AN ALGORITHM FOR THE SOLUTION OF THE

WORD PROBLEM FOR

$N_{p,q} = \langle y, x \mid \bar{y}x^p y \bar{x}^q \rangle$

IF $(p,q) = 1$, $p > 1$, AND $|q| \neq 1$

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ABSTRACT OF THE DISSERTATION

AN ALGORITHM FOR THE SOLUTION OF THE WORD PROBLEM FOR $N_{p,q} = \langle y,x \mid \bar{y}x^p y\bar{x}^q \rangle$ IF $(p,q) = 1$, $p > 1$, AND $|q| \neq 1$

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The word problem for a group with one defining relator was proved solvable by Magnus in 1931[1]. However, he did not give an algorithm for the solution. This paper provides an algorithm for the solution of the word problem for the groups $N_{p,q} = \langle x,y \mid \bar{y}x^p y\bar{x}^q \rangle$ if $(p,q) = 1$, $p > 1$ and $|q| \neq 1$.

The paper uses the fact that the subgroup of $N_{p,q}$ generated by x is isomorphic to $H_{p,q} = \langle b_i : i \in J \mid b_i^q \bar{b}_{i+1}^p : i \in J \rangle$.

A normalization process is given to find a unique representative for each word in $H_{p,q}$. Using the fact that

$S_{p,q} = <\{\frac{m}{p^s q^t} | m \in J, s \in J, t \in J\}, +>$ is a homomorphic image of $H_{p,q}$, and by expressing $H_{p,q}$ as a free product under amalgamation it is shown that every word in $H_{p,q}$ that is equal to 1 in $H_{p,q}$ becomes 1 under normalization.

The definition of when a word $W \in H_{p,q}$ is in normal form is presented here, however the algorithm itself is too lengthy to describe here.

<u>Definition</u>. If $W \in H_{p,q}$, $(p,q) = 1$, $p > 1$, $|q| \neq 1$ then W is in normal form if and only if

i) $W \equiv 1$; or

ii) $W \equiv b_j^{\alpha}$ with $\alpha \not\equiv 0 \bmod |q|$; or

iii) $W \equiv b_j^{\alpha} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ with the following properties

$j \neq j_1$,

$j_i \neq j_{i+1}$,

if $j < j_1$ then $\alpha \not\equiv 0 \bmod |q|$ and $0 < \alpha_1 < p$,

if $j > j_1$ then $\alpha \not\equiv 0 \bmod p$ and $0 < \alpha_1 < |q|$,

if $j_i > j_{i-1}$ then $0 < \alpha_i < p$,

and if $j_i < j_{i-1}$ then $0 < \alpha_i < |q|$.

[1]Magnus, W. 1931, Untersuchungen über einige unendliche diskontinuierliche Gruppen. <u>Math. Ann.</u> 52-74.

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INTRODUCTION

The group $G = \langle x,y,z \mid \bar{y}x^2y\bar{x}^3, \bar{x}z^2x\bar{z}^3, \bar{z}y^2z\bar{y}^3 \rangle$ is either the trivial group or a group of infinite order.[1] [6]

I conjectured in the Spring of 1971 that if W was any non-empty word on the generators of

$$G = \langle x,y,z \mid \bar{y}x^2y\bar{x}^3, \bar{x}z^2x\bar{z}^3, \bar{z}y^2z\bar{y}^3 \rangle$$

and $W = 1$ in G then if $W \not\equiv 1$ it, or a short conjugate (cyclic permutation) of W, has an initial segment of length 4 that is also an initial segment of a short conjugate of one of the defining relators or its inverse.

My conjecture was based upon Greendlinger's findings. [2] He defined a 1/k group (more precisely, the presentation of the group), as follows: if R_u and R_v are any two defining relators or their inverses, then whenever $1/k^{th}$ the symbols of R_u or in R_v can be deleted by free reduction, then R_uR_v is freely equal to the empty word.

He proved if W was any non-empty word on the generators of G, G being a less than 1/6 group, then W or some short conjugate of W has an initial segment that is also a major initial segment (more than half) of a short conjugate of one of the defining relators or its inverse.

I generalized my conjecture by defining a (1/k, 1/h)

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[1] See appendix page 50 for proof.

presentation when pairs of distinct defining relators (not short conjugates of each other nor inverses) have the 1/k property, and pairs of non-distinct relators (short conjugates and their inverses) a 1/h property. My conjecture was for a (1/k,1/h) presentation the word problem was solvable whenever $k > 6$, $h > 3$ and the length of each defining relator was odd.

In the Fall of 1972 I proved my conjecture to be false by the following counter example.

$$G = \langle x,y,z \mid \bar{y}x^2y\bar{x}^3, \bar{x}z^2x\bar{z}^3, \bar{z}y^2z\bar{y}^3 \rangle$$

is a (1/7,2/7) presentation and each defining relator has length 7. The word $\bar{y}^2x^4y^2\bar{x}^2\bar{y}\bar{x}^4y\bar{x}$ is 1 in G and does not contain a subword, (nor does any short conjugate of $\bar{y}^2x^4y^2\bar{x}^2\bar{y}\bar{x}^4y\bar{x}$), of length four that is also a major initial segment of a defining relator or its inverse.

We can see that $\bar{y}^2x^4y^2\bar{x}^2\bar{y}\bar{x}^4y\bar{x}$ is 1 in G by using $\bar{y}x^2y\bar{x}^3 = 1$.

Hence $\bar{y}x^2y = x^3$; $\bar{y}\bar{x}^2y = \bar{x}^3$

$\bar{y}x^4y = x^6$; $\bar{y}\bar{x}^4y = \bar{x}^6$

$\bar{y}^2x^4y^2 = \bar{y}x^6y = x^9$;

whence

$$x^9\bar{x}^9 = 1$$

$$\bar{y}^2x^4y^2\bar{x}^9 = 1$$

$$\bar{y}^2x^4y^2\bar{x}^2\bar{x}^6\bar{x} = 1$$

$$\bar{y}^2x^4y^2\bar{x}^2\bar{y}\bar{x}^4y\bar{x} = 1$$

I then decided to find the solution to the word problem for $\langle y,x \mid \bar{y}x^2y\bar{x}^3\rangle$ so as to continue my research into the word problem for

$$\langle x,y,z \mid \bar{y}x^2y\bar{x}^3, \bar{x}z^2x\bar{z}^3, \bar{z}y^2z\bar{y}^3\rangle.$$

I have generalized my finding to the solution to the word problem for

$$\langle y,x \mid \bar{y}x^p y\bar{x}^q\rangle,\ (p,q) = 1,\ p > 1,\ |q| \neq 1.$$

The word problem was formulated by Max Dehn in 1911.

Let a group G be defined by means of a presentation. The word problem is: For an arbitrary word W in the generators, decide in a finite number of steps whether W defines the identity element of G, or not.

Novikov [5] proved in 1955 that there is some finite presentation for which the word problem is not solvable.

Magnus [3] proved in 1931 that the word problem is solvable for any group with one defining relator.

"However, the general method for solving the word problem in groups with a single defining relator is already a rather complicated process,..." [4] page 400.

The group $G = \langle y,x \mid \bar{y}x^2y\bar{x}^3\rangle$ is the simplest known example of a non-Hopfian finitely presented group. [1] This group is the special case when p = 2 and q = 3.

Finding an algorithm for the solution to the word problem of

$$\langle y,x \mid \bar{y}x^{p}y\bar{x}^{q}\rangle,\ (p,q) = 1,\ p > 1,\ |q| \neq 1$$

is of interest for the following reasons.

i) It is useful in studying the properties of $\langle y,x \mid \bar{y}x^{2}y\bar{x}^{3}\rangle$, the simplest known non-Hopfian finitely presented group.

ii) Magnus did not provide an algorithm for the solution of the word problem.

iii) Groups with one defining relator are an important subject in combinatorial group theory. For example the fundamental groups of closed two dimensional orientable surfaces of a genus ≥ 2 are groups with a single defining relator. [4] page 398.

iv) It is useful in the understanding of groups of infinite order. For these are not free groups, free abelian groups, nor the product of two free groups under amalgamation. Hence they provide examples of groups of infinite order with just one defining relator with an algorithm for the solution to the word problem, where the groups are neither abelian nor the product of two free groups under amalgamation.

Given a group $G = \langle a_1,a_2,\ldots,a_n : R(a_1,\ldots,a_n)\rangle$ then the word problem is solvable is what Magnus proved. [3] To have some understanding of Magnus' proof, a sketch of a particular case when $R(a_1,a_2,\ldots,a_n)$ has zero exponent sum on one of the generators is provided. The particular case will be

$G_{p,q} = \langle y,x \mid \bar{y}x^p y\bar{x}^q\rangle$ where $\sigma_y(\bar{y}x^p y\bar{x}^q) = 0$.

The normal subgroup of $G_{p,q}$ generated by x is isomorphic to $H_{p,q} = \langle b_i : i \in J \mid b_i^q \bar{b}_{i+1}^p : i \in J\rangle$.

On this pattern, one then defines subgroups

$$N_i = \langle b_i, b_{i+1} \mid b_i^q \bar{b}_{i+1}^p\rangle,\ i \in J.$$

For example $N_0 = \langle b_0, b_1 \mid b_0^q \bar{b}_1^p\rangle$

$N_1 = \langle b_1, b_2 \mid b_1^q \bar{b}_2^p\rangle$

One then sets $N_{0,1} = \langle b_0, b_1, b_2 \mid b_0^q \bar{b}_1^p, b_1^q \bar{b}_2^p\rangle$ which is the free product of N_0 and N_1 with the free subgroup in each generated by b_1, amalgamated under the identity mapping.

Now $N_0 \subset N_{0,1}$.

Similarly one would define

$$N_{-1,1} = \langle b_{-1}, b_0, b_1 \mid b_{-1}^q \bar{b}_0^p, b_0^q \bar{b}_1^p, b_1^q \bar{b}_2^p\rangle,$$

etc., obtaining a chain of groups.

$N_0 \subset N_{0,1} \subset N_{-1,1} \subset \ldots \subset N_{-i+1,i} \subset N_{-i,i} \subset N_{-i,i+1} \subset \ldots$ where $N_{-i,i}$ is the free product of $N_{-i+1,i}$ and N_{-i} with an amalgamated free subgroup (b_{-i+1} under the identity mapping), and $N_{-i,i+1}$ is the free product of $N_{-i,i}$ and N_{i+1} with an amalgamated free subgroup.

$H_{p,q}$ is the union of this chain of groups.

Setting $Q_1 = N_0$, $Q_2 = N_{0,1}$, $Q_3 = N_{-1,1}$ etc. One would show by induction on j that in each Q_j, if the generators of N_i are among the generators of Q_j, then it can be decided if an element of Q_j is in N_1, and if so, express it in N_1. Then one shows if an element of $H_{p,q}$ is an element of N_1, and if so to express it in N_1.

This paper presents an algorithm for the solution to the word problem for

$$H_{p,q} = \langle b_i : i \in J \mid b_i^p \bar{b}_i^q : i \in J \rangle \text{ if } (p,q) = 1,\ p > 1 \text{ and } |q| \neq 1.$$

For example, no chain of groups is used. The proof is different from Magnus'.

Following Magnus, the group $H_{p,q}$ will be presented by

$$H_{p,q} = \langle b_i : i \in J \mid b_i^p \bar{b}_i^q : i \in J \rangle.$$

In Section II a normalization process is given to find a unique representative for every word in $H_{p,q}$.

In Section I it is shown that the group $S_{p,q} = \langle \{ \frac{m}{p^s q^t} \mid m \in J, s \in J, t \in J \}, + \rangle$ is a homomorphic image of $H_{p,q}$. Using this homomorphism and expressing $H_{p,q}$ as a free product with amalgamated subgroups, it is proven in Section III that every word in $H_{p,q}$ that is equal to 1 in $H_{p,q}$ becomes 1 under the normalization.

This solves the word problem for $H_{p,q}$ and hence for $G_{p,q}$ as shown in Section IV.

SECTION 1

SOME PRELIMINARY THEOREMS

Definition 1. $F = \langle a, b \mid \rangle$.

Definition 2. $K_{p,q} = F(\bar{a}b^p a \bar{b}^q)$, $(p,q) = 1$, $p > 1$, $|q| \neq 1$ the normal closure of $\bar{a}b^p a\bar{b}^q$ in F.

Definition 3. $B = F(b)$ the normal closure of b in F.

Definition 4. If $W \in F$ then $\sigma_a(W)$ is the exponent sum of the a's.

Theorem 1. If $W \in F$ and $W = 1 \bmod K_{p,q}$ then $W = 1 \bmod B$.

Proof. $W = 1 \bmod B$ if and only if $\sigma_a(w) = 0$. If $W \in F$ such that $W = 1 \bmod K_{p,q}$ then $\sigma_a(w) = 0$, whence $W = 1 \bmod B$.

Definition 5. $N_{p,q} = \langle y, x \mid \bar{y}x^p y \bar{x}^q \rangle$, $(p,q) = 1$, $p > 1$ and $|q| \neq 1$.

Definition 6. $M_{p,q} = \langle x_i : i \in J \mid x_i^q \bar{x}_{i+1}^p : i \in J \rangle$ $(p,q) = 1$, $p > 1$, $|q| \neq 1$, J being the set of integers.

Theorem 2. $N_{p,q} \cong \langle y, M_{p,q} \mid x_i = \bar{y}^i x_0 y^i : i \in J \rangle$.

Proof. $N_{p,q} = \langle y, x \mid \bar{y}x^p y \bar{x}^q \rangle$

$$\Rightarrow N_{p,q} \cong \langle y, x, x_0 \mid \bar{y}x^p y\bar{x}^q, x = x_0, \bar{y}x_0^p y \bar{x}_0^q \rangle$$

$$\Rightarrow N_{p,q} \cong \langle y, x_0 \mid \bar{y}x_0^p y \bar{x}_0^q \rangle$$

$$\bar{y}x_0^p y \bar{x}_0^q = 1 \Rightarrow \bar{y}x_0^p y = x_0^q.$$

$$\Rightarrow \bar{y}^{i+1}x_0^p y^{i+1} = \bar{y}^i x_0^q y^i$$
$$\Rightarrow (\bar{y}^i x_0 y^i)^q (\bar{y}^{i+1} x_0 y^{i+1})^{-p} = 1$$
$$\Rightarrow N_{p,q} \cong \langle y, x_0 \,|\, (\bar{y}^i x_0 y^i)^q (\bar{y}^{i+1} x_0 y^{i+1})^{-p} : i \in J \rangle$$
$$\Rightarrow N_{p,q} \cong \langle y, x_i : i \in J \,|\, (\bar{y}^i x_0 y^i)^q (\bar{y}^{i+1} x_0 y^{i+1})^{-p}, \; x_i = \bar{y}^i x_0 y^i : i \in J \rangle$$
$$\Rightarrow N_{p,q} \cong \langle y, x_i : i \in J \,|\, (\bar{y}^i x_0 y^i)^q (\bar{y}^{i+1} x_0 y^{i+1})^{-p}, \; x_i = \bar{y}^i x_0 y^i, x_i^q \bar{x}_{i+1}^p : i \in J \rangle$$
$$\Rightarrow N_{p,q} \cong \langle y, x_i : i \in J \,|\, x_i^q \bar{x}_{i+1}^p, x_i = \bar{y}^i x_0 y : i \in J \rangle$$
$$M_{p,q} = \langle x_i : i \in J \,|\, x_i^q \bar{x}_{i+1}^p : i \in J \rangle$$
$$\Rightarrow N_{p,q} \cong \langle y, M_{p,q} \,|\, x_i = \bar{y}^i x_0 y^i : i \in J \rangle.$$

Definition 7. $F_b = \langle b_i : i \in J \,|\, \rangle$.

Definition 8. $\rho(w)$ is the freely reduced word equal to w in the free group on the generators.

Definition 9. $U \equiv V$ means that the word U is identical to the word V. For example $\bar{a}b^2a\bar{b}^3 \equiv \bar{a}bba\bar{b}\bar{b}\bar{b}$ in any group, but $\bar{a}b^2a\bar{b}^3 \not\equiv \bar{a}bbb\bar{b}a\bar{b}\bar{b}\bar{b}$ and $\bar{a}b^2a\bar{b}^3 \not\equiv 1$ in $\langle a,b \,|\, \bar{a}b^2a\bar{b}^3 \rangle$.

Definition 10. $\emptyset : B \to F_b$, $\emptyset$ is a mapping such that if $W \in B$, $\sigma_a(w) = 0$ and

i) if $\rho(w) = \prod_{i=1}^{n} a^{\alpha_i} b^{\beta_i}$ then $\emptyset(w) = \prod_{i=1}^{n} b^{\beta_i}_{(\sum_{j=1}^{i} \alpha_j)}$;

or

ii) if $\rho(w) \equiv 1$ then $\emptyset(w) = 1$.

[Note: $\emptyset(\bar{a}^i b a^i) = b_i$ and completely defines $\emptyset$, as $\sigma_a(w) = 0$ for all $w \in B$.]

Theorem 3. $\emptyset : B \to F_b$ is a homomorphism.

Proof. Let $U, V \in B$ such that

$$\rho(U) \equiv \prod_{i=1}^{n} a^{\alpha_i} b^{\beta_i} \text{ and } \rho(V) \equiv \prod_{i=1}^{m} a^{\gamma_i} b^{\delta_i}.$$

$$\sigma_a(U) = \sigma_a(V) = \sigma_a(UV) = 0$$

$$\emptyset(\bar{a}^i b a^i) = b^i.$$

Hence $\emptyset(U)\emptyset(V) = \emptyset(UV) = [\prod_{i=1}^{n} b^{\beta_i}_{(\sum_{j=1}^{i}\alpha_j)}][\prod_{i=1}^{m} b^{\delta_i}_{(\sum_{j=1}^{i}\gamma_j)}]$.

Theorem 4. If $W \in F$ and $W = 1 \bmod K_{p,q}$ then $W \in B$ and $W = 1$ in $B/K_{p,q}$.

Proof. $F = \langle a, b | \rangle$, $K_{p,q} = F(\bar{a}b^p a\bar{b}^q)$, $B = F(b)$. By Theorem 1 if $W \in F$ and $W = 1 \bmod K_{p,q}$ then $W = 1 \bmod B$.

Hence if $W \in H$ then $W \in B$.

Therefore, if $W \in F$ and $W = 1 \bmod K_{p,q}$ then $W = 1$ in $B/K_{p,q}$.

Definition 11. $\theta : F_b \to B$ under $\theta(1) = 1$ and $\theta(b_i) = \bar{a}^i b a^i$.

Theorem 5. $\theta : F_b \to B$ under $\theta(1) = 1$ and $\theta(b_i) = \bar{a}^i b a^i$ is an isomorphism between the group F_b and B, and $\theta = \emptyset^{-1}$.

Proof. $F_b = \langle b_i : i \in J | \rangle$, $F = \langle a, b | \rangle$, $B = F(b)$, $\emptyset(\bar{a}^i b a^i) = b_i$, $\emptyset(1) = 1$, $\theta(b_i) = \bar{a}^i b a^i$, $\theta(1) = 1$.

θ is clearly an isomorphism onto B and $\theta = \emptyset^{-1}$.

Definition 12. $H_{p,q} = \langle b_i : i \in J \mid b_i^q \bar{b}_{i+1}^p : i \in J \rangle$.

Theorem 6. If $W \in B$ and $\emptyset(w) = W_b \in F_b$, then $W = 1 \bmod K_{p,q} \Leftrightarrow W_b = 1$ in $H_{p,q}$.

Proof. $F = \langle a,b \mid \rangle$, $B = F(b)$, $K_{p,q} = F(\bar{a}b^p a\bar{b}^q)$, $F_b = \langle b_i : i \in J \mid \rangle$,
$H_{p,q} = \langle b_i : i \in J \mid b_i^q \bar{b}_{i+1}^p : i \in J \rangle$
$\emptyset : B \to F_b$ under $\emptyset(\bar{a}^i b a^i) = b_i$, $\emptyset(1) = 1$.

By Theorem 5, $\emptyset$ is an isomorphism between F_b and B.

$$\emptyset(\bar{a}b^p a\bar{b}^q) = b_1^p \bar{b}_0^q = 1 \text{ in } H_{p,q}.$$

$$\begin{aligned}\emptyset^{-1}(b_i^q \bar{b}_{i+1}^p) &= \bar{a}^i b^q a^i \bar{a}^{i+1} \bar{b}^p a^{i+1} \\ &= \bar{a}^i b^q \bar{a} \bar{b}^p a a^i \\ &= \bar{a}^i (\bar{a}b^p a\bar{b}^q)^{-1} a^i = 1 \bmod H_{p,q}.\end{aligned}$$

Hence $W = 1 \bmod K_{p,q} \Leftrightarrow W_b = 1$ in $H_{p,q}$.

Definition 13. $S_{p,q} = \langle \{\frac{m}{p^s q^t} \mid m \in J, s \in J, t \in J\}, + \rangle$.

Definition 14. $\psi_{p,q} : M_{p,q} \to S_{p,q}$ is a mapping under $\psi(x_j^\varepsilon) = \varepsilon(\frac{q}{p})^j$, $\varepsilon = \pm 1$ and $\psi(1) = 0$.

Theorem 7. $\psi_{p,q} : M_{p,q} \to S_{p,q}$ is a homomorphism.

Proof. $M_{p,q} = \langle x_i : i \in J \mid x_i^q \bar{x}_{i+1}^p : i \in J \rangle$
$S_{p,q} = \langle \{\frac{m}{p^s q^t} \mid m \in J, s \in J, t \in J\}, + \rangle$.

[Note: $S_{p,q} \cong \langle x_i : i \in J \mid x_i^q \bar{x}_{i+1}^p, x_k x_j \bar{x}_k \bar{x}_j : i,j,k \in J \rangle$. That is $S_{p,q}$ is isomorphic to the commutator quotient group. This fact is not used in this paper.]

$\psi_{p,q}(x_j^{\varepsilon}) = \varepsilon(\frac{q}{p})^j$, $\psi(1) = 0$.

$S_{p,q}$ is clearly a subgroup of the rationals under addition. It is obvious that if $U \in M$ and $V \in M$ then $\psi_{p,q}(UV) = \psi_{p,q}(U) + \psi_{p,q}(V)$.

Furthermore $\psi_{p,q}(x_i^q x_{i+1}^{-p}) = q(\frac{q}{p})^i - p(\frac{q}{p})^{i+1} = \frac{q^{i+1}}{p^i} - \frac{q^{i+1}}{p^i} = 0$.

<u>Definition 15</u>. $P_{p,q}: F_b \to M$ under $P_{p,q}(b_j) = x_j$ and $P_{p,q}(1) = 1$.

<u>Theorem 8</u>. If $W_b \in F_b$ and $W_b = 1$ in $H_{p,q}$ then $\psi_{p,q}(P_{p,q}(W_b)) = 0$.

<u>Proof</u>.

$$F_b = \langle b_i : i \in J | \rangle$$
$$H_{p,q} = \langle b_i : i \in J | b_i^q b_{i+1}^{-p} : i \in J \rangle$$
$$M_{p,q} = \langle x_i : i \in J | x_i^q x_{i+1}^{-p} : i \in J \rangle$$
$$S_{p,q} = \langle \{ \frac{m}{p^s q^t} | m \in J, s \in J, t \in J \}, + \rangle$$

$P_{p,q}: F_b \to M_{p,q}$, $P(b_j) = x_j$, $P(1) = 1$

$\psi : M_{p,q} \to S_{p,q}$, $\psi(x_j^{\varepsilon}) = \varepsilon(\frac{q}{p})^j$, $\psi(1) = 0$.

If $W_b \in F_b$ and $W_b = 1$ in $H_{p,q}$, then obviously $P_{p,q}(W_b) = 1$ in $M_{p,q}$. And by Theorem 7, $\psi_{p,q}(P_{p,q}(W_b)) = \psi(1) = 0$.

SECTION II

THE NORMAL FORM, N(W)

Definition 16. If $W \in H_{p,q}$, $(p,q) = 1$, $p > 1$, $|q| \neq 1$, then W is in normal form, $(N(W) \equiv W)$, if and only if

i) $W \equiv 1$; or

ii) $W \equiv b_j^{\alpha}$ with $\alpha \not\equiv 0 \bmod |q|$; or

iii) $W \equiv b_j^{\alpha} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ with the following properties:

$j \neq j_1$,

$j_i \neq j_{i+1}$,

if $j < j_1$ then $\alpha \not\equiv 0 \bmod |q|$ and $0 < \alpha_1 < p$,

if $j > j_1$ then $\alpha \not\equiv \bmod p$ and $0 < \alpha_1 < |q|$,

if $j_i > j_{i-1}$ then $0 < \alpha_i < p$,

and if $j_i < j_{i-1}$ then $0 < \alpha_i < |q|$.

FINDING N(W)

Let $W \in H_b$.

If $W \equiv 1$ then $N(W) \equiv 1$.

If $W \equiv b_t^{\varepsilon}$, $\varepsilon = \pm 1$, then $N(W) \equiv b_t^{\varepsilon}$.

Let $W \equiv b_t^{\varepsilon} U$, $\varepsilon = \pm 1$, U in normal form.

That is: i) $U \equiv 1$; or

ii) $U \equiv b_j^{\alpha}$ with $\alpha \not\equiv 0 \bmod |q|$; or

iii) $U \equiv b_j^{\alpha} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ with the following properties:

$j \neq j_1$,

$j_i \neq j_{i+1}$,

if $j < j_1$ then $\alpha \not\equiv 0 \bmod |q|$, and $0 < \alpha_1 < p$

if $j > j_1$ then $\alpha \not\equiv 0 \bmod p$, and $0 < \alpha_1 < |q|$

if $j_i > j_{i-1}$ then $0 < \alpha_i < p$

and if $j_i < j_{i-1}$ then $0 < \alpha_i < |q|$.

Then there are four cases to consider. They are

A: $U \equiv 1$

B: $t > j$

C: $t < j$

D: $t = j$.

Case A. If $U \equiv 1$ then

$$W \equiv b_t^{\varepsilon} \cdot 1$$
$$W = b_t^{\varepsilon}$$

and $N(W) \equiv b_t^{\varepsilon}$ is the normal form of W.

For example: If $p = 2$, $q = 3$, $W \equiv b_3^{-1} \cdot 1$, $U \equiv 1$, then $N(W) \equiv b_3^{-1}$.

Case B. $W \equiv b_t^{\varepsilon} b_j^{\alpha}$ or $W \equiv b_t^{\varepsilon} b_j^{\alpha} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ and $t > j$. There are two cases to consider. They are:

B-1: $0 < \alpha < |q|$

B-2: $\alpha < 0$ or $\alpha \geq |q|$.

Case B-1. $0 < \alpha < |q|$ and $t > j$. Then W is in normal form and $N(W) \equiv W$.

For example: $p = 2$, $q = 3$, $W \equiv b_3^1 b_1^2 b_2^1$, $U \equiv b_1^2 b_2^1$ and $N(W) \equiv b_3^1 b_1^2 b_2^1$.

Case B-2. $t > j$ and $\alpha < 0$ or $\alpha \geq |q|$.

Let $\alpha \equiv \gamma_1 \bmod |q|$, that is $\alpha = qm_1+\gamma_1$ with $0 \leq \gamma_1 < |q|$. Since U is in normal form, $m_1 \neq 0$. Using the relator $b_j^{qm_1} = b_{j+1}^{pm_1}$, one gets $b_j^{\alpha} = b_{j+1}^{pm_1} b_j^{\gamma_1}$ and $W = b_t^{\varepsilon} b_{j+1}^{pm_1} b_j^{\gamma_1} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ or $W = b_t^{\varepsilon} b_{j+1}^{pm_1} b_j^{\gamma_1}$.

There are two cases to consider. They are:

B-2-A: $\gamma_1 = 0$

B-2-B: $\gamma_1 \neq 0$.

Case B-2-A. $t > j$, $W = b_t^{\varepsilon} b_{j+1}^{pm_1} b_j^{\gamma_1} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ and $\gamma_1 = 0$ or $W = b_t^{\varepsilon} b_{j+1}^{pm_1} b_j^{\gamma_1}$ and $\gamma_1 = 0$.

Hence q divided α. Hence $W \neq b_t^{\varepsilon} b_{j+1}^{pm_1} b_j^{\gamma_1}$ and $j > j_1$. Now we have $t > j > j_1$.

If $t = j+1$ then $W = b_{j+1}^{pm_1+\varepsilon} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ this is in normal form. $N(W) \equiv b_{j+1}^{pm_1+\varepsilon} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$.

If $t > j+1$ then the process is iterated as follows: For $t = j+2$, $\alpha = qm_1+\gamma_1$, $b_j^{\alpha} = b_{j+1}^{pm_1}$ for $\gamma_1 = 0$; and $pm_1 = qm_2+\gamma_2$, $0 \leq \gamma_2 < |q|$; $b_j^{\alpha} = b_{j+2}^{pm_2} b_{j+1}^{\gamma_2}$ whence

$$W = b_{j+2}^{pm_2+\varepsilon} b_{j+1}^{\gamma_2} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}, \quad 0 \leq \gamma_2 < |q|.$$

Similarly, for all $t = j+k > j$

$$b_j^{\alpha} = b_{j+k}^{pm_k} b_{j+k-1}^{\gamma_k} \cdots b_{j+1}^{\gamma_2}, \quad 0 \leq \gamma_i < |q|.$$

Hence $b_t^{\varepsilon} b_j^{\alpha} = b_{j+k}^{pm_k+\varepsilon} b_{j+k-1}^{\gamma_k} \cdots b_{j+1}^{\gamma_2}$ and

$N(W) \equiv \rho(b_{j+1}^{pm_k+\varepsilon} b_{j+k-1}^{\gamma_k} \cdots b_{j+1}^{\gamma_2}) \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ is the normal form of W.

Case B-2-B. $\gamma_1 \neq 0$, $t > j$, $W = b_t^{\varepsilon} b_{j+1}^{pm_1} b_j^{\gamma_1} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$.

Hence $0 < \gamma_1 < |q|$. If $t = j+1$ then $W = b_{j+1}^{pm_1+\varepsilon} b_j^{\gamma_1} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ when this word is freely reduced one gets $\rho(b_{j+1}^{pm_1+\varepsilon} b_j^{\gamma_1} \prod_{i=1}^{n} b_{j_i}^{\alpha_i})$ and this is W in normal form.

$$N(W) \equiv \rho(b_{j+1}^{pm_1+\varepsilon} b_j^{\gamma_1} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}).$$

If $t > j+1$ then the process is iterated as follows: For $t = j+2$, $\alpha = qm_1+\gamma_1$, $b_j^{\alpha} = b_{j+1}^{pm_1} b_j^{\gamma_1}$ as before; and $pm_1 = qm_2+\gamma_2$, $0 \le \gamma_2 < |q|$, $b_j^{\alpha} = b_{j+2}^{pm_2} b_{j+1}^{\gamma_2} b_j^{\gamma_1}$, whence $W = b_{j+2}^{pm_2+\varepsilon} b_{j+1}^{\gamma_2} b_j^{\gamma_1} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$, $0 < \gamma_1 < |q|$, $0 \le \gamma_2 < |q|$. Similarly, for all $t = j+k > j$

$$b_j^{\alpha} = b_{j+k}^{pm_k} b_{j+k-1}^{\gamma_k} \cdots b_{j+1}^{\gamma_2} b_j^{\gamma_1}, \quad \gamma_1 \neq 0, \ 0 \le \gamma_i < |q|.$$

Hence $b_t^{\varepsilon} b_j^{\alpha} = b_{j+k}^{pm_k+\varepsilon} b_{j+k-1}^{\gamma_k} \cdots b_{j+1}^{\gamma_2} b_j^{\gamma_1}$, and

$N(W) \equiv \rho(b_{j+k}^{pm_k+\varepsilon} b_{j+k-1}^{\gamma_k} \cdots b_{j+1}^{\gamma_2} b_j^{\gamma_1}) \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ is the normal form of W.

For example: $p = 2$, $q = 3$, $W \equiv b_4^{-1} b_{-2}^{17} b_3^{1}$

$$\begin{aligned} b_j^{\alpha} &\equiv b_{-2}^{17} = b_{-2}^{3\cdot 5} b_{-2}^{2} = b_{-1}^{2\cdot 5} b_{-2}^{2} = b_{-1}^{3\cdot 3} b_{-1}^{1} b_{-2}^{2} \\ &= b_0^{2\cdot 3} b_{-1}^{1} b_{-2}^{2} = b_0^{3\cdot 2} b_0^{0} b_{-1}^{1} b_{-2}^{2} = b_1^{2\cdot 2} b_0^{0} b_{-1}^{1} b_{-2}^{2} \\ &= b_1^{3\cdot 1} b_1^{1} b_0^{0} b_{-2}^{1} b_{-2}^{2} = b_2^{2\cdot 1} b_1^{1} b_0^{0} b_{-1}^{1} b_{-2}^{2} \\ &= b_2^{2\cdot 0} b_2^{2} b_1^{1} b_0^{0} b_{-1}^{1} b_{-2}^{2} = b_4^{0} b_3^{0} b_2^{2} b_1^{1} b_0^{0} b_{-1}^{1} b_{-2}^{2} \end{aligned}$$

$$N(W) \equiv \rho(b_4^{-1} b_4^{0} b_3^{0} b_2^{2} b_1^{1} b_0^{0} b_{-1}^{1} b_{-2}^{2}) b_3^{1}$$

$$N(W) \equiv b_3^{-1} b_2^{2} b_1^{1} b_{-1}^{1} b_{-2}^{2} b_3^{1}.$$

Case C. $W \equiv b_t^{\varepsilon} b_j^{\alpha}$ or $W \equiv b_t^{\varepsilon} b_j^{\alpha} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$, and $t < j$. There are two cases to consider. They are:

C-1: $0 < \alpha < p$

C-2: $\alpha < 0$ or $\alpha \geq p$.

Case C-1. $t < j$ and $0 < \alpha < p$.

Then W is in normal form and $N(W) \equiv W$.

Case C-2. $t < j$ and $\alpha < 0$ or $\alpha \geq p$.

Let $\alpha \equiv \gamma_1 \bmod p$, that is $\alpha = pm_1 + \gamma_1$ with $0 \leq \gamma_1 < p$. Since U is in normal form, $m_1 \neq 0$. Using the relator $b^{pm_1} = b_{j-1}^{qm_1}$, one gets $b_j^{\alpha} = b_{j-1}^{qm_1} b_j^{\gamma_1}$, and $W = b_t^{\varepsilon} b_{j-1}^{qm_1} b_j^{\gamma_1} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ or $W = b_t^{\varepsilon} b_{j-1}^{qm_1} b_j^{\gamma_1}$.

There are two cases to consider. They are:

C-2-A: $\gamma_1 = 0$

C-2-B: $\gamma_1 \neq 0$.

Case C-2-A. $t < j$, $W = b_t^{\varepsilon} b_{j-1}^{qm_1} b_j^{\gamma_1} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ or $W = b_t^{\varepsilon} b_{j-1}^{qm_1} b_j^{\gamma_1}$ and $\gamma_1 = 0$. Hence p divided α hence $W = b_t^{\varepsilon} b_{j-1}^{qm_1}$ or $j < j_1$.

If $t = j-1$ then $W = b_{j-1}^{qm_1}$ and this is the normal form of W, or $W = b_{j-1}^{qm_1+\varepsilon} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ and this is the normal form of W.

$N(W) \equiv b_{j-1}^{qm_1+\varepsilon}$ or $N(W) \equiv b_{j-1}^{qm_1+\varepsilon} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$.

If $t < j-1$ then the process is iterated as follows:

For $t = j-2$, $\alpha = pm_1 + \gamma_1$, $b_j^{\alpha} = b_{j-1}^{qm_1}$ for $\gamma_1 = 0$; and $qm_1 = pm_2 + \gamma_2$, $0 \leq \gamma_2 < p$; $b_j^{\alpha} = b_{j-2}^{qm_2} b_{j-1}^{\gamma_2}$ whence

$$W = b_{j-2}^{qm_2+\varepsilon} b_{j-1}^{\gamma_2} \prod_{i=1}^{n} b_{j_i}^{\alpha_i} \text{ or } W = b_{j-2}^{qm_2+\varepsilon} b_{j-1}^{\gamma_2}, \; 0 \le \gamma_2 < p.$$

Similarly for all $t = j-k < j$

$$b_j^{\alpha} = b_{j-k}^{qm_k} b_{j-k+1}^{\gamma_k} \cdots b_j^{\gamma_2}, \; 0 \le \gamma_i < p.$$

Hence $b_t^{\varepsilon} b_j^{\alpha} = b_{j-k}^{qm_k+\varepsilon} b_{j-k+1}^{\gamma_k} \cdots b_j^{\gamma_2}$ and

$$N(W) \equiv \rho(b_{j-k}^{qm_k+\varepsilon} b_{j-k}^{\gamma_k} \cdots b_j^{\gamma_2}) \prod_{i=1}^{n} b_{j_i}^{\alpha_i} \text{ or } N(W) \equiv \rho(b_{j-k}^{qm_k+\varepsilon} b_{j-1}^{\gamma_k} \cdots b_j^{\gamma_2})$$

and this is the normal form of W.

<u>Case C-2-B.</u> $\gamma_1 \neq 0$, $t < j$, $W = b_t^{\varepsilon} b_{j-1}^{qm_1} b_j^{\gamma_1} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$.

Hence $0 < \gamma_1 < p$. If $t = j-1$ then $W = b_{j-1}^{qm_1+\varepsilon} b_j^{\gamma_1} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$

when this word is freely reduced one gets $\rho(b_{j-1}^{qm_1+\varepsilon} b_j^{\gamma_1} \prod_{i=1}^{n} b_{j_i}^{\alpha_i})$

and this is W in normal form. $N(W) \equiv \rho(b_{j-1}^{qm_1+\varepsilon} b_j^{\gamma_1} \prod_{i=1}^{n} b_{j_i}^{\alpha_i})$.

If $t < j-1$ then the process is iterated as follows:

For $t = j-2$, $\alpha = pm_1+\gamma_1$, $b_j^{\alpha} = b_{j-1}^{qm_1} b_j^{\gamma_1}$ as before; and

$qm_1 = pm_2+\gamma_2$, $0 \le \gamma_2 < p$, $b_j^{\alpha} = b_{j-2}^{qm_2} b_{j-1}^{\gamma_2} b_j^{\gamma_1}$, whence

$$W = b_{j-2}^{qm_2+\varepsilon} b_{j-1}^{\gamma_2} b_j^{\gamma_1} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}, \; 0 < \gamma_1 < p \text{ and } 0 \le \gamma_2 < p.$$

Similarly for all $t = j-k < j$, $b_j^{\alpha} = b_{j-k}^{qm_k} b_{j-k+1}^{\gamma_k} \cdots b_{j-1}^{\gamma_2} b_j^{\gamma_1}$,

$\gamma_1 \neq 0$, $0 \le \gamma_i < p$.

Hence $b_t^{\varepsilon} b_j^{\alpha} = b_{j-k}^{qm_k+\varepsilon} b_{j-k+1}^{\gamma_k} \cdots b_{j-1}^{\gamma_2} b_j^{\gamma_1}$, and

$N(W) \equiv \rho(b_{j-k}^{qm_k+\varepsilon} b_{j-k+1}^{\gamma_k} \cdots b_{j-1}^{\gamma_2} b_j^{\gamma_1}) \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ is the normal form of W.

For example: $p = 2$, $q = 3$, $W \equiv b_3^1 b_5^8 b_6^1 b_4^1$.

$W = b_3^1 b_4^{12} b_5^0 b_6^1 b_4^1 = b_3^1 b_3^{18} b_4^0 b_5^0 b_6^1 b_4^1$

$N(W) \equiv b_3^{19} b_6^1 b_4^1$.

Case D. $W \equiv b_t^{\varepsilon} b_j^{\alpha}$ or $W \equiv b_t^{\varepsilon} b_j^{\alpha} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ and $t = j$. There are three cases to consider. They are:

D-1: $W \equiv b_t^{\varepsilon} b_j^{\alpha}$

D-2: $j < j_1$

D-3: $j > j_1$.

Case D-1. $t = j$ and $W \equiv b_t^{\varepsilon} b_j^{\alpha}$.

There are two cases to consider. They are:

D-1-A: $\varepsilon + \alpha = 0$

D-1-B: $\varepsilon + \alpha \neq 0$.

Case D-1-A. $t = j$, $W \equiv b_t^{\varepsilon} b_j^{\alpha}$, and $\varepsilon + \alpha = 0$.

Then $W = 1$. 1 is the normal form of W. $N(W) \equiv 1$.

Case D-1-B. $t = j$, $W \equiv b_t^{\varepsilon} b_j^{\alpha}$, and $\varepsilon + \alpha \neq 0$.

Then there exist m and s such that $\alpha_1 + \varepsilon = q^m \cdot s$ where $s \neq 0$ mod $|q|$. Hence $W = b_t^{\alpha_1 + \varepsilon} = b_t^{q^m \cdot s} = b_{t+m}^{p^m \cdot s}$.

$N(W) \equiv b_{t+m}^{p^m \cdot s}$ and is the normal form of W.

For example: $W \equiv b_3^1 b_3^{-1}$, $N(W) \equiv 1$

and $W \equiv b_3^1 b_3^{-55}$, $p = 2$, $q = 3$, $W = b_3^{-54}$

$W = b_3^{-54} = b_3^{3^3(-2)} = b_{3+3}^{2^3(-2)} = b_6^{-16}$

$N(W) \equiv b_6^{-16}$.

Case D-2. $t = j$ and $j < j_1$.

There are three cases to consider. They are:

D-2-A: $\varepsilon + \alpha = 0$

D-2-B: $\varepsilon + \alpha \not\equiv 0 \bmod |q|$

D-2-C: $\varepsilon + \alpha \equiv 0 \bmod |q|$, $\alpha \neq -\varepsilon$.

<u>Case D-2-A</u>. $t = j$, $j < j_1$ and $\varepsilon + \alpha = 0$.

Then $W \equiv b_t^{\varepsilon} b_t^{-\varepsilon} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$.

Hence $W = \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$.

There are two cases to consider. They are:

D-2-A-1: $\prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ is in normal form.

D-2-A-2: $\prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ is not in normal form.

<u>Case D-2-A-1</u>. $t = j$, $j < j_1$, $\varepsilon + \alpha = 0$ and $\prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ is in normal form.

Then $\prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ is the normal form of W. $N(W) \equiv \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$.

<u>Case D-2-A-2</u>. $t = j$, $j < j_1$, $\varepsilon + \alpha = 0$ and $\prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ is not in normal form.

Then $0 < j_1 < p$ hence $j_1 < j_2$ and $\alpha_1 \equiv 0 \bmod |q|$.
$W = \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$.

Let $W' = b_{t'}^{1} b_k^{\beta} \prod_{i=1}^{m} b_{k_i}^{\beta_i}$ where $t' = j_1$, $k = j_1$, $\beta = \alpha_1 - 1$, $m = n-1$, $k_i = j_{i+1}$, $\beta_i = \alpha_{i+1}$, and $N(W) = N(W')$.

$b_k^{\beta} \prod_{i=1}^{m} b_{k_i}^{\beta_i}$ is in normal form and N(W) is found in a finite number of steps.

<u>For example</u>: $p = 4$, $q = 3$, $W \equiv b_2^{-1} b_2^{1} b_3^{3} b_4^{2}$,
$W = b_3^3 b_4^2 = b_3^1 b_3^2 b_4^2 = b_3^3 b_4^2 = b_4^4 b_4^2 = b_4^6 = b_4^{3\cdot 2} = b_5^{4\cdot 2} = b_5^8$.

<u>Case D-2-B.</u> $t = j$, $j < j_1$, and $\varepsilon + \alpha \not\equiv 0 \bmod |q|$.

Then $W = b_t^{\varepsilon+\alpha} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ and $N(W) \equiv b_t^{\varepsilon+\alpha} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ is the normal form of W.

<u>Case D-2-C.</u> $t = j$, $j < j_1$, $\varepsilon + \alpha \equiv 0 \bmod |q|$ and $\alpha \neq -\varepsilon$.

Then there exist m and s such that $\varepsilon + \alpha = q^m \cdot s$ where $s \not\equiv 0 \bmod |q|$. There are two cases to consider. They are:

D-2-C-1: $m < (j_1 - j)$

D-2-C-2: $m \geq (j_1 - j)$.

<u>Case D-2-C-1.</u> $t = j$, $j < j_1$, $\varepsilon + \alpha \equiv 0 \bmod |q|$, $\varepsilon + \alpha = q^m \cdot s$ where $s \not\equiv 0 \bmod |q|$, and $m < (j_1 - j)$.

Then $W = b_t^{q^m \cdot s} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$.

Hence $W = b_{t+m}^{p^m \cdot s} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ and $N(W) \equiv b_{t+m}^{p^m \cdot s} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ and is the normal form of W.

<u>Case D-2-C-2.</u> $t = j$, $j < j_1$, $\varepsilon + \alpha \equiv 0 \bmod |q|$, $\varepsilon + \alpha = q^m \cdot s$ where $s \not\equiv 0 \bmod |q|$, and $m \geq (j_1 - j_2)$.

Then there exists an x such that $\alpha + \varepsilon = q^{j-j_1} \cdot x$.

Hence $W = b_j^{q^{j_1 - j} \cdot x} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$, $j < j_1$ implies $0 < \alpha_1 < p$.

Hence $W = b_{j_1}^{p^{j_1 - j} \cdot x + \alpha_1} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$.

There are three cases to consider. They are:

D-2-C-2-A: $W = b_{j_1}^{p^{j_1-j}\cdot x+\alpha_1}$, (n=1)

D-2-C-2-B: $j_1 > j_2$

D-2-C-2-C: $j_1 < j_2$

Case D-2-C-2-A. $W = b_{j_1}^{p^{j_1-j}\cdot x+\alpha_1}$, $n = 1$.

Then there exist k and h such that $p^{j_1-j}\cdot x+\alpha_1 = q^k\cdot h$ where $h \not\equiv 0 \bmod |q|$.

Hence $W = b_{j_1}^{q^k\cdot h} = b_{j_1+k}^{p^k\cdot h}$ and $N(W) \equiv b_{j_1+k}^{p^k\cdot h}$ and is the normal form of W.

For example: $p = 2$, $q = 3$, $W \equiv b_2^{-1}b_2^{19}b_3^1$

$W = b_2^{18}b_3^1 = b_2^{3^1\cdot 6}b_3^1 = b_3^{2^1\cdot 6}b_3^1 = b_3^{13}$.

For example: $W \equiv b_2^{-1}b_2^{19}b_4^1$, $p = 2$, $q = 3$

$W = b_2^{18}b_4^1 = b_2^{3^2\cdot 2}b_4^1 = b_4^{2^2\cdot 2}b_4^1 = b_4^9 = b_4^{3^2} = b_6^{2^2} = b_6^4$.

Case D-2-C-2-B. $W = b_{j_1}^{p^{j_1-j}\cdot x+\alpha_1} \prod_{i=2}^{n} b_{j_i}^{\alpha_i}$, $0 < \alpha_1 < p$ and $j_1 > j_2$.

Then $N(W) \equiv b_{j_1}^{p^{j_1-j}\cdot x+\alpha_1} \prod_{i=2}^{n} b_{j_i}^{\alpha_i}$ and is the normal form of W.

Case D-2-C-2-C. $W = b_{j_1}^{p^{j_1-j}\cdot x+\alpha_1} \prod_{i=2}^{n} b_{j_i}^{\alpha_i}$, $0 < \alpha_1 < p$ and $j_1 < j_2$.

Then there are two cases to consider. They are:

D-2-C-2-C-1: $p^{j_1-j}\cdot x+\alpha_1 \not\equiv 0 \bmod |q|$.

D-2-C-2-C-2: $p^{j_1-j} \cdot x+\alpha_1 \equiv 0 \bmod |q|$.

<u>Case D-2-C-2-C-1.</u> $W = b_{j_1}^{p^{j_1-j} \cdot x+\alpha_1} \prod_{i=2}^{n} b_{j_i}^{\alpha_i}$, $0 < \alpha_1 < p$, $j_1 < j_2$ and $p^{j_1-j} \cdot x+\alpha_1 \not\equiv 0 \bmod |q|$.

Then $N(W) \equiv b_{j_1}^{p^{j_1-j} \cdot x+\alpha_1} \prod_{i=2}^{n} b_{j_i}^{\alpha_i}$ and is the normal form of W.

<u>Case D-2-C-2-C-2.</u> $W = b_{j}^{p^{j_1-j} \cdot x+\alpha_1} \prod_{i=2}^{n} b_{j_i}^{\alpha_i}$, $0 < \alpha_1 < p$, $j_1 < j_2$ and $p^{j_1-j} \cdot x+\alpha_1 \equiv 0 \bmod |q|$.

Let $p^{j_1-j} \cdot x+\alpha_1 = z+1$.

Then $W = b'_{j_1} b_{j_1}^{z} \prod_{i=2}^{n} b_{j_i}^{\alpha_i}$ and $b_{j_1}^{z} \prod_{i=2}^{n} b_{j_i}^{\alpha_i}$ is in normal form.

Let $W' = b'_{t'} b_{j'}^{\alpha'} \prod_{i=1}^{n'} b_{j_i'}^{\alpha_i'}$

where $t' = j_1$

$j' = j_1$

$\alpha' = z$

$n' = n-1$

$j_i' = j_{i+1}$

and $\alpha_i' = \alpha_{i+1}$

$N(W) \equiv N(W')$.

Clearly in every case the normal form N(W) is found in a finite number of steps.

For example: $W \equiv b_3^1 b_3^{26} b_6^1 b_7^1$, $p = 2$, $q = 3$.

$W = b_3^{3^3} b_6^1 b_7^1 = b_6^{2^3} b_6^1 b_7^1 = b_6^9 b_7^1$

$N(W) = N(b_6^1 b_6^8 b_7^1)$

$W' \equiv b_6^1 b_6^8 b_7^1 = b_6^{3^2 \cdot 1} b_7^1 = b_6^{3^1 \cdot 3} b_7^1 = b_7^{2^1 \cdot 3} b_7^1 = b_7^7$

$N(W') \equiv b_7^7$

$N(W) \equiv N(W') \equiv b_7^7$.

Case D-3. $t = j$ and $j > j_1$.

There are three cases to consider. They are:

D-3-A: $\varepsilon + \alpha = 0$

D-3-B: $\varepsilon + \alpha \not\equiv 0 \mod p$

D-3-C: $\varepsilon + \alpha \equiv 0 \mod p$, $\alpha \not\equiv -\varepsilon$.

Case D-3-A. $t = j$, $j > j_1$, and $\varepsilon + \alpha = 0$.

Then $W \equiv b_t^{\varepsilon} b_t^{-\varepsilon} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$.

Hence $W = \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ and there are two cases to consider. They are:

D-3-A-1: $\prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ is in normal form.

D-3-A-2: $\prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ is not in normal form.

Case D-3-A-1. $\prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ is in normal form.

Then $N(W) \equiv \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ and is the normal form of W.

Case D-3-A-2. $\prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ is not in normal form, $j > j_1$ implies $0 < \alpha_1 < |q|$.

Hence $j_1 > j_2$ and $\alpha_1 \equiv 0 \mod p$.

Let $W' \equiv b_{j'}^{\varepsilon'} b_{j_1'}^{\alpha_1'} \prod_{i=1}^{n'} b_{j_i'}^{\alpha_i'}$

where $\varepsilon' = 1$

$j' = j_1' = j_1$

$\alpha_1' = \alpha_1 - 1$

$n' = n-1$

$j_i' = j_{i+1}$

$\alpha_i' = \alpha_{i+1}$

$W = W'$ and $b_{j_1'}^{\alpha_1'} \prod_{i=1}^{n'} b_{j_i'}^{\alpha_i'}$ is in normal form.

$N(W) \equiv N(W')$ and again $N(W)$ is found in a finite number of steps.

Case D-3-B. $t = j$, $j > j_1$ and $\varepsilon+\alpha \not\equiv 0 \mod p$.

Then $W = b_t^{\alpha+\varepsilon} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ and $N(W) \equiv b_t^{\alpha+\varepsilon} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ and is the normal form of W.

Case D-3-C. $t = j$, $j > j_1$, $\varepsilon+\alpha \equiv 0 \mod p$ and $\alpha \neq -\varepsilon$.

Then there exist m and s such that $\varepsilon+\alpha = p^m \cdot s$ where $s \not\equiv 0 \mod p$.

There are two cases to consider. They are:

D-3-C-1: $m < (j-j_1)$

D-3-C-2: $m \geq (j-j_1)$.

<u>Case D-3-C-1.</u> $t = j,\ j > j_1,\ \varepsilon+\alpha \equiv 0 \bmod p,\ \alpha \not\equiv -\varepsilon,$ $\varepsilon+\alpha = p^m \cdot s,\ s \not\equiv 0 \bmod p$ and $m < (j-j_1)$.

Then $W = b_t^{p^m \cdot s} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$, hence $W = b_{t-m}^{q^m \cdot s} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ and $N(W) \equiv b_{t-m}^{q^m \cdot s} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ and is the normal form of W.

<u>Case D-3-C-2.</u> $t = j,\ j > j_1,\ \varepsilon+\alpha = 0 \bmod p,\ \varepsilon+\alpha = p^m \cdot s$ where $s \not\equiv 0 \bmod p$, and $m \geq (j-j_1)$.

Then there exists an x such that $\alpha+\varepsilon = p^{j-j_1} \cdot x$.

Hence $W = b_j^{p^{j-j_1} \cdot x} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ which implies $0 < \alpha_1 < |q|$ for $j > j_1$.

Hence $W = b_j^{q^{j-j_1} \cdot x} \prod_{i=1}^{n} b_{j_i}^{\alpha_i} = b_j^{q^{j-j_1} \cdot x+\alpha_1} \prod_{i=2}^{n} b_{j_i}^{\alpha_i}$.

There are three cases to consider. They are:

D-3-C-2-A: $W = b_{j_1}^{q^{j-j_1} \cdot x+\alpha_1}$, $(n=1)$

D-3-C-2-B: $j_1 < j_2$

D-3-C-2-C: $j_1 > j_2$.

<u>Case D-3-C-2-A.</u> $W = b_{j_1}^{q^{j-j_1} \cdot x+\alpha_1}$.

Then $N(W) \equiv b_{j_1}^{q^{j-j_1} \cdot x+\alpha_1}$ and is the normal form of W.

<u>Case D-3-C-2-B.</u> $W = b_{j_1}^{q^{j-j_1} \cdot x+\alpha_1} \prod_{i=2}^{n} b_{j_i}^{\alpha_i}$, $0 < \alpha_1 < |q|$ and $j_1 < j_2$.

Then $N(W) \equiv b_{j_1}^{q^{j-j_1} \cdot x+\alpha_1} \prod_{i=2}^{n} b_{j_i}^{\alpha_i}$ and is the normal form of W.

<u>Case D-3-C-2-C.</u> $W = b_{j_1}^{q^{j-j_1}\cdot x+\alpha_1} \prod_{i=2}^{n} b_{j_i}^{\alpha_i}$, $0 < \alpha_1 < |q|$ and $j_1 > j_2$.

Then there are two cases to consider. They are:

D-3-C-2-C-1: $q^{j-j_1}\cdot x+\alpha_1 \not\equiv 0 \bmod p$

D-3-C-2-C-2: $q^{j-j_1}\cdot x+\alpha_1 \equiv 0 \bmod p$.

<u>Case D-3-C-2-C-1.</u> $W = b_{j_1}^{q^{j-j_1}\cdot x+\alpha_1} \prod_{i=2}^{n} b_{j_i}^{\alpha_i}$, $0 < \alpha_1 < |q|$, $j_1 > j_2$ and $q^{j-j_1}\cdot x+\alpha_1 \not\equiv 0 \bmod p$.

Then $N(W) \equiv b_{j_1}^{q^{j-j_1}\cdot x+\alpha_1} \prod_{i=2}^{n} b_{j_i}^{\alpha_i}$ and is the normal form of W.

<u>Case D-3-C-2-C-2.</u> $W = b_{j_1}^{q^{j-j_1}\cdot x+\alpha_1} \prod_{i=2}^{n} b_{j_i}^{\alpha_i}$, $0 < \alpha_1 < |q|$, $j_1 > j_2$ and $q^{j-j_1}\cdot x+\alpha_1 \equiv 0 \bmod p$.

Let $z+1 = q^{j-j_1}\cdot x+\alpha_1$.

Hence $W = b_{j_1}^{1} b_{j_1}^{z} \prod_{i=2}^{n} b_{j_i}^{\alpha_i}$ and $b_{j_1}^{z} \prod_{i=2}^{n} b_{j_i}^{\alpha_i}$ is in normal form.

Let $W' \equiv b_{t'}^{\prime} b_{j'}^{\alpha'} \prod_{i=1}^{n'} b_{j_i'}^{\alpha_i'}$

where $t' = j_1$;

$j' = j_1$;

$\alpha' = z$;

$n' = n-1$;

$j_i' = j_{i+1}$;

and $\alpha_i' = \alpha_{i+1}$.

$W = W'$.

$N(W) = N(W')$.

Again the normal form $N(W)$ is found in a finite number of steps.

This covers every word of the form $W = b_t^{\varepsilon}U$, $\varepsilon = \pm 1$, with U already in normal form.

To find the normal form $N(W)$ of any word $W \in H_b$ one successively normalizes its terminal segment.

SECTION III

THE UNIQUENESS OF THE NORMAL FORM

Theorem 9. If $W \in H_{p,q}$, $(p,q) = 1$, $p > 1$, $|q| \neq 1$, then W and N(W) represent the same word in $H_{p,q}$. $[W \underset{H_{p,q}}{=} N(W)]$.

Proof. This is immediate from the definition of N(W).

Definition 17. $K_t = \langle \{x_i : i \in J, i < t\} \mid \{x_{i-1}^q \bar{x}_i^p : i \in J, i < t\} \rangle$.

Definition 18. $T_t = \langle \{x_i : i \in J, i \geq t\} \mid \{x_i^q \bar{x}_{i+1}^p : i \in J, i \geq t\} \rangle$.

Definition 19. A_t the subgroup of K_t generated by x_{t-1}^q.

Definition 20. B_t the subgroup of T_t generated by x_t^p.

Definition 21. $L_t : A_t \to B_t$ under $L_t(x_{t-1}^q) = x_t^p$.

[Note. In Definitions 17-21, it is understood that p and q are fixed, otherwise one would write $K_{(t,p,q)}$ instead of K_t.]

Theorem 10. For a fixed p,q where $(p,q) = 1$, $p > 1$, $|q| \neq 1$ and for any $t \in J$, $M_{p,q} = (K_t, T_t, A_t, B_t, W = L_t(W))$ is the free product of K_t and T_t with A_t and B_t amalgamated.

Proof. $M_{p,q} = \langle x_i : i \in J \mid x_{i-1}^q \bar{x}_i^p : i \in J \rangle$

$K_t = \langle x_i : i \in J, i < t \mid x_i^q \bar{x}_{i+1}^p : i \in J, i < t \rangle$ is the subgroup of $M_{p,q}$ generated by x_i, $i < t$.

$T_t = \langle x_i : i \in J, i \geq t \mid x_i^q \bar{x}_{i+1}^p : i \in J, i \geq t \rangle$ is the subgroup of $M_{p,q}$ generated by x_i, $i \geq t$.

A_t is the subgroup of K_t generated by x^q_{t-1}.

B_t is the subgroup of T_t generated by x^p_t.

K_t and T_t are disjoint and L_t is an isomorphism between A_t and B_t.

q.e.d.

Let $M_{p,q} = \langle x_i : i \in J \mid x_i^q x_{i+1}^{-p} : i \in J\rangle$

$F_b = \langle b_i : i \in J \mid \rangle$

$H_{p,q} = \langle b_i : i \in J \mid b_i^q b_{i+1}^{-p} : i \in J\rangle$

$P_{p,q} : F_b \to M_{p,q}$ under $P(b_j) = x_j$ and $P(1) = 1$.

Theorem 11. If $W, W' \in F_b$ then the following three statements are equivalent.

(1) $P(W) \underset{M_{p,q}}{=} P(W')$

(2) $N(W) \underset{H_{p,q}}{=} N(W')$

(3) $W \underset{H_{p,q}}{=} W'$

Proof. (1) $\Leftrightarrow$ (3) is obvious. (1) and (3) are equivalent under a change of notation.

(2) $\Leftrightarrow$ (3) By Theorem 9 $N(W) = W$ in $H_{p,q}$. Hence $W \underset{H_{p,q}}{=} W' \Leftrightarrow N(W) = N(W')$ so (2) and (3) are equivalent. This proves Theorem 11.

Theorem 12. For any fixed p,q with $(p,q) = 1$, $p > 1$, $|q| \neq 1$.

If i) W is in normal form, and

ii) b_t^γ is in normal form or $\gamma = 0$, then $W \underset{H_{p,q}}{=\!=} b_t^{\alpha}$ if and only if

iii) $W \equiv 1$ and $\gamma = 0$, or

iv) $W \equiv b_t^{\gamma}$.

Proof. The proof follows in the form of five lemmas.

Lemma 12-1. Suppose

i) $W \equiv 1$, and

ii) b_t^{γ} is in normal form or $\gamma = 0$.

Then $W \underset{H_{p,q}}{=\!=} b_t^{\gamma}$ if and only if $\gamma = 0$.

Proof of Lemma 12-1. Let $W \equiv 1$, and b_t^{γ} is in normal form or $\gamma = 0$, and $W \underset{H_{p,q}}{=\!=} b_t^{\gamma}$.

Then
$$1 \underset{H_{p,q}}{=\!=} b_t^{\gamma}$$
$$P(1) \underset{M_{p,q}}{=\!=} P(b_t^{\gamma})$$
$$1 \underset{M_{p,q}}{=\!=} x_t^{\gamma}$$

by Theorem 7 $\psi(1) = \psi(x_t^{\gamma})$
$$0 = \gamma(\tfrac{q}{p})^t, \quad \psi(x_t^{\varepsilon}) = \varepsilon(\tfrac{q}{p})^t$$
So $\gamma = 0$.

Conversely, if $\gamma = 0$ then $b_t^{0} \underset{H_{p,q}}{=\!=} 1 \equiv W$.

Lemma 12.2. Suppose

i) $W \equiv b_j^{\alpha}$ is in normal form, and

ii) b_t^{γ} is in normal form or $\gamma = 0$.

Then $W \underset{H_{p,q}}{=\!=} b_t^{\gamma}$ if and only if $W \equiv b_t^{\gamma}$.

Proof of Lemma 12.2. $W \equiv b_j^{\alpha}$ is in normal form implies $\alpha \not\equiv 0 \bmod |q|$ and b_t^{γ} in normal form implies $\gamma \not\equiv 0 \bmod |q|$.

If $W \underset{H_{p,q}}{=} b_t^{\gamma}$

then $b_j^{\alpha} \underset{H_{p,q}}{=} b_t^{\gamma}$

$\alpha \neq 0$ as W is in normal form.

$$P(b_j^{\alpha}) \underset{M_{p,q}}{=} P(b_t^{\gamma})$$

$$x_j^{\alpha} \underset{M_{p,q}}{=} x_t^{\gamma}$$

$$\psi(x_j^{\alpha}) = \psi(x_t^{\gamma})$$

$$\alpha(\tfrac{q}{p})^{j} = \gamma(\tfrac{q}{p})^{t}.$$

Now $\alpha\ q^j p^t = \gamma\ q^t p^j$

$\alpha \neq 0$ implies $\gamma \neq 0$.

Moreover $\alpha \not\equiv 0 \bmod |q|$, $\gamma \not\equiv 0 \bmod |q|$.

Therefore as $(p,q) = 1$

$$t = j$$

$$\alpha = \gamma$$

and $W \equiv b_t^{\gamma}$.

The converse is trivial! If $W \equiv b_t^{\gamma}$ then $W \underset{H_{p,q}}{=} b_t^{\gamma}$.

Lemma 12.3. Let $N(b_t^{\gamma}) \equiv b_t^{\gamma}$ or $\gamma = 0$ and $N(W) \equiv b_j^{\alpha} b_{j_1}^{\alpha_1}$, $j \neq j_1$.

Then $W \underset{H_{p,q}}{\neq} b_t^{\gamma}$.

Proof of Lemma 12.3. Let $N(b_t^{\gamma}) \equiv b_t^{\gamma}$ or $\gamma = 0$. b_t^{γ} is in normal form iff $\gamma \not\equiv 0 \bmod |q|$. $W \equiv b_j^{\alpha} b_{j_1}^{\alpha_1}$, $j \neq j_1$ is in normal form.

Then for $j < j_1$, $\alpha \not\equiv 0 \bmod |q|$, and $0 < \alpha_1 < p$ while for $j > j_1$, $\alpha \not\equiv 0 \bmod p$, and $0 < \alpha_1 < |q|$. (Definition 16).

Suppose to the contrary that $W \underset{H_{p,q}}{=} b_t^\gamma$, $W \equiv b_j^\alpha b_{j_1}^{\alpha_1}$ with $j \neq j_1$ and W is in normal form, and b_t^γ is in normal form or $\gamma = 0$.

Then $b_j^\alpha b_{j_1}^{\alpha_1} \underset{H_{p,q}}{=} b_t^\gamma$

$$P(b_j^\alpha b_{j_1}^{\alpha_1}) \underset{M_{p,q}}{=} P(b_t^\gamma)$$

$$x_j^\alpha x_{j_1}^{\alpha_1} \underset{M_{p,q}}{=} x_t^\gamma$$

$$\psi(x_j^\alpha x_{j_1}^{\alpha_1}) = \psi(x_t^\gamma)$$

$$\alpha(\tfrac{q}{p})^j + \alpha_1(\tfrac{q}{p})^{j_1} = \gamma(\tfrac{q}{p})^t.$$

$$(*) \quad \alpha\, q^j p^{j_1+t} + \alpha_1 q^{j_1} p^{j+t} = \gamma\, q^t p^{j+j_1}$$

It will be shown that (*) is impossible.

Case A. If $j < j_1$, $j < t$ then in (*) q is a divisor of $\alpha\, p^{j_1+t}$, $(p,q) = 1$, hence $q|\alpha$.

But W is in normal form and as $j < j_1, \alpha \not\equiv \bmod |q|$ - a contradiction.

Case B. If $j > j_1$ and $t > j_1$, then in (*) q is a divisor of $\alpha_1 p^{j+t}$, $(p,q) = 1$ hence $q|\alpha_1$. But W is in normal form so $\alpha_1 \not\equiv 0 \bmod |q|$ as $j > j_1$ - a contradiction.

Case C. If $t < j$ and $t < j_1$, then in (*) q is a divisor of

γp^{j+j_1}. $(p,q) = 1$ hence $q|\gamma$. But b_t^{γ} is in normal form or $\gamma = 0$.

Hence $\gamma = 0$

and $b_j^{\alpha} b_{j_1}^{\alpha_1} \underset{H_{p,q}}{=} 1$

$P(b_j^{\alpha} b_{j_1}^{\alpha_1}) \underset{M_{p,q}}{=} P(1)$

$x_j^{\alpha} x_{j_1}^{\alpha_1} \underset{M_{p,q}}{=} 1$

$\psi(x_j^{\alpha} x_{j_1}^{\alpha_1}) = \psi(1)$

$\alpha(\frac{q}{p})^{j} + \alpha_1(\frac{q}{p})^{j_1} = 0$

$\alpha\ q^{j} p^{j_1} + \alpha_1 q^{j_1} p^{j} = 0.$

If $j < j_1$ then $q|\alpha\ p^{j_1}$, $(p,q) = 1$ hence $q|\alpha$. But $j < j_1$ and W is in normal form hence $\alpha \not\equiv 0 \bmod |q|$ - a contradiction.

If $j > j_1$ then $p|\alpha\ q^{j}$, $(p,q) = 1$ hence $p|\alpha$. But $j > j_1$ and W is in normal form hence $\alpha \not\equiv 0 \bmod p$ - a contradiction.

<u>Case D</u>. $t = j_1$

$$b_j^{\alpha} b_{j_1}^{\alpha_1} \underset{H_{p,q}}{=} b_{j_1}^{\gamma}$$

$$b_j^{\alpha} \underset{H_{p,q}}{=} b_{j_1}^{\gamma-\alpha_1}.$$

If $\gamma-\alpha_1 = 0$ then $b_j^{\alpha} \underset{H_{p,q}}{=} b_{j_1}^{0}$

$$P(b_j^{\alpha}) \underset{M_{p,q}}{=} P(b_{j_1}^{0})$$

$$x_j^{\alpha} \underset{M_{p,q}}{=} x_{j_1}^{0}$$

$$\psi(x_j^{\alpha}) = \psi(x_{j_1}^{0})$$

$$\alpha(\tfrac{q}{p})^{j} = 0$$

$$\alpha = 0 \text{ - a contradiction.}$$

If $\gamma-\alpha_1 \neq 0$ then $b_j^{\alpha} \underset{H_{p,q}}{=} b_{j_1}^{\gamma-\alpha_1}$, $\alpha \neq 0$, $\gamma-\alpha_1 \neq 0$

$$P(b_j^{\alpha}) \underset{M_{p,q}}{=} P(b_{j_1}^{\gamma-\alpha_1})$$

$$x_j^{\alpha} \underset{M_{p,q}}{=} x_{j_1}^{\gamma-\alpha_1}$$

$$\alpha(\tfrac{q}{p})^{j} = (\gamma-\alpha_1)(\tfrac{q}{p})^{j_1}$$

$$\alpha\, q^{j}p^{j_1} = (\gamma-\alpha_1)q^{j_1}p^{j}.$$

If $j < j_1$ then $q|\alpha p^{j_1}$, $(p,q) = 1$ hence $q|\alpha$. But $j < j_1$ and W is in normal form implies $\alpha \not\equiv 0 \bmod |q|$ - a contradiction.

If $j > j_1$ then $p|\alpha\, a^{j}$, $(p,q) = 1$ hence $p|\alpha$. But $j > j_1$ and W is in normal form implies $\alpha \not\equiv 0 \bmod p$ - a contradiction.

<u>Case E</u>. $t = j$

$$b_j^{\alpha}b_{j_1}^{\alpha_1} \underset{H_{p,q}}{=} b_j^{\gamma}$$

$$b_{j_1}^{\alpha_1} \underset{H_{p,q}}{=} b_j^{\gamma-\alpha}$$

$$P(b_{j_1}^{\alpha_1}) \underset{M_{p,q}}{=} P(b_j^{\gamma-\alpha})$$

$$x_{j_1}^{\alpha_1} \underset{M_{p,q}}{=} x_j^{\gamma-\alpha}$$

$$\psi(x_{j_1}^{\alpha_1}) = \psi(x_j)^{\gamma-\alpha}$$

$$\alpha_1(\tfrac{q}{p})^{j_1} = (\gamma-\alpha)(\tfrac{q}{p})^{j}.$$

If $\gamma-\alpha = 0$ then $\alpha_1 = 0$, but $W \equiv b_j^{\alpha} b_{j_1}^{\alpha_1}$ is in normal form and $\alpha_1 \neq 0$ - a contradiction.

If $\gamma-\alpha \neq 0$ then $\alpha_1 p^j q^{j_1} = (\gamma-\alpha) p^{j_1} q^j$.

If $j < j_1$ then $p | \alpha_1 q^{j_1}$ but $(p,q) = 1$ implies $p | \alpha_1$ but $b_j^{\alpha} b_{j_1}^{\alpha_1}$ is in normal form and $j < j_1$, hence $0 < \alpha_1 < p$ - a contradiction.

If $j > j_1$ then $q | \alpha_1 p^j$ but $(p,q) = 1$ implies $q | \alpha_1$ but $b_j^{\alpha} b_{j_1}^{\alpha_1}$ is in normal form and $j > j_1$, hence $0 < \alpha_1 < |q|$ - a contradiction.

This proves Lemma 12.3.

<u>Lemma 12.4</u>. If $W \equiv b_j^{\alpha} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ is in normal form, then $\prod_{i=k}^{m} b_{j_i}^{\alpha_i}$ is in normal form where $1 \leq k \leq m \leq n$ whenever: (1) $k = 1$, $j < j_1$ and $p < |q|$, or

(2) $k = 1$, $j > j_1$ and $p > |q|$, or

(3) $k \neq 1$, $j_{k-1} < j_k$ and $p < |q|$, or

(4) $k \neq 1$, $j_{k-1} > j_k$ and $p > |q|$.

<u>Proof</u>. If $W \equiv b_j^{\alpha} \prod_{i=1}^{n} b_{j_i}^{\alpha_i}$ is in normal form, then:

i) if $j < j_1$, then $\alpha \not\equiv 0 \bmod |q|$ and $0 < \alpha_1 < p$, and

ii) if $j > j_1$, then $\alpha \not\equiv 0 \bmod p$ and $0 < \alpha_1 < |q|$, and

iii) if $j_i > j_{i-1}$ then $0 < \alpha_i < p$, and

iv) if $j_i < j_{i-1}$ then $0 < \alpha_i < |q|$.

Let $W' \equiv \prod_{i=k}^{m} b_{j_i}^{\alpha_i}$ where $1 \leq k \leq m \leq n$.

Hence by the definition of being in normal form, the only conditions to check to see if W' is in normal form are:

a) if $j_k < j_{k+1}$ is $\alpha_k \neq 0 \bmod |q|$

b) if $j_k > j_{k+1}$ is $\alpha_k \neq 0 \bmod p$

and c) if $k = m$ is $\alpha_k \neq 0 \bmod |q|$.

Now if (1): $k = 1$, $j < j_1$, and $p < |q|$ then $0 < \alpha_k = \alpha_1 < p < |q|$ and $\alpha_k \neq 0 \bmod p$, $\alpha_k \neq 0 \bmod |q|$.

Or if (2): $k = 1$, $j > j_1$ and $p > |q|$ then $0 < \alpha_k = \alpha_1 < |q| < p$ and $\alpha_k \neq 0 \bmod p$, $\alpha_k \neq 0 \bmod |q|$.

Or if (3): $k \neq 1$, $j_{k-1} < j_k$ and $p < |q|$ then $0 < \alpha_k < p < |q|$ and $\alpha_k \neq 0 \bmod p$, $\alpha_k \neq 0 \bmod |q|$.

Or if (4): $k \neq 1$, $j_{k-1} > j_k$ and $p > |q|$ then $0 < \alpha_k < |q| < p$ and $\alpha_k \neq 0 \bmod p$, $\alpha_k \neq 0 \bmod |q|$.

Hence if condition (1), (2), (3) or (4) are met, then W' is in normal form.

This proves Lemma 12.4.

<u>Lemma 12.5</u>. Let $W \equiv b_j^{\alpha} \prod_{i=1}^{v} b_{j_i}^{\alpha_i}$, $j \neq j_1$, $j_i \neq j_{i-1}$ be in normal form, and let b_t^{γ} be in normal form or $\gamma = 0$. Then $W \underset{H_{p,q}}{\neq} b_t^{\gamma}$ for $v \geq 2$.

<u>Proof of Lemma 12.5</u>. Induction hypothesis: $W \underset{H_{p,q}}{\neq} b_t^{\gamma}$ for $1 \leq v < n$. Assume the contrary the induction hypothesis holds and $W \underset{H_{p,q}}{=} b_t^{\gamma}$, $v = n$

$$b_j^{\alpha} \prod_{i=1}^{n} b_{j_i}^{\alpha_i} \underset{H_{p,q}}{=} b_t^{\gamma}.$$

There are two cases to consider. They are:

Case 1. $\gamma = 0$

Case 2. $\gamma \neq 0$.

Case 1. If $\gamma = 0$, then

$$b_j^{\alpha} \prod_{i=1}^{n} b_{j_i}^{\alpha_i} \underset{H_{p,q}}{=} 1 \qquad n > 1$$

$$b_j^{\alpha} \prod_{i=1}^{n-1} b_{j_i}^{\alpha_i} \underset{H_{p,q}}{=} b_{j_n}^{-\alpha_n}$$

Let $-\alpha_n = q^x \cdot s$, $s \not\equiv o \bmod |q|$.

Then $b_j^{\alpha} \prod_{i=1}^{n-1} b_{j_i}^{\alpha_i} \underset{H_{p,q}}{=} b_{j_n+s}^{p^x \cdot s}$ and $b_j^{\alpha} \prod_{i=1}^{n-1} b_{j_i}^{\alpha_i}$ is in normal form and $b_{j_n+s}^{p^x \cdot s}$ is in normal form.

Hence by the induction hypothesis

$$b_j^{\alpha} \prod_{i=1}^{n-1} b_{j_i}^{\alpha_i} \underset{H_{p,q}}{=} b_{j_n+s}^{p^x \cdot s} \text{ - a contradiction.}$$

Case 2. If $\gamma \not\equiv 0$ then

$$b_j^{\alpha} \prod_{i=1}^{n} b_{j_i}^{\alpha_i} \underset{H_{p,q}}{=} b_t^{\gamma}, \quad \gamma \not\equiv 0 \bmod |q|.$$

There exist s and x such that $j, j_i \in [s,x]$.

By Theorem 10, for any z

$M_{p,q} = (K_z, T_z, A_z, B_z, W = L_z(W)))$.

There are two cases to consider. They are

2A: $p < |q|$

2B: $p > |q|$.

Case 2A. If $p < |q|$ then set

$M_{p,q}' = (K_{s+1}, T_{s+1}, A_{s+1}, B_{s+1}, W = L_{s+1}(W))$.

Now $b_j^{\alpha} \in K_{s+1}$ if and only if $j = s$ and $b_{j_i}^{\alpha_i} \in K_{s+1}$ if and only if $j_i = s$. As $j, j_i \in [s,x]$.

There exist $U_1, U_2, \ldots, U_r$; $V_1, V_2, \ldots, V_r$ such that

$W \equiv U_1 V_1 U_2 V_2, \ldots, U_r V_r$

where $U_h \in K_{s+1}$, $V_h \in T_{s+1}$, $1 \le h \le r$; and

$V_h \not\equiv 1$ if $h < r$; and

$U_h \not\equiv 1$ if $1 < h$; and

for some h, $U_h \not\equiv 1$; and

for some k, $V_k \not\equiv 1$.

Also $b_t^{\gamma} \in K_{s+1}$ or $b_t^{\gamma} \in T_{s+1}$.

Hence for some h either $U_h \in A_{s+1}$ and $U_h \not\equiv 1$; or $V_h \in B_{s+1}$ and $V_h \not\equiv 1$.

But if $U_1 \not\equiv 1$ then $U_1 \equiv b_j^{\alpha}$, $j = s$, $\alpha \not\equiv 0 \bmod |q|$ hence $U_1 \notin A_{s+1}$.

And if $h > 1$ then $U_h \equiv b_s^{\beta_h}$ where $0 < \beta_h < |q|$ hence $U_h \notin A_{s+1}$.

And if $U_1 \equiv 1$ then $V_1 \equiv b_j^{\alpha}$ or $V_1 \equiv b_j^{\alpha} \prod_{i=1}^{n_1} b_{j_i}^{\alpha_i}$ and V_1 is in normal form hence $V_1 \underset{H_{p,q}}{\not\equiv} b_{s+1}^{p \cdot m}$ by the induction hypothesis as the normal form of $b_{s+1}^{p \cdot m}$ is of the form b_v^{δ}.

And if $U_1 \not\equiv 1$ then $V_1 \equiv \prod_{i=k_1}^{n_1} b_{j_i}^{\alpha_i}$ and V_1 is in normal form and $V_1 \underset{H_{p,q}}{\not\equiv} b_{s+1}^{p \cdot m}$.

And for $h > 1$ if $V_h \not\equiv 1$ then $V_h \equiv \prod_{i=k_h}^{n_h} b_{j_i}^{\alpha_i}$ and is in normal form and $V_h \underset{H_{p,q}}{\not\equiv} b_{s+1}^{p \cdot m}$.

Therefore for all h where $U_h \not\equiv 1$, $U_h \notin A_{s+1}$; and for all h where $V_h \neq 1$, $V_h \notin B_{s+1}$ - a contradiction.

Case 2B. If $p > |q|$ then set

$M_{p,q} = (K_x, T_x, A_x, B_x, W = L_x(W))$.

Now $b_j^{\alpha} \in T_x$ if and only if $j = x$ and $b_{j_1}^{\alpha_1} \in T_x$ if and only if $j_1 = x$ as $j, j_1 \in [s,x]$.

There exist $U_1, U_2, \ldots, U_r; V_1, V_2, \ldots, V_r$ such that

$W \equiv U_1V_1U_2V_2, \ldots, U_rV_r$

where $U_h \in T_x$, $V_h \in K_x$, $1 \leq h \leq r$; and

$U_h \neq 1$ if $h > 1$; and

$V_h \neq 1$ if $h < r$; and

for some h, $U_h \neq 1$; and

for some k, $V_k \neq 1$.

Also $b_t^{\gamma} \in K_x$ or $b_t^{\gamma} \in T_x$.

Hence for some h either $U_h \in B_x$ and $U_h \neq 1$, or $V_h \in A_x$ and $V_h \neq 1$.

But if $U_1 \neq 1$ then $U_1 \equiv b_j^{\alpha}$, $j = t$. Hence $\alpha \neq 0 \bmod p$. Hence $U_1 \notin B_x$.

And if $h > 1$ then $U_h \equiv b_t^{\beta_h}$ where $0 < \beta_h < p$ hence $U_h \notin B_x$.

And if $U_1 \equiv 1$ then $V_1 \equiv b_j^{\alpha}$ or $V_1 \equiv b_j^{\alpha} \prod_{i=1}^{n_1} b_{j_i}^{\alpha_i}$ and is in normal form. Hence $V_1 \underset{H_{p,q}}{\neq} b_{x-1}^{qm}$ by the induction hypothesis as the normal form of b_{x-1}^{qm} is of the form b_v^{δ}.

And if $U_1 \neq 1$ then $V_1 \equiv \prod_{i=k_1}^{n_1} b_{j_i}^{\alpha_i}$ is in normal form and

$V_1 \underset{H_{p,q}}{\neq} b_{x-1}^{qm}$.

And for $h > 1$ if $V_h \not\equiv 1$ then $V_h \equiv \prod_{i=k_h}^{n_h} b_{j_i}^{\alpha_i}$ and is in normal form and $V_h \underset{H_{p,q}}{\neq} b_{x-1}^{qm}$.

Therefore, for all h where $U_h \not\equiv 1$ $U_h \notin B_x$ and for all h where $V_h \not\equiv 1$, $V_h \notin A_x$ - a contradiction.

This proves Lemma 12.5 and completes the proof of Theorem 12.

Theorem 13. If $W \in H_{p,q}$ then $W \underset{H_{p,q}}{=} 1$ if and only if $N(W) \equiv 1$.

Proof. This follows immediately from Theorem 12.

SECTION IV

CONCLUDING THEOREM

Let $N_{p,q} = \langle y,x \mid \bar{y}x^p y \bar{x}^q \rangle$, $(p,q) = 1$, $p > 1$, $|q| \neq 1$.

$F = \langle a,b \mid \rangle$

$K_{p,q} = F(\bar{a}b^p a \bar{b}^q)$, $(p,q) = 1$, $p > 1$, $|q| \neq 1$.

$N_{p,q} \simeq F \bmod K_{p,q}$

and δ the obvious isomorphism from $N_{p,q}$ to $F/K_{p,q}$.

Also let $F_b = \langle b_i : i \in J \rangle$

$G = F(b)$

and $\emptyset : G \to F_b$ the isomorphism given as follows:

$$\text{if } W \in G, \text{ and } \rho(W) \equiv \prod_{i=1}^{n} a^{\alpha_i} b^{\beta_i}$$

$$\text{then } \emptyset(W) = \prod_{i=1}^{n} b^{\beta_i}_{\left(\sum_{j=1}^{i} \alpha_j\right)}$$

and if $\rho(W) \equiv 1$ then $\emptyset(W) \equiv 1$.

For $W \in F_b$, $N(W)$ designates the normal form of W as defined in Section II.

<u>Theorem 14</u>. If U is a word on the generators of $N_{p,q}$ then $U \underset{N_{p,q}}{=} 1$ if and only if

i) $\sigma_a(\delta(U)) = 0$, and

ii) $N(\emptyset(\delta(U))) \equiv 1$.

<u>Proof</u>. Clearly $U \underset{N_{p,q}}{=} 1$ if and only if $\delta(U) = 1 \bmod K_{p,q}$.

If $W = 1 \bmod K_{p,q}$ then $\sigma_a(W) = 0$ since then W is the product of conjugates of $\bar{a}b^p a \bar{b}^q$. [4] pp. 71-72.

Hence if $U \underset{N_{p,q}}{=} 1$ then $\sigma_a(\delta(U)) = 0$.

By Theorem 6, $W = 1 \bmod K_{p,q}$ if and only if $\emptyset(W) \underset{K_{p,q}}{=\!=} 1$, hence $U \underset{N_{p,q}}{=} 1$ if and only if

$$\emptyset(\delta(W)) \underset{H_{p,q}}{=\!=} 1.$$

By Theorem 13, $W \underset{H_{p,q}}{=\!=} 1$ if and only if

$$N(W) \equiv 1.$$

Therefore, $U \underset{N_{p,q}}{=} 1$ if and only if

i) $\sigma_a(\delta(U)) = 0$, and

ii) $N(\emptyset(\delta(U))) \equiv 1$.

This solves the word problem for

$$\langle y,x \mid \bar{y}x^p y\bar{x}^q\rangle,\ (p,q) = 1,\ p > 1,\ q \neq 1.$$

For example $y^2x^3\bar{y}^3x^4y\bar{x}^6y\bar{x}^2\bar{y} = 1$.

<u>Proof of example.</u>

$$\delta(y^2x^3\bar{y}^3x^4y\bar{x}^6y\bar{x}^2\bar{y}) \equiv a^2b^3\bar{a}^3b^4a\bar{b}^6a\bar{b}^2\bar{a}$$

$$\sigma_a(a^2b^3\bar{a}^3b^4a\bar{b}^6a\bar{b}^2\bar{a}) = 0$$

$$\emptyset(a^2b^3\bar{a}^3b^4a\bar{b}^6a\bar{b}^2\bar{a}) = b^3_{-2}b^4_1\bar{b}^6_0\bar{b}^2_{-1}$$

Normalizing $\emptyset(\delta(y^2x^3\bar{y}^3\bar{x}^4yx^6y\bar{x}^2\bar{y}))$ one gets

$$N(\emptyset(\delta(y^2x^3\bar{y}^3\bar{x}^4yx^6y\bar{x}^2\bar{y}))) \equiv N(b^3_{-2}\bar{b}^4_1b^6_0\bar{b}^2_{-1}).$$

One normalizes $N(b^3_{-2}\bar{b}^4_{-1}b^6_0\bar{b}^2_{-1})$ by successively normalizing its terminal segment.

$$N(\bar{b}^1_{-1}) \equiv \bar{b}^1_{-1}$$

$$N(\bar{b}^2_{-1}) \equiv N(\bar{b}^1_{-1} \cdot N(\bar{b}^1_{-1})) \equiv N(\bar{b}^1_{-1}\bar{b}^1_{-1}) \equiv \bar{b}^2_{-1}$$

$$N(b^1_0\bar{b}^2_{-1}) \equiv N(b^1_0 \cdot N(\bar{b}^2_{-1})) \equiv N(b^1_0\bar{b}^2_{-1})$$

$$\equiv N(b^1_0\bar{b}^3_{-1}b^1_{-1}) \equiv N(b^1_0\bar{b}^2_0 b^1_{-1}) \equiv \bar{b}^1_0 b^1_{-1}$$

$$N(b^2_0\bar{b}^2_{-1}) \equiv N(b^1_0 \cdot N(b^1_0\bar{b}^2_{-1})) \equiv N(b^1_0\bar{b}^1_0 b^1_{-1}) \equiv b^1_{-1}$$

$$N(b^3_0\bar{b}^2_{-1}) \equiv N(b^1_0 \cdot N(b^2_0\bar{b}^2_{-1})) \equiv N(b^1_0 b^1_{-1}) \equiv b^1_0 b^1_{-1}$$

$$N(b^4_0\bar{b}^2_{-1}) \equiv N(b^1_0 \cdot N(b^3_0\bar{b}^2_{-1})) \equiv N(b^1_0 b^1_0 b^1_{-1})$$

$$\equiv N(b^2_0 b^1_{-1}) \equiv N(b^3_{-1} b^1_{-1}) \equiv b^4_{-1}$$

$$N(b^5_0\bar{b}^2_{-1}) \equiv N(b^1_0 \cdot N(b^4_0\bar{b}^2_{-1})) \equiv N(b^1_0 b^4_{-1})$$

$$\equiv N(b^1_0 b^3_{-1} b^1_{-1}) \equiv N(b^1_0 b^2_0 b^1_{-1}) \equiv b^3_0 b^1_{-1}$$

$$N(b^6_0\bar{b}^2_{-1}) \equiv N(b^1_0 \cdot N(b^5_0\bar{b}^2_{-1})) \equiv N(b^1_0 b^3_0 b^1_{-1})$$

$$\equiv N(b^4_0 b^1_{-1}) \equiv N(b^6_{-1} b^1_{-1}) \equiv b^7_{-1}$$

$$N(\bar{b}^1_1 b^6_0\bar{b}^2_{-1}) \equiv N(\bar{b}^1_1 \cdot N(b^6_0\bar{b}^2_{-1})) \equiv N(\bar{b}^1_1 b^7_{-1})$$

$$\equiv N(\bar{b}^1_1 b^6_{-1} b^1_{-1}) \equiv N(\bar{b}^1_1 b^4_0 b^1_{-1})$$

$$\equiv N(\bar{b}^1_1 b^3_0 b^1_0 b^1_{-1}) \equiv N(\bar{b}^1_1 b^2_1 b^1_0 b^1_{-1}) \equiv b^1_1 b^1_0 b^1_{-1}$$

$$N(\bar{b}^2_1 b^6_0\bar{b}^2_{-1}) \equiv N(\bar{b}^1_1 \cdot N(\bar{b}^1_1 b^6_0 b^2_{-1})) \equiv N(\bar{b}^1_1 b^1_1 b^1_0 b^1_{-1}) \equiv b^1_0 b^1_{-1}$$

$$N(b^{-3}_1 b^6_0\bar{b}^2_{-1}) \equiv N(\bar{b}^1_1 \cdot N(\bar{b}^2_1 b^6_0\bar{b}^2_{-1})) \equiv N(\bar{b}^1_1 b^1_0 b^1_{-1}) \equiv \bar{b}^1_1 b^1_0 b^1_{-1}$$

$$N(\bar{b}^4_1 b^6_0 \bar{b}^2_{-1}) \equiv N(\bar{b}^1_1 \cdot N(\bar{b}^3_1 b^6_0 \bar{b}^2_{-1})) \equiv N(\bar{b}^1_1 \bar{b}^1_1 b^1_0 b^1_{-1})$$

$$\equiv N(\bar{b}^2_1 b^1_0 b^1_{-1}) \equiv N(\bar{b}^3_0 b^1_0 b^1_{-1}) \equiv N(\bar{b}^2_0 b^1_{-1})$$

$$\equiv N(\bar{b}^3_{-1} b^1_{-1}) \equiv \bar{b}^2_{-1}$$

$$N(b^1_{-2} \bar{b}^4_1 b^6_0 \bar{b}^2_{-1}) \equiv N(b^1_{-2} \cdot N(\bar{b}^4_1 b^6_0 \bar{b}^2_{-1})) \equiv N(b^1_{-2} \bar{b}^2_{-1})$$

$$\equiv N(b^1_{-2} \bar{b}^2_{-1} b^0_{-1}) \equiv N(b^1_{-2} \bar{b}^3_{-2} b^0_{-1}) \equiv N(\bar{b}^2_{-2} b^0_{-1}) \equiv \bar{b}^2_{-2}$$

$$N(b^2_{-2} \bar{b}^4_1 b^6_0 \bar{b}^2_{-1}) \equiv N(b^1_{-2} \cdot N(b^1_{-2} \bar{b}^4_1 b^6_0 \bar{b}^2_{-1}))$$

$$\equiv N(b^1_{-2} \bar{b}^2_{-2}) \equiv \bar{b}^1_{-2}$$

$$N(b^3_{-2} \bar{b}^4_1 b^6_0 \bar{b}^2_{-1}) \equiv N(b^1_{-1} \cdot N(b^2_{-2} \bar{b}^4_1 b^6_0 \bar{b}^2_{-1}))$$

$$\equiv N(b^1_{-2} \bar{b}^1_{-2}) \equiv N(b^0_{-2}) \equiv 1.$$

Hence $y^2 x^3 \bar{y}^3 x^4 y \bar{x}^6 y \bar{x}^2 y \underset{N_{p,q}}{=} 1.$

q.e.d.

Note: Set $R = \bar{y} x^2 y \bar{x}^3$, $R^{-1} = x^3 \bar{y} \bar{x}^2 y$. $y^2 x^3 \bar{y}^3 x^4 y \bar{x}^6 y \bar{x}^2 y = 1$ in $N_{p,q}$ hence it can be written as a product of conjugates of R and R^{-1}. As a check on the computation, such a product will be shown.

$$y^2 x^3 \bar{y}^3 x^4 y \bar{x}^6 y \bar{x}^2 \bar{y}$$

$$= y^2 x^3 \bar{y}^3 x^2 y \bar{y} x^2 y \bar{x}^6 y \bar{x}^2 \bar{y}$$

$$= y^2 x^3 \bar{y}^3 x^2 y \cdot \bar{y} x^2 y \bar{x}^3 \cdot \bar{x}^3 y \bar{x}^2 \bar{y}$$

$$= y^2 x^3 \bar{y}^3 x^2 y \cdot R \cdot \bar{x}^3 y \bar{x}^2 \bar{y}$$

$$= y^2 x^3 \bar{y}^3 x^2 y \bar{x}^3 x^3 \cdot R \cdot \bar{x}^3 y \bar{x}^2 \bar{y}$$

$$= y^2 x^3 \bar{y}^2 \cdot \bar{y} x^2 y \bar{x}^3 \cdot x^3 R \bar{x}^3 \cdot y \bar{x}^2 \bar{y}$$

$= y^2x^3\bar{y}^2 \cdot R \cdot x^3R\bar{x}^3 \cdot y\bar{x}^2\bar{y}$

$= y^2x^3\bar{y}\bar{x}^2y\bar{y}^2yx^2\bar{y} \cdot R \cdot x^3R\bar{x}^3 \cdot y\bar{x}^2\bar{y}$

$= y^2 \cdot x^3\bar{y}\bar{x}^2y \cdot \bar{y}^2yx^2\bar{y} \cdot R \cdot x^3R\bar{x}^3y\bar{x}^2\bar{y}$

$= y^2R^{-1}\bar{y}^2 \cdot yx^2\bar{y}Rx^3R\bar{x}^3y\bar{x}^2\bar{y}$

$= y^2R^{-1}\bar{y}^2 \cdot yx^2\bar{y}Ry\bar{x}^2\bar{y} \cdot y\bar{x}^2\bar{y} \cdot yx^2\bar{y}x^3R\bar{x}^3y\bar{x}^2\bar{y}$

$= AR^{-1}A^{-1} \cdot BRB^{-1} \cdot CRC^{-1}$

$A = y^2, \; B = yx^2\bar{y}, \; C = yx^2\bar{y}x^3$

Theorem 15. No initial segment (subword) in $\bar{y}x^2y\bar{x}^3$ or any short conjugate, K, of $\bar{y}x^2y\bar{x}^3$ is equal to 1 in N.

Proof. $\sigma_y(U) = \sigma_a(\delta(U))$ and Theorem 14 gives a necessary condition for U = 1 in N. It is that $\sigma_a(\delta(U)) = 0$ hence one needs only to consider those initial segments U of $\bar{y}x^2y\bar{x}^3$, or K, with $\sigma_y(U) = 0$.

Hence, the only words to consider are:

i) $\bar{y}x^2y\bar{x}^2$

ii) $\bar{y}x^2y\bar{x}$

iii) $\bar{y}x^2y$

iv) $xy\bar{x}^3\bar{y}$

v) $y\bar{x}^3\bar{y}x$

vi) $y\bar{x}^3\bar{y}$

vii) $\bar{x}^2\bar{y}x^2y$

viii) $\bar{x}\bar{y}x^2y$

i) $\delta(\bar{y}x^2y\bar{x}^2) = \bar{a}b^2a\bar{b}^2$

$\phi(\bar{a}b^2a\bar{b}^2) = b_1^2\bar{b}_0^2$

$$N(b_1^2\bar{b}_0^2) = N(b_1^2\bar{b}_0^1 \cdot N(\bar{b}_0^1)) = N(b_1^2 \cdot N(\bar{b}_0^1\bar{b}_0^1))$$

$$= N(b_1^1 \cdot N(b_1^1\bar{b}_0^2) = N(b_1^1 \cdot N(b_1^1\bar{b}_0^3 b_0^1))$$

$$= N(b_1^1 \cdot N(b_1^1\bar{b}_1^2 b_0^1)) = N(b_1^1\bar{b}_1^1 b_0^1) = b_0^1 \neq 1.$$

ii) $\delta(\bar{y}x^2y\bar{x}) = \bar{a}b^2a\bar{b}$

$\emptyset(\bar{a}b^2a\bar{b}) = b_1^2\bar{b}_0^1$

$$N(b_1^2\bar{b}_0^1) = N(b_1^2 \cdot N(\bar{b}_0^1)) = N(b_1^1 \cdot N(b_1^1\bar{b}_0^1))$$

$$= N(b_1^1 \cdot N(b_1^1\bar{b}_0^3 b_0^2)) = N(b_1^1 \cdot N(b_1^1\bar{b}_1^2 b_0^2)$$

$$= N(b_1^1\bar{b}_1^1 b_0^2) = b_0^2 \neq 1.$$

iii) $\delta(\bar{y}x^2y) = \bar{a}b^2a$

$\emptyset(\bar{a}b^2a) = b_1^2$

$$N(b_1^2) = N(b_1^1 \cdot N(b_1^1)) = N(b_1^1 b_1^1) = b_1^2 \neq 1.$$

iv) $\delta(xy\bar{x}^3\bar{y}) = ba\bar{b}^3\bar{a}$

$\emptyset(ba\bar{b}^3\bar{a}) = b_0^1\bar{b}_{-1}^3$

$$N(b_0^1\bar{b}_{-1}^3) = N(b_0^1\bar{b}_{-1}^2\ N(\bar{b}_{-1}^1)) = N(b_0^1\bar{b}_{-1}^1\ N(\bar{b}_{-1}^1\bar{b}_{-1}^1))$$

$$= N(b_0^1\ N(\bar{b}_{-1}^1\bar{b}_{-1}^2)) = N(b_0^1\ N(\bar{b}_{-1}^3)) = N(b_0^1\ N(\bar{b}_0^2))$$

$$= N(b_0^1\bar{b}_0^2) = \bar{b}_0^1 \neq 1.$$

v) $\delta(y\bar{x}^3\bar{y}) = a\bar{b}^3\bar{a}b$

$\emptyset(a\bar{b}^3\bar{a}b) = \bar{b}_{-1}^3 b_0^1$

$$N(\bar{b}^3_{-1}b^1_0) = N(\bar{b}^3_{-1}\cdot N(b^1_0)) = N(\bar{b}^2_{-1}\cdot N(\bar{b}^1_{-1}b^1_0))$$

$$= N(\bar{b}^1_{-1}\cdot N(\bar{b}^2_{-1}b^1_0)) = N(\bar{b}^1_{-1}\bar{b}^2_{-1}b^1_0) = N(\bar{b}^3_{-1}b^1_0)$$

$$= N(\bar{b}^2_0 b^1_0) = \bar{b}^1_0 \neq 1.$$

vi) $\delta(y\bar{x}^3\bar{y}) = a\bar{b}^3\bar{a}$

$\emptyset(a\bar{b}^3\bar{a}) = \bar{b}^3_{-1}$

$$N(\bar{b}^3_{-1}) = N(\bar{b}^2_{-1}\cdot N(\bar{b}^1_{-1})) = N(\bar{b}^1_{-1}\cdot N(\bar{b}^1_{-1}\bar{b}^1_{-1}))$$

$$= N(\bar{b}^1_{-1}\bar{b}^2_{-1}) = N(\bar{b}^3_{-1}) = N(\bar{b}^2_0) = \bar{b}^2_0 \neq 1.$$

vii) $\delta(\bar{x}^2\bar{y}x^2y) = \bar{b}^2\bar{a}b^2a$

$\emptyset(\bar{b}^2\bar{a}b^2a) = \bar{b}^2_0 b^2_1$

$$N(\bar{b}^2_0 b^2_1) = N(\bar{b}^2_0 b^1_1\cdot N(b^1_1)) = N(\bar{b}^2_0\cdot N(b^1_1 b^1_1))$$

$$= N(\bar{b}^1_0\cdot N(\bar{b}^1_0 b^2_1)) = N(\bar{b}^1_0\cdot N(\bar{b}^1_0 b^3_0)) = N(\bar{b}^1_0 b^2_0) = b^1_0 \neq 1.$$

viii) $\delta(\bar{x}\bar{y}x^2y) = \bar{b}\bar{a}b^2a$

$\emptyset(\bar{b}\bar{a}b^2a) = \bar{b}^1_0 b^2_1$

$$N(\bar{b}^1_0 b^2_1) = N(\bar{b}^1_0 b^1_1\cdot N(b^1_1)) = N(\bar{b}^1_0\cdot N(b^1_1 b^1_1))$$

$$= N(\bar{b}^1_0 b^2_1) = N(\bar{b}^1_0 b^3_0) = b^2_0 \neq 1.$$

THE END

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[6] Strasser, E. Rapaport (see appendix).

APPENDIX

The following theorem was proved by E. Rapaport Strasser.

<u>Theorem</u>. Let $G = (a,b,c;\bar{a}b^2a\bar{b}^3,\bar{b}c^2b\bar{c}^3,\bar{c}a^2c\bar{a}^3)$. If G/N is finite then G/N = 1.

<u>Proof</u>. Let $b^j = a^k = c^\ell = 1$ in G/N, $jk\ell > 0$.

Then $\bar{a}b^2a = b^3$, $\bar{a}b^{2^t}a = b^{3.2^{t-1}}$

$$\bar{a}^2b^{2^t}a^2 = b^{3^2.2^{t-2}},\ldots,\bar{a}^tb^{2^t}a^t = b^{3^t}.$$

If $t = k$ then $b^{3^k} = b^{2^k}$ or $b^{3^k-2^k} = 1$.

Then $3^k-2^k = nj$ and is prime to 2 and to 3, so $(j,2) = (j,3) = 1$. Similarly $(k,2) = (k,3) = (p,2) = (p,3) = 1$.

Then b^2 has order j, and so $\exists\ u \ni (b^2)^u = b$, $(u,j) = 1$ and so $(3u,j) = 1$. Thus $\bar{a}ba = (\bar{a}b^2a)^u = b^{3u} = b^t$ and $(t,j) = 1$, $t > 1$.

Let $t^v \equiv 1 \bmod j$, v least such positive number. Then $\bar{a}ba = b^t$ implies $\bar{a}^2ba^2 = b^{t^2} \ldots \bar{a}^vba^v = b^{t^v} = b$, whence $[a^v,b] = 1$ (a^v commutes with b). If $(g,k) = d$ then a^d commutes with b, so $\bar{a}^dba^d = b^{t^d} = b$ and $d = v$. So g divides k. $k = v\tilde{k}$. Similarly $[b^r,c] = 1$ with $j = r\tilde{j}$

and $[c^s,a] = 1$ with $\ell = s\tilde{\ell}$.

Let j_0,k_0,ℓ_0 be the least prime factors of j,k,ℓ respectively. If one of these is 1 we are through, since e.g. $[a,b] = 1$ implies G/N = 1. So let $j_0 > 1$. Since $t^v \equiv 1 \bmod j$,

$t^v \equiv 1 \bmod j_o$. Let $t^{v_o} \equiv 1 \bmod j_o$ for the least positive integer v_o. Then $v_o < j_o$ and $v = v_o m$. But $k = v\widetilde{k}$, so v_o divides k, whence $k_o \leq v_o < j_o$.

Similarly, one get $j_o < \ell_o$ and $\ell_o < k_o$, whence $j_o < k_o$ - a contradiction. Thus $j_o \not> 1$ and $j = 1$, whence $G/N = 1$.

q.e.d.