to both sides of this relation we have $S = g\left(g^{-1}(S)\right) \subseteq g(S)$. qed

The two most important uses of the second part of this proposition are: i) S_a is an f-invariant (fixed) point and $S = FP(f)$ and ii) S_a is a f-invariant circle and $S = C(f)$ (as defined in subsection 4.2).

5.2 As in section 4 we will assume whenever possible that the transformations we examine are in normal form. This is clearly sufficient since conjugating a transformation f into normal form by an element h and then finding all elements g such that $[hfh^{-1},g] = id$ is equivalent to finding all transformations $h^{-1}gh$ such that $[f,h^{-1}gh] = id$. We also will continue the convention of letting T, D and R denote translation, dilation and rotation respectively.

In this subsection we describe the set of transformations which commute with a parabolic transformation. In the succeeding 3 subsections we do the same for loxodromic elliptic and bielliptic elements respectively.

<u>Proposition</u>: Let f be a parabolic element of G^3 of the form $f = RT$ where $0 \neq T(0) \in FP(R)$. If $R = id$ then $g \in G^3$ and $[f,g] = id$ if and only if:

1) g is a translation (i.e., g is purely parabolic and $FP(f) = FP(g) = \infty$),
2) g is a rotation with axis $(g) \in C(f)$, or
3) g is a composition of elements of these first two types.

If $R \neq id$ then $g \in G^3$ and $[f,g] = id$ if and only if:

4) g is a translation and $C(f) \subseteq C(g)$,
5) g is a rotation with $\{FP(g)\} = C(f)$, or
6) g is a composition of transformations of type 4) and 5) above.

<u>Proof</u>: Let g be a transformation in G^3 such that $[f,g] = id$. By proposition 5.1 $[f,g] = id$ implies that $g\big(FP(f)\big) = FP(f)$, therefore $\infty \in FP(g)$. By the symmetry of proposition 5.1 $f\big(FP(g)\big) = FP(g)$. This implies that g must either have the unique fixed point ∞, or it has an infinite number of fixed points. Thus g must either be parabolic or a rotation.

Assume first that $R = id$. If g is purely parabolic then it is a translation and clearly $[f,g] = id$. If g is elliptic then $f\big(FP(g)\big) = FP(g)$ or equivalently axis $(g) \in C(f)$. This establishes the necessity of 2). Conversely if axis $(g) \in C(f)$ then in some basis for R^3 $T(0) = (0,0,a)$ for $a \in R$ and axis $(g) = \{(0,0,r) \in R^3\}$. A direct computation shows that $fg = gf$.

It is clear that a composition of elements of type 1) and 2) must commute with f. It is also clear that this is the only type of parabolic transformation with a unique invariant circle which can commute with f, for the transformation g must satisfy $FP(f) = FP(g)$ because $FP(g)$ is a single point, and it must also satisfy $C(g) \subseteq C(f)$ as $C(g)$ is a single circle.

We next assume that $f = RT$ with $R \neq \mathrm{id}$. Part 4) is clearly the same as part 3). If g is elliptic then it must satisfy the condition of statement 5), i.e., $\{FP(g)\} = C(f)$ since $f\big(FP(g)\big) = FP(g) = \text{axis } (g)$, implying that axis $(g) \subseteq C(f)$. But as we have noted $C(f)$ has a unique element. The sufficiency of the condition $\{FP(g)\} = C(f)$ is easily seen by representing g in matrix form by a rotation with axis equal to axis (R) and computing. Part 6) follows immediately in the same manner as part 3) above. qed

5.3 In this subsection we consider the loxodromic case.

Proposition: Let $f = RD$ where $FP(D) \subseteq FP(R)$ then $g \in G^3$ satisfies $[f,g] = id$ if and only if g is a dilation D_1 such that $FP(D) = FP(D_1)$ or g is a rotation R_1 satisfying $[R,R_1] = id$ or g is a composition of two such elements.

Proof: The assumption $[f,g] = id$ implies that $g\big(FP(f)\big) = g(\{0,\infty\}) = \{0,\infty\}$. This implies that $\{0,\infty\} \subseteq FP(g)$ or $g(0) = \infty$ and $g(\infty) = 0$. Since f is loxodromic this second case is impossible. To see this note that for any $v \in \overline{R^3} - \{0,\infty\}$ $\lim_{l\to\infty} \|f^l(v)\|$ is either 0 or ∞ depending on whether $\|D(1,0,0)\|$ is less than one or greater than one. Assume that the limit is infinite.

Since we are assuming that $[f,g] = id$ we have $g^{-1}f^l g(v) = f^l(v)$ for all integers l and $v \in \overline{R^3} - \{0,\infty\}$. But $\lim_{l\to\infty} \|g^{-1}f^l g(v)\| = \|g^{-1}\big(\lim_{l\to\infty} f^l g(v)\big)\| = 0$ if $g(0) = \infty$ and $g(\infty) = 0$. This does not agree with the assumption that $\lim_{l\to\infty} \|f^l(v)\| = \infty$, therefore $\{0,\infty\} \subseteq FP(g)$. Thus g must be a dilation D_1 with $FP(D_1) = \{0,\infty\}$ or a rotation R_1 with $\{0,\infty\} \subseteq FP(R_1)$ or the composition of two such elements. We let $f = R_1D_1$

where either R_1 or D_1 might be trivial. Clearly $[f,g] = [RD,R_1D_1] = [R,R_1]$, i.e., $[f,g] = id$ if and only if $[R,R_1] = id$. If R or $R_1 = id$ then the condition $[R,R_1] = id$ is trivially satisfied and $[f,g] = id$. In any case when $[f,g] = id$ we have shown that g is either a dilation D_1 with $FP(D) = FP(D_1)$ or a rotation R_1 such that $[R,R_1] = id$ or g is a composition of two such transformations. The converse is clear.

qed

5.4 In the previous two subsections we saw that elliptic transformations could commute with parabolic and with loxodromic transformations. In this subsection we give necessary and sufficient conditions for two elliptic elements to commute.

Proposition: Let f and g be elliptic elements of G^3 then $[f,g] = id$ if and only if axis $(f) \in C(g)$ and axis $(g) \in C(f)$.

Proof: Assume $[f,g] = id$. By proposition 5.1 $FP(f) = \text{axis } (f) \in C(g)$ and $FP(g) = \text{axis } (g) \in C(f)$. The converse is established in two parts, first we assume $f^2 \neq id$ then assume $f^2 = id = g^2$. We will assume for the remainder of this proof that f is a rotation with matrix represetation

$$\begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(where $a, b \in R$ and $a^2 + b^2 = 1$) with respect to the standard basis for R^3.

We first assume that $f^2 \neq id$, i.e., that $a \neq -1$. By theorem 4.18, axis (g) $\in C(f)$ implies that axis (g) $= \{(0,0,z) | z \in R\} \cup \{\infty\}$, or that for some $d, k \in R$, axis (g) $= \{(x,y,d) | x^2 + y^2 = k > 0\}$. If axis (g) is of the former type then g has matrix representation $\begin{pmatrix} e & -f & 0 \\ f & e & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (with $e^2 + f^2 = 1$) with respect to the standard basis for R^3 and a simple computation shows that $[f,g] = id$.

If axis (g) is of the second type then g restricted to the set $\overline{R^3}$ - axis (f) can be represented in cylindrical coordinates. To understand this representation consider the following.

Let $P \cup \{\infty\}$ be any extended plane which contains axis (f). P will intersect axis (g) orthogonally in two antipodal points $\{w_1, w_2\}$. $g(P \cup \{\infty\})$ must be a sphere or an extended plane, and must intersect $g\big(\text{axis (g)}\big)$ orthogonally at $\{g(w_1), g(w_2)\}$. Since

$w_1, w_2 \in$ axis $(g) = FP(g)$, $g(P)$ must also be orthogonal to axis (g) at w_1 and w_2. $P \cup \{\infty\}$ is the unique such sphere or extended plane. Thus $g(P) = P$. $g|P$ can be expressed as a fractional linear transformation T_1 with fixed points w_1 and w_2. T_1 must be elliptic as axis $(f) \in C(g|P \cup \{\infty\})$ and axis (f) separates w_1 and w_2. Let $P_1' \cup \{\infty\}$ be any other extended plane which contains axis (f). The same argument shows that restricted to $P_1' \cup \{\infty\}$, g must be an elliptic fractional linear transformation T_1'. Representing $P \cup \{\infty\}$ - axis (f) in cylindrical coordinates as $\left\{(a+bi, \theta) \mid a,b \in R,\ b > 0 \text{ and } \theta \in \{0,\pi\}\right\}$ and $P_1' \cup \{\infty\}$ - axis (f) as $\left\{(a+bi, \theta) \mid a,b \in R,\ b > 0 \text{ and } \theta \in \{\psi, \psi+\pi\}\right\}$ for $\psi \in (0,\pi)$, then $T(a+bi)$ must equal $T_1'(a+bi)$ as T and T_1' agree on axis (f). Thus on $\overline{R^3}$ - axis (f) we can represent every point as $(a+bi, \theta)$ where $a,b \in R$, $b > 0$ and $\theta \in [0,2\pi)$. In these cylindrical coordinates $g(a+bi,\theta) = \big(T(a+bi),\theta\big)$ and $f(a+bi,\theta) = (a+bi, \theta+\theta_0)$ for some $\theta \in (0,2\pi)$.

The computation below now shows that $[f,g] = \text{id}$. $gf(a+bi,\theta) = g(a+bi,\theta+\theta_0)$ $=\big(T(a+bi),\theta+\theta_0\big)$, $fg(a+b,\theta) = f\big(T(a+bi),\theta\big)$ $= \big(T(a+bi),\theta+\theta_0\big)$. This completes the proof of the proposition under the assumption $f^2 \neq id$. Note that the case of $f^2 \neq id$ and $g^2 = id$ is included in the above considerations.

Assume next that $f^2 = id = g^2$. By theorem 4.18 we can see that the only new possibility for axis (f) $\in C(g)$ and axis (g) $\in C(f)$ is that axis (f) and axis (g) intersect orthogonally in two points p_1 and p_2. Conjugating if necessary we can assume that $p_1 = 0$ and $p_2 = \infty$. The matrix representation for f becomes

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

We can assume by conjugation if necessary that the matrix representation of g with respect to the same basis is

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$$

Clearly $[f,g] = id$. qed

In proving the above proposition we derived a useful representation of the action of an elliptic transformation with axis equal to $\{(x,y,d) | x^2 + y^2 = k > 0\}$ for $d \in R$ and having $\{(0,0,r) | r \in R\}$ as an invariant circle. We state this representation in the following corollary for future use. Note that we do not have to assume that $\{(0,0,r) | r \in R\}$ is invariant. In fact we show that this must necessarily follow from the fact that the transformation has axis $\{(x,y,d) \mid x^2 + y^2 = k > 0\}$.

Corollary: Let f be an elliptic transformation of G^3 with axis (f) = $\{(x,y,0) \mid x^2 + y^2 = 1\}$. Represent $R^3 - \{0,0,z) \mid z \in R\}$ in cylindrical coordinates as $\{(w,\theta) \mid \theta \in [0,2\pi),\ w \in \not{C}$ with Im (w) > 0} then $f(w,\theta) = \big(T(w),\theta\big)$ where T is an elliptic fractional linear transformation with fixed points $\pm i$.

Proof: To prove this corollary we must only show that $L = \{(0,0,z) \mid z \in R\} \cup \{\infty\}$ is f-invariant and that $FP(T) = \pm i$; the corollary will then follow from the proof of proposition 5.4. Since f is elliptic we can find a transformation h in G^3 such that hfh^{-1} is a rotation R of $\overline{R^3}$. Let h be such a transformation. We must prove that h(L) is R-invariant. Since $L \cap$ axis $(f) = \phi$ we have $h(L) \cap$ axis $(R) = \phi$ therefore by theorem 4.18 we must show that h(L) is a circle contained in a non-extended Euclidean plane P which is orthogonal to axis (R) and such that h(L) is centered at $P \cap$ axis (R). Let $P_1 \cup \{\infty\}$ and $P_2 \cup \{\infty\}$ be two distinct extended planes containing L.

$h(L) = h\big((P_1 \cup \{\infty\}) \cap (P_2 \cup \{\infty\}\big)$

$= h(P_1 \cup \{\infty\}) \cap h(P_2 \cup \{\infty\})$. P_1 and P_2 intersect axis (f) orthogonally, therefore $h(P_1 \cup \{\infty\})$ and $h(P_2 \cup \{\infty\}$ must be spheres which intersect axis (R) orthogonally. Any sphere which intersects axis (R) orthogonally is certainly R-invariant. Thus h(L) is the intersection of two R-invariant spheres and must itself be R-invariant. Thus by the proof of proposition 5.4 $f(w,\theta) = \left(T(w),\theta\right)$ where T is an elliptic fractional linear transformation and w and θ are as in the statement of the corollary.

The fact that T has fixed points $\pm$ i in $\mathbb{C}$ can be seen by looking at
$\text{axis}(f) \cap \{(w,0)|w \in H^2\} = \{(i,0)\}$. Therefore T is an elliptic fractional linear transformation which keeps the real axis invariant and has a fixed point at +i. T must have the other fixed point at -i. (See Ford [4], 20.) qed

5.5 We conclude section 5 with a proposition which describes the elements of G^3 which can commute with a bielliptic transformation.

Proposition: Let f be a bielliptic transformation in G^3. Given $g \in G^3$ let $\hat{f}$ and $\hat{g}$ be the corresponding elements in the D^4-model of G^3 (see subsection 2.20). If $[f,g] = \text{id}$ then $FP(f) \subseteq FP(g)$ and $\hat{g}$ is elliptic or bielliptic.

<u>Proof</u>: Since f is bielliptic, $FP(f) = \emptyset$. This is also true of $\bar{f}$ the corresponding transformation in the S^3 model of G^3. By corollary 3.11 $\hat{f}$ is therefore conjugate to a rotation of the form $\begin{pmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & c & -d \\ 0 & 0 & d & c \end{pmatrix}$

where $a^2 + b^2 = 1 = c^2 + d^2$ and $a \neq 1 \neq c$. Thus $\hat{f}$ must have a unique fixed point in D^4. Clearly $[f,g] = id$ implies that $[\hat{f},\hat{g}] = id$ and therefore $FP(\hat{f}) \subseteq FP(\hat{g})$. By lemma 3.10, this implies that $\hat{g}$ is conjugate to a rotation of D^4 and g must have a circle of fixed points or no fixed points; i.e., g is elliptic or bielliptic. qed

Section 6

Construction of the Manifold M_0

6.1 In this section a closed, orientable, 3-dimensional manifold M_0 is constructed and its fundamental group is computed. Subsequently, in section 7, it is shown that M_0 has no conformal structure as defined in the introduction.

6.2 Let X be a n-dimensional closed, orientable, connected manifold. Any homeomorphism f of X onto itself determines an equivalence relation, $\sim f$, on the product space $X \times [0,1]$ as follows:

Given (x_0,t_0) and (x_1,t_1) in $X \times [0,1]$, we define $(x_0,t_0) \sim f\ (x_1,t_1)$ if $(x_0,t_0) = (x_1,t_1)$ or if $t_0 = 0$, $t_1 = 1$ and $f(x_1) = x_0$. Let $M = (X \times [0,1]/\sim f)$ be the space of $\sim f$ equivalence classes with the quotient topology M is a closed (n+1)-dimensional manifold. It is orientable if f is orientation preserving.

We have the following procedure for describing $\Pi_1(M)$, the fundamental group of M. Let $\langle x,i \rangle$ denote the equivalence class under $\sim f$ of the pair $(x,i) \in X \times [0,1]$. Let $in: X \to M$ be the map

$in(x) = \langle x,0\rangle$ and let $p: M \to S^1$ be defined by $p(x,t) = e^{2\pi it}$. The map p gives us a weak fibration and we have the short exact sequence

$$1 \to \Pi_1(X) \xrightarrow{in_*} \Pi_1(M) \xrightarrow{p_*} \Pi_1(S^1) \to 1 .$$

(See Spanier [10], 377). (Note we have used the fact that $\Pi_2(S^1) = 1$.) If $FP(f) \neq \emptyset$ then there exists a cross section $j: S^1 \to X$ namely $j(e^{2\pi it}) = \langle x_0,t\rangle$ where $x_0 \in FP(f)$. In this case $pj: S^1 \to S^1$ is the identity and we get a splitting of the sequence

$$1 \to \Pi_1(X) \xrightarrow{in_*} \Pi_1(M) \underset{\xleftarrow[j_*]{}}{\xrightarrow{p_*}} \Pi_1(S^1) \to 1.$$

$\Pi_1(M)$ is therefore a semi-direct product of $\Pi_1(X)$ and $\Pi_1(S^1)$.

6.3 We now use the procedure outlined above to construct M_0 and compute $\Pi_1(M_0)$. Let $X = T^2 = S^1 \times S^1$ and let $f: T^2 \to T^2$ be a Dehn Twist about a loop representing a generator of $\Pi_1(T^2) = Z \times Z$. Let $M_0 = (X \times [0,1]/\sim f)$. f can be taken to be the identity on all of T^2 except a "small" collared neighborhood of a "nice" loop representing the chosen generator. If we call this neighborhood N and let $h: S^1 \times (0,2) \to T^2$ be a homeomorphism onto N then f can be defined by: $f|(T^2 - N)$ is the identity and

$$f\bigl(h(e^{i\theta},t)\bigr) = \begin{cases} h(e^{i\theta},t) & \text{for } t \in (0,\tfrac{1}{2}] \cup [3/2,2) \\ h(e^{i\theta}\cdot e^{2\pi i(t-\frac{1}{2})},t) & \text{for } t \in [\tfrac{1}{2},3/2]. \end{cases}$$

Let $b \in \Pi_1(T^2)$ be represented by the loop $h_1(e^{i\theta}) = h(e^{i\theta},1)$ for $\theta \in [0,2\pi]$. Let $a \in \Pi_1(T^2)$ be represented by a loop which intersects the loop h_1 exactly once and such that a and b together generate $\Pi_1(T^2)$. Then $f_*(a) = ab$ and by the remarks in subsection 6.2, $\Pi_1(M_0)$ is a semi-direct product of $\Pi_1(T^2)$ and $\Pi_1(S^1)$.

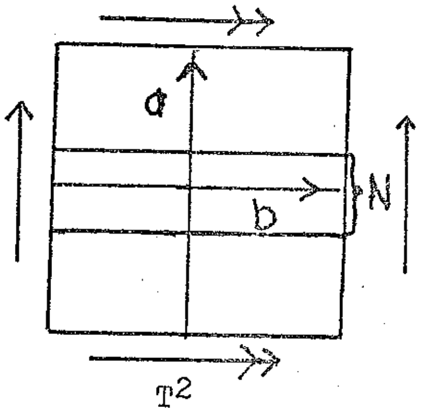
The sequence

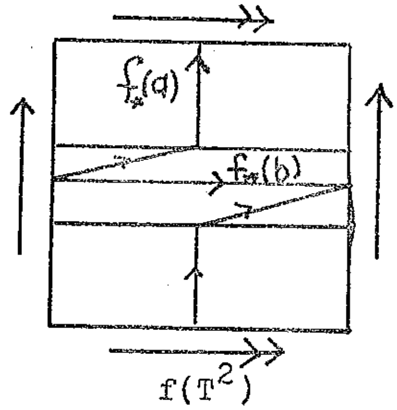
$$1 \to \Pi_1(T^2) \xrightarrow{in_*} \Pi_1(M_0) \underset{j_*}{\overset{p_*}{\rightleftarrows}} \Pi_1(S^1) \to 1$$

is split-exact. If c is a generator for $\Pi_1(S^1)$ then $\Pi_1(M_0)$ is generated by $in_*(a)$, $in_*(b)$ and $j_*(c)$. Since $[a,b] = id$ we must have $[in_*(a),in_*(b)] = id$. The only other relations in $\Pi_1(M_0)$ are given by the action of $j_*(c)$ on $in_*(a)$ and $in_*(b)$. Denote this action by m. $in_*(a) \overset{m}{\longmapsto} j_*(c)\, in_*(a) j_*(c)^{-1}$ and

$$in_*(b) \overset{m}{\longmapsto} j_*(c)\, in_*(b) j_*(c)^{-1}.$$

(See diagram 4 .) Clearly $m\bigl(in_*(a)\bigr) = in_*\bigl(f_*(a)\bigr)$ and $m\bigl(in_*(b)\bigr) = in_*\bigl(f_*(b)\bigr)$. The action of m therefore





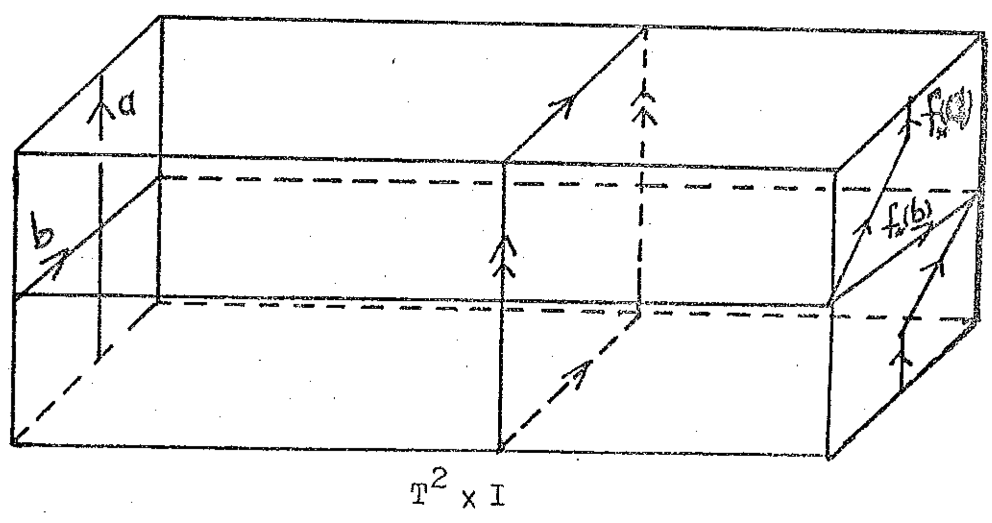


Diagram 4. The manifold M_0 is constructed by making the indicated identifications on $T^2 \times I$.

gives us the two additional relations

$$j_*(c)in_*(a)j_*(c)^{-1} = in_*\big(f_*(a)\big) = in_*(a)in_*(b) \text{ and}$$

$$j_*(c)in_*(b)j_*(c)^{-1} = in_*\big(f_*(b)\big) = in_*(b)$$

in $\Pi_1(M_0)$.

Summarizing: $\Pi_1(M_0)$ is generated by $in_*(a)$, $in_*(b)$ and $j_*(c)$ and a full set of relations is given by $[in_*(a), in_*(b)] = id$, $[in_*(b), j_*(c)] = id$ and $[j_*(c), in_*(a)] = in_*(b)$.

gives us the two additional relations

$j_*(c)in_*(a)j_*(c)^{-1} = in_*\big(f_*(a)\big) = in_*(a)in_*(b)$ and

$j_*(c)in_*(b)j_*(c)^{-1} = in_*\big(f_*(b)\big) = in_*(b)$

in $\Pi_1(M_0)$.

Summarizing: $\Pi_1(M_0)$ is generated by $in_*(a)$, $in_*(b)$ and $j_*(c)$ and a full set of relations is given by $[in_*(a), in_*(b)] = id$, $[in_*(b), j_*(c)] = id$ and $[j_*(c), in_*(a)] = in_*(b)$.

Section 7

Proof that M_0 Does Not Admit a Strong Conformal Structure

7.1 This final section is devoted to proving the following:

Theorem: M_0 has no conformal structure.

Before proving this we recall our definition of conformal structure and we establish some notation.

Definition: Given a closed orientable n-dimensional manifold M, a conformal structure for M, denoted S(M), is a pair (G,D) where D is an open connected subset of S^n and G is a discrete subgroup of G^n which acts freely on D, where D/G is homeomorphic to M.

Notation: Given G, a discrete subgroup of G^n, $\Omega(G)$ will be the set of free discontinuity of G, i.e., $\Omega(G)$ is the set of all points v, in S^n, such that there exist a neighborhood N_v of v for which $N_v \cap g(N_r) = \emptyset$ for all $g \in G^3 - \{id\}$.

For the same group G let $\Lambda(G)$ denote the limit set of G, i.e., $\Lambda(G)$ is the set of v_0 for which there exists an infinite collection of distinct elements $g_1, g_2, \ldots$ in G and a point v_1 in R^n such that $v_0 = \lim_{k \to \infty} \left(g_k(v_1)\right)$. Note that $\Omega(G) \cap \Lambda(G) = \emptyset$;

$\Omega(G)$ clearly can also not contain the fixed point of any $g \in G$.

The ordered triple $(\hat{M},p,M)$ is said to be a (normal) covering triple if and only if $\hat{M}$ is a (normal) covering space for M with projection $p:\hat{M} \to M$.

If $(\hat{M},p,M)$ is a normal covering triple then $T(\hat{M},M)$ will denote the group of deck transformations for $(\hat{M},p,M)$, i.e., $T(\hat{M},M)$ is the group of all self homeomorphisms h, of $\hat{M}$, such that $p = ph$.

7.2 Theorem 7.1 will be established through a sequence of lemmas and proposition. First we prove a necessary condition for the existence of a conformal structure on a manifold M.

Lemma: If (G,D) is a conformal structure for a closed, orientable, n-dimensional manifold M then there exists a homomorphism $h:\Pi_1(M) \to G^n$ with image (h) = G and such that D is a maximal connected component of $\Omega(G)$.

Proof: The definition of conformal structure implies that D is a normal covering space of M with T(D,M) isomorphic to G. Therefore $G = \Pi_1(M)/\big(\Pi_*(\Pi_1(D)\big)$ where $\Pi:D \to M$ projects a point in D onto its orbit under G. By definition, D is contained in a connected component of $\Omega(G)$. Assume D is not maximal, i.e., assume that there exist $\bar{D}$ connected with $D \subsetneq \bar{D} \subseteq \Omega(G)$. M would then be homeomorphic to a proper sub manifold of the n-dimensional connected manifold $\bar{D}/G$. This contradicts the fact that M is assumed to be closed. qed

7.3 In this section we show that no discrete subgroup of G^3 can contain an elliptic or bielliptic element of infinite order. This lemma utilizes the following theorem:

Theorem: A group of isometries G of a manifold M is discrete if and only if it is discontinuous (i.e., if and only if $\Lambda(G) = \emptyset$). (See Siegel [9], p. 32.)

Lemma: If G is a discrete subgroup of G^3 then G contains no elliptic or bielliptic of infinite order.

Proof: In both the elliptic and bielliptic case there are compact non-empty subsets of S^3 and a Riemannian metric for which the transformation in question is an isometry. It will then follow from the theorem quoted above that the cyclic subgroup of G generated by the given transformation will be discontinuous and therefore discrete if and only if it is of finite order.

If $g \in G^3$ is elliptic then by a suitable conjugation g can be assumed to be a rotation of R^3 with matrix representation $(r_{i,j})$. In subsection 2.10 we showed that g was an isometry of R^3. The orbit of a point v, not contained in FP(g), is contained in a circle in $\overline{R^3} - FP(g)$. Since the circle is compact the orbit of v can only have a finite number of distinct values and therefore g has a finite order.

When g is bielliptic we can use the S^3-model of G^3 to get our results. By corollary 3.11 we know that in this model g is conjugate to a rotation of S^3. Rotations of S^2 are Euclidean isometries and S^3 is compact therefore g must have finite order. qed

7.4 By lemma 7.2 a necessary condition for the existence of a conformal structure on M_0 is the existence of a homomorphism $h: \Pi_1(M_0) \to G^3$ with image (h) discrete. Recall from subsection 6.3 that $\Pi_1(M_0)$ is a group with generators $in_*(a)$, $in_*(b)$ and $j_*(c)$, which we now simplify to a, b and c respectively, and with the three defining relations $[a,b]$ = identity, $[b,c]$ = identity and $[a,c] = b$. In subsection 7.5 we show that there does not exist an injective homomorphism from $\Pi_1(M_0)$ into G^3 whose image is discrete. By lemma 7.2 this means that (G,D) cannot be a conformal structure if $\Pi_1(D)$ is trivial. In lemma 7.6 we establish a necessary relationship between D and the elliptic fixed points of G. In the final four subsections we show that for any homomorphism not previously examined from $\Pi_1(M_0)$ to G^3 with discrete image, $\Omega(G)$ is connected and either simply connected or does not satisfy lemma 7.6. Thus none of these homorphisms yield a conformal structure for M_0.

For simplicity of notation we will denote both an element in $\Pi_1(M_0)$ and its image under a homomorphism h by the same symbol whenever this does not lead to confusion. For example we will write, "Let c be elliptic." This should be understood to mean, "Let h(c) be elliptic."

7.5 Proposition: There does not exist a homomorphism $h:\Pi_1(M_0) \to G^3$ where h(a), h(b) and h(c) are all of infinite order.

Proof: By lemma 7.3 h(a), h(b) and h(c) cannot be elliptic or bielliptic. We divide the rest of this proof into two cases; case 1, h(b) is parabolic and case 2, h(b) is loxodromic. (We will simplify h(a), h(b) and h(c) to a, b and c respectively for the remainder of this proof.)

Case 1: b is parabolic. We can assume that $FP(b) = \{\infty\}$. Since $[a,b] = id = [b,c]$ and a and c cannot be elliptic, proposition 5.2 implies that a and c are parabolic with $FP(a) = FP(c) = \infty$. Both a and c cannot be translations as this would imply that $b = [a,c] =$ identity. Assume that a is not a translation. Then by theorem 4.18, a has a unique invariant circle which we call L_a. The fact that $[a,b] =$ identity implies that L_a is b-invariant; therefore $L_a = b(L_a) = cac^{-1}a^{-1}(L_a) = cac^{-1}(L_a)$ or equivalently $c^{-1}(L_a) = ac^{-1}(L_a)$. Since $\{L_a\} = C(a)$, the circle $c^{-1}(L_a)$ must be L_a; i.e., L_a is also c-invariant. By proposition 5.2 this also implies that $b = [a,c] =$ identity. Thus if b is parabolic it must be trivially parabolic, i.e., b = identity. This completes case 1.

Case 2: b is loxodromic. We can assume that $FP(b) = \{0,\infty\}$. Since a and b cannot be elliptic and $[a,b] = id = [b,c]$, proposition 5.3 implies that a and c must also be loxodromic with $FP(a) = FP(c) = \{0,\infty\}$. We know by theorem 3.7 that we can let $a = D_1R_1$, $b = D_2R_2$ and $c = D_3R_3$ where the D_i are non-trivial dilations and the R_i are possibly trivial rotations, for $i = 1,2,3$. We must have

$b = [a,c]$. The commutator $[a,c]$ however equals $[R_1, R_3]$ which is an isometry of R^3 with the usual metric and therefore is certainly not loxodromic. This completes case 2. qed

Corollary 1: There does not exist a homomorphism $h: \Pi_1(M_0) \to G^3$ with image (h) discrete and trivial kernel.

Corollary 2: If (G,D) is a conformal structure for M_0 then D is not simply connected.

7.6 Lemma: Let G be a discrete subgroup of G^3 and D' an invariant, maximal path component of $\left(\overline{R^3} - \Lambda(G)\right)$ such that D' contains fixed points of elliptic elements. If D equals $D' - \{v \in D' | v$ is the fixed point of an elliptic element of g$\}$ then D/G is not compact.

Proof: By an appropriate conjugation of G we can assume that $\{0\} \in D' - D$ and that G contains an elliptic element g where

$FP(g) = \{(0,0,z) \in R^3 | z \in R\}$.

Let $N_\epsilon = \{(x,y,z) \in R^3 | x^2 + y^2 + z^2 < \epsilon\}$.

For sufficiently small $\epsilon > 0$, $N_\epsilon \subseteq D$ and

$h(N_\epsilon) \cap N_\epsilon = \emptyset$ or $h(N_\epsilon) = N_\epsilon$ for all $h \in G$.

If $h(N_\epsilon) \cap N_\epsilon = \emptyset$ then $h(N_\epsilon)$ is homeomprhic N_ϵ and the closure of N_ϵ is homeomorphic to h(closure of N_ϵ) for all $h \in G$. G acts on the set of elliptic fixed points in D'. If p is the fixed point of an elliptic element e then heh^{-1} is elliptic and has fixed point at h(p). Let $S_k = \bigcup_{h \in G} \left(h(\text{closure}(N_{\epsilon/k}))\right)$ for $k = 1,2,3,\ldots$. Let $C_k = D - S_k$. $C = \{C_1, C_2, C_3, \ldots\}$ is an open cover for D. Each C_k is invariant under the action of G; therefore we can form the cover $C/G = \{C_1/G, C_2/G, C_3/G, \ldots\}$ for D/G. Clearly these covers have no finite subcover. The C_i are properly nested (i.e., $j < k$ implies that C_j is properly contained in C_k) and $N_{\epsilon(k+1)}$ is not contained in C_k thus C has no finite subcover of D. By the invariance of G on the C_i this implies that C/G has no finite subcover for D/G and D/G cannot be compact.

qed

<u>Corollary</u>: Let G be a discrete subgroup of G^3 and let H be a subgroup of G of finite index. If $\Lambda(H)$ is of finite cardinality, then for any maximal connected subsets D of $\Omega(G)$, the pair (G,D) cannot be a conformal structure for M_0.

<u>Proof</u>: Since H is of finite index in G there exist $g_1, \ldots, g_k$ in G such that $G = g_1(H) \cup \ldots \cup g_k(H)$. This implies that $\Lambda(G)$ $g_1(\Lambda(H)) \cup \ldots g_k(\Lambda(H))$ and

$\Lambda(G)$ must also be a finite set. This implies that $\Omega(G)$ is simply connected which is not acceptable by corollary 7.5-2 or it implies that D/G is not compact by the lemma above which is also not acceptable. qed

7.7 Lemma: 1) Every element of $\Pi_1(M_0)$ can be written in the form $a^i b^j c^k$ where i, j and k are integers. 2) Multiplication in $\Pi_1(M_0)$ is given by the formula

$(a^i b^j c^k)\cdot(a^p b^q c^r) = a^{i+p} b^{(j+q-h\cdot 1)} c^{k+r}$ where

i, j, k, p, q and r are integers.

Proof: Part 1): Since b commutes with both a and with c it suffices to prove the following two statements: Let w be any word in the letters a, b and c, then there exists a word w' with no more occurrences of the letter a than there are in w and such that wa = aw'. Similarly there exists a word w" with no more occurrences of the letter c than there are in w, such that wc = cw". We prove the existence of w'. The existence of w" follows from the symmetry of a and c in $\Pi_1(M_0)$. Recall that $b = cac^{-1}a^{-1}$ or equivalently ca = bac. We do an induction on the number of occurrences of c. If there are no occurrences of c then wa = aw since w is a string of powers of a and of b which commutes with a. Assume we can find w' if w has n-1 occurrences of c. Let w have n occurrences, w = xcy where x is a word with n-1 occurrences of c and y is a word using only the letters a and b. Then wa = xcya = xcay = xbacy = xabcy. By the induction hypothesis xa = ax', therefore wa = ax'bcy and w' = x'bcy satisfies the requirement on w'.

Part 2): It suffices to prove that $c^k a^p = b^{-kp} a^p c^k$. This is a simple induction on k. When k = 0 this is trivial. When k = 1 we have $ca^p = caa^{p-1} = baca^{p-1} = \dots = b^p a^p c$. Assume that $c^{k-1} a^p = b^{-(k-1)p} a^p c^{k-1}$ for some $k \geq 2$ and for all integers p. We have the following simple calculation.

$$c^k a^p = c\left(c^{(k-1)} a^p\right) = c\left(b^{-(k-1)p} a^p c^{k-1}\right)$$

$$= b^{-(k-1)p} (ca^p) c^{k-1} = b^{-(k-1)p} (b^{-p} a^p c) c^{k-1} = b^{-kp} a^p c^k.$$

A similar induction proves the statement for $k < 0$. qed

7.8 Proposition: If h(a), h(b) or h(c) is bi-elliptic then image (h) = G is finite and therefore for any $D \subseteq S^3$ (G,D) is not a conformal structure for M_0.

Proof: If h(b) is bielliptic then [h(a),h(b)] = id and [h(c),h(b)] = id imply that h(a) and h(c) must be elliptic or bielliptic by proposition 5.6. Lemma 7.3 implies that h(a), h(b) and h(c) must have finite order. By lemma 7.7 this means that the order of image (h) would be finite.

If h(a) is bielliptic, let A, B and C be the extensions of h(a), h(b) and h(c) respectively to D^4 (see subsection 2.20). By corollary 3.7, A has a unique fixed point, say v_0. $[a,b] = id$ implies that $v_0 \in FP(b)$ therefore $v_0 = b(v_0) = cac^{-1}a^{-1}(v_0)$ $= cac^{-1}(v_0)$ or equivalently $c(v_0) \in FP(a) = \{v_0\}$. Therefore $v_0 \in FP(c)$ and by lemma 3.7, c is elliptic or bielliptic. Again image (h) must be finite. The case h(c) bielliptic is the same as h(a) bielliptic by the symmetry of $\Pi_1(M_0)$. qed

7.9 Proposition: There does not exist a homomorphism $h: \Pi_1(M_0) \to G^3$ with h(b) of infinite order and image (h) discrete.

Proof: By proposition 7.5, a, b and c in G^3 cannot all have infinite order. By corollary 7.7, a and c can be assumed to be elliptic whenever they have finite order. We will first examine what happens when b is loxodromic and subsequently when b is parabolic.

Let b be loxodromic and such that $FP(b) = \{0,\infty\}$. Either a or c must be elliptic as a, b and c cannot all have infinite order. Assume a is elliptic. By proposition 5.3, $[b,c] = id$ implies that c is either elliptic or loxodromic and $\{0,\infty\} \subseteq FP(c)$. In either case $b = [a,c]$ is a (perhaps trivial) rotation, contradicting the assumption that b is loxodromic. If we start with the assumption that c is elliptic rather than a, the symmetry of $\Pi_1(M_0)$ gives us the same result.

Next consider the case where b is parabolic. Again we must assume that a or c is elliptic. Assume a is. We also can assume that $FP(b) = \{\infty\}$. This implies that a and c which both commute with b have a fixed point at ∞. We will show that FP(a) is c-invariant and using proposition 5.2 we can show that c must be a translation or a rotation with axis (c) parallel to axis (a), or a composition of two such transformations. This will suffice to show that $[a,c]$ cannot be a non-trivial parabolic transformation.

Since $[a,b] = id$, FP(a) is b-invariant. We also have the equation $a^{-1}c^{-1} = c^{-1}a^{-1}b^{-1}$ and therefore $a^{-1}c^{-1}\big(FP(a)\big) = c^{-1}a^{-1}b^{-1}\big(FP(a)\big) = c^{-1}\big(FP(a)\big)$. Thus we have that $c^{-1}\big(FP(a)\big)$ is a-invariant. Since ∞

is in FP(a) and in FP(c), $\{\infty\} \in c^{-1}(FP(a))$. If $a^2 \neq id$ then by theorem 4.18 FP(a) is the unique a-invariant circle containing ∞, i.e., $c^{-1}(FP(a)) = FP(a)$. If $a^2 = id$, $c^{-1}(FP(a))$ could be an extended line which intersects FP(a) orthogonally. However since $[b,c] = id$ and b is parabolic, proposition 5.2 implies that c^{-1} is either a translation, or a rotation with axis $(c^{-1}) \in C(b)$ and therefore parallel to FP(a), or a composition of two such transformations. In all of these three cases $c^{-1}(FP(a))$ is clearly not orthogonal to FP(a). Thus even in the case $a^2 = id$ we have $c^{-1}(FP(a)) = FP(a)$.

Finally we see that all three possibilities for c imply that $b = [a,c]$ is not a non-trivial parabolic transformation. When c is a translation proposition 5.2 implies that $[a,c] = id$. If c is a rotation with axis parallel to axis (a) then any plane P orthogonal to axis (a) (and therefore to axis (c)) is kept invariant by $[a,c]$. Proposition 5.2 however, clearly implies that P cannot be b-invariant if $[a,b] = id$. Finally if $c = TR$ where T is a translation as in the first case above and R is a rotation as in the second case, then $[c,a] = [TR,a] = [R,a]$ and we have reduced this case to the previous one.

This completes the proof of the proposition under the assumption that b is parabolic and a is elliptic. The case that c is elliptic is the same by the symmetry of $\Pi_1(M_0)$. qed

7.10 The proof of theorem 7.1 is finally completed using corollary 7.6 and the following proposition:

Proposition: Let $h: \Pi_1(M_0) \to G^3$ be a homomorphism with h(b) of finite order. Let G = image (h). If G is discrete then $\Lambda(G)$ has finite cardinality.

Proof: We prove this proposition by attempting to find an h such that $\Lambda(G)$ is not finite. We will see that no such h exists. By proposition 7.8 we can assume that b is elliptic of finite order, or that b is the identity. In either case both a and c must be of infinite order. If they were not both of infinite order, then G would have a cyclic subgroup H of finite index. The limit set of H would equal the fixed point set of the generator of H which contains one or two points and $\Lambda(G)$ would be finite. Thus we must let both a and c be of infinite order if $\Lambda(G)$ is to be an infinite set.

If b is the identity then a and c must commute and must therefore be simultaneously parabolic or loxodromic. In either case $\Lambda(G)$ $\bigl(= FP(a) = FP(c)\bigr)$ is finite again.

The only remaining possibility is that b elliptic of finite order. Since both a and c must commute with b, both FP(a) and FP(c) are contained in FP(b). Restricted to FP(b) a and c must commute, and therefore $a\bigl(FP(c)\bigr) = FP(c)$ and $c\bigl(FP(a)\bigr) = FP(a)$. This implies that $FP(a) = FP(c)$ and therefore a and c must both be parabolic or both be loxodromic.

First we assume that a and c are parabolic. Since a and c commute with b both a and c must be invariant on axis (b). By proposition 5.2 two parabolic transformations with a common fixed point and a common invariant circle must commute. Thus the map sending a, b and c to non-trivial transformations with a and c parabolic and b elliptic cannot be a homomorphism.

Next we assume that a and c are both loxodromic. Recall that we have shown that $FP(a) = FP(c)$. We can conjugate G if necessary to get $FP(a) = \{0,\infty\} = FP(c)$. A typical element in G must therefore be a rotation of $\overline{R^3}$ followed by a dilation of $\overline{R^3}$ with fixed points 0 and ∞. Certainly $\{0,\infty\} \subseteq \Lambda(G)$ and in fact we will see that

$\{0,\infty\} = \Lambda(G)$. Let $a = D_a R_a$, $b = R_b$ and $c = D_c R_c$ where D_a and D_c are non-trivial dilations with fixed points at 0 and ∞ and R_a, R_b and R_c are possibly trivial rotations which have fixed points at 0 and ∞. By propositions 5.3 and 5.4 if $R_b^2 \neq$ id then $FP(R_b)$ must be contained in $FP(R_a)$ and $FP(R_c)$. (Note we only have inclusion and not equality of fixed point sets as R_a or R_c may in general be trivial.) If $R_b^2 =$ id then either $FP(R_a)$, $FP(R_b)$ and $FP(R_c)$ are related as above or $FP(R_b)$ is perpendicular to R_a and $R_a^2 =$ id or $FP(R_b)$ is perpendicular to $FP(R_c)$ and $R_c^2 =$ id. (This, of course, includes the possibility that both R_a and R_c may have order 2 and have their axis perpendicular to $FP(R_b)$.

In order to deal with these various possibilities all together we define a subgroup H of finite index in G which satisfies $\Lambda(H) = \{0,\infty\}$ for these possible definitions of G. Let H be the subset of G consisting of all elements of the form $a^i b^j c^k$ where i, j and k are integers and where i is even if axis (R_a) is perpendicular to axis (b) and k is even if axis (R_c) is perpendicular to axis (b). By lemma 7.7 H is a subgroup of G of index 1, 2 or 4. Note that axis (R_a) perpendicular to axis (b) implies that $a^2 = (D_a R_a)^2 = D_a^2 R_a^2 = D_a^2$. A similar statement can be made for c. Thus H acts as a group of conformal motions on the extended plane $P \cup \{\infty\}$ which is orthogonal to axis (b) at the origin.

Restricted to $P \cup \{\infty\}$ 0 and ∞ are certainly limit points of H. Since the set $\{0,\infty\}$ is closed in $P \cup \{\infty\}$, contains more than one point and is invariant under $H|P \cup \{\infty\}$ we have $\Lambda(H|P \cup \{\infty\}) = \{0,\infty\}$. (See Ford [4], theorem 5, 43.) This implies that $\Lambda(H) = \{0,\infty\}$. To see this we assume on the contrary that there exists another limit point x_0 and get a contradiction.

For notational convenience we assume that axis (b) $= \{(0,0,z)|z \in R\}$ and that $P \cup \{\infty\} = \{(x,y,z)|z = 0\} \cup \{\infty\}$. We will write the point (x,y,z) in R^3 in "modified spherical coordinates" as $(x+iy, \arctan(z/(x^2+y^2)))$ when $x^2 + y^2 \neq 0$ and as $(0,z)$ when $x^2 + y^2 = 0$, i.e., for points not on axis (b) we write $(x + iy, \phi)$ where ϕ is the "angle of inclination" of the point (x,y,z) from the plane $P \cup \{\infty\}$. The point $(0,z) \neq (0,0)$ is assigned the angle of inclination $(z/|z|)\cdot\Pi/2$. The importance of this representation is that the angle of inclination is left invariant by all elements of H. Thus if

$x_0 \neq 0,\infty$ is a limit point of H then $x_0 = (w_0, \phi_0)$ where $w_0 \in C - \{0\}$, $x_0 = \lim_{k\to\infty} g_k(w_1, \phi_0)$ and $\{g_k\}_{k=1}^{\infty}$ is a sequence of distinct transformations in H. This implies that $\lim_{k\to\infty} g_k(w_1, 0) = (w_0, 0)$ which contradicts the fact that $\Lambda(H|P \cup \{\infty\}) = \{0,\infty\}$. Thus we must have $\Lambda(H) = \{0,\infty\}$. Finally since a, c and ac fix 0 and ∞, not only must $\Lambda(G)$ be finite but clearly $\Lambda(H) = \Lambda(G)$. This completes the proof of theorem 7.1. qed

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