Moduli of Polarized Abelian Varieties

and

Complex Multiplications

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Abstract of the Dissertation

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The object of this thesis is to study the moduli scheme of polarized abelian varieties, its stratification according to p-rank and the endomorphism rings of its fibers.

One knows that an abelian variety X defined over a finite field has sufficiently many complex multiplications and the rank r of End(X) satisfies the

inequality: $2g \leq r \leq 4g^2$, where $g = \dim(X)$. If X is defined over an infinite field, then the rank r could be any integer satisfying the inequality: $1 \leq r \leq 4g^2$. One can ask whether there are abelian varieties having trivial endomorphism ring. For elliptic curves Deuring's theory proves that a "generic" elliptic curve has no complex multiplications. In this thesis, we generalize this result to any dimension using Serre-Tate's theory of lifting of ordinary abelian varieties.

Now we shall describe briefly the contents of various sections. First we fix some notations. k will denote a field of characteristic $p > 0$. If X is an abelian variety defined over k, $\mathrm{End}^\circ(X) = \mathrm{End}_k(X) \otimes \mathbb{Q}$. $M_{g,1,n}$ will denote an irreducible component of the fine moduli scheme of principally polarized abelian varieties of dimension g with level n structure defined over k.

In §1, we define the index of good reduction of abelian varieties. If v is a discrete rank one valuation of k with inertia group I, then the index of good reduction $\underline{s}$ of an abelian variety X is the rank over Z_ℓ of $T_\ell(X)/T_\ell(X)^I$, where $T_\ell(X)$ is the ℓ-adic Tate module of X, $\ell \neq p$ and $T_\ell(X)^I$ is the submodule of elements of $T_\ell(X)$ invariant under I. Then we prove:

i) X has good reduction iff $\underline{s} = o$

ii) X has very bad reduction iff $\underline{s} = o$

iii) X has stable reduction iff $\underline{s} = g\text{-}\dim(Y)$, where Y is the abelian part of the connected component of the Néron model of X.

In §2 we explain Serre-Tate's theory of lifting of ordinary abelian varieties and using that we give counter examples to two questions on Barsotti-Tate groups.

In §3, we define special, completely special and supersingular abelian varieties and study the connections between them. We also prove that the endomorphism algebra of any abelian variety with p-rank one never contains a simple algebra which is not a field.

In §4, we prove that the fine moduli scheme $M_{2,1,n}$ contains a projective curve with singularities. Note that this is opposed to the situation in char.zero where the moduli space $M_{2,1,n}$ is affine as Igusa has shown.

In §5, we prove the following theorems:

Theorem I: The generic fiber of $M_{g,1,n}$ has no complex multiplications.

Theorem II: Any principally polarized abelian variety A defined over an algebraically closed field k is a specialization of an abelian variety defined over $k((t_1,\ldots,t_{1/2\ g(g+1)}))^{ac}$ having no complex multiplications(here the superscript ac denotes the algebraic closure).

As a consequence, we see that the generic fiber is simple and has Picard number one.

In § 6 and §7, we consider the following problem: Given a division algebra L and an integer g, does there exists a simple abelian variety X of dimension g such that $End^\circ(X) = L$? In characteristic zero, Albert and Shimura have solved this problem. In characteristic $p > o$, I have some partial answer to this question when $g = 3$. When $g = 2$, R.Fisher has discussed this question in his Harvard thesis. In fact, we determine algebras L occur and compute the dimension of the moduli space M_p in the case of dimension 3. Here M_p is an irreducible component of the moduli scheme $M_{\omega,n}$,where $\omega=(L,\rho,\Lambda,g,d)$ is an ordered set consisting of a finite simple $\mathbb{Q}$-algebra L,a positive involution ρ, Λ an order in L and g,d are integers such that $d^2\Lambda\subset\Lambda$. $M_{\omega,n}$ is the analogue of Shimura's moduli variety V_Ω in char.$p > o$.

Table of Contents

§ 0. Introduction

Let AV(k) be the category of abelian varieties defined over an algebraically closed field k of characteristic $p > 0$. For any object X in AV(k), let $\mathrm{End}^\circ(X) = \mathrm{End}_k(X) \otimes \mathbb{Q}$. Then $\mathrm{End}^\circ(X)$ is a semi-simple $\mathbb{Q}$-algebra of rank $\leq 4g^2$, where g = dimension of X and End(X) is an order in $\mathrm{End}^\circ(X)$. If X is polarized, then $\mathrm{End}^\circ(X)$ is endowed with a totally positive involution.

The rank of $\mathrm{End}^\circ(X)$ depends on the field of definition and on the structure of X itself and it satisfies the inequality: $1 \leq \mathrm{rank}\ \mathrm{End}^\circ(X) \leq 4g^2$. If Xis defined over a finite field, then $2g \leq \mathrm{rank}\ \mathrm{End}^\circ(X) \leq 4g^2$. For example, see [26]. But if X is not defined over a finite field, we can find plenty of abelian varieties X such that $\mathrm{End}^\circ(X) \simeq \mathbb{Q}$. In fact we prove that generic principally polarized abelian varieties do not have complex multiplications.

If L is a division algebra with an involution which is totally positive and $\theta: L \hookrightarrow \mathrm{End}^\circ(X)$ is an embedding compatible with the involutions, one can construct the analogue of Shimura's moduli space V_Ω in positive characteristic and compute the dimension of the moduli space $M_{\omega,n}$ for various simple algebras L. In generalizing Shimura!s results to characteristic $p > 0$ one must observe the important difference. In char 0 the order End(X)$\cap$L does not change when X varies in a continous family whereas this is not always true in the case of positive characteristic.In the latter case one must fix the order Λ and determine the embeddings $\Lambda \hookrightarrow$ End(X). The dimension of the moduli space will depend on the order Λ.

In § 1, we give an account of the reduction of abelian varieties and complex multiplications. In §2, we explain briefly Serre-Tate's theory of lifting of ordinary abelian varieties. We also give counter examples to two questions on Barsotti-Tate groups.

In § 3, we define special, completely special and suersingular abelian varieties and observe that completely special is equivalent to special. In dimenson 2 these three notions are equivalent.A special two dimensional abelian variety is a product of supersingular elliptic curves. In dimension ≥ 3, this need not be the case. One has only construct a special simple abelian variety as in Lenstra and Oort[7] or one must find an abelian variety whose Picard number is not the maximum possible.

We also show that the endomorphism algebra $\mathrm{End}^{\circ}(X)$ of any abelian variety with p-rank one never contains a simple algebra which is not a field.

In §4, we show that the fine moduli scheme of principally polarized abelian surfaces contains a projective curve with singularities. Note that this is opposed to the situation in characteristic zero where the moduli space $M_{2,1,n}$ is affine as Igusa has shown[5].

In § 5, we prove the following theorems:

Theorem I. If $M_{g,1,n}$ denote an irreducible component of the fine moduli scheme of principally polarized abelian varieties, then its generic fiber has no complex multiplications.

Theorem II. Any principally polarized abelian variety A defined over k is a specialization of an abelian variety defined over $k((t_1,,,,t_{1/2\ g(g+1)}))^{ac}$ having no complex multiplications (here the superscript ac denotes algebraic closure.)

As a consequence, we see that the generic fiber is simple and has Picard number one.

In § 6, we collect the results of Shimura on the moduli variety V_{Ω} and its analogue in positive characteristic as described by Fisher[27].

In §7, we determine which algebras L occur and compute the dimension of the moduli space M_p in dim 3. This is done in the case of dimension two by Fisher[27]. In both the cases the results are only partial.

I wish to express my sincere thanks to Professor M.Fried for his help and encouragement during the preparation of this thesis.

1. Reduction of abelian varieties and complex multiplications

Let k be a field of characteristic $p > 0$. Let $\underline{AV}(k)$ denote the category of abelian varieties defined over k. $\underline{AV}^\circ(k)$ will denote the category of abelian varieties upto isogeny. $\underline{AV}^\circ(k)$ is a semi-simple $\mathbb{Q}$-linear category. It is the category of "effective motives of weight one."

In $\underline{AV}^\circ(k)$ isogenies are isomorphisms. For X,Y in $\underline{AV}(k)$, $\mathrm{Hom}_{\underline{AV}^\circ(k)}(X,Y) = \mathrm{Hom}_{\underline{AV}(k)}(X,Y)\otimes\mathbb{Q}$. If X=Y, then $\mathrm{End}^\circ(X)=\mathrm{End}(X)\otimes\mathbb{Q}$ is a semi-simple $\mathbb{Q}$-algebra of rank $\leq 4g^2$ where g=dimension of X. In fact by Poincare-Weil complete reducibility theorem X is isogenous to a product of simple abelian varieties:

$$X \sim X_1^{n_1} \times X_2^{n_2} \times \dots \times X_r^{n_r}$$

where X_i are not isogenous to each other. If $D_i=\mathrm{End}^\circ(X_i)$ then $\mathrm{End}^\circ(X)=M_{n_1}(D_1)\times\dots\times M_{n_r}(D_r)$, where $M_{n_i}(D_i)$ is an $n_i \times n_i$-matrix with coefficients in D_i. Note that End(X) is an order in $\mathrm{End}^\circ(X)$.

If X^t denote the dual of X, then any polarization $\lambda: X \longrightarrow X^t$ induces an involution $u \longrightarrow u'$ on $\mathrm{End}^\circ(X)$ which is totally positive, i.e. for any non-zero u in $\mathrm{End}^\circ(X)$, $\mathrm{tr}_{\mathrm{End}^\circ(X)/\mathbb{Q}}(uu') > o$.

Let X be an object in AV(k) of dimension g and let $G=\mathrm{Gal}(k_s/k)$, where k_s is the separable closure of k. For any prime $\ell \neq p$, let $T_\ell(X)$ denote the ℓ-adic Tate module of X. $T_\ell(-)$ is a covariant functor from AV(k) into the category of Z_ℓ-modules. $T_\ell(X)$ is free of rank 2g over Z_ℓ. Put $V_\ell(X)=T_\ell(X)\otimes\mathbb{Q}$. Both $T_\ell(X)$ and $V_\ell(X)$ are G-modules.

If X_{ℓ^∞} is the ℓ-primary part of the torsion subgroup of X, then we have the following exact sequence of G-modules:

$$o \longrightarrow T_\ell(X) \longrightarrow V_\ell(X) \longrightarrow X_{\ell^\infty} \longrightarrow o$$

Taking Galois cohomology, we have a long exact sequence:

$$o \longrightarrow H^\circ(G_\ell, T_\ell(X)) \longrightarrow H^\circ(G_\ell, V_\ell(X)) \longrightarrow$$
$$\longrightarrow H^\circ(G_\ell, X_{\ell^\infty}) \longrightarrow H^1(G_\ell, T_\ell(X)) \longrightarrow \ldots\ldots$$

where G_ℓ is the closed subgroup of $\mathrm{Aut}_{Z_\ell}(T_\ell(X))$ identified with the Galois group $\mathrm{Gal}(k_{\ell^\infty}/k)$, k_{ℓ^∞} being the field of rationality of X_{ℓ^∞}. Since $H^n(G_\ell, V_\ell(X))=o$, [20], we have the following

Proposition 1.1. $H^\circ(G_\ell, T_\ell(X))=o$ and

$$H^n(G_\ell, T_\ell(X))=H^{n-1}(G\ \ , X_{\ell^\infty}).$$

Remark 1.2. I do not know the geometric significance of the above cohomology groups.

Let us now recall the reduction of abelian varieties. Let k be a field with discrete rank one valuation v, of valuation ring R and residue field k. Let S=Spec(R) with closed point s and generic point t. Let X be an abelian variety in AV(k). Then according to Néron and Raynaud, X has a Néron model N over S. N is a group scheme smooth and of finite type over S such that the generic fiber N_t is isomorphic to X and N represents the functor : $T \longrightarrow \mathrm{Hom}_k(T_t, N_t)$ on the category of group schemes smooth over S or equivalently, N satisfies the following functorial property:

$$\mathrm{Hom}_S(T,N) \longrightarrow \mathrm{Hom}_k(T_t, N_t) \text{ is bijective.}$$

Definition 1.3. (a) X has good reduction at v if N is an abelian scheme.

(b) X has stable reduction at v if (after a finite separable extension of k) the connected component of the special fiber N_s of N is an extension of an abelian variety by a torus.

If I is the inetia group of v, let $T_\ell(X)^I$ denote the submodule of elements of $T_\ell(X)$ invariant under I. It is clear that $T_\ell(X)/T_\ell(X)^I$ is a free Z_ℓ-module of rank say s.

Definition 1.4. s is called the index of good reduction of X with respect to v. Note that $o \leq s \leq 2g$.

Let N be the Néron model of X and N_s° the connected component of the special fiber N_s of N. Let T_o be the maximal subtorus of N_s°. Then N_s°/T_o is an extension of an abelian variety Y by a unipotent group U_o(which is smooth and connected):

$$o \longrightarrow U_o \longrightarrow N_s^\circ/T_o \longrightarrow Y \longrightarrow o$$

Note that $\dim N_s^\circ = \dim X = g$; let $r = \dim U_o$, $u = \dim T_o$ $v = \dim Y$. Then $g = r+u+v$. Also $\mathrm{rank}_{Z_\ell} T_\ell(T_o) = u$ $\mathrm{rank}\, T_\ell(N_s^\circ/T_o) = \mathrm{rank}\, T_\ell(Y) = 2v$; $\mathrm{rank}\, T_\ell(N_s^\circ) = u+2v$ $= \mathrm{rank}\, T_\ell(N_s) = \mathrm{rank}\, T_\ell(X)^I$.

<u>Definition 1.5</u>. r is called the unipotent rank and u the reductive rank of X.

If g = r, i.e. the reductive rank of X is g, then we say that X has very bad reduction at v.

<u>Proposition 1.6</u>. i) X has good reduction at v if and only if s = o

ii) X has very bad reduction at v if and only if s = 2g

iii) X has stable reduction at v if and only if s = g– dim Y.

<u>Proof:</u> i) follows from the theorem 1 of Serre-Tate [21].

ii) If X has very bad reduction, then $g = r$. Therefore $g = g+u+v$ or $u+v = o$ and hence $u = o$ and $v = o$. This implies that $N_s^\circ - U_o$ and rank $T_\ell(N_s^\circ)$ = rank $T_\ell(X)^I$ which is zero. This implies $s = 2g$.

Conversely, $s = 2g$ implies $T_\ell(X)^I = o$. Therefore

$$\begin{aligned}\text{rank } T_\ell(X) &= \text{rank } T_\ell(X)^I \\ &= \text{rank } T_\ell(N_s^\circ) \\ &= o\end{aligned}$$

So $u+2v = o$, i.e. $u = o$, $v = o$; hence $g = r$ Q.E.D.

iii) can be proved in a similar fashion.

Remark 1.7. If X is an elliptic curve, then the index of good reduction is either o, 1 or 2. The case $s = o$ corresponds to good reduction; $s = 1$ implies that X has stable reduction or equivalently X has nodal reduction; $s = 2$ means that X has very bad reduction or X has cuspidal reduction.

Let X be an object of AV(k) of dimension g. Then as we mentioned before $\text{rank}_Z \text{ End }(X) \leqslant 4g^2$.

Definition 1.8. Suppose that $\text{End}(X) \simeq Z$ for any finite extension of k. Then we say that X has no complex multiplications; otherwise X is said to have complex multiplications. We say that X has sufficiently many complex multiplications over k if End°(X) contains a semi-simple commutative $\mathbb{Q}$-algebra of rank 2g.

Proposition 1.9. Assume i) X is k-simple abelian variety defined over k

ii) X has sufficiently many complex multiplications by a field L. Then

1. if char k = o, L is a CM-field, i.e. a totally imaginary quadratic extension of a totally real field.
2. if char k $\neq$ o and if L is stable under the Rosati involution φ defined by an ample line bundle, L is a CM-field.
3. Either X has good reduction or very bad reduction.

Proof. For the proof of 1, see [24] and for the proof of 3, see [18]. We prove the statement 2.

Since X is simple, End°(X) is a division algebra over $\mathbb{Q}$. Then by proposition 3, chapter II[24], the commutant of L in End°(X) coincides with L itself and L contains the center of End°(X). Then L is a maximal subfield. Since L is stable under φ, we can restrict φ to L and denote the restriction again by φ. Then L is not elementwise fixed; otherwise 2g must divide g by the corollary on pp.191[11] which is impossible. Let L_o be the subfield of L consisting of elements fixed by φ. Since φ is totally positive on End°(X), its restriction is also totally positive on L and so $Tr_{L/\mathbb{Q}}(xx^{\varphi}) > o$ for every x in L. Then by lemma 2 pp 41[24], L is CM-field.

<u>Theorem 1.10</u>. Let X be an abelian variety in <u>AV</u>(k) having sufficiently many complex multiplications. Then there exists an abelian variety B defined over a finite extension of the prime field of k and an isogeny between X and B (hence B is defined over a number field or a finite field).

Conversely, an abelian variety defined over a finite field has sufficiently many complex multiplications.

<u>Proof</u>. The proof of the second part is a consequence of a theorem of Tate[25]. For the first part see[29].

<u>Remark 1.11</u>. From the second part of the above theorem one can ask whether there are abelian varieties defined over infinite fields and not having sufficiently many complex multiplications. In fact there are lots of abelian varieties having no complex multiplications at all, see § 4. For the sake of fun we give here an example

of an abelian variety defined over a number field which does not have sufficiently many complex multiplications.

Consider the following congruence subgroup

$$\Gamma_o(23) \overset{\text{def}}{=\!=} \{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z) \quad \text{such that}$$

$$c \equiv o \ (\text{mod}23)\}.$$

Let H* be the union of the upper half plane and the cusps of $\Gamma_o(23)$. The quotient space H*/ $\Gamma_o(23)$ is a compact Riemann surface and so corresponds to an algebraic curve. It has a model C defined over $\mathbb{Q}$. It is easily seen that the genus of C is 2. Let $J_{\Gamma_o(23)}$ be the Jacobian variety of C. It is known that $J_{\Gamma_o(23)}$ is a simple abelian surface and $\text{End}^\circ(J_{\Gamma_o(23)})$ is a real quadratic field. Hence by i) of proposition1.11 $J_{\Gamma_o(23)}$ does not have sufficiently many complex multiplications.

2. Barsotti-Tate groups.

Hereafter we assume that k is algebraically closed and char $k = p > o$. Let X be an object in AV(k). For any integer n, let $X_{p^n} = \mathrm{Ker}(p^n : X \longrightarrow X)$. X_{p^n} is a finite commutative locally free group scheme of rank 2g. X_{p^n} are never etale(as opposed to the case for $\ell \neq p$) unless $X = 0$.

Decompose $X_{p^n} = X^{\circ}_{p^n} \times X^{et}_{p^n}$ into connected and etale parts. Let $X(\infty) = \{ X_{p^n} , i_n \}$ be the p-Barsotti-Tate group associated to X. Denote by $T_p(X)$ the projective system associated to $X(\infty)$. We shall also call $T_p(X)$ the p-Barsotti-Tate group of X. The decomposition of X_{p^n} induces a decomposition of $T_p(X)$. Define $T_p(X)_{red} = \varprojlim X^{et}_{p^n}$. $T_p(X)_{red}$ is a free Z_p-module of rank, say, r and call $T_p(X)_{red}$ the p-adic Tate module of X. Also $\varinjlim X^{\circ}_{p^n}$ is the formal group associated to X.

Definition 2.1. The integer "r" is called the p-rank of X and denoted by pr(X).

Remark 2.2. Observe that $pr(X) \leqslant g$ and that the p-rank is an isogeny invariant. By looking at the Lie algebra of X, we see that pr(X) = dimension of the semi-simple part of Lie(X) with respect to the p-th power = dimension of the semi-simple part of $\mathrm{Lie}(X^t)$ = dimension of the semi-simple part of $H^1(X,O_X)$. Let

$$F : H^1(X,O_X) \longrightarrow H^1(X,O_X)$$

be the Hasse-Witt transformation induced by the Frobenius homomorphism $F : O_X \longrightarrow O_X$ defined by $a \longrightarrow a^p$. By theorem 3, pp 148[11], under the isomorphism

$$\mathrm{Lie}(X^t) \longrightarrow H^1(X,O_X)$$

the p-th power map in $\mathrm{Lie}(X^t)$ goes over into the Frobenius map in $H^1(X,O_X)$. Hence pr(X) = dimension of the semi-simple part of the Hasse-Witt transformation of X.

<u>Definition 2.3</u>. We define rank(X) to be the rank of the Hasse-Witt transformation of X.

<u>Definition 2.4</u>. An abelian variety X is called <u>ordinary</u> if the p-rank of X is equal to dim(X).

From the remark 2.2, we see that the Hasse-Witt matrix of an ordinary abelian variety is invertible.

We shall now briefly explain the Serre-Tate's theory of lifting of ordinary abelian varieties.

Let A be an ordinary abelian variety in <u>AV</u>(k) of dimension g. Then $A_{p^n} = (\mu_{p^n})^g \oplus (Z/pZ)^g$ and hence $T_p(A) = T_p(A)^\circ \oplus T_p(A)^{et} \simeq \hat{\mathbb{G}}^g_{m,k} \oplus (\mathbb{Q}_p/Z_p)^g$. In particular, the Barsotti-Tate groups associated to all ordinary abelian varieties are isomorphic.

For any artin local ring R with residue field k Z/p^nZ has a unique lifting to R and hence by Cartier duality μ_{p^n} has a unique lifting to R. By taking direct limits and sums, $\mathbb{G}_m^g$ and $(\mathbb{Q}_p/Z_p)^g$ have unique liftings to R. Let them be M and E respectively. So among the lifts of A/k to R there is a natural lift X given by setting $T_p(X) = M \oplus E$. X is called the canonical lifting of A/k to R. For more details on these liftings and for the proof of the following theorem see Messing[10].

Theorem 2.5. An ordinary abelian variety X_o in AV(k) has a canonical lifting to characteristic zero, i.e. there exists a complete noetherian local ring R of characteristic zero with residue field k and an abelian scheme X/R such that $X \otimes_{\mathcal{O}} Spec(k) \simeq X_o$ and the canonical lifting is the unique lifting X_o/k to R such that every endomorphism (and hence every polarization) of X_o lifts to an endomorphism (to a polarization) of X/R.

Corollary 2.6. If an ordinary abelian variety has sufficiently many complex multiplications, so does its canonical lifting.

Remark 2.7. From the existence and the formal smoothness of the separably polarized abelian varieties, it follows that a separably polarized abelian variety can be lifted to characteristic zero. From the general principle that specialization of liftable varieties are liftable, one can prove that any polarized abelian variety can be lifted to characteristic zero.

We now discuss a few questions on Barsotti-Tate groups of abelian schemes.

Question 1. Let $\mathbb{P}$ be the set of all natural primes. For an abelian variety X in AV(k) and for every prime p in $\mathbb{P}$, let $T_p(X)$ be the p- Barsotti-Tate group of X. Consider now two objects X and Y in AV(k). Suppose $T_p(X)$ is isomorphic to $T_p(Y)$. Then is X isomorphic to Y? In other words, is the natural injection $\mathrm{ISOM}(X,Y) \longrightarrow \mathrm{ISOM}(T_p(X),T_p(Y))$ bijective?

We give an example to show that the answer to this question is no in general.

Let X be an object in AV(k) with a ring of endomorphisms $\underline{O}$ which is a finite Z-module. Let N be an invertible $\underline{O}$-module, projective of rank one over $\underline{O}$. Let $Y = N \otimes Z_p$; $\underline{O} \otimes Z_p = \underline{O}_p$. But $N_p (= N \otimes Z_p)$ is

$\underline{O}_p$-isomorphic to $\underline{O}_p$ and hence $T_p(X)$ is isomorphic to $T_p(Y)$, since $T_p(Y)$ is isomorphic to $N_p \otimes T_p(X)$. But it may happen that X is not isomorphic to Y. For example consider the ring of integers $\mathcal{O}$ in an imaginary quadratic field having class number $h > 1$. There are h different elliptic curves defined over $\mathbb{C}$ having $\mathcal{O}$ as the ring of endomorphisms. They are all defined over a suitable number field. Let E be one such curve . Choose a non-trivial projective $\mathcal{O}$-module of rank one, say, R and let $E' = E \otimes R$. Then E is not isomorphic to E' since Hom(E,E') is canonically isomorphic to R and hence not a free $\mathcal{O}$-module. But $T_p(E) = T_p(E')$ for any p in $\mathbb{P}$.

<u>Question 2</u>. Let v be a discrete rank one valuation, of ring R and let S = Spec(R) with generic point t and closed point s. Let X and Y be two abelian schemes over S. Let $T_p(X)$, $T_p(Y)$ be the corresponding p-Barsotti-Tate groups of X and Y respectively. Assume X_s and Y_s are isogenous and $T_p(X)$ is isogenous to $T_p(Y)$ i.e. $T_p(X) \otimes \mathbb{Q}_p$ is isomorphic to $T_p(Y) \otimes \mathbb{Q}_p$. Then are X and Y isogenous?

Here again the answer is no in general. Here is an example. Let X_o be an ordinary elliptic curve defined over a perfect field of characteristic $p \neq o$, say over a finite field k. Consider its liftings over the ring $W(k)$ of Witt vectors as explained above. There is a one-to-one correspondence between the isomorphism classes of liftings of X_o and the elements of the group $W(k)^*$ of the ring $W(k)$, see[10]. Also recall that lifting X_o is equivalent to lifting $T_p(X_o)$. Let u be a unit of $W(k)^*$ and let X_u be a lifting corresponding to u. Now if $u' = u^n$ where $u, u' \equiv 1 \pmod p$ and n is a p-adic unit $T_p(X_u)$ and $T_P(X_{u'})$ are isomorphic. However the curves X_u and $X_{u'}$ are not always isomorphic; in fact there are too many of them.

Another question for which I have no answer at present is the following: Characterize the system $\{G_p\}$ $p \in \mathbb{P}$, of Barsotti-Tate groups which come from abelian schemes.

3. Abelian varieties with p-ranks o and 1.

Definition 3.1. Let X be an object in AV(k). X is called special if pr(X) = o; X is called super singular if the formal group X* associated to X is isogenous to $(G_{1,1})^g$, i.e. X* is isogenous to E^{*g} where E* is the formal group of a super singular elliptic curve; X is called completely special if all the invariant differentials are exact.

The following theorem of Manin[9] gives the structure of the formal group of an abelian variety upto isogeny.

Theorem 3.2. Suppose X is an object of AV(k) of dimension g. Then X*, the formal group of X is isogenous to :

$$X^* \sim P = r\, G_{1,0} + \Sigma(G_{n_i,m_i} + G_{m_i,n_i}) + h\, G_{1,1}$$

$$g = r + \Sigma(n_i + m_i) + h; \qquad 1 < m,n < \infty$$

$$r = \text{p-rank of } X \qquad (m,n) = 1$$

Proposition 3.3. 1. X is completely special if and only if X is special.

2. X super-singular implies that X is special. The converse is always true if the dim X=2.

3. If X is defined over a finite-

field, then X is completely special if and only if X^* is isogenous to $hG_{1,1}$ for some h.

<u>Proof</u>. 1. $H^1(X, O_X)$ has a basis consisting of invariant differential forms which are exact by our assumption. Since the Cartier operator vanishes on exact differentials and since it induces the Frobenius map on $H^1(X, O_X)$, the semi-simple rank of the Hasse-Witt transformation h: $H^1(X, O_X) \longrightarrow H^1(X, O_X)$ is zero. Hence pr(X) = o. The converse follows by retracing the steps.

2. Suppose now that X is super singular. Then by the theorem 4.2 of Oort[17], X is isogenous to E^g where E is a super singular elliptic curve. But the pr(E) = o (note that for elliptic curves super singularity is equivalent to special). Since p-rank is additive $pr(E^g) = o$. Therefore E^g and hence X is special.

Let X be a special 2-dimensional abelian variety. For a 2-dimensional abelian variety, by the theorem 3.2 X^* is isogenous to $2(G_{1,0} + G_{0,1})$ or $G_{1,0} + G_{0,1} + G_{1,1}$ or $2G_{1,1}$. In our case since pr(X) = o, X^* is isogenous to $2G_{1,1}$ and so X is super singular.

3. is proved by Manin[8].

Remark 3.4. If X has dimension $\leqslant 2$, all the three notions are equivalent and so the distinction arises only in dim $\geqslant 3$.

In case dim $\geqslant 3$, there exists special abelian variety which is not supre-singular. Lenstra and Oort prove in [7] that for any isogeny type of a formal group having at least one factor different from $G_{1,1}$ there exists a simple(even absolutely simple) abelian variety having the given isogeny type as formal group. They use Honda-Serre's classification of simple abelian varieties over finite fields[4]. From this we see that we can construct a special simple abelian variety.

Theorem 3.5. Let X be an abelian variety in AV(k) of dimension $g \geqslant 2$ with p-rank one. Then End°(X) will never contain a simple subalgebra which is not a field.

Proof. Suppose that End°(X) contais a simple algebra L such that the identity of L maps into the identity of End°(X). Let K be the center of L so that $[L:K]=d^2$ and $[K:\mathbb{Q}] = e$.

Consider the representation of L on the Dieudonné module of the formal group X* of X. This representation has degree 2g over W(k). It splits into three parts corresponding to the splitting of X^* into X^{*et}, $X^{*loc,et}$, $X^{*loc,loc}$.

The first representation is the representation over Z_p of L on the p-adic Tate module $T_p(X)_{red}$. Since the p-rank of X is one, $T_p(X)_{red} \simeq Z_p$. Extend this representation to a representation of $L \otimes \mathbb{Q}_p$ on $V_p(X) = T_p(X)_{red} \otimes \mathbb{Q}_p \simeq \mathbb{Q}_p$.

Write $L_p \otimes \mathbb{Q}_p \simeq \prod_1^e M_d (\mathbb{Q}_p)$. Since the identity of L maps into the identity of End°(X), this representation does not comtain the zero representation. Hence d = 1, i.e. L is a field. Hence the only simple algebra contained in End°(X) is a field.

<u>Corollary 3.6</u>. Let M be the one dimensional family of abelian varieties of dimension 2 defined over $\mathbb{C}$ whose endomorphism ring is an order in an indefinite quaternion algebra over $\mathbb{Q}$. Then closed fiber of M is either special or ordinary, i.e. its reduction at any prime is either special or ordinary.

4. Moduli scheme of principally polarized abelian varieties and its subspaces.

Definition 4.1. Let $p : X \longrightarrow S$ be an abelian scheme of relative dimension g. Assume that n is invertible in S, i.e. the residue characteristics of all s in S are prime to n. For $n \geqslant 2$, a level n structure on X/S is a set of 2g sections $\sigma_1, \ldots, \sigma_{2g}$ of X/S, such that i) for all geometric points s of S, the images $\sigma_i(s)$ form a basis for the group of points of order n on the fiber $\overline{X}_s$ and ii) $n_X \cdot \sigma_i = \epsilon$ where $n_X: X \longrightarrow X$ is multiplication by n and ϵ is the identity section. X/S by itself is called a level 1-structure.

Definition 4.2. If S is any locally noetherian scheme, let $\mathcal{M}_{g,d,n}(S)$ be the set of triples:

i) an abelian scheme X over S of dimension g

ii) a polarization : $X \longrightarrow X^t$ of degree d^2

iii) a level n structure $\sigma_1, \ldots, \sigma_{2g}$ over S all

upto isomorphism.

Definition 4.3. If $\mathcal{M}_{g,d,n}$ is represented by a scheme $M_{g,d,n}$, then $M_{g,d,n}$ will be called the fine moduli scheme of level n, for g-dimensional polarized abelian varieties of polarization degree d^2.

Definition 4.4. Let M be a scheme and φ is a morphism from $\mathcal{M}_{g,d,n}$ to the functor represented by M. Then M is <u>called a coarse moduli scheme</u> if

i) for all algebraically closed fields Ω, $\varphi(\mathrm{Spec}(\Omega)): \mathcal{M}_{g,d,n}(\mathrm{Spec}(\Omega)) \longrightarrow h_M(\mathrm{Spec}(\Omega))$ is an isomorphism

ii) for all other morphisms Ψ from $\mathcal{M}_{g,d,n}$ to representable functors h_B, there is a unique morphism $\chi : h_M \longrightarrow h_B$ such that $\Psi = \varphi \cdot \chi$.

<u>Theorem 4.5</u>. a) If $n > 6^g \cdot d \cdot \sqrt{g!}$, then the fine moduli scheme $M_{g,d,n}$ exists. It is quasi-projective over Spec(Z).

b) For all g,d,n, the coarse moduli scheme $M_{g,d}$ exists. It is quasi-projective over every open set Spec(Z)-(p) in Spec(Z).

For the proof, see Mumford's book[12].

<u>Remark 4.6</u>. If the fine moduli scheme exists, then it is also a coarse moduli scheme and there is a proper morphism from $M_{g,d,n} \longrightarrow M_{g,d}$.

It is known that the generic fiber of $M_{g,1,n}$ is irreducible. This is seen as follows: let $M_{g,1,n} \otimes \mathbb{C}$ denote the generic fiber with base extended to $\mathbb{C}$ and let $M^{an}_{g,1,n}$ denote the analytic space associated to the variety $M_{g,1,n} \otimes \mathbb{C}$. Since $M_{g,1,n} \otimes \mathbb{C}$ is of finite type over $\mathbb{C}$, so is $M^{an}_{g,1,n}$ and hence it is quotient of the Siegel's upper half space by a simplectic modular group. Consequently, $M^{an}_{g,1,n}$ is irreducible and therefore $M_{g,1,n} \otimes \mathbb{C}$ is irreducible. This proves that the generic fiber of $M_{g,1,n}$ is irreducible.

Remark 4.7. It is not known whether the fiber at any closed point is irreducible or not.

Hereafter we shall fix a prime p. Denote $M_{g,1,n}$ an irreducible component of the fiber at p of the fine moduli scheme of the principally polarized abelian varieties. Then $M_{g,1,n}$ is quasi-projective, of finite type and hence noetherian over $\mathbb{F}_p$. Let $A_{g,1,n}$ be the universal abelian scheme over $M_{g,1,n}$. The fiber at any closed point is defined over a finite field.

Proposition 4.8. $A_{g,1,n}$ is projective over $M_{g,1,n}$.

Proof. First note that $M_{g,1,n}$ is smooth. This is seen as follows: take the level n structure such that $(n,p)=1$. Then $M_{g,1,n}$ is etale over the coarse moduli scheme. To prove that $M_{g,1,n}$ is smooth, it suffices to prove that the coarse moduli scheme $M_{g,1}$ is smooth. Since smoothness is a local property and the local moduli functor of a principally polarized abelian variety is formally smooth, $M_{g,1}$ is smooth. If t is the generic point of $M_{g,1,n}$, then on the generic fiber X_t of $A_{g,1,n}$ take an ample invertible sheaf $\mathbb{L}_t$. Then there exists an invertible symmetric sheaf $\mathbb{L}$ on $A_{g,1,n}$ such that $\mathbb{L}_t$ is algebraically equivalent to $\mathbb{L}_t^2$. Since $\mathbb{L}_t$ is ample, $\mathbb{L}$ is $M_{g,1,n}$-ample. Consequently, $A_{g,1,n}$ is projective over $M_{g,1,n}$.

Remark 4.9. The smoothness property holds also in characteristic zero for the same reason given in the above proposition.

The fine moduli scheme of abelian varieties with separable polarization or inseparable polarization whose kernal contains no local-local component is smooth because in those cases the local moduli functor is formally smooth[2,16]. As a result, in those cases the conclusion of the above proposition holds.

As before let $A_{g,1,n}$ be the universal abelian scheme over $M_{g,1,n}$ with generic point t. For any closed point s, let X_s be the fiber at s. For the proof of the following theorem, see[6] and [17].

Theorem 4.10. a) $pr(X_s) \leqslant pr(X_t)$ for any closed point s in $M_{g,1,n}$

b) Let W be the subset of $M_{g,1,n}$ over which the fibers have p-rank $\leqslant$ g-1. This set is closed by corollary 1.5 in[17]. W is non-empty because it contains a fiber which is supersingular. Then each component of W has codimension one in $M_{g,1,n}$

c) Let $M_{g,1,n,r} = \{ s \in M_{g,1,n} \mid pr(X_s)$

is $\leqslant r \}$ for any integer $r \leqslant g$.

Then $M_{g,1,n,r}$ is pure of codimension g-r in $M_{g,1,n}$.

d) $M_{g,1,n,r}$ is smooth at those points where the fiber has rank g-1.

Corollary 4.11. 1. The generic fiber of $M_{g,1,n}$ is ordinary.

2. The closed subset of special abelian varieties in $M_{g,1,n}$ is pure of dim=1/2 g(g-1).

3. The set of supersingular abelian varieties has dimension $\leqslant$ 1/2 g(g-1). In fact strict inequality may hold in dimension $\geqslant$ 3, cf. Remark 3.4.

Note also that from this corollary we can deduce the well-known fact that the set of isomorphism classes of supersingular elliptic curves is finite.

Proof. 2) and 3) follow from the theorem and from the fact that the dimension of the moduli scheme $M_{g,1,n}$ is 1/2 g(g+1) by the theorems of Grothendieck and Mumford[16].

In[28], Grothendieck has mentioned that in characteristic $p > o$, the fine moduli scheme $M_{2,1,n}$ could contain a projective line as opposed to the situation in characteristic zero where $M_{2,1,n}$ is affine as Igusa has shown[5]. The following theorem asserts only the existence of a projective curve which might have singularities. Although this theorem is a particular case of a more general theorem(see [17]),we give a slightly different proof. But first some definition.

Definition 4.12. Let α_p br the local group scheme defined by $\alpha_p = \mathrm{Ker}(F: \mathbb{G}_a \longrightarrow \mathbb{G}_a)$, $\mathbb{G}_a$ being the additive group scheme and f is the Frobenius map. Then $\mathrm{Hom}(\alpha_p, X)$ is a k-vector space for any abelian variety X in AV(k). Then define $a(X) = \dim_k \mathrm{Hom}(\alpha_p, X)$.

Some facts on a(X). For details see[15].

1. a(X) is not an isogeny invariant.
2. If X and Y are isogenous abelian varieties then a(X) = o if and only if a(Y) = o.

3. If $a(X) = \dim(X)$, then $pr(X) = 0$; but the converse is not true.
4. In general $a(X) + pr(X) \leq \dim(X)$. If $\dim(X) - pr(X) = 2$, then there exists an abelian variety B isogenous to X such that $pr(B) + a(B) = \dim(B)$.

Lemma(Oort)4.13. If $a(X) = \dim(X)$, then X can be defined over a finite field.

For the proof see[17].

Corollary 4.14. Let X be a two dimensional special abelian variety. Then X has sufficiently many complex multiplications.

Proof. Under the assumptions on X and by the fact 4) above, there exists an abelian surface B isogenous to X such that $a(B) = \dim(B) = 2$. Then by the lemma of Oort B can be defined over a finite field. The corollary follows by theorem 1.10.

Theorem 4.15. $M_{2,1,n}$ contains a projective curve possibly with singularities.

Proof. Since the dimension of $M_{2,1,n}$ is 3, by theorem 4.10

the closed subscheme P of special abelian surfaces has dimension 3 - 2 = 1. To show that P is projective, it suffices to prove the following:

i) P is quasi-projective

ii) P is proper

The quasi-projectivity follows because $M_{2,1,n}$ is quasi-projective and P is closed in $M_{2,1,n}$.

To prove that P is proper, we use the valuation criterion for properness and so we have to prove that the canonical injection:

$$\mathrm{Hom}_k(\mathrm{Spec}(R),P) \longrightarrow \mathrm{Hom}_k(\mathrm{Spec}(K),P)$$

is bijective for any discrete valuation ring R which is a k-algebra, K being the field of fractions of R with residue field k. Let u belong to $\mathrm{Hom}_k(\mathrm{Spec}(K),P)$. We associate to u an abelian scheme over R.

Now u in P(K) defines a special abelian surface A over K. Since any abelian variety has stable reduction there exists a finite separable extension L of K such that if S is the integral closure of R in L and if N_L is the Néron model of $A \times_K L$ over S, the closed fiber of N_L has no unipotent radical. We show that N_L is actually an abelian scheme over S. For that we show that the closed fiber $N_{L,s}$ of N_L is an abelian variety over k. We have the following exact sequence:

$$o \longrightarrow \mathbb{G}_m^r \longrightarrow N^\circ_{L,s} \longrightarrow B \longrightarrow o$$

where $r \leqslant 2$ and B is an abelian variety of dim $\leqslant 2$. We can assume that all the groups in the exact sequence are defined over k.

Every endomorphism f in $\mathrm{End}_K(A)$ induces an endomorphism of $N_{L,s}$. Since every endomorphism is continuous in the Zarisky topology, f induces an endomorphism f' of $N^\circ_{L,s}$. Moreover since $\mathbb{G}_m^r$ is the maximal linear sub group of $N^\circ_{L,s}$ and since the homomorphic image of a linear group is linear, f' maps $\mathbb{G}_m^r$ into itself. Hence f induces an endomorphism of $\mathbb{G}_m^r$ and an endomorphism of B. We thus have $\chi : \mathrm{End}_K(A) \ni f \longrightarrow f_1 \in \mathrm{End}_k(\mathbb{G}_m^r)$ a ring homomorphism such that $\chi(1_A) = 1_{\mathbb{G}_m^r}$. Since $\mathrm{End}_k(\mathbb{G}_m^r) \simeq M_r(Z)$, χ induces a homomorphism of $\mathrm{End}^\circ(A)$ into $M_2(\mathbb{Q})$. The central $\mathbb{Q}$-algebra $M_r(\mathbb{Q})$ of r x r matrices over $\mathbb{Q}$ can have subfields of rank at most r over $\mathbb{Q}$. But $r = \dim(\mathbb{G}_m^r) \leqslant \dim(A) \leqslant 2 \dim(A) = [F:\mathbb{Q}]$ since A has sufficiently many complex multiplications by corollary 4.14, say, by the field F. Note that $A \neq o$. Hence $1 \in F$ maps to o = identity of $\mathbb{G}_m^r$ and hence $\mathbb{G}_m^r = o$. Therefore $N^\circ_{L,s}$ and hence N_L is an abelian scheme/S. Lifting the polarization of A to N_L, we deduce that the section u can be extended to a section

$$v\colon \operatorname{Spec}(R) \longrightarrow M_{2,1,n}.$$

Since P is closed in $M_{2,1,n}$, v factors through P. This proves the surjectivity of the map in the valuation criterion. The possibility of P having singularities is discussed in the following

Remark 4.16. Let X be an abelian variety in AV(k) of dimension g. Let $h\colon H^1(X,O_X) \longrightarrow H^1(X,O_X)$ be the Hasse-Witt transformation induced by the Frobenius. Let $v_1,\ldots,v_g$ be a basis of the g-dimensional vector space $H^1(X,O_X)$. If pr(X) = r, then r elements of the basis will be fixed by h; let them be $v_1,\ldots,v_r$. Then for a suitable choice of $v_{r+1},\ldots,v_g$ the p-linear action of h on $H^1(X,O_X)_{nilp}$ has matrix of the form

$$N = \begin{bmatrix} N_{g_1} & 0 & \ldots & 0 \\ 0 & N_{g_2} & \ldots & 0 \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ 0 & 0 & & N_{g_e} \end{bmatrix}$$

where N_{g_i} is the $g_i \times g_i$-matrix with nilpotent rank g_i-1 of the form

$$N_{g_i} = \begin{bmatrix} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & 0 & & \\ & 0 & 0 & 1 & 0 & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & 0 & 1 \\ & & & & 0 & 0 \end{bmatrix}$$

Hence there is a one-to-one correspondence between isomorphism types of p-linear endomorphisms of $H^1(X, O_X)$ and partitions $\pi = (r, g_1, \ldots, g_e)$ of g such that $r \geqslant 0,\ g_1 \geqslant g_2 \geqslant \ldots \geqslant g_e \geqslant 1,\ r + \Sigma\, g_i = g$, given by

$$(r, g_1, \ldots, g_e) \longleftrightarrow H = \begin{bmatrix} I_r & 0 & . & 0 \\ 0 & N_{g_1} & . & . \\ 0 & . & . & 0 \\ 0 & . & . & N_{g_e} \end{bmatrix}$$

Where I_r is r x r-identity matrix. Clearly rank(X)=g-e.

In our case g being 2, the possible types of the Hasse-Witt matrices are:

$T_1 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ rank(X) = pr(X) = 2

$T_2 : \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ rank(X) = pr(X) = 1

$T_3 : \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ rank(X) = 1, pr(X) = 0

$T_4 : \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ rank(X) = pr(X) = 0

The projective curve P consists of abelian surfaces of types T_3 and T_4. By theorem 4.10 d), the singularities of the projective curve P occur at those abelian surfaces of type T_4.

Question: Does P have a projective line as a component ?

5. A generic abelian variety has no complex multiplications

Proposition 5.1. Let X be an abelian scheme over an algebraic k-scheme S. Assume S is irreducible with generic point t. Then for any two points s,s' where s' is a specialization of s, $End(X_s) \subseteq End(X_{s'})$.

Proof. By E.G.A. II 7.1.4, there exists a discrete valuation ring R such that if T = Spec(R) with generic point a and closed point b and a morphism $f: T \longrightarrow S$ with f(a) = s, f(b) = s'. Thus we can consider $X_{s'}$ as a specialization of X_s with respect to R. Consequently $End(X_s) \subseteq End(X_{s'})$.

Corollary 5.2. If some closed fiber has no complex multiplications, so does the generic fiber.

Corollary 5.3. The map $s \longrightarrow r_s = rank_Z End(X_s)$ is lower semi-continuous.

Proposition 5.4. Let $M_{g,1,n} \otimes \mathbb{C}$ denote the fine moduli scheme of principally polarized abelian varieties defined over $\mathbb{C}$. Then the generic fiber of the universal abelian scheme over $M_{g,1,n} \otimes \mathbb{C}$ has no complex multiplications.

Proof. It follows from a theorem of Weil[26] that for any given integer g, there exists an abelian variety defined over $\mathbb{C}$ of dimension g having trivial endorphism ring. From this we deduce that some special member and hence the generic fiber of $M_{g,1,n} \otimes \mathbb{C}$ has no complex multiplications.

Remark 5.5. We can also give a dirct proof of the proposition 5.4. without using Weil's theorem by applying equicharacteristic formal deformation theory.

Theorem 5.6. Let $M_{g,1,n}$ denote an irreducible component of the fine moduli scheme of principally polarized abelian varieties defined over an algebraically closed field k of positive characteristic. Then the generic fiber of $M_{g,1,n}$ has no complex multiplications.

Proof. By corollary to theorem 4.10, the generic fiber A_t is ordinary. A_t is defined over k(t), the residue field of t. k(t) is a function field of transcendance degree 1/2 g(g+1) and it is not a perfect field. Denote by k again the algebraic closure of k(t) and let $A_o = A_t \otimes k$. Since A_o is ordinary over a perfect field k, Serre-Tate theory (cf. § 2) assures the existence of the canonical lifting of A_o. In other words, there is a projective abelian scheme $\mathcal{A}$ over W(k) whose reduction is A_o such that $\mathrm{End}_{W(k)}(\mathcal{A}) \longrightarrow \mathrm{End}_k(A_o)$ is bijective.

Also recall that $\mathcal{A}$ is the unique(upto isomorphism) lifting of A_o such that the above map is bijective. Taking a suitable level n structure, for example, (n,p)=1 the level structure and the polarization can be lifted to A.

Let K be the field of fractions of W(k). The generic fiber A_K of A/W(k) can be considered as the generic fiber of a fine moduli scheme of principally polarized abelian varieties in characteristic zero. In fact let $M'_{g,1,n}$ be a fine moduli scheme of abelian varieties over K. Then $M'_{g,1,n}$ is irreducible K-scheme. If A_K is not the generic member of $A'_{g,1,n} \longrightarrow M'_{g,1,n}$ we can consider it (by a suitable finite extension of K if necessary) as a closed point of $M'_{g,1,n}$. Then by reduction mod p, the generic member of $M'_{g,1,n}$ is mapped to A_o and A_K maps to a proper specialization of A_o which is impossible. In short, we can say that the generic point t lifts to the generic point in characteristic o. Since the generic fiber in characteristic zero has no complex multiplications, by proposition 5.4, and since $\mathrm{End}_{W(k)}(\mathcal{A}) \simeq \mathrm{End}_K(A_K)$, the conclusion of our theorem follows.

<u>Corollary 5.7</u>. The generic fiber A_t of $M_{g,1,n}$ is simple.

Proof. If not by Poincare-Weil theorem on the decomposability of an abelian variety into a product of simple abelian varieties(cf. § 1), the rank of the endomorphism ring of the generic fiber would be > 1, contradiction.

Corollary 5.8. The Picard number of the generic fiber is one.

Proof. The Picard number is, by definition, the rank of the Neron-Severi group NS(X) for any variety X. Since $NS(X) \otimes \mathbb{Q}$ is contained in $End^{\circ}(X)$ consisting of symmetric elements, the Picard number is $\leq$ the rank of the endomorphism ring. In our case, since the rank of the generic fiber is one, the Picard number is also one.

Theorem 5.9.. Let k be an algebraically closed field as before. Then any principally polarized abelian variety A defined over k is a specialization of an abelian variety B defined over $k((t_1,\ldots,t_{1/2\, g(g+1)}))^{ab}$ such that B has no complex multiplications. (here the superscript ab denotes the algebraic closure)

Proof. Consider A as a fiber at a closed point s of the fine moduli scheme $M_{g,1,n}$. Then the generic point t of $M_{g,1,n}$ belongs to $Spec(\widehat{O_{M_{g,1,n},s}})$; here $\wedge$ denotes the completion.-

Note that t is also the generic point of $\mathrm{Spec}(\widehat{O_{M_{g,1,n},s}})$, s being the unique closed point.

Now $\mathrm{Spec}(\widehat{O_{M_{g,1,n},s}})$ is the algebrization of the formal moduli of A_s and so we have an abelian scheme over $\mathrm{Spec}(\widehat{O_{M_{g,1,n},s}})$. Since A_s is principally polarized, the formal moduli scheme is formally smooth and hence $\mathrm{Spec}(\widehat{O_{M_{g,1,n},s}})$ is isomorphic to $k[[t_1,\ldots,t_{1/2\, g(g+1)}]]$. Here we take equicharacteristic formal deformation. The closed and the generic fibers of the abelian scheme over $k[[t_1,\ldots,t_{1/2\, g(g+1)}]]$ being A respectively $B/k((t_1,\ldots,t_{1/2\, g(g+1)}))$, the conclusion of our theorem follows from theorem 5.7.

6. Analogue of Shimura's moduli space in char $p > o$.

Let $\omega = (L, \rho, \Lambda, g, d)$ be an ordered set consisting of:

L ; a finite simple $\mathbb{Q}$-algebra

ρ : a positive involution on L

Λ : a Z-order in L

g, d : integers such that $(d^2 \Lambda \subset \Lambda)$

Definition.6.1. For any locally noetherian scheme S, let $\mathcal{M}_{\omega,n}(S)$ be the set of isomorphism classes of objects $(X/S, \lambda, \{\sigma_i\}, \bar{\theta})$ where

i) X is an abelian scheme over S of relative dimension g

ii) $\lambda : X \longrightarrow X^t$ a polarization of degree d^2

iii) $\{\sigma_i\}$ is a level n-structure on X/S

iv) $\theta : L \longrightarrow \mathrm{End}^{\circ}_S(X)$ is an injective algebra homomorphism such that a) $\theta(\Lambda) \quad \mathrm{End}_S(X)$

b) $\theta(1_\Lambda) = 1_X$

c) $\theta(x^\rho) = (\theta(x))'$ for all x in L

where $x \longrightarrow x'$ is the involution induced by λ .

Theorem 6.2. 1. For $n > 6^g \cdot d \cdot \sqrt{g!}$, $\mathcal{M}_{\omega,n}$ is represented by a scheme $M_{\omega,n}$. $M_{\omega,n}$ is quasi-projective over Z.

2. for all n, the coarse moduli scheme $M_{\omega,n}$

exists. It is quasi-projective over every open set $Spec(Z)-(p)$ in $Spec(Z)$

3. The forgetful morphism

$$p_{\omega,n}: M_{\omega,n} \longrightarrow M_{g,d,n}$$

is a finite morphism(i.e. we forget the endomorphism ring)

<u>Proof</u>. The proof of 2) can be given by imitating the proof of Mumford for constructing the usual coarse moduli scheme.

For the proof of 1) and 3), see Fisher[27].

<u>Remark 6.3</u>. If L is not a division algebra, decompose L into a direct sum of simple algebras: $L = \bigoplus_{i=1}^{k} L_i$. Let $\{e_i\}$ be the corresponding central idempotents of L_i. Let Λ be an order in L. Assume that $\Lambda_i = L_i \cap \Lambda$ contains e_i. If ρ is a positive involution on L, then $\rho = (\rho_1, \ldots, \rho_k)$. Let $(X/S, p)$ be a polarized abelian scheme over S and $\theta : L \longrightarrow End_S^{\circ}(X)$ be an injection such that $\theta(\Lambda)$ is contained in $End_S(X)$ and $\theta(x^{\rho}) = \theta(x)'$, $\theta(1_{\Lambda}) = 1_X$. Then $1_X = \Sigma\, \theta(e_i)$ and we get $X = \Sigma\, \theta(e_i)X$. Then X splits into a product of abelian schemes X_i, where $X_i = \theta(e_i)X$. Also θ induces an injection $\theta_i : \Lambda_i \longrightarrow End_S(X_i)$ and $\theta_i(e_i) = 1_{X_i}$. The polarization p will induce a polarization p_i on X_i and p_i

induces the involution ' on $\mathrm{End}^{\circ}_{S}(X_i)$. If $\omega=(L,\lambda,\rho,g,d)$ with $L = \oplus L_i$, then

$$M_{\omega,n}(S) = \bigcup \left(\prod_1^k M_{\omega_i,n}(S) \right)$$

where the union is taken over all sets of integers g_i and d_i such that $\Sigma g_i=g$, $\Sigma d_i=d$ and $\omega_i=(L_i,\rho_i,\Lambda_i,g_i,d_i)$ and so $M_{\omega,n} = \bigcup M_{\omega_i,n}$.

Now let us compare the moduli scheme $M_{\omega,n}$ with Shimura's moduli space V_{Ω}.

Let L be a simple algebra over $\mathbb{Q}$ such that $[L:\mathbb{Q}]$ divides 2g and let ρ be a positive involution on L. The condition $[L:\mathbb{Q}]|2g$ is automatically satisfied when L is a division algebra. A <u>PEL-type</u> is a collection $\Omega = (L,\varphi,\rho;V,T,\mathfrak{m},x_1,\ldots,x_s)$ where

- φ is a representation of L into $M_n(\mathbb{C})$
- V a left L-module of rank $m=2g/[L:\mathbb{Q}]$
- T a non-degenerate L-valued ρ-antihermitian form on V
- $\mathfrak{m}$ a free Z-submodule of V of rank 2g
- $x_1,\ldots,x_s$ are elements of V

Note that the order Λ in our ω is present here implicitly as $\Lambda = \{\lambda \in L \mid \lambda\mathfrak{m} \subset \mathfrak{m}\}$.

A <u>PEL-structure</u> is a collection $\underline{Q} =(A,\lambda,\theta;t_1,\ldots,t_s)$

where (A, λ) is a polarized abelian variety over $\mathbb{C}$, θ is an isomorphism of L into $\mathrm{End}^\circ(A)$ and $t_1, \ldots, t_s$ are points of finite order on A.

We say that $\underline{O}$ is of type Ω if there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{M} & \longrightarrow & V_R & \longrightarrow & V_R/\mathfrak{M} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow f & & \downarrow & & \\ 0 & \longrightarrow & D & \longrightarrow & \mathbb{C}^g & \xrightarrow{\epsilon} & A & \longrightarrow & 0 \end{array}$$

such that ϵ gives a holomorphic isomorphism of $\mathbb{C}^g/D$ onto A, f is R-linear and $f(\mathfrak{M}) = D$, $f(\alpha x) = \varphi(\alpha) f(x)$ and $\varphi(\alpha)$ defines $\theta(\alpha)$ for α in L, λ determines a Riemann form E on $\mathbb{C}^g/D$ such that for $(x,y) \in V \times V$,

$$E(f(x), f(y)) = \mathrm{Tr}_{L/\mathbb{Q}}(T(x,y))$$

and $t_i = \epsilon(f(x_i))$, $i = 1, \ldots, s$.

A PEL-type Ω is called admissible if there is at least one PEL-structure of type Ω. For Ω to be admissible it is necessary that $\varphi + \overline{\varphi}$ is equivalent to a rational representation of L.

For an admissible PEL-type Ω, Shimura Σ_Ω, the maximal family of PEL-structures of type Ω. Σ_Ω is parametrized by a bounded symmetric domain S on which acts an arithmetic group Γ in such a way that the coset space $\Gamma \backslash S$ is the moduli space for PEL-structures of type Ω. Except for a few Ω's, $\Gamma \backslash S$ has a quasi-projective algebraic model V_Ω. V_Ω is defined over the field of moduli of Ω. V_Ω is the moduli -variety for PEL-structures of type Ω.

When L is division algebra, Shimura computes the dimension of S. Hereafter assume that L is division algebra.

Let K = center of L, K_o = the subfield of K fixed by ρ. Put $[L:K] = d^2$, $e = [K:\mathbb{Q}]$, $e_o = [K_o:\mathbb{Q}]$, $m = 2g/[L:\mathbb{Q}]$. If L is type IV, i.e. L is a central division algebra over a CM field, let $\{\sigma_1, \ldots, \sigma_{e_o}, \rho\sigma_1, \ldots, \rho\sigma_{e_o}\}$ be a complete set of isomorphisms of K into $\mathbb{C}$. Let dr_ν, respectively ds_ν be the multiplicity of σ_ν resp. $\rho\sigma_\nu$ in φ restricted to K. Since $\varphi + \bar{\varphi}$ is equivalent to a rational representation, we have $r_\nu + s_\nu = md$.

The dimension, N, of S is:

(Type I) $\quad N = \dfrac{m(m+2)e}{8} \quad g = me/2$

(Type II) $\quad N = \dfrac{m(m+1)e}{2} \quad g = 2me$

(Type III) $\quad N = \dfrac{m(m-1)e}{2} \quad g = 2me$

(Type IV) $N = \sum_{\nu=1}^{e_0} r_\nu s_\nu$, $r_\nu + s_\nu = md = \frac{g}{e_0 d}$.

Shimura also proves the following

Theorem 6.4. If (A,λ,θ) is ageneric member of Σ_Ω, then $\theta(L) = \mathrm{End}(A) \otimes \mathbb{Q}$ except in the following cases:

a) L is of type III and $m = 1$.

b) L is of type III, $m = 2$ and there exists a totally positive element α such that $N(T) = \alpha^2$, where N is the reduced norm of $M_2(L)$ to K.

c) L is of type IV, $\sum_{\nu=1}^{e_0} r_\nu s_\nu = 0$.

d) L is of type IV, $m = 1$, $d = 2$, $r_\nu = s_\nu = 1, \nu = 1, \ldots, e_0$.

e) L is of type IV, $m = 2$, $d = 1$, $r_\nu = s_\nu = 1, \nu = 1, \ldots, e_0$.

In the cases a,b,d, if if (A,λ,θ) is of type (L,φ,ρ), then A is isogenous to a product of two copies an abelian variety B. In case e), if (A,λ,θ) is a generic member, then A is isogenous to a product of two copies of a simple abelian variety B such that $\mathrm{End}^\circ(B)$ is a totally indefinite quaternion algebra over K_0. In case c), A is isogenous to $k=md^2$ copies of an abelian variety B of dimension e_0 belonging to the CM-type $(K;\{\sigma_i\})$. B may or may not be simple.

Remark 6.5. Let $\Omega = (L, \varphi, \rho; V, T, \mathfrak{m})$ be a PEL-type and $\underline{O} = (A, \lambda, \theta)$ a PEL-structure of type Ω. Let $M_{\omega,1}$ be the coarse moduli scheme for a given $\omega = (L, \rho, \Lambda, g, d)$. Assume that the order Λ satisfies $\Lambda\mathfrak{m} \subset \mathfrak{m}$. Then $\underline{O}$ determines a point of $M_{\omega,1}(\mathbb{C})$. Conversely, a point of $M_{\omega,1}(\mathbb{C})$ determines a PEL-structure $\underline{O}$ of type Ω for some type Ω with $\mathfrak{m}$ a lattice in V. Infact $\underline{O}$ determines a unique equivalence class of such Ω's. If P is the complete set of representatives of admissible equivalence classes of Ω's, then P is finite by theorem 6.2 the morphism $p_{\omega,1} : M_{\omega,1} \longrightarrow M_{g,d,1}$ being finite. Hence $M_{\omega,1} \times \mathrm{Spec}(\mathbb{C}) \simeq \bigcup V_{\Omega}$ with Ω in P, the union being disjoint.

If we assume that Λ is maximal, then all the components of the generic fiber $M_{\omega,1} \otimes \mathbb{Q}$ have the same dimension, say, N. One can show that this statement does not depend on Λ being maximal.

In the case of dimension 3, we compute and tabulate the possible endomorphism algebras and compute the dimension of the moduli space $M_{\omega,n} \otimes \mathbb{Q} = M_o$. In the case of dimension 2 Fisher has computed the same [27]. We use the following table which gives restrictions on the invariants e, d, g, When the division algebra $L = \mathrm{End}^{\circ}(X)$

where X is a g-dimensional abelian variety which is simple.

TABLE I

TYPE	e	d	Restrictions in char 0; g=dim X L = End°(X)	Restrictions in char $p > 0$; g=dim X L = End°(X)
I	e_o	1	$e \mid g$	$e \mid g$
II	e_o	2	$2e \mid g$	$2e \mid g$
III	e_o	2	$2e \mid g$	$e \mid g$
IV	$2e_o$	d	$e_o d^2 \mid g$	$e_o d \mid g$

Next we shall give tables for the possible endomorphism algebras in diemnsions 2 and 3. Note that by theorem 6.4, quadratic imaginary field does not occur in our tables. Also note that we have included only those algebras where generically End°(X) = L.

Table II $\dim(X) = 2$

L	dimensions
$\mathbb{Q}$	3
$\mathbb{Q}(\sqrt{d})$, $d > 0$	2
D, quaternion algebra	1
K, imaginary quadratic extension of a real quadratic field	0
$\mathbb{Q} \times \mathbb{Q}$	2
$\mathbb{Q} \times \mathbb{Q}(\sqrt{d})$, $d < 0$	1
$\mathbb{Q}(\sqrt{d}) \times \mathbb{Q}(\sqrt{e})$, $d,e < 0$	0
$M_2(\mathbb{Q})$	1
$M_2(\mathbb{Q}(\sqrt{d}))$, $d < 0$	0

Table III $\dim(X) = 3$

L	dimensions
$\mathbb{Q}$	6
K_0, totally real cubic field	3
K, totally imaginary quadratic extension of totally real cubic field	0
$\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$	3
$\mathbb{Q} \times \mathbb{Q} \times Q(\sqrt{d})$, $d < 0$	2
$\mathbb{Q} \times \mathbb{Q}(\sqrt{d}) \times \mathbb{Q}(\sqrt{e})$, $d, e < 0$	1
$\mathbb{Q}(\sqrt{d}) \times \mathbb{Q}(\sqrt{e}) \times \mathbb{Q}(\sqrt{f})$, $d, e, f < 0$	0
$M_3(\mathbb{Q})$	1
$M_3(\mathbb{Q}(\sqrt{d}))$, $d < 0$	0
$\mathbb{Q} \times \mathbb{Q}$	2
$\mathbb{Q} \times F_0$, F_0 a real quadratic field	3
$\mathbb{Q} \times F$, F a CM-field of degree 4 over $\mathbb{Q}$	1
$\mathbb{Q} \times D$, D a quaternion algebra over $\mathbb{Q}$	2

7. Characteristic $p > 0$ and dimension = 3.

In this section we shall compute the dimension of the moduli space $M_{\omega,1}$ in cáharacteristic $p > 0$ and the dimension of the abelian varieties is 3; the results are only partial. In dimension 2 Fisher[27] describes the possible division algebras and computes the dimension of the moduli space $M_{\omega,n}$ for various p-ranks. At least in p-rank o, the difference between dimensions 2 and 3 is that in dimension 2, one such abelian variety is isomorphic to a product of supersingular elliptic curves whereas in dimension 3, this is not the case. In the latter case, the set of supersingular abelian varieties is a proper subset of the set of special abelian varieties. This fact makes the computation harder in dimension 3 than in dim 2.

Let X be an abelian variety of dimension 3 in AV(k) where k is algebraically closed of char $p > 0$. Then the formal group X* of X is isogenous to one of the following five possible types (upto isogeny): $3(G_{1,0} + G_{0,1})$ $2(G_{1,0} + G_{0,1}) + G_{1,1}$; $G_{1,0} + G_{0,1} + 2\, G_{1,1}$; $3\, G_{1,1}$; $G_{1,2} + G_{2,1}$.

(A). p-rank 3. Let M_p be an irreducible component of the fine moduli scheme $M_{\omega,n}$ at the prime p. Let $A_{\omega,n}$ be the universal abelian scheme over $M_{\omega,n}$. If M_p contains an ordinary point, i.e. a point corresponding to an ordinary abelian variety, then its generic point, say, t is also an ordinary point. The set of such points being dense, M_p is a component of the closure of points in characteristic zero and hence dim M_p is the same as in characteristic zero. Also since the generic point is ordinary, it has a canonical lifting to characteristic zero(cf.§ 2), the latter having the same endomorphism ring as A_t, i.e. $\mathrm{End}^{\circ}(A_t)=L$ whenever this is true in characteristic zero-that is, for the algebras in Table III. Thus we have to determine which of the algebras in Table III actually occur. Thus in the case of p-rank three, we have only to show the existence of an ordinary point.

i) $\mathbb{Q}$. This algebra obviously occurs and so the dimension of M_p is six.

ii) $M_3(\mathbb{Q})$; $M_3(\mathbb{Q}(\sqrt{d}))$; $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}(\sqrt{d})$; $\mathbb{Q} \times \mathbb{Q}(\sqrt{d}) \times \mathbb{Q}(\sqrt{e})$; $\mathbb{Q}(\sqrt{d}) \times \mathbb{Q}(\sqrt{e}) \times \mathbb{Q}(\sqrt{f})$; $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$; $\mathbb{Q} \times \mathbb{Q}$.

The first six of them occur as endomorphism algebras of A_t, when A_t is isogenous to products of elliptic curves with the restrictin that the prime p splits in each of the quadratic fields that occur. The algebra $\mathbb{Q} \times \mathbb{Q}$ occurs as product of a generic elliptic curve and an ordinary generic abelian surface. In these cases the dimension is the same as in char 0 and $\mathrm{End}^{\circ}(X) = L$ generically.

iii) K, totally imaginary quadratic extension of a totally real cubic field. In this case we can show the existence of ordinary point as in the two dimensional case. Since the degree of K is four over $\mathbb{Q}$ and since the representation R_1+R_2 is equivalent to a rational representation of K and is degree six, it follows that R_1+R_2 contains each of the irreducible representation of K over $\overline{\mathbb{Q}}_p$ exactly once. Recall that R_1 is a representation of Λ over Z_p and R_2 is equivalent to $(R_1 \cdot \rho)^*$, * representing the linear dual. If q is a prime of K lying over p and if R_1 contains any representation over $\mathbb{Q}_p$ of the completion K_q it must contain all the representations of K_q. It follows that if (K,ρ) can embedded in the endomorphism algebra of an ordinary 3-dimensional abelian variety, it follows that the following condition must hold:

(*) For every prime q of K lying over p, $q \neq \overline{q}$.

Now suppose condition $(*)$ holds. Let $p\, O_K = \prod \wp^{e_i} \bar{\wp}^{e_i}$ and set $\underline{a} = \prod \wp^{e_i}$. Then $\underline{a}$ determines an isogeny class of simple abelian varieties which are ordinary with endomorphism ring ,say,K',see Honda[4]. Then K' determines ordinary points in our moduli scheme M_p.

iv) $\mathbb{Q} \times F_o$, F_o a real quadratic field;
$\mathbb{Q} \times F$, F a CM-field of degree 4 over $\mathbb{Q}$
$\mathbb{Q} \times D$, D a quaternion algebra over $\mathbb{Q}$.

These cases occur as one can see from the two dimensional case.

v) K_o. a real quadratic field. This case will also occur but I am not able to prove this at present.

(B) p-rank 2.

Proposition 7.1. Let X be an object of AV(k) of dimension 3 and p-rank 2. Assume that End°(X) is a simple algebra. Then X itself is simple.

Proof. Let$[\mathrm{End}^\circ(X):K] = d^2$,where K is the center of End°(X). Then arguing as in theorem 3.5,we see that d must divide 2 and hence d = 1 or d =2. So End°(X) is a division algebra and so X is simple.

If X is a product three elliptic curves $E_1 \times E_2 \times E_3$, then End°(X) is of the form:

$$D_p \times \mathbb{Q} \times \mathbb{Q}; \quad D_p \times \mathbb{Q}(\sqrt{d}) \times \mathbb{Q}(\sqrt{e});$$
$$D_p \times M_2(\mathbb{Q}); \quad D_p \times M_2(\mathbb{Q}(\sqrt{d})),$$

where one of the curves is supersingular and the other two are ordinary. The dimensions of the moduli spaces are 2,0,1,0 respectively.

If X is a product a supersingular curve and an ordinary simple abelian surface, then End°(X) is either $D_p \times \mathbb{Q}$ or $D_p \times \mathbb{Q}(\sqrt{d})$ and so the dimension of the moduli space is 1 or 0 . Here D_p denotes the quaternion algebra over $\mathbb{Q}$ ramified only at p and ∞ and d,e are < 0.

The simple algebras which may occur are given by the Table I: $\mathbb{Q}$; totally real cubic field
Definite quaternion algebra over $\mathbb{Q}$;
Definite quaternion algebra over a totally real cubic field;
CM-field

i) K is a CM-field. In this case, as in p-rank three, Honda's theory shows that the moduli consists of simple abelian varieties having sufficiently many complex multiplications and hence they are all defined over a finite field. In this case the dimension of the moduli space is 0.

Since $\dim(X) = 3$ and $pr(X) = 2$, the isogeny type of the formal group of X is $2(G_{1,0} + G_{0,1}) + G_{1,1}$. Since we can deform such an abelian variety to an ordinary one, we conclude that the abelian varieties with p-rank 2 are contained in the closure of ordinary points and hence any component of the space of abelian varieties of p-rank 2 has codimension one in M_p.

ii) $\mathbb{Q}$. By reasoning as above we see that the points of M_p with p-rank two and $\mathrm{End}^\circ(X) = \mathbb{Q}$ are contained in the closure of ordinary points ans so the moduli space has dimension 5 in this case.

(C) p-rank one. The isogeny type of the formal group of such an abelian variety is $G_{1,0} + G_{0,1} + G_{1,1}$.

If X is a product of elliptic curves, then $\mathrm{End}^\circ(X)$ is of the form: $\mathbb{Q} \times D_p \times D_p$ or $\mathbb{Q}(\sqrt{d}) \times D_p \times D_p$ or $\mathbb{Q} \times M_2(D_p)$ or $\mathbb{Q}(\sqrt{d}) \times M_2(D_p)$. The corresponding dimensions of the moduli space are 1, 0, 1, 0, 0 respectively.

In case $\mathrm{End}^\circ(X)$ is a simple algebra, it is a field of one of the following types: $\mathbb{Q}$; toatlly real cubic field; CM-field of degree six; $\mathbb{Q}(\sqrt{d})$, $d < 0$

If is a product of an ordinary simple abelian surface and a supersingular curve, then End°(X) is:

$D_p \times \mathbb{Q}$; $D_p \times \mathbb{Q}(\sqrt{d})$, $d < 0$

$D_p \times \mathbb{Q}(\sqrt{d})$, $d > 0$; $D_p \times K$, K a CM-field.

The corresponding dimensions of the moduli space are: 1, 0, 0, 0,

By deforming $G_{1,0} + G_{0,1} + 2\, G_{1,1}$ to p-rank 2 and then to p-rank 3, we see that if M_p contains a point of p-rank one, then the dimension of the moduli space is four.

(D) p-rank 0. If X has dimension 3 and p-rank 0, then X* is isogenous to one of the following types: $3\, G_{1,1}$ or $G_{1,2} + G_{2,1}$.

Suppose X* is isogenous to $3\, G_{1,1}$. Then X is isogenous to a product of a supersingular elliptic curve and so $\mathrm{End}^\circ(X) \simeq M_3(D_p)$. Further if $a(X) = 3$, then the isomorphism classes of supersingular 3-dimensional abelian varieties is a finite set. This we can see as follows: If $a(X) = 3$, then X is isomorphic to a product of supersingular elliptic curves. Since the isomorphism classes of supersingular elliptic curves is finite, our claim is proved. Consequently, the dimension of the moduli space M_p is 0.

If X is simple, then X* is isogenous to $G_{1,2} + G_{2,1}$ and the possible endomorphism algebras in this case are: $\mathbb{Q}$; $M_2(\mathbb{Q})$; Division algebra over an imaginary quadratic field; a CM-field of degree 6.

If these algebras occur, then the set of abelian varieties whose formal group has isogeny type $G_{1,2} + G_{2,1}$ form a component by themselves.

TABLE IV $\dim(X) = 2$

L	p-rank	dim
$\mathbb{Q}$	2	3
	1	2
$Q(\sqrt{d})$, $d > 0$	2	2
,, p splits	1	1
D, quaternion, splits at p and ∞	2	1
K, a CM-field, for any prime q over p, $q \neq \bar{q}$	2	0
,, $pO_K=$ or $pO_K=$ where Nm =Nm =p, Nm $=p^2$	1	0
$\mathbb{Q} \times \mathbb{Q}$	2	2
,,	1	1
$\mathbb{Q} \times \mathbb{Q}(\sqrt{d})$, $d < 0$, $(\frac{d}{p}) = 1$	2	1
,,	1	0
$\mathbb{Q}(\sqrt{d}) \times \mathbb{Q}(\sqrt{e})$, $d, e < 0$, $(\frac{d}{p})=(\frac{e}{p})=1$	2	1
$\mathbb{Q} \times D_p$,,	1	1
$\mathbb{Q}(\sqrt{d}) \times D_p$, $d < 0$, $(\frac{d}{p}) = 1$	1	0
$M_2(\mathbb{Q})$	2	1
$M_2(\mathbb{Q}(\sqrt{d}))$, $d < 0$, $(\frac{d}{p}) = 1$	2	0
$M_2(D_p)$, p maximal	0	0
,, p sufficiently small	0	1

The above table is computed by Fisher[27].

TABLE V $\dim(X) = 3$

L	p-rank	dim
$\mathbb{Q}$	3	6
,,	2	5
,,	1	4
K, a CM-field of degree 6	3	0
,,	2	0
,,	1	0
$\mathbb{Q} \times \mathbb{Q}$	3	2
$\mathbb{Q} \times K_0$, K_0 a real quadratic field	3	4
,,	2	3
,,	1	2
$\mathbb{Q} \times K$, K a CM-field of degree 4	3	1
,,	2	0
,,	1	0
$M_3(D_p)$ and $a(X)=3$, maximal	0	0
$\mathbb{Q} \times D$, D a quaternion algebra/$\mathbb{Q}$	3	2
$\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$	3	3
$\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}(\sqrt{d}), d < 0$	3	2
$\mathbb{Q} \times \mathbb{Q}(\sqrt{d}) \times \mathbb{Q}(\sqrt{e}), d,e < 0$	3	1
$\mathbb{Q}(\sqrt{d}) \times \mathbb{Q}(\sqrt{e}) \times \mathbb{Q}(\sqrt{f}), d,e,f < 0$	3	0
$M_3(\mathbb{Q})$	3	1
$M_3(\mathbb{Q}(\sqrt{d}))$	3	0

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Errata

page	line	from	read	for
ii	9	bottom	supersingular	suersingular
3	5	,,	inertia	inetia
14	4	top	$W(k)^*$ of the units of the ring $W(k)$	$W(k)^*$ of the ring $W(k)$
18	10	top	contain	comtain
18	3	bottom	any closed fiber	closed fiber
21	9	top	Denot by $M_{g,1,n}$	Denote $M_{g,1,n}$
24	7	bottom	be	br
24	7	,,	F	f
38	4	,,	Shimura constructs Σ	Shimura Σ
47	10	top	six	four
51	3	,,	If X is a product	If is product