# Deformations of Fuchsian Groups

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Abstract of the Dissertation

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In this dissertation we generalize the results of Kra on deformations of Fuchsian groups to include deformations which have branching. We derive necessary and sufficient conditions for the existence of such deformations. For the case of finitely generated Fuchsian groups of the first kind we obtain the necessary condition on the total branching order of the deformation for the existence of such deformations. This generalizes a similar result of Mandelbaum for the case of compact Riemann surfaces. We also give two proofs of the uniqueness of a deformation under certain restrictions. One of the proofs shows that the problem of uniqueness of a deformation with branching can be reduced to that of a deformation without branching.

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## Introduction

The problem of uniformization is related to the problem of finding all complex analytic structures, in which the transition functions are fractional linear transformations, subordinate to a given complex analytic structure. Gunning studied this problem by cohomological methods. Kra showed that it could also be studied by classical methods, some of which are contained in the works of Poincare.

Mandelbaum extended Gunning's work to include structures which were branched. In this dissertation we shall show that this extension can also be treated classically, and in particular by the methods used by Kra.

In Section 1 we give some preliminary definitions and statements of some known results concerning Fuchsian groups and Riemann surfaces. The general definition of a deformation which will cover the case considered by Mandelbaum is given in Section 2 and some of its elementary properties are derived in Sections 2 and 3. In Section 4 we prove the existence of deformations for finitely generated Fuchsian groups of the first kind, under certain restrictions. Mandelbaum had this result for compact Riemann surfaces.

In Sections 5 and 6 we give two proofs of a uniqueness theorem for deformations. One is a direct proof and the other is obtained by reducing the problem to that of deformations which do not have any branching.

In Section 7 we consider 1-deformations in more detail and obtain uniqueness theorems for it, and in Section 8 we show that divisors used to define deformations can be extended to include parabolic fixed points.

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## Section 1 Preliminaries

In this section we summarize some basic facts. We shall be studying groups $\Gamma$ whose elements are Möbius transformations, that is, mappings

$$\gamma : \quad z \longmapsto \frac{az + b}{cz + d}, \; ab - bc = 1.$$

Hence the elements of $\Gamma$ are conformal self-mappings of $\mathbb{C} \cup \{\infty\}$. If $c = 0$ then the mapping is called affine.

Let $\Gamma$ be the subgroup of the group of all Möbius transformations, then for $z \in \mathbb{C} \cup \{\infty\}$ we let $\Gamma_z$ denote the stabilizer of z; that is,

$$\Gamma_z = \{\gamma \in \Gamma ; \gamma z = z\}.$$

We shall say that $\Gamma$ is discontinuous at z if

(i) $\Gamma_z$ is finite and

(ii) there is a neighborhood U of z such that

$\gamma(U) = U$ for all $\gamma \in \Gamma_z$ and

$\gamma(U) \cap U$ is empty for $\gamma \in \Gamma - \Gamma_z$.

We set $\Omega = \Omega(\Gamma) = \{z \in \mathbb{C} \cup \{\infty\} \mid \Gamma \text{ is discontinuous at } z\}$, and call $\Omega$ the region of discontinuity of $\Gamma$. The group $\Gamma$ is called discontinuous if $\Omega$ is not empty. The limit set $\Lambda$ is defined by

$$\Lambda = \mathbb{C} \cup \{\infty\} - \Omega.$$

Obviously $\Omega$ is an open, $\Gamma$-invariant set ($\gamma\Omega = \Omega$, all $\gamma \in \Gamma$).

It can be shown that card $\Lambda = 0, 1, 2,$ or $\infty$. If card $\Lambda \leq 2$, then $\Gamma$ is called elementary; otherwise it is called a nonelementary Kleinian group. For a Kleinian group $\Lambda$ is a closed, perfect, nowhere dense subset of $\mathbb{C} \cup \{\infty\}$.

We can classify the elements of $\Gamma$ according to the following scheme. Define

$\text{trace}^2 \gamma = (a + d)^2$. If $\gamma \neq$ identity, $\gamma$ is

elliptic iff $0 \leq \text{trace}^2 \gamma < 4$,

parabolic iff $\text{trace}^2 \gamma = 4$,

loxodromic iff $\text{trace}^2 \gamma \notin [0, 4]$.

Those loxodromic elements $\gamma$ with $\text{trace}^2 \gamma > 4$ are called hyperbolic. An element is parabolic if and only if it has one fixed point.

If $\Gamma$ is Kleinian, and if there is a circle C in the extended complex plane (a straight line is a circle through $\infty$) such that the interior of C is fixed by $\Gamma$, then $\Gamma$ is called Fuchsian. In this case $\Lambda \subset C$. If $\Lambda = C$, $\Gamma$ is called of the first kind; of the second kind otherwise. A Fuchsian group cannot contain non-hyperbolic loxodromic elements.

We can always choose the circle to be the real line which is the boundary of the upper half plane U.

For a Fuchsian group G we define a q-<u>form</u> $\phi$ as a meromorphic function on U which satisfies the condition

$$\phi(Az)A'(z)^q = \phi(z) \quad \text{for } A \in G \text{ and } z \in U.$$

Suppose p is a parabolic fixed point of an element $A \in G$. Then by conjugation we may take $p = \infty$ and $Az = z + 1$. Then

$$\phi(z + 1) = \phi(z)$$

and $\phi$ has a Fourier series expansion at $\infty$

$$\phi(z) = \sum_{-\infty}^{\infty} a_n e^{2\pi i n z}$$

If $\phi(z) = \sum_{m}^{\infty} a_n e^{2\pi i n z}$ where $a_m \neq 0$ and m is positive then we say that $\phi$ has a zero of order m at p; if m is negative then $\phi$ has a pole of order m at p.

More generally if f is a meromorphic function on U and

$$f \circ A^d = f \qquad Az = z + 1$$

and f has the Fourier series expansion at $\infty$

$$f(z) = \sum_{m}^{\infty} a_n e^{\frac{2\pi i n z}{d}}, \; a_m \neq 0$$

then the <u>order</u> of f at $\infty$ is $\frac{m}{d}$.

The function f is <u>holomorphic</u> at the fixed points if $m \geq 0$ at the fixed points.

If G is a Fuchsian group and $\Omega$ is its region of discontinuity then $\Omega/G$ has a natural conformal structure which makes each component of $\Omega/G$ a Riemann surface.

Suppose M is a compact Riemann surface of genus G.

A <u>divisor</u> a on M is a finite formal sum

$$a = \sum_{j=1}^{n} n_j x_j, \quad n_j \in \mathbb{Z},\ x_j \in M.$$

A divisor is called <u>positive</u> ($a \geq 0$) if $n_j \geq 0$ for all j.

The <u>degree</u> of the divisor a is

$$\deg a = \sum_{j=1}^{n} n_j.$$

If f is a non-zero meromorphic function on M, then f determines a divisor (f) by

$$(f) = \sum_{x \in M} (\mathrm{ord}_x f)\, x.$$

(If z is a local coordinate vanishing at x and if $f(z)=z^n g(z)$ near x, with $g(0) \neq 0$ and $g(0) \neq \infty$ then $n = \mathrm{ord}_x f$.)

A meromorphic q-<u>differential</u> on M is an assignment of a meromorphic function $\mu(z)$ to each local coordinate z such that $\mu(z)dz^q$ is a conformal invariant. To any meromorphic q-differential $\alpha$ we similarly assign a divisor $(\alpha)$. This divisor is called a q-<u>canonical</u> divisor.

Let K(M) be the field of meromorphic functions on M. If a is a divisor on M the <u>space of the divisor</u> a is defined by

$$L(a) = \{f \in K(M);\ f = 0 \text{ or } (f) + a \geq 0\}.$$

We set $\dim a = \dim L(a)$.

The theorem of Riemann-Roch states:

Theorem: If a is any divisor on M then $\dim a = \deg a + \dim(w-a) + 1 - g$ where w is any 1-canonical divisor.

Later we shall also refer to the structure theorem for finitely generated Fuchsian groups of the first kind. It states that any such group G has a set of generators

$$A_1,\ldots,A_{2g};\ B_1,\ldots,B_s;\ C_{s+1},\ldots,C_n$$

satisfying the relations

$$A_1A_2A_1^{-1}A_2^{-1}\ldots A_{2g-1}A_{2g}A_{2g-1}^{-1}A_{2g}^{-1}B_1\ldots B_sC_{s+1}\ldots C_n = I$$

and $C_{s+i}^{m_i} = I$.

The $A_i$ may be chosen as hyperbolic transformations, the $B_i$ as parabolic, and the $C_i$ as elliptic transformations of finite order $m_i$, $m_i \geq m_{i+1}$.

## Section 2 Deformations of Fuchsian Groups.

Let G be a discrete group of Möbius transformations acting on D, where D is either the complex plane $\mathbb{C}$, or the unit disc (upper half plane) U. Let $M_1$ denote the group of all affine transformations and $M_2$ denote the group of all Möbius transformations.

The pair $(\chi, f)$ is called a $\nu$-deformation ($\nu = 1$ or 2) of G if (i) $\chi$ is a homomorphism of G into $M_\nu$, and (ii) f is a local homeomorphism except at a finite number of points in a fundamental domain of G. Let these points be $z_1,\ldots,z_n$ and suppose that in the neighborhood of these points f is an $m_1$ to $1,\ldots,m_n$ to 1 map respectively, and f is meromorphic on D such that

$$f \circ A = \chi(A) \circ f \text{ for } A \in G.$$

It is clear from the last relation that if f is an $m_i$ to 1 map at $z_i$ then f is also $m_i$ to 1 at $Az_i$. Hence $z_i$ could have been replaced by $Az_i$. The definition is therefore meaningful. Two $\nu$-deformations $(\chi_1, f_1)$ and $(\chi_2, f_2)$ are $\nu$-equivalent if there is an element $B \in M_\nu$ such that

$$f_2 = B \circ f_1$$

and

$$\chi_2(A) = B \circ \chi_1(A) \circ B^{-1} \text{ for } A \in G.$$

Let f be a meromorphic function on a domain S and define

$$\theta_1 f = \frac{f''}{f'}$$

and

$$\theta_2 f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

Then it is an easy calculation that for any meromorphic function h on f(S)

$$\theta_\nu(h \circ f) = [(\theta_\nu h) \circ f] f'^{\nu} + \theta_\nu f$$

and

$\theta_\nu f = 0$ if and only if $f \in M_\nu$.

A meromorphic function $\phi$ on D is called a <u>multiplicative-q-form</u> (where q is any integer) for G if

$$\phi(Az)A'(z)^q = C_A\phi(z) \text{ for } z \in D, A \in G,$$

where $C_A \in \mathbb{C}^* = \mathbb{C} - \{0\}$ depends only on A. If $C_A = 1$ for all $A \in G$, then we have a q-form for G.

It is easy to see that if $(\mathcal{X}, f)$ is a 2-deformation then $\theta_2 f$ is holomorphic except at the points where f is not one to one. Hence $\theta_2 f$ has only a finite number of poles in the fundamental domain.of G. Similarly if $(\mathcal{X}, f)$ is a 1-deformation then $\theta_1 f$ has residues $\neq -2$ only at a finite number of points in the fundamental domain. These are actually the points where f is not one ot one. Since f could have a pole of the form $\frac{1}{z}$ + ... and still be one to one, $\theta_1 f$ can have residues = -2 at as many points as one wants.

This leads us to define a connection. A meromorphic function $\phi$ on D is called a 1-connection for G if

$$(\phi \circ A)A' + \theta_1 A = \phi \text{ for all } A \in G$$

and $\phi$ has residues $\neq -2$ at only a finite number of points in a fundamental domain of G. A meromorphic function $\phi$ on D is called a 2-connection for G if

$$(\phi \circ A)A'^2 = \phi \text{ for all } A \in G$$

and has poles at only a finite number of points in the fundamental domain of G. We see that a 2-connection is a 2-form. In any case the difference between two $\nu$--connections ($\nu$ = 1, 2) is a $\nu$-form.

Following Mandelbaum (11): A $\nu$-connection is said to be integrable if we can find a meromorphic function f on D such that $\theta_\nu f = \phi$.

With these definitions we have

Proposition A: There is a canonical one-one correspondence between the set of equivalence classes of $\nu$-deformations of G and the set of integrable $\nu$-connections for G.

Remark: Every group admits an integrable 2-connection, namely $\phi = 0$.

Proof: Let $(\chi, f)$ be a $\nu$-deformation of G. Then $\theta_\nu f$ is a meromorphic function of the type described above. Since

$$f \circ A = \chi(A) \circ f \text{ for all } A \in G$$

and

$$\chi(A) \in M_\nu$$

we obtain by applying $\theta_\nu$ to both sides of this equation

$$(\theta_\nu f) \circ AA'^{\nu} + \theta_\nu A = \theta_\nu f \text{ for all } A \in G.$$

Hence $\theta_\nu f$ is the required integrable $\nu$-connection.

If $(\chi, f)$ is equivalent to $(\chi_1, f_1)$ there exists an element $B \in M_\nu$ such that

$$f_1 = B \circ f.$$

Hence $\theta_\nu f_1 = \theta_\nu f$ and we get the same integrable $\nu$-connection.

Now suppose $\phi$ is an integrable $\nu$-connection, then there exists a meromorphic function f such that $\theta_\nu f = \phi$. Obviously f is not one to one at only a finite number of points in the fundamental domain of G. Now

$$\theta_\nu(f \circ A) = (\theta_\nu f) \circ AA'^{\nu} + \theta_\nu A = (\phi \circ A)A'^{\nu} +$$

$$+ \theta_\nu A = \phi = \theta_\nu f \text{ for all } A \in G.$$

Hence there exists an element $\chi(A) \in M_\nu$ such that

$$f \circ A = \chi(A) \circ f.$$

This implies immediately that $\chi(A \circ B) = \chi(A) \circ \chi(B)$ for $A, B \in G$. Thus $\chi$ is a group homomorphism and $(\chi, f)$ is a

$\nu$-deformation.

Suppose there is another meromorphic function g such that $\theta_\nu g = \phi$. This gives another deformation $(\chi_1, g)$. Then

$$\theta_\nu f = \theta_\nu g$$

which implies that

$$f = B \circ g \text{ for } B \in M_\nu.$$

Also

$$B \circ g \circ A = \chi(A) \circ B \circ g \text{ or}$$

$$B \circ \chi_1(A) \circ g = \chi(A) \circ B \circ g.$$

Hence

$$\chi(A) = B \circ \chi_1(A) \circ B^{-1};$$

that is, $(\chi, f)$ and $(\chi_1, g)$ are equivalent $\nu$-deformations.

Proposition B: The following are equivalent conditions for a Fuchsian Group acting on D:

(a) G admits a multiplicative 1-form all of whose residues vanish and it has poles of order greater than two and zeros at only a finite number of points in the fundamental domain of G.

(b) G admits an integrable 1-connection.

(c) There exists a 1-deformation of G.

Proof: (a) implies (b). Suppose f is a multiplicative 1-form described in (a). Then

$$f(Az)A'(z) = C_A f(z) \qquad (A)$$

and the equation $\frac{dh}{dz} = f(z)$ has a meromorphic solution $h(z)$.

Now differentiating (A) we get

$$f'(Az)A'(z)^2 + f(Az)A''(z) = C_A f'(z). \qquad (B)$$

Dividing (A) by (B) we get

$$\frac{f'}{f} \circ AA' + \frac{A''}{A'} = \frac{f'}{f}$$

and since $h' = f$

$$\frac{h''}{h'} \circ AA' + \frac{A''}{A'} = \frac{h''}{h'},$$

that is,

$$(\theta_1 h) \circ AA' + \theta_1 A = \theta_1 h.$$

From the condition on f it follows that $\theta_1 h$ has residues $\neq -2$ at only a finite number of points in the fundamental domain of G. Hence $\phi = \theta_1 h$ is the required integrable 1-connection.

(b) implies (c). This follows from Proposition A.

(c) implies (a). Let $(\chi, f)$ be a 1-deformation. Then

$$f \circ A = \chi(A) \circ f \text{ where } \chi(A) \in M_1.$$

Hence

$$\chi(A)z = \alpha_A z + \beta_A \text{ and therefore}$$

$$f \circ A = \alpha_A f + \beta_A.$$

Differentiating this equation we get

$$(f' \circ A)A' = \alpha_A f'.$$

Now f' is a multiplicative 1-form of the required type.

Corollary: There exists a 1-deformation of G.

Proof: If $D = \mathbb{C}$ then every $A \in G$ is an affine mapping. Thus $\phi \equiv 0$ is an integrable 1-connection for G and there exists a 1-deformation of G in this case. If D = U then

Case 1. U/G is a compact Riemann surface. Then G is a finitely generated Fuchsian group of the first kind without parabolic elements. For such groups there exists a nonconstant meromorphic function such that

$$f \circ A = f \text{ for } A \in G.$$

By differentiating we get $(f' \circ A)A' = f'$ for $A \in G$. Thus f' is a 1-form all of whose residues vanish.

Case 2. U/G is an open Riemann surface. In this case Kra (6) has shown that there exists a multiplicative 1-form which is holomorphic and does not vanish at any point.

## Section 3 The Type of a Deformation

Let R be the fundamental domain of the group G acting on the upper half plane U.. If $z_1, \dots, z_n \in U$ are points in R and associated with them are integers $m_1-1, \dots m_n-1$ respectively, then we call $D = \sum_{i=1}^{n}(m_i-1)z_i$ a <u>divisor</u> on the fundamental domain. The points zi are not parabolic fixed points of G. A 2-deformation $(\mathcal{X}, f)$ is of type $D = \sum(m_i-1)z_i$ if and only if f is a homeomorphism at every point except in the neighborhood of $z_i$ $(i = 1, \dots, n)$ and points equivalent to them, where it is $m_i$ to 1 $(i = 1, \dots, n)$ respectively. A 2-deformation is of <u>zero type</u> if $m_i = 1$ for all $i = 1, \dots, n$.

Suppose G is finitely generated and of the first kind. Then a 2-deformation $(\mathcal{X}, f)$ satisfies the <u>cusp condition</u> if $\theta_2 f(z) = \phi(z) \longrightarrow 0$ as $z \longrightarrow$ parabolic fixed points of G through a cusp region belonging to the fixed point. A 2-form is of <u>type</u> D if and only if in the neighborhood of each $z_j$ $(j = 1, \dots, n)$

$$\phi(z) = \frac{1 - m_j^2}{2(z-z_j)^2} + \frac{t_{-1}}{z-z_j} + \sum_{i=0}^{\infty} t_i (z-z_j)^i$$

and $\phi$ is holomorphic at all other points of the fundamental domain. Since

$$\lim_{Az \to Az_j} (Az - Az_j)^2 \phi(Az)$$

$$= \lim_{Az \to Az_j} \frac{(Az - Az_j)^2}{(z - z_j)^2} \frac{(z-z_j)^2 \phi(Az) A'(z)^2}{A'(z)^2}$$

$$= \lim_{Az \to Az_j} \left(\frac{Az - Az_j}{z - z_j}\right)^2 \frac{(z-z_j)^2 \phi(z)}{A'(z)^2}$$

$$= \frac{1 - m_j^2}{2},$$

the 2-form $\phi$ has the correct expansion at all equivalent points and the definition is therefore meaningful. A 2-form $\phi$ satisfies the cusp condition at the puncture if $\phi(z) \longrightarrow 0$ as $z \longrightarrow$ parabolic fixed points of G.

We can now state the following

__Theorem 3.1__: For each positive divisor D ($m_i \geq 1$) there exists a canonical bijection between the equivalence classes of the 2-deformations of type D and the integrable 2-forms of type D.

__Proof__: It is sufficient to show that if f is a meromorphic function which is $m \rightarrow 1$ in a neighborhood of $z_0$ then

$$\theta_2 f(z) = \frac{1 - m^2}{2(z-z_0)^2} + \frac{t_{-1}}{z-z_0} + A(z)$$

where $t_{-1} = 0$ if $m = 1$ and $A(z)$ is an analytic function. We may assume $z_0 = 0$. Then we can write locally

$f(z) = g(z)^{\pm m}$ where $g(0) = 0$; $g'(0) \neq 0$. Then

$\frac{f''}{f'} = (\pm m - 1)\frac{g'}{g} + \frac{g''}{g'}$; hence

$$\theta_2 f = \frac{1-m^2}{2}\left(\frac{g'}{g}\right)^2 + \theta_2 g$$

by an easy calculation. But $\frac{g'(z)}{g(z)} = \frac{1}{z}$ + holomorphic function while $\theta_2 g$ = holomorphic function.

<u>Corollary (Kra)</u>: There is a canonical bijection between equivalence classes of the 2-deformations of type zero and the 2-forms of type zero.

<u>Proof</u>: A 2-form $\phi$ of type zero is a holomorphic function and hence there exists a meromorphic function f such that $\theta_2 f = \phi$. Since f is locally one-one $(\mathcal{X}, f)$ is a deformation of type zero. Conversely if $(\mathcal{X}, f)$ is a 2-deformation of zero type, then f is locally one-one at each point and hence $f' \neq 0$. This implies that $\theta_2 f$ is holomorphic and hence of zero type.

Later we shall be concerned with solutions of the equation $\theta_2 f(z) = \phi(z)$ where $\phi(z)$ is a meromorphic function. It is known, (Hille, 4 ), that the problem of solving this equation is equivalent to the simpler problem of finding two linearly independent solutions $w_1$ and $w_2$ of the linearly homogeneous equation of second order

$$w''(z) + \tfrac{1}{2}\phi(z)w(z) = 0.$$

If $w_1$ and $w_2$ solve this equation then $\frac{w_1}{w_2}$ solves the original equation. However $\frac{w_1}{w_2}$ need not be a meromorphic function and in general could be a multivalued function. This difficulty occurs only at the poles of $\phi(z)$. Hence we shall determine the condition when the solution is meromorphic in the neighborhood of the poles.

Suppose $\phi(z) = \frac{1-m^2}{2z^2} + \frac{t_{-1}}{z} + \sum_0^\infty t_i z^i$ for $|z| < 1$;

Then we have the following

Lemma 3.2: There exists a polynomial A(m) in $C[x_1,\ldots x_m]$ such that

$$f'' + \tfrac{1}{2}\phi f = 0$$

has two linearly independent solutions whose ratio is meromorphic if and only if

$$A(m)(t_{-1},\ldots t_{m-2}) = 0.$$

Proof: Assume a solution of the form

$$f = z^s(A_0 + A_1 z + \ldots + A_n z^n + \ldots).$$

Then by substituting in the equation we get

$$(s(s-1)A_0 z^s + \ldots + (n+s)(n+s-1)\,A_n z^{n+s} + \ldots) +$$
$$\tfrac{1}{2}\left(\frac{1-m^2}{2} + t_{-1}z + \ldots\right)z^s(A_0 + A_1 z + \ldots) = 0.$$

By equating coefficients we get the relations

$$s(s-1)A_0 + \frac{m^2-1}{4}A_0 = 0 \qquad (1)$$

and

$$((n+s)(n+s-1) + \frac{1-m^2}{4})A_n +$$
$$\tfrac{1}{2}(A_{n-1}t_{-1} + A_{n-2}t_0 + \ldots + A_0 t_{n-2}) = 0 \text{ for } n = 1, 2, \ldots \qquad (2)$$

From (1), $s = \frac{1 \pm m}{2}$ so that one of the solutions is

$$u(z) = z^{\frac{1+m}{2}}(A_0 + A_1 z + \ldots)$$

where the $A_i$ can be determined from (2). For the other solution we assume $f(z) = u(z)v(z)$ which on being substituted gives the relation

$$\frac{dv'}{v'} = -2\frac{du}{u} \text{ or}$$

$$v' = \frac{1}{cu^2} = \frac{1}{c} z^{-m-1}(A_0 + A_1 z + \ldots + A_n z^n + \ldots)^{-2}.$$

The condition that v be meromorphic, that is, have no logarithmic term, is that the coefficient of $z^m$ in

$$(1 + A_1 z + \ldots + A_m z^m)^{-2}$$

should be zero. The coefficient is obviously a polynomial in $A_i$ and hence by (2) it is a polynomial in $t_i$, $i = -1, \ldots, m-2$. This proves the lemma.

Suppose $\phi$ is a 2-form of type $D = \sum(m_i - 1)z_i$. Then we have

<u>Corollary</u>: There exist polynomials $A(m_i)$ in $C[x_1, \ldots x_n]$ such that $\theta_2 f = \phi$ has a meromorphic solution in U if and only if

$$A(m_i)(t^i_{-1}, \ldots t^i_{m-2}) = 0, \; i = 1, \ldots, n.$$

<u>Proof</u>: By the lemma, local meromorphic solutions at each point exist. Now since U is simply connected, a global meromorphic function can be obtained from the family of

locally defined functions by the Monodromy theorem.

We now give another form to the corollary.

Proposition C: The following are equivalent conditions for a Fuchsian group acting on U.

(a) There exists a 2-deformation of type $D = \sum (m_i - 1) z_i$.

(b) G admits an integrable 2-form of type D.

(c) G admits a 2-form $\phi$ which is holomorphic except at $z_1, \ldots z_n$, the points in the fundamental domain of G in the neighborhood of which points

$$\phi(z) = \frac{1 - m_j^2}{2(z-z_j)^2} + \frac{t_{-1}}{z-z_j} + \sum_{i=0}^{\infty} t_i^j (z - z_j)^i$$

and the coefficient of $z^{m_j}$ in

$$(1 + A_1^j z + \ldots + A_{m_j}^j z^{m_j})^{-2}$$

vanishes, where $A_i^j$ are defined by

$$((n_j + s_j)(n_j + s_{j-1}) + \frac{1-m_j^2}{4} A \quad + \\ + \tfrac{1}{2}(A_{n_j-1}^j t_{-1}^j + \ldots + t_{n_j-2}^j) = 0;$$

$n_j = 1, \ldots m_j$; $j = 1, \ldots n$; and $s_j = \frac{m_j+1}{2}$.

Proof: (a) implies and is implied by (b). This follows from Theorem 3.1.

(b) implies (c). This follows from Lemma 3.2.

(c) implies (b). This follows from the Corollary.

## Section 4 Deformations of Finitely Generated Fuchsian Groups of the First Kind

Let G be a finitely generated Fuchsian group of the first kind with signature $(g; \nu_1, \ldots, \nu_N)$, where g is an integer greater than or equal to zero and $2 \leq \nu_1 \leq \cdots \leq \nu_N \leq \infty$. Let n be the largest integer less than or equal to N such that $\nu_n < \infty$ (if N = 0 set n = 0). Then $S = \overline{U/G}$ (the compactification of the Riemann surface U/G) has genus g, and the generators of the group (as given in section 1) contain n elliptic elements and N - n parabolic elements. In this section we shall only be concerned with groups as described above.

We have already seen that the existence of 2-deformations of type D is equivalent to the existence of integrable 2-forms of type D. Since the latter are easier to work with we shall be concerned with them. We shall work with 2-forms of type D satisfying the cusp condition since these will be seen to have finite dimensionality as a affine spaces. In any case the distinction between 2-forms satisfying the cusp condition and 2-forms exists only for groups which have parabolic elements.

Let $D = \sum_{i=1}^{k} (m_i - 1)z_i$ and s be the number of $z_i$ which are not elliptic fixed points of the group G.

Theorem 4.1: (1) The integrable 2-connections of type $D = \sum_{i=1}^{k} (m_i - 1)$ satisfying the cusp condition for a group G have the structure of a complex affine subvariety of the $3g - 3 + N + s$ dimensional complex linear manifold of all 2-connections of type D satisfying the cusp condition.

(2) The subvariety is nonempty if

(a) $k = 0$ or $1$ or

(b) none of the $z_i$ are fixed points and

$\sum (m_i - 1) \leq 2g - 2 + N$ or

(c) $m_i \leq 2$, $i = 1, \ldots k$, and $k < 3g - 3 + N + s$

or (d) for fixed points $z_i, m_i \leq 3$ and

$k < 3g - 3 + N + s$ and $\sum (m_i - 1) < 2g - 2 + N$.

Proof: We first show the existence of a 2-form with a double pole at $z_i$. Let $S = U/G$ and let z be the local coordinate on U and Z on U/G. Then if e is an elliptic fixed point of order b,

$$Z = (z - e)^b.$$

A 2-form $\phi(z)$ projects to a quadratic differential $\Phi$ on U/G according to the formula

$$\phi(z) = \Phi(Z)\left(\frac{dZ}{dz}\right)^2.$$

$\pi$: $U \rightarrow U/G$ is the projecting map. If $\mathrm{ord}_e\, \phi = r$ and $\mathrm{ord}_{\pi(e)}\, \Phi = R$ then we have from the above relations

$$r = Rb + 2(b - 1) \quad \text{or} \quad r + 2 = b(R + 2).$$

At a parabolic fixed point, which may be assumed to be $\infty$, we can take $Az = z + 1$ as the parabolic transformation. We then get $\phi(z + 1) = \phi(z)$ and $\phi$ has a Fourier series expansion at $\infty$ which is given by

$$\phi(z) = \sum_1^\infty a_n e^{2\pi inz} \text{ since } \phi(z) \longrightarrow 0 \text{ as } z \rightarrow i\infty.$$

Here the uniformising variable is $Z = e^{2\pi iz}$ and

$$\frac{dZ}{dz} = 2\pi i e^{2\pi iz}.$$

We therefore get

$$r = R + 2.$$

Now suppose $\phi$ has a pole of order at most 2 at $z_i$ which is a fixed point of order $\nu_i$. At every other point it is holomorphic. This gives us the following conditions:

$$R_{\pi(z_i)} \geqslant -2$$

$$R \geqslant -[2(1 - \tfrac{1}{\nu_j})] = -1 \text{ at all fixed points} \neq z_i.$$

(If x is a real number then $[x]$ is the greatest integer in x) and

$$R \geqslant -1 \text{ at the parabolic fixed points.}$$

We therefore define a divisor $\alpha$ on U/G

$$\alpha = \sum_{p \in U/G} n(p)p \text{ where } n(p) = -1$$

for $p \in U/G$, $p \neq \pi(z_i)$ and $p$ an elliptic fixed point, $n(p) = -2$ for $p = \pi(z_i)$, $n(p) = -1$ for $p \in \overline{U/G} - U/G$, and $n(p) = 0$ for all other points. Hence every 2-form $\phi$ with at most a double pole at $z_i$ projects to a meromorphic quadratic differential $\Phi$ on $\overline{U/G}$ such that

$$(\Phi) - \alpha \geqslant 0, \text{ and conversely.}$$

Let $\theta$ be any meromorphic quadratic differential on U/G and let $w = (\theta)$ and let $a = 2w - \alpha$.

Then $f \in L(a)$, the space of meromorphic functions whose divisors are multiples of the divisor a if and only if

$$(f\theta^2) - \alpha = (f) + 2w - \alpha = (f) + 2w - \alpha \geqslant 0.$$

Thus $L(a) \cong$ space of 2-forms with at most a double pole at $z_i$. By the Riemann Roch Theorem

$$\dim a = \deg a - g + 1 + \dim (w - a).$$

But

$$\deg a = 4g - 4 + \sum_{\substack{p \neq \pi(z_i) \\ p = \text{ell. fixed pt.}}} 1 + N - n + 2$$

where $N - n$ is the number of points in $\overline{U/G} - U/G$ and $n$ is the number of elliptic fixed points and

$$\deg (w - a) = -2(g - 1) - N - 2 + n - \sum_{\substack{p \neq \pi(z_i) \\ p = \text{fixed pt.}}} 1.$$

It can be easily shown that $\deg (w - a) < 0$ and we get dim (w - a) = 0. Therefore

$$\dim a = 3g - 3 + N - n + 2 + \sum_{p \neq \pi(z_i)} 1.$$

In the same manner we can calculate the dimension of the space of 2-forms with a simple pole at $z_i$. This dimension turns out to be

$$3g - 3 + N - n + 1 + \sum_{p \neq \pi(z_i)} 1.$$

Hence there exists a 2-form satisfying the cusp condition with a double pole at $z_i$. Call it $\phi_i$. We can always choose $\phi_i$ so that

$$\phi_i(z) = \frac{1-m_i^2}{2(z-z_i)^2} + \frac{\phi_{i-1}}{z-z_i} + \sum_{j=1}^{\infty} \phi_{ij}(z-z_i)^j$$

Now let $\phi = \sum_1^k \phi_i$; then $\phi$ is a 2-form of type D satisfying the cusp condition. All the others are found by obtaining all 2-forms which have at most simple poles at $z_i$ where $i = 1,\ldots,k$. The dimension of this space X can be shown to be 3g -3 + N + s where s is the number of points among the $z_i$ which are not elliptic fixed points. Then for any $x \in X$, $\phi + x$ is also a 2-form of type D satisfying the cusp condition. Thus the dimension T of the affine space of all 2-forms of type D with the cusp condition is 3g - 3 + N + s. Now

$$3g - 3 + N \leq T \leq 3g - 3 + N + k$$

where 3g - 3 + N is the dimension of the space of cusp forms.

We also note that there is no 2-form (with cusp condition) with a pole of order 1 at $z_i$ if $z_i$ is an elliptic fixed point. By rearranging $z_i$ we can assume that $z_i$ where $i = 1,\ldots s$ are not elliptic fixed points. Thus there exist 2-forms $\psi_i$ $(i = 1,\ldots s)$ with simple poles at $z_i$ $(i = 1,\ldots s)$ respectively. Hence any 2-form $\mu$ of type D with the cusp condition can be written in the form

$$\mu = \phi + \sum_{i=1}^{s} t_i\psi_i + \sum \sigma_i\theta_i \quad \text{where } \theta_i \text{ are the}$$

cusp forms. Let $\psi = (\psi_1, \ldots, \psi_s)$ and $\theta = (\theta_1, \ldots, \theta_{3g-3+N})$ and

$$(t, \sigma) = (t_1,\ldots t_s, \sigma_1, \ldots, \sigma_{3g-3+N})$$

and for each

$$(t, \sigma) \in \mathbb{C}^s \times \mathbb{C}^{3g-3+N} \quad \text{we let}$$

$$(t, \sigma)\mu = \phi + \psi \cdot t + \theta \cdot \sigma.$$

F(D) denotes the space of all 2-forms satisfying the cusp condition so that

$$\mathrm{IF}(D) = \{\text{integrable 2-forms of type D satisfying the cusp condition}\}$$

is equivalent to

$$\{(t, \sigma) \in \mathbb{C}^s \times \mathbb{C}^{3g-3+N} \mid \theta_2 f = (t, \sigma)\mu \text{ has a meromorphic solution}\}.$$

By the corollary to Lemma 3.2 we know that this is equivalent to

$\{(t,\sigma) \in \mathbb{C}^s \times \mathbb{C}^{3g-3+N} \mid f''(z) + \frac{1}{2}\,{}_{(t,\sigma)}\mu(z) f(z) = 0$

has two linearly independent solutions whose ratio is meromorphic$\}$.

By Lemma 3.2 there exist polynomials $A(m_i) \in \mathbb{C}[x_1,\ldots x_{m_i}]$ with leading term $\alpha_1 x_1^{m_i}$ $(\alpha_1 \neq 0)$ such that $f''(z) + \frac{1}{2}\,{}_{(t,\sigma)}\mu(z) f(z) = 0$ has linearly independent solutions whose ratio is meromorphic if and only if

$$A(m_i)({}_{t,\sigma}\mu_{-1}, \ldots, {}_{t,\sigma}\mu_{m_i-2}) = 0$$

where ${}_{(t,\sigma)}\mu_i$ are obtained from

$${}_{(t,\sigma)}\mu(z) = \frac{1-m_i^2}{2(z-z_i)^2} + \frac{{}_{t,\sigma}\mu_{-1}}{z-z_i} + \sum_0^\infty {}_{(t,\sigma)}\mu_j (z-z_i)^j$$

Now ${}_{(t,\sigma)}\mu(z) = \phi + t\cdot\psi + \sigma\cdot\theta$. Hence at the points $z_i$, $(i = 1,\ldots s)$, that is, the points which are not elliptic fixed points,

$${}_{(t,\sigma)}\mu_{-1} = \phi_{-1} + t_j \; ; \text{ for } h \geq 0 \;\; {}_{(t,\sigma)}\mu_h = \phi_h + \theta_h\cdot\sigma + \psi_h\cdot t \;,\; \phi_h, \theta_h, \psi_h$$

are the coefficients of $(z - z_i)^h$ in the Laurent expansion of $\phi$, $\theta$, $\psi$ respectively. At the other points $z_i$ ${}_{(t,\sigma)}\mu_{-1} = \phi_{-1}$ but

$${}_{(t,\sigma)}\mu_h = \phi_h + \theta_h\cdot\sigma + \psi_h\cdot t \;.$$

Therefore the polynomials $A(m_i)({}_{t,\sigma}\mu_{-1}, \ldots, {}_{t,\sigma}\mu_{m_i-2})$ are actually polynomials in $t_1,\ldots t_s$, $\sigma_1,\ldots,\sigma_{3g-3+N}$. Mandelbaum (11) has pointed out that the polynomial

$$A(m_i)(x_1,\ldots x_{m_i}) = \sum_{(j_1\ldots j_{m_i})} \alpha_{(j_1\ldots j_{m_i})} x_1^{j_1} \ldots x_{m_i}^{j_{m_i}}$$

is such that $\sum_{k=1}^{m_i} k\, j_k = m_i$. Therefore we have n polynomials $K_1, \ldots K_n \in \mathbb{C}[x_1, \ldots x_{3g-3+N}]$ where polynomials $K_i$ corresponding to the points $z_i$ which are not fixed points have leading term of the form $\alpha_0^i t_i^{m_i-1}$ and all other terms are of total degree $\leq [\frac{m_i}{2}]$ and at the other points $z_i$ the polynomial has all terms of degree $\leq [\frac{m_i}{2}]$, and these polynomials are such that

$$A(m_j)(_{t,\sigma}\mu_{-1}, \ldots, {}_{t,\sigma}\mu_{m_j-2}) = 0$$

if and only if $K_j(t, \sigma) = 0$.

If we not let

$$V_D = \{ (t,\sigma) \in \mathbb{C}^s \times \mathbb{C}^{3g-3+N} \mid K_i(t,\sigma) = 0 \;,\; i = 1, \ldots, k \}$$

then $IF(D) \simeq V_D$ which is the required subvariety. If n = 0 or 1 the subvariety is obviously nonempty.

We now assume that the $z_i$, $i = 1, \ldots k$, are not elliptic fixed points. In this case there exist 2-forms with simple poles at $z_i$. Suppose that we could choose 2-forms $\psi_j$, $j = 1, \ldots k$, such that they are holomorphic everywhere except that $\psi_j$ has a pole of order 1 at $z_j$ and zeros of order $m_i - 1$ at $z_i \neq z_j$. Then the system $\theta_j$, $j = 1, \ldots 3g-3+N$ and $\psi_j$, $j = 1, \ldots k$, will again form a basis for 2-forms with at most simple poles at $z_i$. If we calculate $K_i(t, \sigma)$ with this basis, it is easy to see that since the $\psi_j$, $j \neq i$, have no term of degree less than $m_i-1$ at $z_i$,

$$K_i(t,\sigma) = K_i((0,\ldots t_j\ldots),\sigma).$$

Thus for each i, $K_i$ will be a polynomial in $t_i$ and $\sigma$ with leading term $\alpha_0^i t^{m_i} \neq 0$. Thus for any fixed $\sigma$, if we let $t_i^*$ be a root of $K_i(t,\sigma) = 0$ for $i = 1,\ldots k$ we immediately see that

$$(t^*,\sigma) \in V \quad \text{where} \quad t^* = (t_1^*,\ldots t_k^*).$$

Thus $V_D \neq \emptyset$.

We must of course show that the $\psi_j$ defined above exist. Suppose the points $z_i$ $(i = 1,\ldots k)$ project down to $p_i$ on U/G. We define a divisor

$$\alpha_1 = \sum_{p \in \overline{U/G}} n(p)p$$

where

$$n(p) = \left[2 - \frac{2}{\nu(p)}\right] = 1 \text{ at the projection of}$$

a fixed point of order $\nu(p)$.

$$n(p) = -1 \text{ if } p \in \overline{U/G} - U/G.$$

and

$$n(p_i) = m_i - 1 \quad i \neq j$$

$$n(p_j) = -1.$$

Define $\alpha_2 = \sum n(p)p$ where everything is the same as in $\alpha_1$ except that $n_1(p_j) = 0$.

Call $a_s = 2w - \alpha_s$ $(s = 1, 2)$, where w has the same meaning as it had before. Consider the spaces $L(a_s)$ where

$f \in L(a_s)$ if and only if f is a meromorphic function on U/G such that $(f) + a_s \geqslant 0$. To show the existence of $\psi_j$ it is enough to show that $\dim L(a_1) - \dim L(a_2) \geqslant 0$. Now

$$\dim L(a_i) = \deg a_i - g + 1 + \dim (w - a_i).$$

But

$$\deg a_1 - \deg a_2 = 1 \quad \text{so that}$$

$$\dim L(a_1) > \dim L(a_2) \quad \text{provided}$$

$$\dim (w - a_i) = 0 \quad \text{and this is so if}$$

$$\deg (w - a_i) < 0.$$

This last inequality is true if

$$\Sigma(m_i - 1) \leqslant 2g - 2 + N.$$

If $m_i$ is at most 2, that is $m_i - 1 \leqslant 1$, then the polynomials $K_1(t,\sigma), \ldots K_k(t,\sigma)$ are of degree 1 in $\sigma_i$ and $t_i$. In this case if $k < 3g - 3 + N + s$ then we have n equations of the degree 1 in $3g - 3 + N + s$ variables and they certainly have nontrivial solutions so that $V_D \neq 0$. Again, if s of the points $z_i$, $i = 1, \ldots k$, are not elliptic fixed points (say, $z_1, \ldots z_s$), then the polynomial $K_i \in (K_1, \ldots K_s)$ has degree $m_i - 1$ and the polynomial $K_j \in (K_{s+1}, \ldots K_n)$ has degree $\leqslant [\frac{m_i}{2}]$. Hence if $z_i$ is an elliptic fixed point and $m_i \leqslant 3$ then the polynomial $K_i$ has degree $\leqslant [\frac{3}{2}] = 1$ in $\sigma_i$ and $t_i$. Moreover if we choose $\psi_i$, $i = 1,\ldots s$ (as before), so it has a pole of order 1 at $z_i$ and zero of order $m_j$ at $z_j \neq z_i$ then $K_i \in (K_1,\ldots K_s)$ is a polynomial in $t_i$ and $\sigma_1, \ldots, \sigma_{3g-3+N}$ and $K_j \in (K_{s+1},\ldots K_n)$ is a polynomial of degree 1 in $\sigma_i$. Hence if $k - s < 3g - 3 + N$ then the first dsgree equations are solvable for $\sigma_i$ and, substituting these values in $K_i \in (K_1,\ldots K_s)$, we can solve for $t_i$. So we again get a nonempty variety provided

(i) $k < 3g - 3 + N + s$

(ii) $m_i \leq 3$ if $z_i$ is an elliptic fixed point

(iii) $\Sigma (m_i - 1) < 2g - 2 + N$.

We need condition (iii) for the existence of $\psi_i$ as we showed earlier.

## Section 5 Poincare's Theorem

We now consider the following problem: Given a deformation $(\chi, f)$, to what extent is the map f determined by the homomorphism $\chi$? We have the following theorem due to Kra (8).

Let G be a finitely generated Fuchsian group of the first kind. If $(\chi, f)$ and $(\chi, g)$ are 2-deformations of type zero satisfying the cusp condition then $f = g$.

We shall be concerned with generalizations of this theorem to deformations of non-zero type.

We call a deformation $(\chi, f)$ parabolic if whenever $A \in G$ is a parabolic element of G, $\chi(A)$ is either parabolic or the identity.

Lemma 5.1: Let $(\chi, f)$ be a 2-deformation of G of type D satisfying the cusp condition and which is parabolic. Let $A \in G$ be parabolic with fixed point $p \in R \cup \{\infty\}$; then

(a) $\chi(A)$ is parabolic

(b) f and f' have limits as $z \to p$ in any cusped region belonging to p. (We assume A generates the stability subgroup of p.)

Proof: We may assume by conjugation that $p = \infty$ and $Az = z + 1$. Since $\chi$ is parabolic we can conjugate again to get $\mathcal{X}(A)z = z + b$. Since

$$f \circ A = \mathcal{X}(A) \circ f \quad \text{we get}$$

$$f(z + 1) = f(z) + b \quad \text{and}$$

$$f'(z + 1) = f'(z).$$

Now f is not locally one-one at only a finite number of points in a given fundamental domain. Also by Ahlfors ( 1 ) there exists a domain $U_c = \{z \in \mathbb{C} \,;\, \text{Im} z > c\}$ such that two points z and z' in $U_c$ are equivalent under G if and only if $z' = z + n$ for some integer n, that is, if and only if z' and z are equivalent under the group generated by A. We can choose c large enough so that f is locally univalent in $U_c$. Hence in a cusped region belonging to $\infty$, f' is a nonzero meromorphic function all of whose poles are of order 2, and all its residues vanish. Thus there exists a well defined holomorphic function $y = (f')^{-\frac{1}{2}}$ in $U_c$. It is well known that y satisfies the second order differential equation

$$2y'' + \phi(z)y(z) = 0 \qquad \text{(A)}$$

for $z \in U_c$; $\phi(z) = \theta_2 f(z)$. Obviously $\phi(z + 1) = \phi(z)$. Also either $y(z + 1) = y(z)$ or

$$y(z + 1) = -y(z).$$

We may assume the first equation to be true without loss of generality since we could have worked with $A^2$ instead of A.

The cusped region $\{z \in U_c;\ z = x + iy,\ 0 \leq x < 1\}$ can be mapped conformally onto the punctured disc by the map $z \to e^{2\pi i z}$. Hence we have functions $\tilde{y}$ and $\tilde{\phi}$ on $\Delta - \{0\}$ such that

$$y(z) = \tilde{y}(e^{2\pi i z}) \quad \text{and} \quad \phi(z) = \tilde{\phi}(e^{2\pi i z}).$$

Letting $\varsigma = e^{2\pi i z}$, then $\frac{dy}{dz} = 2\pi i \frac{d\tilde{y}}{d\varsigma}$.

$$\frac{d^2y}{dz^2} = (2\pi i)\varsigma \frac{d\tilde{y}}{d\varsigma} + (2\pi i)^2 \varsigma^2 \frac{d^2\tilde{y}}{d\varsigma^2}.$$

Hence equation (A) becomes

$$\frac{d^2\tilde{y}}{d\varsigma^2} + \frac{1}{\varsigma}\frac{d\tilde{y}}{d\varsigma} - \frac{1}{8\pi^2}\frac{\tilde{\phi}(\varsigma)}{\varsigma^2}\tilde{y}(\varsigma) = 0. \qquad \text{(B)}$$

Since $(\chi, f)$ is a deformation satisfying the cusp condition, $\theta_2 f \to 0$ as $z \to \infty$, $\tilde{\phi}(0) = 0$ and $\tilde{\phi}$ is actually holomorphic in $\Delta$. It is known (Boyce-DiPrima, 3) that the general solution is of the form

$$\tilde{y}(\varsigma) = C_1\phi_1(\varsigma) + C_2(\phi_2(\varsigma) + \phi_1(\varsigma)\log\varsigma),$$

where $\phi_1$ and $\phi_2$ are analytic functions. Now since y is a single valued holomorphic function in $\Delta - \{0\}$, $C_2 = 0$. Now suppose that

$$\phi_1(\varsigma) = A_0 + A_1\varsigma + \cdots \quad ; \quad \tilde{\phi}(\varsigma) = a_1\varsigma + a_2\varsigma^2 + \cdots ,$$

then on substituting these relations in the equation (B) we get

$$n^2 A_n = \frac{1}{8\pi^2}(a_1 A_{n-1} + \cdots + a_n A_0) \text{ for } n \geq 2 \text{ and}$$

$$A_1 = \tfrac{1}{2}a_1 A_0.$$

Obviously if $A_0 = 0$ then $A_n = 0$ for all n and $\phi_1(\varsigma) \equiv 0$

and $\tilde{y}(\zeta) \equiv 0$ which is false and hence $A_0 \neq 0$, that is, $\phi_1(0) \neq 0$. Thus $\tilde{y}(\zeta) = C_1\phi_1$ with $C_1 \neq 0$ and $\tilde{y}(0) \neq 0$. Now $\lim\limits_{z\to i\infty} f'(z)^{-1} = \lim \tilde{y}^2(0) = C_1^2 A_0^2 \neq 0$. Hence $f'$ has a finite nonzero limit as $z \to \infty$ through a cusped region belonging to $\infty$. Since $f'(z+1) = f'(z)$ for $z \in U$, $f'$ has

$$f'(z) = \Sigma\, a_n e^{2\pi i n z}, \quad z \in U$$

as a Fourier series expansion.

Since $f'$ has a finite nonzero limit as $z \to \infty$, we have $a_n = 0$ for $n < 0$ and $a_0 \neq 0$, that is,

$$f'(z) = \sum_0^\infty a_n e^{2\pi i n z}, \quad z \in U. \text{ Thus}$$

$$f(z) = \sum_1^\infty b_n e^{2\pi i n z} + bz \text{ with } b \neq 0. \text{ Thus}$$

$$|f(z)| = O(|z|) \text{ as } z \to \infty \text{ and}$$

$$\frac{1}{|f'(z)|} = O(1) \text{ as } z \to \infty.$$

We can extend Lemma 5.1 to a more general situation. Let $(\chi, f)$ be a deformation of G such that $\theta_2 f$ is holomorphic. Let $p_1, \ldots p_r$ be a complete set of inequivalent fixed points of parabolic elements of G. Let $A_1, \ldots A_r$ be the parabolic elements with fixed points $p_1, \ldots p_r$ respectively. Suppose $B_i$ is a Möbius transformation which fixes U and has the following properties:

(i) $B_i(\infty) = p_i$

(ii) $B_i^{-1} \circ A_i \circ B_i(z) = z + 1$

(iii) if $U_c = \{z \in \mathbb{C} \,;\, \mathrm{Im} z > c\}$, then for sufficiently large c two points of $B(U_c)$ are equivalent under G if and only if they are equivalent under the cyclic subgroup generated by $A_i$.

These $B_i$ exist (Ahlfors, 1 ).

Then $(\mathcal{X}, f)$ is called a special deformation if $\lim_{z \to i\infty} \theta_2(f \circ B_i) = 2\pi^2 m_i^2$ for i = 1,...r where $m_i$ are integers. We shall also say that the limit of $(\mathcal{X}, f)$ at $p_i$ is $m_i$. The limit is independent of the $B_i$.

Lemma 5.2: If $(\mathcal{X}, f)$ is a parabolic special deformation and if its limit at p is m then

$$f'(z) = \sum_{-m}^{\infty} a_n e^{2\pi i n z} \quad z \in U \text{ around } \infty \text{ and}$$

$$f(z) = \sum_{-m}^{\infty} b_n e^{2\pi i n z} + bz \quad z \in U.$$

In the expression for f(z) b might be zero.

Conversely if f has the above Fourier expansion then the limit of $(\mathcal{X}, f)$ is m.

Proof: We calculate exactly as in Lemma 5.1 until we come to the equation

$$\frac{d^2\tilde{y}}{d\zeta^2} + \frac{1}{\zeta}\frac{d\tilde{y}}{d\zeta} - \frac{1}{8\pi^2}\frac{\tilde{\phi}(\zeta)}{\zeta^2}\tilde{y}(\zeta) = 0 .$$

$\theta_2 f = \phi$ is a holomorphic function even at the fixed points and hence is actually holomorphic on $\Delta$ and $\tilde{\phi}(0) = 2\pi^2 m^2$.

The point $\zeta = 0$ is then a regular singular point and we can obtain solutions of the equation which are of the form $\sum a_n z^{n+s}$. Substituting this for $\tilde{y}$ in the equation we get

$$s = \pm\sqrt{\frac{2\pi^2 m^2}{8\pi^2}} = \pm \frac{m}{2}.$$

Arguing as in Lemma 5.1

$$\tilde{y}(\zeta) = c_1\phi_1(\zeta) \quad \text{where} \quad \phi_1\zeta = \sum a_n \zeta^{n+\frac{m}{2}}, \; a_0 \neq 0.$$

Thus
$$\frac{1}{|f'(z)|} = O(e^{-2\pi m y}) \qquad z \to i\infty$$

Since $f'(z+1) = f'(z)$, $f'$ has a Fourier expansion

$f'(z) = \sum_{-\infty}^{\infty} a_n e^{2\pi i n z}$ and the growth condition on $f'(z)$ gives $f'(z) = \sum_{-m}^{\infty} a_n e^{2\pi i n z}$. $z \in U$ and hence $f(z) = \sum_{-m}^{\infty}{}' b_n e^{2\pi i n z} + bz$.

For the converse we notice that

$$\theta_2 f = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = (2\pi i m)^2 - \frac{3}{2}(2\pi i m)^2 = 2\pi^2 m^2.$$

This proves the lemma.

Let $(\mathcal{X}, f)$ be a deformation of type $D = \sum_{i=1}^{n} (m_i - 1) z_i$

We denote the total branching order $\sum \frac{(m_i-1)}{\nu(z_i)}$ by $O(f)$.

Here $\nu(z_i)$ is the ramification order of $z_i$ (ord $G_{z_i}$).

We can now prove the following generalization of Poincare's theorem (Kra, 8 ).

<u>Theorem 5.3</u>: Let $(\mathcal{X}, f)$ and $(\mathcal{X}, g)$ be parabolic deformations of type $D_f = \sum_1^n (m_i - 1) z_i$ and $D_g = \sum_1^k (d_j - 1)\zeta_j$ respectively, satisfying the cusp condition, of a finitely generated

Fuchsian group G of the first kind. Let k be the genus of G and $A(G) = 2k - 2 + \sum_{p \in \bar{U}/G} (1 - \frac{1}{\nu(p)})$ where $\nu(p)$ is the ramification index of p. If $O(f) + O(g) < 2A(G)$ then $f = g$.

Proof: Consider the function

$$V(z) = \frac{(f(z) - g(z))^2}{f'(z)g'(z)}$$

This is actually the square of the function considered in Kra ( 8 ). By following the proof given there, it can be shown that V is a multiplicative 2-form. This, however, involves multivalued functions so I shall give the simpler proof suggested to me by Professor Kra.

It is easy to verify that if A is a Möbius transformation; then

$$(A\zeta - Az)^2 = (\zeta - z)^2 A'\zeta A'z. \qquad (A)$$

Since $f(Az) = \chi(A)f(z)$ we get

$$f'(Az)A'(z) = (\chi(A)'\ f)(z)\ f'(z) \qquad (B)$$

and similarly

$$g'(Az)A'(z) = (\chi(A)'\ g)(z)\ g'(z) \qquad (C)$$

for A G. Now

$$V(Az) = \frac{(\chi(A)f(z) - \chi(A)g(z))^2}{f'(Az)\,g'(Az)}.$$

From (A), (B), and (C).

$$V(Az) = \frac{(f(z) - g(z))^2 \chi(A)'f(z)\chi(A)'g(z)}{f'(Az)\,g'(Az)} = \frac{(f(z) - g(z))^2}{f'(z)g'(z)} A'(z)^2$$

$= V(z)A'(z)^2$, that is, $V(Az)A'(z)^{-2} = V(z)$, that is, V is a -2-form.

We shall determine the behavior of V in the neighborhood of the parabolic fixed points. Let q be a puncture on U/G. It involves no loss of generality to assume that the puncture q is generated by the parabolic element $Az = z + 1$. (Merely replace G by $B \circ G \circ B^{-1}$ for a suitable self-map B of U.) In this case

$$V(z + 1) = V(z).$$

Lemma 5.1 also gives the following:

$$|f(z)| = O(|z|) = \text{ as } z \to \infty \text{ and}$$

$$\frac{1}{|f'(z)|} = O(1) = \frac{1}{|g'(z)|} \text{ as } z \to \infty. \quad \text{Hence}$$

$$V(z) = O(|z|^2) \text{ as } z \to \infty. \qquad \text{(D)}$$

V also has a Fourier series expansion around

$$V(z) = \sum_{-\infty}^{\infty} a_n e^{2\pi i n z} \quad z \in U.$$

(D) gives $a_n = 0$ for $n < 0$. Hence V has a zero of order $L \geq 0$ at the parabolic fixed points.

$D_f = \sum_{i=1}^{n} (m_i - 1) z_i$. Suppose $z_i$ is not an elliptic fixed point. If f has a pole of order $m_i$ at $z_i$ then f' has a zero of $m_i - 1$ at $z_i$ and again V has a pole of order at most $m_i - 1$. If $z_i$ is a fixed point we have to compute the order by dividing the ordinary order by the order of the fixed point. Hence even at a fixed point V has a pole of order at most $\frac{m_i - 1}{v_i}$. Hence the total order of poles of V is at most O(f) + O(g).

Since V is a -2-form we have from Lehner (10) that unless $V \equiv 0$ deg V = -2A(G). Hence V has poles of order at least 2A(G) so that if O(f) + O(g) < 2A(G) then $V \equiv 0$, that is, $f \equiv g$ and the theorem is proved.

Corollary: (Kra) If ($\mathcal{X}$, f) and ($\mathcal{X}$, g) are two deformations of type zero satisfying the cusp condition then f = g.

Proof: In case ($\mathcal{X}$, f) satisfies the cusp condition and is of type zero then it has been shown by Bers ( 2 ) and Kra ( 7 ) that ($\mathcal{X}$, f) is parabolic and hence we have the corollary.

We now give a simple example of two 2-deformations which have the same homomorphism but are defined by two different functions. Let G be the covering group of a compact Riemann surface of genus g > 1 and let f and g be two automorphic functions for G. Project f and g to meromorphic functions f and g on the Riemann surface. Then f and g give the Riemann surface as an n-sheeted covering surface of the sphere. (For f and g n may be different.) It is well known (Springer,13)

$$O(f) = O(g) = 2(n + g - 1) > 2g - 2.$$ Hence

$$O(f) + O(g) > 4g - 4.$$

But here we have 2-deformations of G, (I, f), (I, g) where I is the trivial homomorphism and $f \neq g$. We may generalize our last theorem in the following way: Suppose $D = \sum (m_i - 1) z_i$ then we denote by deg D the sum $\sum \underset{\nu(z_i)}{\underline{(m_i - 1)}}$.

Theorem 5.4: Suppose $(\chi, f)$ and $(\chi, g)$ are deformations which satisfy the hypothesis of the previous theorem. Let $D + D_f$ and $D + D_g$ be the types of f and g respectively where $D = \sum_{i=1}^{n} (m_i - 1) z_i \geq 0$. If

s $\deg D_f + \deg D_g < 2(A(G) + n)$ then $f = g$ provided $f(z_i) = g(z_i)$, $i = 1, \ldots n$.

Proof: $$V(z) = \frac{(f-g)^2(z)}{f'(z)g'(z)} = \left(\frac{1}{f(z)} - \frac{1}{g(z)}\right)^2 \frac{f^2(z)\, g^2(z)}{f'(z)\, g'(z)}$$

Suppose f and g have a pole at $z_i$ of order $m_i$. Then $\left(\frac{1}{f(z)} - \frac{1}{g(z)}\right)^2$ has a zero of order $\geq 2m_i$ at $z_i$ and $\frac{f^2 g^2}{f' g'}$ has a pole of order $2m_i - 2$ and V has a zero of order of at least 2.

Now if $f(z_i) = g(z_i) = a \neq \infty$ then $(f - g)^2$ has a zero of order at least $2m_i$ at $z_i$ and $\frac{1}{f'(z)g'(z)}$ has a pole of order $2m_i - 2$ and again V has a zero of order at least 2 at $z_i$. In any case the points $z_i$ contribute zeros of order 2 to V. Hence altogether the total order of zeros contributed by $z_i$ is 2n. The result now follows from the previous theorem.

## Section 6 Deformations Without the Cusp Condition

We shall now approach the problem of uniqueness in a slightly different manner. Suppose $(\mathcal{X}, f)$ is a 2-deformation of type $D = \sum (m_i - 1) z_i$ of a Fuchsian group G acting on the upper half plane U and $(\mathcal{X}, f)$ satisfies the cusp condition. Let $U_0 = U - \{\bigcup_i G z_i\}$. Then G acts invariantly on $U_0$ and f is locally univalent on $U_0$ since the points where it is not a local homeomorphism have been removed. Then with the help of the universal covering map $\pi: U \to U_0$ we lift the group G to a Fuchsian group F acting on U and the deformation $(\mathcal{X}, f)$ of G is lifted to deformation $(\tilde{\mathcal{X}}, \tilde{f})$ of F. Now $\tilde{f}$ is locally univalent, which makes things simpler, though $\theta_2 \tilde{f}$ is never a cusp form. However, $\theta_2 \tilde{f}$ is a holomorphic 2-form and the difficulty which arises from the fact that it is not a cusp form can in some cases be overcome.

For the time being we consider a slightly more general case. Let $\Gamma$ be a non-elementary Kleinian group with an invariant domain D. Let $D_0 = D - \{\bigcup_{i=1}^{n} \Gamma z_i\}$ where $z_i$ $(i = 1, \ldots n)$ are points of D inequivalent under $\Gamma$. Then $\Gamma$ acts invariantly on $D_0$. Let $\pi: U \to D_0$ be the universal holomorphic covering map. Since $D_0$ has more than two points on the boundary, $U$ can be taken as the upper half plane. Let $G$ be the covering group of $\pi$; that is $G$ consists of those conformal self-mappings $g$ of $U$ for which $\pi \circ g = \pi$. We lift $\Gamma$ to a discrete group $F$ of conformal self-mappings of $U$ as follows: Let $\gamma \in \Gamma$ and $z \in U$. We construct a self-mapping $h \in F$ of $U$ such that the diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & U \\ \pi \downarrow & & \downarrow \pi \\ D_0 & \xrightarrow{\gamma} & D_0 \end{array}$$

commutes. For $z \in U$, consider the points $\pi(z)$ and $\gamma(\pi(z))$. The mapping $\pi \circ \gamma \circ \pi^{-1}$ is defined in a neighborhood of $z$. Hence we have a family of locally defined holomorphic functions and since $U$ is simply connected they define a global map by the Monodromy theorem. The map $h$ obtained is a conformal self-map of $U$. It is obviously holomorphic. Now suppose $h(z_1) = h(z_2)$. Then $\pi(z_1) = \pi(z_2)$. Hence there exists $g \in G$ such that $gz_1 = z_2$. Then in the neighborhood of $z_1$ $h \circ g = h$ and hence $g$ is the identity and $z_1 = z_2$. This proves that $h$ is conformal. We have in fact constructed an exact sequence of groups:

$$\{1\} \longrightarrow G \xrightarrow{i} F \xrightarrow{j} \Gamma \longrightarrow \{1\},$$

where i is the inclusion map, and if $h \in F$ then $j(h)$ is determined via

$$\pi \circ h = j(h) \circ \pi \ ,$$

that is, j is the inverse of the lifting map used in the construction.

Now suppose $D/\Gamma$ is of finite type, that is, it is a compact Riemann surface with a finite number of punctures. We denote the compact Riemann surface from which $D/\Gamma$ is derived by $\overline{D/\Gamma}$. Then $\overline{D/\Gamma} - D/\Gamma$ is a set with a finite number of points. Also $D/\Gamma$ has a finite number of distinguished (ramified) points which arise from elliptic elements of $\Gamma$. In this case $D_0/\Gamma$ is also of finite type. Since $U/F$ is conformal to $D_0/\Gamma$, $U/F$ is of finite type so that F is a finitely generated Fuchsian group of the first kind. The details are in Kra ( 5 ).

Now consider the pair $(j, \pi)$, $j\colon F \longrightarrow \Gamma$ and $\pi$ is a locally univalent map on U such that $\pi \circ h = j(h) \circ \pi$ for all $h \in F$. Hence $(j, \pi)$ is a deformation of F. We shall determine what $\theta_2 \pi$ looks like. If $D_0 = D$ then it was shown by Kra ( 7 ) that $\theta_2 \pi$ is a cusp form. With the above notation we have

__Theorem 6.1__: If $D/\Gamma$ is of finite type, then $\theta_2 \pi$ is a

2-form which is holomorphic at every point including the parabolic fixed points of F.

Proof: The punctures on $D_0/\Gamma$ arise either from parabolic fixed points of $\Gamma$ or from "elliptic" elements of order $d \geq 1$. Of course if d = 1 then it is not an elliptic element but we are calling it that for convenience.

Case 1: Punctures arising from parabolic elements:

We calculate as in Kra ( 7 ). Let p be a puncture on $D_0/\Gamma$ and q the corresponding puncture on U/F. According to Ahlfors ( 1 ), to the puncture q there corresponds a cyclic subgroup of G generated by a parabolic element $A \in F$. We may assume $Az = z + 1$. In this case there is a $c > 0$ such that

$$U_c = \{z \in \mathbb{C} ; \text{Im} z > c\}$$

factored by the subgroup generated by A is conformally equivalent to a deleted neighborhood of q in U/F. Similarly, by conjugating $\Gamma$ if necessary, we may assume that the puncture p on $D_0/\Gamma$ corresponding to q is also generated by A. We may also assume that in this case $D \supset U_c$ for sufficiently high c. Since $\pi$ induces a conformal map $\pi^*$ U/F to $D_0/\Gamma$, we have, for some integer k,

$$\pi(z + 1) = \pi(z) + k, \; z \in U. \quad \text{Since}$$

$\pi'(z+1) = \pi'(z)$, $z \in U$. We have for some $c > 0$,

$$\pi'(z) = \sum_{-\infty}^{\infty} a_n e^{2\pi i n z}, \quad z \in U_c, \text{ and thus}$$

$$\pi(z) = \sum_{n \neq 0} a_n (2\pi i n)^{-1} e^{2\pi i n z} + kz + b, \quad z \in U_c.$$

We can choose a local parameter $\varsigma$ at q such that in terms of this local parameter, the projection map $U \longrightarrow U/F$ is given by $\varsigma = e^{2\pi i z}$ for $z \in U_c$. Similarly, $Z = e^{2\pi i z}$ is such a local parameter at $\pi^*(q) = p$.

$$\begin{array}{ccccl} z \in U_c & \longrightarrow & U_c & \ni & x = \pi(z) = \sum_{n \neq 0} a_n (2\pi i n)^{-1} \varsigma^n + \frac{k}{2\pi i} \log \varsigma + b \\ \downarrow & & \downarrow & & \\ e^{2\pi i z} = \varsigma \in \overline{U}/F & \longrightarrow & \overline{D}_0/\Gamma & \ni & Z = e^{2\pi i x} \end{array}$$

The mapping $\pi^*$ yields Z as a holomorphic function of $\varsigma$ with $Z(0) = 0$ and $Z'(0) \neq 0$.

By looking at the diagram we get

$$Z(\varsigma) = \varsigma^k \exp\left(b + \sum_{n \neq 0} a_n (2\pi i n)^{-1} \varsigma^n\right).$$

Since Z is holomorphic $a_n = 0$ for $n < 0$ and since $Z'(0) \neq 0$, $k = 1$. Hence,

$$\pi'(z) = \sum_{1}^{\infty} a_n e^{2\pi i n z} + 1.$$

$$\phi = \theta_2 \pi = \frac{\pi'''}{\pi'} - \frac{3}{2}\left(\frac{\pi''}{\pi'}\right)^2. \quad \text{Now}$$

$$\phi(z+1) = \phi(z), \quad z \in U$$

so that $\phi$ has a Fourier expansion $\phi(z) = \sum_{-\infty}^{\infty} b_n e^{2\pi i n z}$ at $\infty$.

Obviously

$$\lim_{z \to i\infty} \theta_2 \pi = 0 \text{ and hence } b_n = 0 \text{ for } n < 1, \text{ i.e.,}$$

$$\phi(z) = \sum_{1}^{\infty} b_n e^{2\pi i n z}, \quad z \in U_c$$

and $\phi$ is holomorphic and vanishes at the puncture.

Case 2: Punctures which arise from elliptic elements of $\Gamma$: Again let q be a puncture on U/F. This comes from a parabolic element Az = z + 1. Let p be the corresponding puncture on $D_0/\Gamma$ which comes from an elliptic element $B \in \Gamma$ of order d. By conjugating $\Gamma$ if necessary we may assume that the fixed points of B are 0 and $\infty$ and that the point corresponding to q is the origin. Hence $D_0$ contains a deleted neighborhood of the origin and B divides the disc into d sections each of angle $2\pi/d$.

Since $\pi$ induces a conformal map $\pi^*$: $\overline{U}/F \longrightarrow \overline{D}_0/\Gamma$, we get

$$\pi(z + 1) = e^{2\pi i k/d}\pi(z) \quad \text{for } z \in U \text{ where } 0 < k < d. \qquad (A)$$

(Note that this already shows that $\theta_2\pi$ cannot be a cusp form since if it is, then it follows from Kra ( 7 ) that j: $F \longrightarrow \Gamma$ takes parabolic elements to parabolic elements which is obviously not the case.)

(A) gives us the following Fourier expansion for

$$\pi(z) = \sum_{-\infty}^{\infty} a_n e^{2\pi i(n+k/d)z} = e^{2\pi i k/d z}\sum_{-\infty}^{\infty} a_n e^{2\pi i n z}, \quad z \in U_c.$$

Let $\zeta$ be the local parameter at q such that the projection $U \longrightarrow U/G$ is given by $\zeta = e^{2\pi i z}$ for $z \in U_c$. Similarly $Z = z^d$ is the local parameter at $\pi^*(q) = p$. We get the diagram

$$\begin{array}{ccc} z \in U_c & \longrightarrow & D_0 \ni \pi(z) \\ \downarrow & & \downarrow \\ e^{2\pi i z} = \zeta \in U/F & \longrightarrow & D_0/\Gamma \ni Z(\zeta) = \pi(z)^d. \end{array}$$

Once again $\pi^*$ is conformal and Z is a holomorphic function of $\zeta$ with

$$Z(0) = 0 \text{ and } Z'(0) \neq 0.$$

Looking at the diagram we get

$$Z(\zeta) = (\zeta^{k/d} \sum_{-\infty}^{\infty} a_n \zeta^n)^d = \zeta^k (\sum_{-\infty}^{\infty} a_n \zeta^n)^d.$$

Since $Z(0) = 0$ and Z is holomorphic

$$Z = \zeta^k (\sum_{-N}^{\infty} a_n \zeta^n)^d \text{ with leading term } a_{-N}^d \zeta^{k-dN}$$

so that $k \geq dN$ but $k < d$. Hence $N = 0$ and $k = 1$.

Also $a_n = 0$ for $n < 0$ and

$$\pi(z) = \sum_0^{\infty} a_n e^{2\pi i(n+1/d)z}$$

$$\pi'(z) = \sum_0^{\infty} a_n (n + \tfrac{1}{d}) 2\pi i e^{2\pi i(n+\frac{1}{d})z}$$ and similarly

we determine $\pi''$ and $\pi'''(z)$.

$$\begin{aligned} \lim_{z\to i\infty} \phi(z) &= \lim_{z\to i\infty} \left[ \frac{\pi'''}{\pi'} - \frac{3}{2} \left(\frac{\pi''}{\pi'}\right)^2 \right] \\ &= - \frac{4\pi^2 d^4}{d^6} + \frac{3}{2} \frac{4\pi^2 d^4}{d^6} \\ &= \frac{2\pi^2}{d^2}. \end{aligned}$$

Hence $\theta_2\pi$ always has finite limits at the fixed points, that is, it is a holomorphic 2-form.

Now let $(\mathcal{X}, f)$ be a restricted 2-deformation of type $D = \sum_{i=1}^{n} (m_i - 1) z_i$ of a Fuchsian group G, which is finitely generated and of the first kind, acting on the upper half plane U. Let $U_0 = U - \{\bigcup_i^n Gz_i\}$. $\pi: U \to U_0$ is the universal holomorphic covering map.

Suppose G lifts to the Fuchsian group H. Then H is of the first kind and finitely generated. There is a homomorphism $j: H \to G$ such that

$$\pi \circ h = j(h) \circ \pi \text{ for all } h \in H.$$

Now $(\chi \circ j, f \circ \pi)$ is a deformation of H, since

$$f \circ \pi \circ h = f \circ j(h) \circ \pi \quad h \in H$$
$$= \chi(j(h)) \circ f \circ \pi.$$

Because $f \circ \pi$ is locally univalent, $(\chi \circ j, f \circ \pi)$ is of zero type. However $(\chi \circ j, f \circ \pi)$ obviously does not satisfy the cusp condition.

Since $(\chi, f)$ satisfies the cusp condition $\theta_2 f \to 0$ as $z \to$ parabolic fixed points of G. The punctures on $U_0/G$ arising from these points lift to punctures on U/H. We have already shown that $\theta_2 \pi \to 0$ as $z \to$ a fixed point arising from these punctures on U/G and that $\pi' \to 1$. Now $\theta_2(f \circ \pi) = (\theta_2 f) \circ \pi \ \pi'^2 + \theta_2 \pi$ and hence $\theta_2(f \circ \pi) \to 0$ at these points. Now consider punctures on U/H coming from the points $z_i$ $i = 1, \ldots n$. As already mentioned we may assume $z_i = 0$. Then $\theta_2 f$ has the following expansion around $z_i$ (which is a fixed point of order $d_i$):

$$\theta_2 f = \frac{1-m_i^2}{2z^2} + \frac{t-1}{z} + \sum_0^{\infty} t_i z^i. \quad \text{Hence}$$

$$\lim_{z\to i\infty} \{f, \pi(z)\} \pi'(z)^2 + \{\pi, z\}$$

$$= \lim_{z\to i\infty} \left[ \frac{1-m_i^2}{2\pi(z)^2} \pi'(z)^2 + t_{-1} \frac{\pi'(z)^2}{\pi(z)} + \cdots \right] + \frac{2\pi^2}{d_i^2}$$

$$= \frac{2\pi^2 m_i^2}{d_i^2}$$

Hence we now have to consider deformations $(\chi, f)$ which are of zero type but $\theta_2 f$ are not cusp forms. The $\theta_2 f$, however, are holomorphic. We now have

Theorem 6.2: If $(\chi, f)$ is a 2-deformation with the cusp condition of type zero and $(\chi, g)$ is of type zero such that $\theta_2 g$ is holomorphic and is either non-special or a cusp form then $f = g$.

Proof: Since $(\chi, f)$ satisfies the cusp condition $\chi$ maps parabolic elements to parabolic elements whence $\theta_2 g$ must be special so that it is a cusp form by hypothesis. It now follows from the corollary to Theorem 5.3 that $f = g$.

Now let $(\chi, f)$ and $(\chi, g)$ be two deformations of type $D = \sum_{i=1}^{n} (m_i - 1) z_i$ and $\sum_{i=1}^{k} (d_i - 1) \zeta_i$ of a group G. Let $U_0 = U - \{ \bigcup_1^n G z_i \ \bigcup_1^k G \zeta_i \}$ and $\pi : U \to U_0$ be the universal holomorphic covering map. If none of the points $z_i$, $i=1,\ldots n$ or $\zeta_i$, $i = 1,\ldots k$ is an elliptic fixed point then we have already seen that $(\chi \circ j, f \circ \pi)$ and $(\chi \circ j, g \circ \pi)$ are special deformations of F, the group to which G is lifted.

Now if $\chi$ is parabolic then $\chi \circ j$ is also parabolic since the homomorphism j takes parabolic elements to parabolic elements or the identity. Hence if $(\chi, f)$ and $(\chi, g)$ are parabolic so are $(\chi \circ j, f \circ \pi)$ and $(\chi \circ j, g \circ \pi)$.

Let $A(G) = 2g - 2 + \sum(1 - \frac{1}{\nu_i})$ as before. Then $A(F) = A(G) + n + k$ since F has n + k extra parabolic elements arising from $z_i$ and $\zeta_i$. We have of course assumed that $\zeta_i$ and $z_i$ are distinct but it is not necessary to do so. Denote $(\chi \circ j, f \circ \pi)$ and $(\chi \circ j, g \circ \pi)$ by $(\tilde{\chi}, \tilde{f})$ and $(\tilde{\chi}, \tilde{g})$ respectively and consider the function

$$V(z) = \frac{(\tilde{f}(z) - \tilde{g}(z))^2}{\tilde{f}'(z)\tilde{g}'(z)}.$$

V is a -2-form as we proved earlier. V is holomorphic in U and we have only to check its behavior at the parabolic fixed points. Lemma 5.2 tells us that

$$\begin{aligned} V(z) &= \sum_{-(m_i+1)}^{\infty} a_n e^{2\pi i n z} \text{ at points arising from } z_i \text{ and} \\ &= \sum_{-(d_i+1)}^{\infty} b_n e^{2\pi i n z} \text{ at points arising from } \zeta_i \text{ and} \\ &= \sum_{0}^{\infty} c_n e^{2\pi i n z} \text{ at all other punctures.} \end{aligned}$$

Hence $V \equiv 0$ provided

$$\sum_{i=1}^{n}(m_i + 1) + \sum_{i=1}^{k}(d_i + 1) < 2(A(G) + n + k$$

$$\text{i.e. } \sum(m_i - 1) + \sum(d_i - 1) < 2A(G),$$

that is, $f = g$ which implies that $f = g$. We have therefore reproved Theorem 5.2 under the condition that none of the

$z_i$ or $\zeta_j$ are elliptic fixed points. We can however generalize to include the case when $z_i$ and $\zeta_j$ are elliptic fixed points. To do this we consider the following generalization of Lemma 5.2.

<u>Lemma 5.2'</u>: Suppose $(\mathcal{X}, f)$ has the limit $\frac{2\pi^2 m^2}{d^2}$ and that $\mathcal{X}(A)$ is an elliptic element of order $d$ (A is the parabolic element $Az = z + 1$). Then

$$f'(z) = \sum_{-m}^{\infty} b_n e^{\frac{2\pi i n z}{d}} \quad \text{and}$$
$$f(z) = \sum_{-m}^{\infty}{}' b'_n e^{\frac{2\pi i n z}{d}} .$$

<u>Proof</u>: Here since $\mathcal{X}(A)^d = \text{Identity}$

$$f(z + d) = f(z)$$

and then calculating exactly as in Lemma 5.2 we have the required result.

<u>Remark</u>: Note that if $(\mathcal{X}, f)$ has a limit different from $\frac{2\pi^2 m^2}{d^2}$ then $\mathcal{X}(A)$ will be loxodromic.

Now suppose that $z_i$ is a fixed point of order $d_i$. Then

$$\tilde{f}'(z) = \sum_{-m_i}^{\infty} b_n e^{\frac{2\pi i n z}{d_i}} \quad \text{and}$$
$$\tilde{f}(z) = \sum b'_n e^{\frac{2\pi i n z}{d_i}} .$$

We have similar relations for $\tilde{g}'(z)$ and $\tilde{g}(z)$ and since

$$V(z + d_i) = V(z)$$

the condition on $\tilde{f}$, $\tilde{f}'$, etc. gives us

$$V(z) = \sum_{-(m_i+1)}^{\infty} a_n e^{\frac{2\pi i n z}{d_i}} .$$

Hence the order of V at the parabolic fixed point arising from an elliptic fixed point is again at most $\frac{-(m_i+1)}{d_i}$ and by the same analysis we get the result in the more general situation.

As an application of this uniqueness theorem we prove a particular case of a theorem due to Maskit (12). The idea of the proof is from Kra.(9 ).

Suppose G is a finitely generated Kleinian group with invariant domain D. Suppose $\pi$: $U \longrightarrow D$ is the holomorphic universal covering map and G is lifted to a Fuchsian group F by this map.

<u>Theorem 6.3</u>: Suppose w: $D \longrightarrow \Delta$ is a holomorphic mapping such that $\Delta$ is invariant under G and $w \circ h = h \circ w$ for all $h \in G$. Suppose also that in a fundamental domain of G w is an $m_i$ to 1 map at $z_i$ $(i = 1, \ldots n)$ such that

$$\sum \frac{(m_i - 1)}{d_i} < A(F)$$

where $d_i$ is the order of $Gz_i$ and it is a local homeomorphism at other points. Then w is the identity mapping.

<u>Proof</u>: Let $D_0 = D - \cup Gz_i$ and $\Delta_0$ be the image of $D_0$ under w.

$$\begin{array}{ccc} U & \xrightarrow{f} & U \\ \pi \downarrow & & \downarrow \pi \\ D_0 & \xrightarrow{h} & D_0 \\ w \downarrow & & \downarrow w \\ \Delta_0 & \xrightarrow{h} & \Delta_0 \end{array}$$

The function $\theta_2\pi$ has the limit $\frac{2\pi^2}{d_i^2}$. at punctures which arise from $z_i$ and vanishes at other punctures, and the function $\theta_2(w \circ \pi)$ has the limit $\frac{2m_i^2\pi^2}{d_i^2}$ at the punctures which arise from $z_i$ and vanishes at other punctures.

Hence as before we can consider the function

$$V(z) = \frac{(\pi(z) - w(\pi(z)))^2}{\pi'(z)(w \circ \pi)'(z)}$$

and from the condition on w

$V \equiv 0$ and hence $\pi = w \circ \pi$ implies

w is the identity.

## Section 7 1-Deformations of Type D

To study 1-deformations $(\mathcal{X}, f)$ of type D we distinguish the points where f has poles. Thus if f is $m_i$ to 1 at $z_i$ $(i = 1,\ldots n)$ and if f has a pole at $z_j$ only $(j = s+1,\ldots n)$ we say f is of type $D = \sum_{1}^{s}(m_i-1)z_i - \sum_{s+1}^{n}(m_i+1)z_i$.
A multiplicative 1-form $\phi$ is of type D if $\phi$ has zeros of order $m_i - 1$ at $z_i$ $(i = 1,\ldots s)$ and poles of order $m_i + 1$ at $z_i$ $(i = s+1,\ldots n)$ and $\phi$ has no other zeros or poles of order greater than two.
A one-connection $\phi$ is of type D if $\phi$ has residues $m_i - 1$ at $z_i$ $(i = 1,\ldots s)$ and $-(m_i + 1)$ at $z_i$ $(i = s+1,\ldots n)$ and $\phi$ has no residues $\neq -2$ at any other point.
We then have

Theorem 7.1: The following are equivalent conditions for a Fuchsian group acting on U:

(a) G admits a 1-deformation of type D

(b) G admits a multiplicative 1-form of type D

(c) G admits a 1-connection of type D which is integrable.

**Proof:** The proof is exactly the proof of Proposition B.

We can prove uniqueness theorems for 1-deformations. Suppose G is a group of the first kind and finitely generated. We call a 1-deformation $(\mathcal{X}, f)$ of G holomorphic if $f \in (\mathcal{X}, f)$ is holomorphic in U.

Theorem 7.2: If $(\mathcal{X}, f)$ and $(\mathcal{X}, g)$ are two holomorphic 1-deformations such that $f(z) = g(z)$ at at least one point $z \in U$ then $f = g$.

Proof: From $f \circ A = \alpha_A f + \beta_A$

$$g \circ A = \alpha_A g + \beta_A \text{ we get}$$

$$(f - g)(Az) = \alpha_A (f - g)(z).$$

Hence $f - g$ is a holomorphic multiplicative 0-form. Since $f - g$ has at least one zero it is identically zero.

We can also prove a theorem similar to Theorem 5.3. Suppose $(\mathcal{X}, f)$ is parabolic and $\theta_1 f \to 0$ as $z \to$ parabolic fixed point through a cusped region. If we assume that a parabolic element $A \in G$ is $Az = z + 1$ then we get

$$f(z + 1) = f(z) + c. \text{ Therefore}$$

$$f'(z + 1) = f'(z)$$

and from the fact that $f'' \to 0$ we get

$$f'(z) = \sum_1^\infty a_n e^{2\pi i n z} + c \text{ and}$$

$$f(z) = \sum_1^\infty a_n' e^{2\pi i n z} + cz.$$

We then have the analogue of Theorem 5.3.

Theorem 7.3: If $(\mathcal{X}, f)$ and $(\mathcal{X}, g)$ are parabolic 1-deformations such that $\theta_1 f \to 0$ and $\theta_1 g \to 0$ at the fixed points and $O(f) + O(g) < 2A(G)$ then $f = g$.

Proof: The proof is the same as the proof of Theorem 5.3.

## Section 8 Divisors Which Include Parabolic Fixed Points

We shall now consider the more general situation where a divisor $D = \sum (m_i - 1) z_i$ is such that $z_i$ may be a parabolic fixed point.

Suppose z is a parabolic fixed point of a Fuchsian group G which is finitely generated and of the first kind. We may assume the fixed points to be $\infty$ and $Az = z + 1$ to be the parabolic element which fixes it. We can map a cusped region belonging to $\infty$ into the punctured unit disc by the map $z \longrightarrow e^{2\pi i z}$. If a function f is m to 1 in the neighborhood of $\infty$ within each cusped region then the function $\tilde{f}(\zeta) = f(z)$, $\zeta = e^{2\pi i z}$ will be m to 1 in the punctured disc. $\tilde{f}$ has an expansion of the form

$$\tilde{f}(\zeta) = a_0 + \sum_{\pm m}^{\infty}{}' a_n \zeta^n .$$

However in this case $\chi(A)$ is not loxodromic. We then define f to be m to 1 in the neighborhood of infinity if

$$f(z) = a_0 + \sum_{\pm m}^{\infty}{}' a_n e^{\frac{2\pi i n z}{d}} .$$

In this case

$$\lim_{z \to i\infty} \theta_2 f = \frac{2\pi^2 m^2}{d^2} .$$

Suppose a deformation $(\chi, f)$ is such that $\chi(A_i)$ $(i = 1, \ldots s)$ is either parabolic, the identity or an elliptic element of order $d_i$ where $A_i$ are parabolic elements of G which fix $z_i$ $(i = 1,\ldots s)$. Then we say that $(\mathcal{X}, f)$ is of type $\sum_1^s m_i z_i + \sum_{s+1}^n (m_i - 1) z_i$ ($z_i$, $i = s + 1,\ldots n$) are not parabolic fixed points) if $\lim (\mathcal{X}, f) = \frac{2\pi^2 m_i^2}{d_i^2}$ and f is $m_i$ to 1 at $z_i$ $(i = 1,\ldots s)$.

If $(\mathcal{X}, f)$ is a deformation of the above type then we define

$$O(f) = \sum_1^s m_i + \sum_{s+1}^n \frac{(m_i - 1)}{v_i}$$

where $v_i$ is the order of $G_{z_i}$.

With this definition we have

<u>Theorem</u> <u>5.3'</u>: If $(\mathcal{X}, f)$ and $(\mathcal{X}, g)$ are 2-deformations of types $D_f$ and $D_g$ respectively, $(\mathcal{X}, f)$ and $(\mathcal{X}, g)$ satisfying the cusp condition at all fixed points not in $D_f$ and $D_g$ respectively, and $O(f) + O(g) < 2A(G)$ then $f = g$.

<u>Proof</u>: We note that if $z_i \in D_f$ is a parabolic fixed point then the function

$$V(z) = \frac{(f(z) - g(z))^2}{f'(z)g'(z)}$$

has a pole of order at most $m_i$ at $z_i$. The result therefore follows.

The existence theorem 4.1 can similarly be generalized to this situation. The only difference is that we shall have to consider 2-forms which have nonzero limits at some parabolic fixed points.

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