ON THE COHOMOLOGY OF MODULES OVER THE KLEIN GROUP

A Dissertation presented

by

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Abstract of the Thesis

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In this paper we calculate the first and second cohomology groups for a certain family of modules over the Klein group $\Phi = Z_2 \times Z_2$. This is done by first finding the structure of these modules when restricted to the cyclic subgroups of Φ. Then, by the Hochschild-Serre spectral sequence, it becomes evident that the cohomology groups are imbedded in exact sequences. In order to eliminate the indeterminacy of the exact sequences, we must do the following. For K cyclic in Φ, we first must determine which elements of $H^2(K,M)$ can be lifted under the restriction map

$$\text{res}_K\colon H^2(\Phi,M) \longrightarrow H^2(K,M)^{\Phi},$$

(where res_K is induced by the inclusion $K \subset \Phi$. We then must decide whether $H^2(\Phi,M)$ has any elements of order four.

The former question is answered more generally, for any group Φ of the form $K \times L$ where K is arbitrary and L is finite cyclic. This is done by associating a family of obstructions to each element α, of $H^2(K,M)^{\Phi}$. Then the lifting of α under res_K is equivalent to the vanishing of one of these obstructions. Returning to $\Phi = Z_2 \times Z_2$, the question of the existence of elements of order 4 is answered by analyzing the "antispecial cohomology classes" -- those elements of $H^2(\Phi,M)$ that vanish when restricted to $H^2(K,M)$ for all cyclic $K \subset \Phi$.

In addition, the notion of "supernormalized cocycles" is discussed. This provides a canonical form for representatives of elements of $H^2(K \times L,M)$ and motivates much of the theory in this paper.

This dissertation is dedicated to my wife Eleanor and son Bret whose love formed a very tangible chapter in this work.

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My family deserves special thanks for their devotion and encouragement throughout this effort.

Finally, I am grateful to Ms. Louise Baggot for beautifully typing this manuscript.

INTRODUCTION

In general, the problem of computing the cohomology of groups for arbitrary groups is difficult. This is made even more so, because the structure of modules is known only for a small family of groups. These include the cyclic p-groups [11], dihedral groups of order 2p [5,6], cyclic groups of square free order [4], Z_4 [4], and the Klein group $Z_2 \times Z_2$.

Calculating the cohomology of cyclic groups is easy and the cohomology of the dihedral group case was done by Parr [10]. The only one of the above groups for which there is an infinite number of non-isomorphic modules is the Klein group, and it is to this case that we address ourselves in this paper.

Heller and Reiner [2] have shown that the number of indecomposable isomorphism classes of modules over Z[G] is infinite if G is a non-cyclic p-group. Nazarova [8] has provided a description in terms of two lists of representation types for the indecomposable modules over $Z[\Phi]$ where $\Phi = Z_2 \times Z_2$, of rank $n \geq 5$. The first list has an infinite number of representation types, while the second has a finite number of types. In this paper we consider those modules describeable in terms of the first list and as we indicate in the final section, we intend to carry out a similar analysis of the second list in a subsequent paper.

In section 3 we analyze the structure of these modules with respect to each of the three cyclic subgroups of $Z_2 \times Z_2$, and we summarize the corresponding first and second cohomology groups. In section 4 we use the results of section 3 and apply the Hochschild-Serre [3] spectral sequence to realize $H^2(\Phi,m)$ as one of the terms in an exact sequence. Section 5 is a discussion of "supernormalized 2-cocycles" for arbitrary groups having a proper normal subgroup, which provides us with a type of canonical form for a cocycle in which the restriction of that cocycle to a normal subgroup appears. This provides a key to deciding (in section 6), whether the corresponding restriction map in cohomology can be lifted, in the case where the group $G \simeq H \times K$ with K cyclic. As a result of section 6, the group $H^2(\Phi,m)$ is usually seen to be the middle term of a short exact sequence with the end terms having exponent 2. In section 7 we discuss antispecial cohomology classes and in section 8, we show how the form of these classes can help decide whether $H^2(\Phi,m)$ has any elements of order 4. In section 9 we apply the results of sections 5 through 8 to calculate $H^1(\Phi,m)$ and $H^2(\Phi,m)$.

In the last section, we remark on an application of these results to flat manifolds.

§1. BRIEF SUMMARY OF NAZAROVA PAPER

In her paper, Nazavova describes canonical forms for representations of the Klein group, in the group of integral n × n matrices, $n \geq 5$. A representation is described by giving a pair of matrices A and B, such that $A^2 = B^2 = E$ = identity matrix, and $AB = BA$. A $\mathbb{Z}$-basis for the kernal of the operator A-B is supplemented to form a $\mathbb{Z}$-basis for the free abelian group of rank n upon which Φ operates. Then, with respect to this basis, the matrices A and B will have the form

$$A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline & A_{22} \end{array}\right), \qquad B = \left(\begin{array}{c|c} A_{11} & B_{12} \\ \hline & -A_{22} \end{array}\right)$$

where a blank indicates a zero submatrix.

The matrices A_{11} and A_{22}, providing representations of the cyclic group of order two, can then be decomposed into diagonal blocks of the forms (+1), (-1), and $\binom{01}{10}$. Nazavova then distinguishes two families of representations where A_{11} and A_{22} can be decomposed into only (+1) and (-1) blocks. The second family of representations has the property that A_{11} or A_{22} has a single block of the form $\binom{01}{10}$.

If a representation is in the first family it can be put into the form

$$A = \left(\begin{array}{cc|cc} E & & & D_4 \\ & -E & D_2 & \\ \hline & & E & \\ & & & -E \end{array}\right) \qquad B = \left(\begin{array}{cc|cc} E & & D_1 & \\ & -E & & D_3 \\ \hline & & -E & \\ & & & E \end{array}\right)$$

Hence any representation of the first family can be described by giving the matrix

$$D = \begin{array}{|c|c|} \hline D_1 & D_4 \\ \hline D_2 & D_3 \\ \hline \end{array}$$

On the other hand, any representation belonging to the second family, (i.e., ker (A-B) or cok (A-B) does not split up into (+1)'s and (-1)'s.) can be put into the general form

$$A = \left(\begin{array}{ccc|ccc} E & & & 0 & A_{15} & A_{16} \\ & -E & & A_{24} & 0 & A_{26} \\ & & \begin{smallmatrix}0&1\\1&0\end{smallmatrix} & A_{34} & A_{35} & 0 \\ \hline & & & E & & \\ & & & & -E & \\ & & & & & \begin{smallmatrix}0&1\\1&0\end{smallmatrix} \end{array}\right) \quad \text{and}$$

$$B = \left(\begin{array}{ccc|ccc} E & & & B_{14} & 0 & B_{16} \\ & -E & & 0 & B_{25} & B_{26} \\ & & \begin{smallmatrix}0&1\\1&0\end{smallmatrix} & B_{34} & B_{35} & 0 \\ \hline & & & -E & & \\ & & & & E & \\ & & & & & \begin{smallmatrix}0&-1\\-1&0\end{smallmatrix} \end{array}\right)$$

Any indecomposable module has a representation in one of the above canonical forms depending on which family it belongs to.

Nazarova describes those indecomposable representations in the first family by giving their D-matrix, and those belonging to the second family in terms of the matrices B_{14}, A_{15}, B_{16}, A_{16}, etc. The first family has four equivalence classes of representations for $n \equiv 1,2$ or 3 (mod 4) where n is the rank of the module. However, when $n \equiv 0 \pmod 4$, there are $3 + \frac{1}{4} \sum_{d|m} \phi(d) 2^{m/d}$ classes of representations where $m = n/4$ and ϕ is the Euler phi function.

The second family, however, has only a finite number of classes of representations within each equivalence class (mod 4), (actually, there are four each for $n \equiv 1$ and $n \equiv 3 \pmod 4$, two each for $n \equiv 2 \pmod 4$, $n \equiv 0$ (mod 8) and $n \equiv 4 \pmod 8$, and one for $n \equiv 0 \pmod 4$)).

§2. REPRESENTATIONS AND NOTATION

In this section we list the representations in Nazavova's first family by giving the matrix

$$D = \left[\begin{array}{c|c} D_1 & D_4 \\ \hline D_2 & D_3 \end{array}\right].$$

Let n = rank of module M as a free abelian group (listed in order of Nazavova's description).

{1} $n \equiv 0 \pmod 4$

$$D = \left[\begin{array}{c|c} \begin{array}{cccc} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{array} & \begin{array}{ccc} 1 & & \\ & \ddots & \\ & & 1 \end{array} \\ \hline \begin{array}{ccc} 1 & & \\ & \ddots & \\ & & 1 \end{array} & \begin{array}{ccc} 1 & & \\ & \ddots & \\ & & 1 \end{array} \end{array}\right]$$

{2a} $n \equiv 1 \pmod 4$

$$D = \left[\begin{array}{c|c} \begin{array}{cccc} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{array} & \begin{array}{ccc} 1 & & \\ & \ddots & \\ & & 1 \end{array} \\ \hline \begin{array}{cccc} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{array} & \begin{array}{ccc} 1 & & \\ & \ddots & \\ & & 1 \end{array} \end{array}\right]$$

{2b} $n \equiv 3 \pmod 4$

$$\left[\begin{array}{c|c} \begin{array}{ccc} 0 & \cdots & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{array} & \begin{array}{ccc} 1 & & \\ & \ddots & \\ & & 1 \end{array} \\ \hline \begin{array}{ccc} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & \cdots & 0 \end{array} & \begin{array}{ccc} 1 & & \\ & \ddots & \\ & & 1 \end{array} \end{array}\right]$$

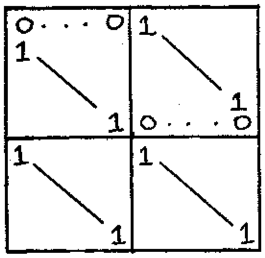
{3a} $n \equiv 2 \pmod 4$

$$\left[\begin{array}{c|c} \begin{array}{ccc} 0 & \cdots & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{array} & \begin{array}{ccc} 1 & & \\ & \ddots & \\ & & 1 \end{array} \\ \hline \begin{array}{ccc} 1 & & \\ & \ddots & \\ & & 1 \end{array} & \begin{array}{cccc} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{array} \end{array}\right]$$

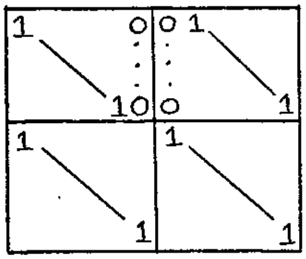
{3a} $n \equiv 2 \pmod 4$

$$\left[\begin{array}{c|c} \begin{array}{cccc} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{array} & \begin{array}{ccc} 1 & & \\ & \ddots & \\ & & 1 \end{array} \\ \hline \begin{array}{ccc} 1 & & \\ & \ddots & \\ & & 1 \end{array} & \begin{array}{ccc} 0 & \cdots & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{array} \end{array}\right]$$

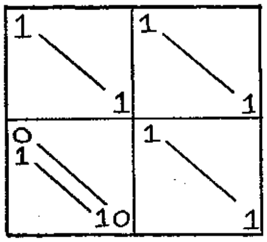
{4a} n ≡ 1 (mod 4)



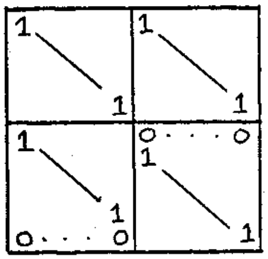
{4b} n ≡ 3 (mod 4)



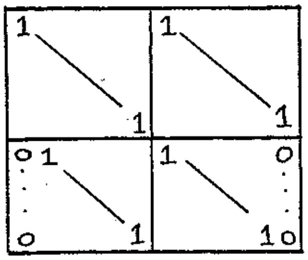
{5} n ≡ 0 (mod 4) D =



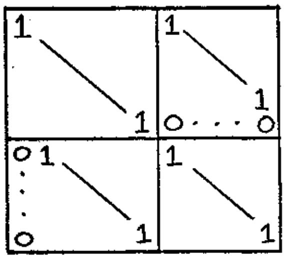
{6a} n ≡ 1 (mod 4)



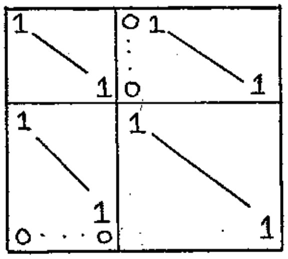
{6b} n ≡ 3 (mod 4)



{7a} n ≡ 2 (mod 4)

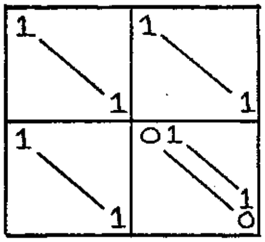


{7b} n ≡ 2 (mod 4)

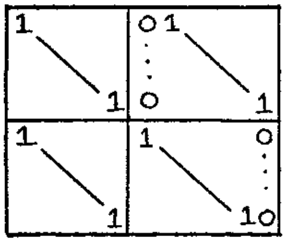


{8} n ≡ 0 (mod 4)

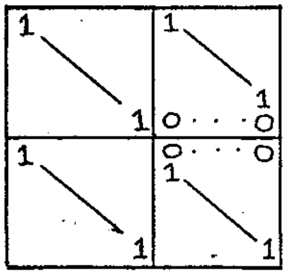
D =



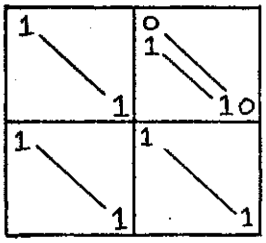
{9a} n ≡ 1 (mod 4)



{9b} n ≡ 3 (mod 4)

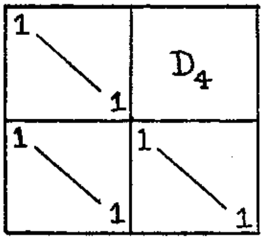


{10} n ≡ 0 (mod 4)



Finally, there is the family of representations of the form

{∞} D =



n ≡ 0 (mod 4)

where D_4 is an indecomposable matrix over Z_2

Remark. D_4 may be varied up to conjugation without changing the equivalence class of the representation.

§3. $\mathbb{Z}_2$-STRUCTURE

The reader may omit the details of this section until they are refered to later.

Each module over Φ can be regarded as a module over the group ring $\Gamma[\mathbb{Z}_2]$ in three ways according to the action of the three generators σ,τ and $\sigma\tau$. In this section we describe these induced $\mathbb{Z}_2$ module structures, and give the cohomology of $\Gamma[\mathbb{Z}_2]$ with coefficients in each module.

As is well known, every module over $\mathbb{Z}_2$ is isomorphic to a direct sum of indecomposeable modules of three types, denoted as follows;

$<+1>$ is $\mathbb{Z}$ with trivial action

$<-1>$ is $\mathbb{Z}$ with action given by $\gamma(n) = -n$, $\gamma \in \mathbb{Z}_2$.

and

Γ is the group ring $\Gamma[\mathbb{Z}_2]$ regarded as a module over itself.

The matrix representations for these modules were mentioned in §1; they are $(+1)$, (-1) and $\binom{01}{10}$ respectively.

The cohomology of the indecomposeable modules over $\mathbb{Z}_2$ is trivial to calculate;

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The cohomology of the indecomposeable modules over $\mathbb{Z}_2$ is trivial to calculate;

$$H^n(Z_2, <+1>) = \begin{cases} 0 & \text{if } n \text{ odd} \\ Z_2 & \text{if } n > 0 \text{ and even} \\ Z & \text{if } n = 0 \end{cases}$$

$$H^n(Z_2, <-1>) = \begin{cases} Z_2 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

and

$$H^n(Z_2,\Gamma) = \begin{cases} 0 \text{ if } n \neq 0 \\ Z \quad\quad n = 0 \end{cases}$$

Since cohomology is additive, the cohomology of any module over Z_2 is the direct sum of the cohomology of each indecomposeable summand.

For most of the modules, one can read the Z_2 structure with respect to σ and τ by simple inspection of the A and B matrix representations. However, the structure with respect to $\sigma\tau$ is less transparent and we give several examples to show how to see through the representation into its module structure.

Before proceding, we would like to establish some notation which will be helpful in the sequel. Each representation has four $\pm$ E blocks down the diagonal where E stands for the identity matrix. Let V_1, V_2, V_3, V_4 be the four subgroups of M corresponding to each of these blocks. So as a free abelian group, $M \simeq V_1 + V_2 + V_3 + V_4$ and each V_i, $i = 1,..,4$ is also a free abelian group on bases

$\{a_1,\dots,a_{n_1}\}$, $\{b_1,\dots,b_{n_2}\}$, $\{c_1,\dots,c_{n_3}\}$ and $\{d_1,\dots,d_{n_4}\}$ respectively. Hence $n_1 + n_2 + n_3 + n_4 = n = \text{rank } M$.

We will describe modules {1} and {2a} in greater detail than the rest, as an illustration.

{1}

$$D = \begin{array}{|c|c|}\hline \begin{matrix}0&1&&\\&\ddots&\ddots&\\&&\ddots&1\\&&&0\end{matrix} & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix}\\ \hline \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix}\\ \hline\end{array}$$

Here σ is given by

$$A = \left(\begin{array}{cc|cc} E&&&E\\ &-E&E&\\ \hline &&E&\\ &&&-E\end{array}\right)$$

M is clearly seen to be $(V_1 + V_4) \oplus (V_2 + V_3)$. Each of the submodules $(V_1 + V_4)$ and $(V_2 + V_3)$ is a direct sum of copies of the group ring Γ. So, with respect to σ, M is a direct sum of copies of Γ. I.e., $M \underset{\sigma}{\cong} \sum\Gamma$. τ is given by

$$B = \left(\begin{array}{cc|c|c} E&&\begin{pmatrix}0&1&&\\&\ddots&\ddots&\\&&\ddots&1\\&&&0\end{pmatrix}&\\ \hline &-E&&E\\ \hline &&-E&\\ &&&E\end{array}\right)$$

Here $M \underset{\tau}{\cong} (V_1 + V_3) \oplus (V_2 + V_4)$. $(V_2 + V_4)$ is a direct sum of Γ's whereas $(V_1 + V_3)$ has a representation

$$\begin{array}{cc} a_{n_1}\rightarrow & \left(\begin{array}{c|c} E & \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{matrix} \\ \hline & -E \end{array}\right). \\ & \begin{array}{cc} & \underset{\uparrow}{c_1} \end{array} \end{array}$$

There is a $<+1>$ summand generated by a_{n_1} and a $<-1>$ summand generated by c_1. The rest of the module breaks up into a number of copies of Γ. Hence with respect to τ, this module $M \underset{\tau}{\cong} \sum\Gamma \oplus <+1> \oplus <-1>$.

Now we view M as a $\Gamma(Z_2)$ module with respect to $\sigma\tau$ by means of the representation

$$AB = \left(\begin{array}{cc|c|c} E & & \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} & E \\ \cline{3-4} & E & -E & -E \\ \hline & & -E & \\ & & & -E \end{array}\right)$$

If we conjugate the above matrix by one of the form $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$, then the resulting matrix $(AB)'$ still has the form $\left(\begin{array}{c|c} E & D' \\ \hline & -E \end{array}\right)$.

In particular, this says we can perform elementary row operations on the top half of AB and elementary column operations on the right hand half of AB, without changing the isomorphism class of the representation.

Now if we add the second set of rows, (D_2D_3) of D to the corresponding rows in the first set (D_1D_4), this changes D as follows;

$$\left(\begin{array}{c|c} \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} & E \\ \hline -E & -E \end{array}\right) \overset{+}{\curvearrowleft} \longrightarrow \underset{\xleftarrow{\;-\;}}{\left(\begin{array}{c|c} \begin{pmatrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & -1 \end{pmatrix} & 0 \\ \hline -E & -E \end{array}\right)}$$

Now substract the second set of columns $\begin{pmatrix} D_4 \\ D_3 \end{pmatrix}$ from the first set of columns $\begin{pmatrix} D_1 \\ D_2 \end{pmatrix}$ of D, this gives

$$\overset{\overset{+}{\curvearrowright\curvearrowright}\cdots\cdots}{\left(\begin{array}{c|c} \begin{pmatrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & -1 \end{pmatrix} & 0 \\ \hline 0 & -E \end{array}\right)}$$

Now by adding the first column in $\begin{pmatrix} D_1 \\ D_2 \end{pmatrix}$ to the second, and then the new second column to the third, etc., we eventually end up with

$$D' = \left(\begin{array}{c|c} \begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix} \end{array}\right)$$

Hence the representation AB is equivalent to

$$\left(\begin{array}{cc|cc} E & & -E & \\ & E & & -E \\ \hline & & -E & \\ & & & -E \end{array}\right)$$

and M is seen to be a direct sum of Γ's. So, with respect to $\sigma\tau$, $M \approx \sum\Gamma$. Summarizing, we have

$$M \approx \begin{cases} \sum\Gamma & \text{with respect to } \sigma \\ \sum\Gamma \oplus \langle +1\rangle \oplus \langle -1\rangle & \text{with respect to } \tau \\ \sum\Gamma & \text{with respect to } \sigma\tau \end{cases}$$

For modules of type {2a} we will shorten the explanation.

$$D = \left[\begin{array}{c|c} \begin{matrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{matrix} & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \\ \hline \begin{matrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{matrix} & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{array}\right]$$

σ is given by

$$A = \left(\begin{array}{cc|c|c} & & \multicolumn{1}{l}{\overset{c_1}{\downarrow}} & \\ E & & & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \\ \cline{3-4} & -E & \begin{matrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{matrix} & \\ \hline & & E & \\ & & & -E \end{array}\right)$$

M is seen to be a sum of copies of Γ along with a $\langle +1\rangle$ summand; $M \underset{\sigma}{\cong} \sum\Gamma \oplus \langle +1\rangle$, where $\langle +1\rangle$ is generated by c_1. τ is given by

$$B = \left(\begin{array}{c|c|c}
 & \begin{matrix} & & & \downarrow \\ & & & c_n \end{matrix} & \\
E & \begin{matrix}1 & & & 0\\ & \ddots & & \vdots \\ & & 1 & 0\end{matrix} & \\ \hline
-E & & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} \\ \hline
 & -E & \\
 & & E
\end{array}\right)$$

τ is a sum of copies of Γ along with a $\langle -1\rangle$ summand generated by the basis element c_{n_3}; $M \underset{\tau}{\cong} \sum\Gamma \oplus \langle -1\rangle$. $\sigma\tau$ is given by

$$AB = \left(\begin{array}{c|c|c}
 & \overset{-}{\curvearrowleft} & \\
E & \begin{matrix}1 & & & 0\\ & \ddots & & \vdots \\ & & 1 & 0\end{matrix} & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} \\ \hline
-E & \begin{matrix}0 & -1 & & \\ \vdots & & \ddots & \\ 0 & & & -1\end{matrix} & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} \\ \hline
 & -E & \\
 & & -E
\end{array}\right)$$

Subtracting $\begin{pmatrix} D_4 \\ D_3 \end{pmatrix}$ from $\begin{pmatrix} D_1 \\ D_2 \end{pmatrix}$ gives

$$D' = \left(\begin{array}{c|c}
0 & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} \\ \hline
\begin{matrix}\cancel{1}\,-1 & & \\ & \ddots\ddots & \\ & & 1\,-1\end{matrix} & \begin{matrix}-1 & & \\ & \ddots & \\ & & -1\end{matrix}
\end{array}\right) \curvearrowright +$$

Adding (D_1D_4) to (D_2D_3) gives

$$D' = \left(\begin{array}{c|c} O & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} \\ \hline \begin{matrix}1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1\end{matrix} & O \end{array}\right)$$

$$\underset{+}{\curvearrowleft\curvearrowright}\cdots\cdots$$

Now adding the first column to the second, then the second to the third etc. gives

$$D' = \left(\begin{array}{c|c} & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} \\ \hline \begin{matrix}1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & \underset{\uparrow}{0}\end{matrix} & \end{array}\right) .$$

So

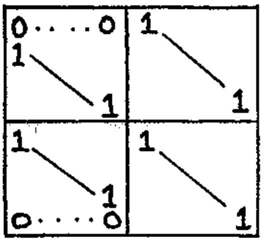
$$AB \sim \left(\begin{array}{c|c} \begin{matrix}E & \\ & E\end{matrix} & \overset{\downarrow}{\begin{array}{c|c} & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} \\ \hline \begin{matrix}1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0\end{matrix} & \end{array}} \\ \hline & \begin{matrix}-E & \\ & -E\end{matrix} \end{array}\right)$$

Hence $M \underset{\sigma\tau}{\cong} \sum\Gamma \oplus \langle -1\rangle$. So the structure of M with respect to σ, τ, and $\sigma\tau$ is

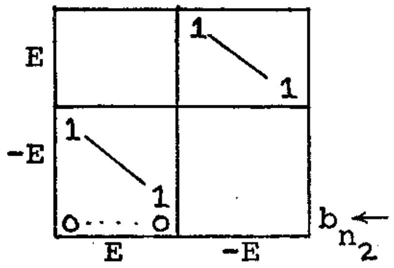
$$\{2a\} \qquad \begin{array}{r|l|} \cline{2-2} \sigma & \sum\Gamma \oplus \langle +1\rangle \\ \tau & \sum\Gamma \oplus \langle -1\rangle \\ \sigma\tau & \sum\Gamma \oplus \langle -1\rangle \\ \cline{2-2} \end{array} .$$

{2b} For these modules and the following, we will compress the discussion further by just indicating how to read the appropriate matrices.

D =

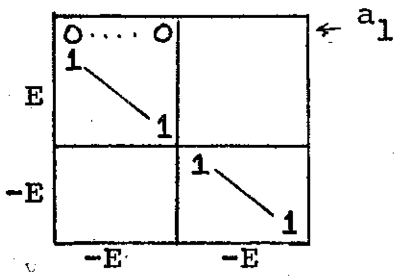


A =



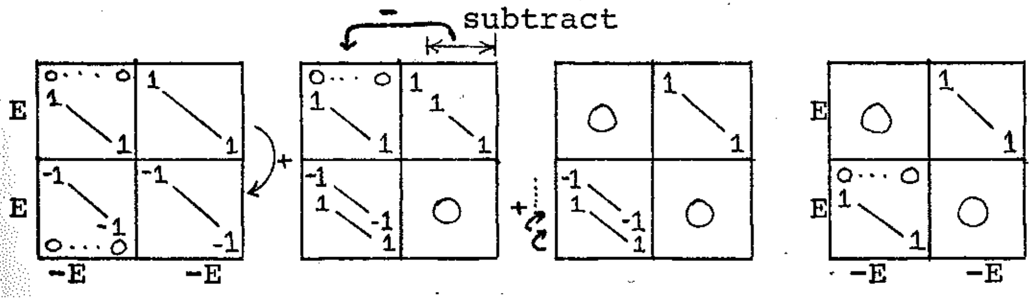
$M \underset{\sigma}{\approx} \sum\Gamma \oplus \langle -1\rangle$ with $\langle -1\rangle$ generated by b_{n_2}.

B =



$M \underset{\tau}{\approx} \sum\Gamma \oplus \langle +1\rangle$ with $\langle +1\rangle = \langle a_1\rangle$

AB =



From this last picture we can read $M \underset{\sigma\tau}{\approx} \sum\Gamma \oplus \langle +1\rangle$

and we have {2b}

$$\boxed{\begin{array}{ll} \sigma & \langle -1\rangle \oplus \sum\Gamma \\ \tau & \langle +1\rangle \oplus \sum\Gamma \\ \sigma\tau & \langle +1\rangle \oplus \sum\Gamma \end{array}}$$

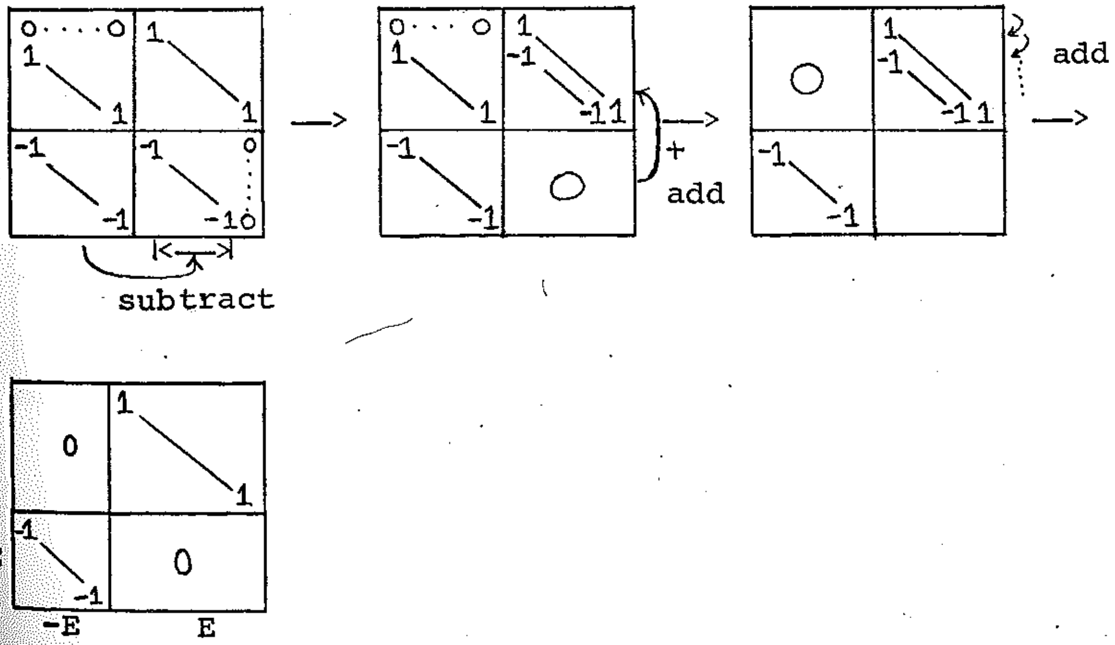
{3a} $D = \left[\begin{array}{c|c} \begin{matrix}0&\cdots&0\\1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} \\ \hline \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}1&&&0\\&\ddots&&\vdots\\&&1&0\end{matrix} \end{array}\right]$ $\quad A = \begin{array}{c|c|c} & & \\ E & & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} \\ \hline -E & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \\ \hline & E & -E \end{array}$

So $M \underset{\sigma}{\cong} \sum\Gamma$

$$B = \begin{array}{c|c|cl} & & d_n^{\downarrow} & \\ E & \begin{matrix}0&\cdots&0\\1&&\\&\ddots&\\&&1\end{matrix} & & \leftarrow a_1 \\ \hline -E & & \begin{matrix}1&&&0\\&\ddots&&\vdots\\&&1&0\end{matrix} & \\ \hline & -E & E & \end{array}$$

Here, there are two <+1> summands. $M \underset{\tau}{\cong} \sum\Gamma \oplus \langle +1\rangle \oplus \langle +1\rangle$.

AB =



So $M \underset{\sigma\tau}{\cong} \sum\Gamma$, and we have

$$\{3a\} \qquad \begin{array}{r|l|} \cline{2-2} \sigma & \sum\Gamma \\ \tau & \sum\Gamma \oplus \langle +1\rangle \oplus \langle +1\rangle \\ \sigma\tau & \sum\Gamma \\ \cline{2-2} \end{array},$$

$$\{3b\} \qquad \left[\begin{array}{c|c}
\begin{matrix}1&&0\\&\ddots&\vdots\\&1&0\end{matrix} & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix}\\ \hline
\begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}0&\cdots&0\\1&&\\&\ddots&\\&&1\end{matrix}
\end{array}\right]
\qquad
A = \begin{array}{r|c|c|}
\cline{2-3}
E & & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix}\\
\cline{2-3}
-E & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \\
\cline{2-3}
\multicolumn{1}{r}{} & \multicolumn{1}{c}{E} & \multicolumn{1}{c}{-E}
\end{array}$$

So $M \underset{\sigma}{\cong} \sum\Gamma$

$$B = \begin{array}{r|c|c|l}
\multicolumn{1}{r}{} & \multicolumn{1}{c}{} & \multicolumn{1}{l}{\overset{\downarrow}{c_n}} & \\
\cline{2-3}
E & \begin{matrix}1&&0\\&\ddots&\vdots\\&1&0\end{matrix} & & \\
\cline{2-3}
-E & & \begin{matrix}0&\cdots&0\\1&&\\&\ddots&\\&&1\end{matrix} & b_1 \leftarrow \\
\cline{2-3}
\multicolumn{1}{r}{} & \multicolumn{1}{c}{-E} & \multicolumn{1}{c}{E} &
\end{array}$$

Here, there are two $\langle -1\rangle$ summands $M \underset{\tau}{\cong} \sum\Gamma \oplus \langle -1\rangle \oplus \langle -1\rangle$ with $\langle -1\rangle$'s generated by c_{n_3} and b_1.

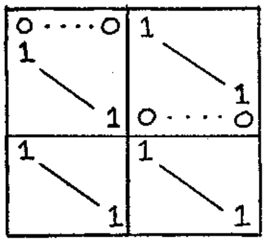
$$AB = \underset{\text{subtract}}{\left[\begin{array}{c|c}
\begin{matrix}1&&0\\&\ddots&\vdots\\&1&0\end{matrix} & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix}\\ \hline
\begin{matrix}-1&&\\&\ddots&\\&&-1\end{matrix} & \begin{matrix}0&\cdots&0\\-1&&\\&\ddots&\\&&-1\end{matrix}
\end{array}\right]}
\longrightarrow
\left[\begin{array}{c|c}
0 & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix}\\ \hline
\begin{matrix}-1&&\\1&\ddots&\\&\ddots&\\&1&-1\end{matrix} & \begin{matrix}0&\cdots&0\\-1&&\\&\ddots&\\&&-1\end{matrix}
\end{array}\right]
\xrightarrow{+\ \text{add}}
\underset{\text{add}}{\left[\begin{array}{c|c}
0 & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix}\\ \hline
\begin{matrix}-1&&\\1&\ddots&\\&\ddots&\\&1&1\end{matrix} & 0
\end{array}\right]}
\longrightarrow$$

$$\begin{array}{r|c|c|}
\cline{2-3}
E & 0 & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix}\\
\cline{2-3}
E & \begin{matrix}-1&&\\&\ddots&\\&&-1\end{matrix} & 0\\
\cline{2-3}
\end{array}$$

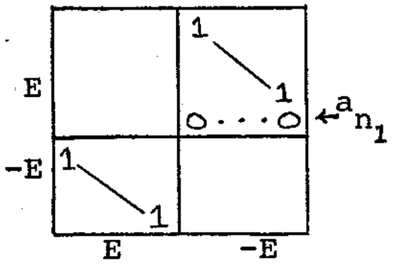
So $M \underset{\sigma\tau}{\cong} \sum\Gamma$, and we have

$$\{3b\} \qquad \begin{array}{r|l|} \cline{2-2} \sigma & \sum\Gamma \\ \tau & \sum\Gamma \oplus <-1> \oplus <-1> \\ \sigma\tau & \sum\Gamma \\ \cline{2-2} \end{array} .$$

{4a} D =

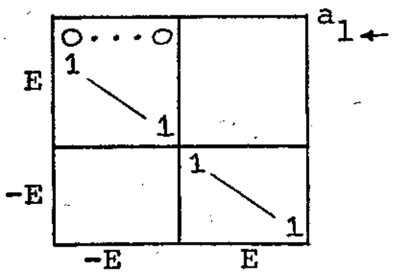


A =



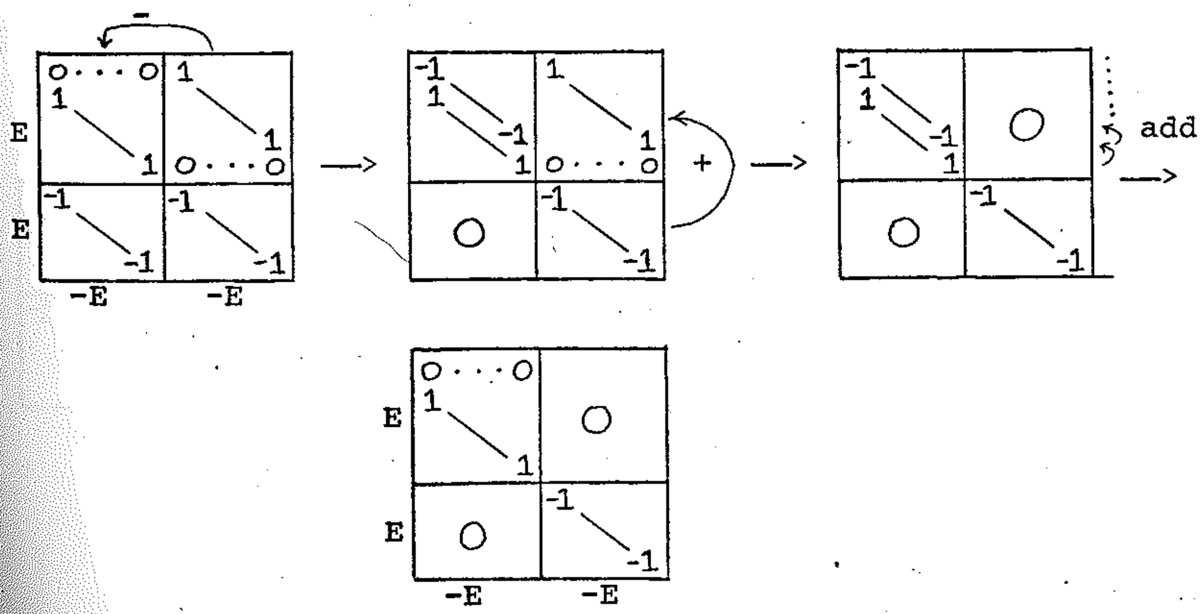
$M \underset{\sigma}{\cong} \sum\Gamma \oplus <+1>$ with $<+1> = <a_{n_1}>$

B =



$M \underset{\tau}{\cong} \sum\Gamma \oplus <+1>$ with $<+1> = <a_1>$

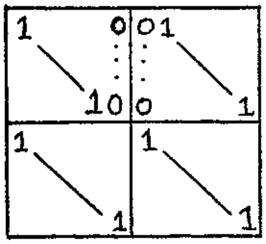
AB =



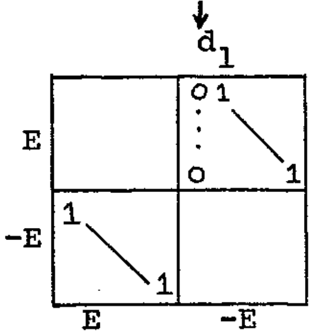
So $M \underset{\sigma\tau}{\cong} \sum\Gamma \oplus <+1>$, and we have

$$\{4a\} \qquad \begin{array}{r|c|} \cline{2-2} \sigma & \sum\Gamma \oplus <+1> \\ \tau & \sum\Gamma \oplus <+1> \\ \sigma\tau & \sum\Gamma \oplus <+1> \\ \cline{2-2} \end{array} ,$$

{4b} D =

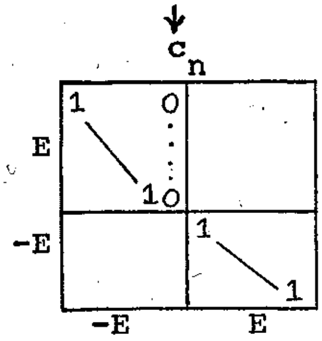


A =



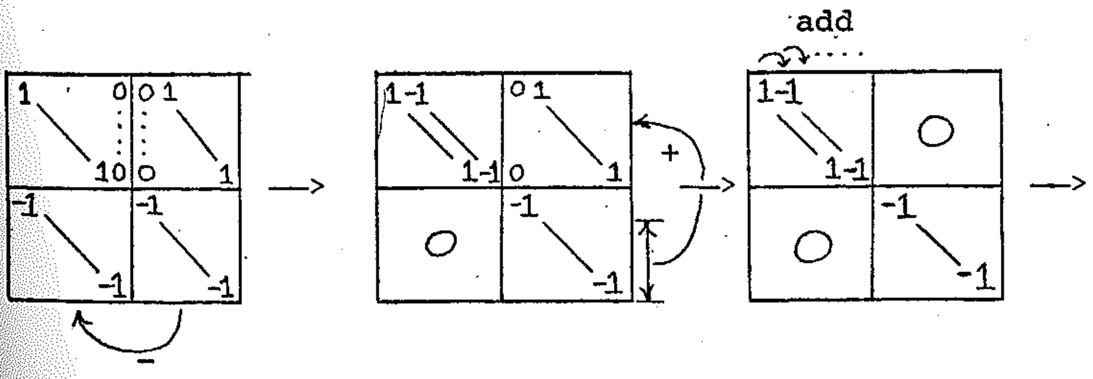
$M \underset{\sigma}{\cong} \sum\Gamma \oplus <-1>$ with $<-1> = <d_1>$

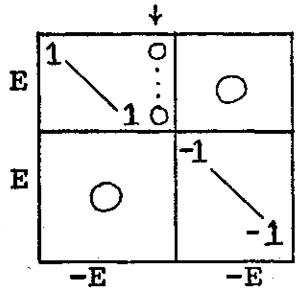
B =



$M \underset{\tau}{\cong} \sum\Gamma \oplus <-1>$ with $<-1> = <c_{n_3}>$

AB =

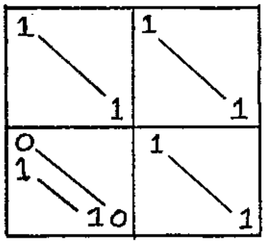




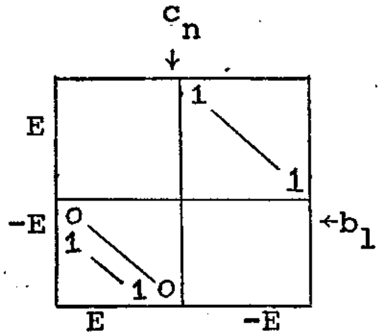
So $M \underset{\sigma\tau}{\approx} \sum\Gamma \oplus <-1>$, and we have

$$\{4b\} \qquad \begin{array}{c|c|} \sigma & \sum\Gamma \oplus <-1> \\ \tau & \sum\Gamma \oplus <-1> \\ \sigma\tau & \sum\Gamma \oplus <-1> \end{array} .$$

{5} D =

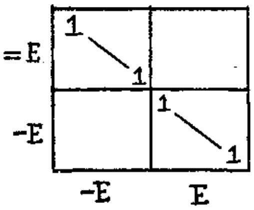


A =



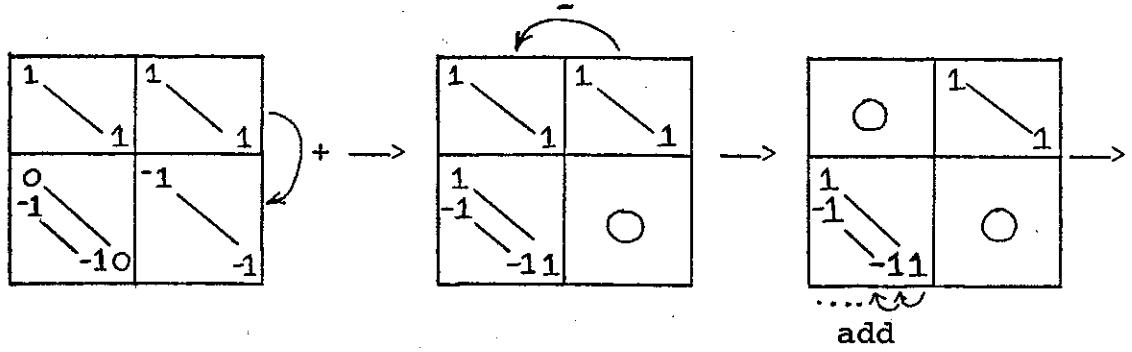
$M \underset{\sigma}{\approx} \sum\Gamma \oplus <+1> \oplus <-1>$ with $<+1> = <c_{n_3}>$, $<-1> = <b_1>$

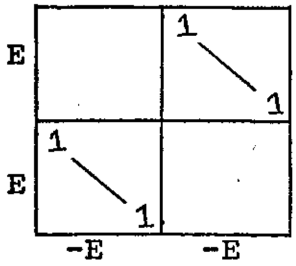
B =



$M \underset{\tau}{\approx} \sum\Gamma$

AB =



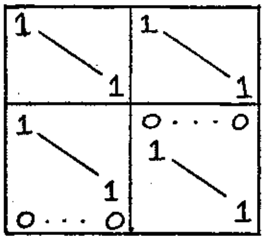


So $M \underset{\sigma\tau}{\cong} \sum\Gamma$, and we have

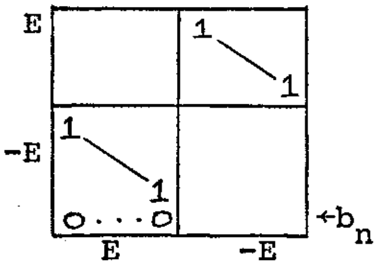
{5}

σ	$\sum\Gamma \oplus <+1> \oplus <-1>$
τ	$\sum\Gamma$
$\sigma\tau$	$\sum\Gamma$

{6a} D =

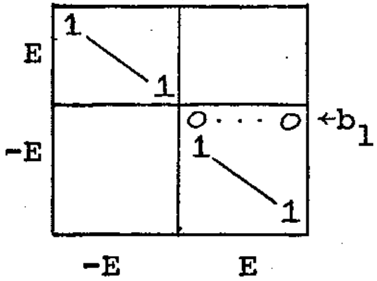


A =



$M \underset{\sigma}{\cong} \sum\Gamma \oplus <-1>$ with $<-1> = <b_{n_2}>$

B =



$M \underset{\tau}{\approx} \sum\Gamma \oplus \langle -1\rangle$ with $\langle -1\rangle = \langle b_1\rangle$

$$AB = \begin{array}{|c|c|}\hline \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} \\ \hline \begin{matrix}-1&&\\&\ddots&\\&&-1\\0&\cdots&0\end{matrix} & \begin{matrix}0&\cdots&0\\-1&&\\&\ddots&\\&&-1\end{matrix} \\ \hline \end{array}$$

By row operations, this can obviously be transformed to the AB matrix of module {4a}.

$$\text{Hence } AB \sim \begin{array}{c|c|c|c} \cline{2-3} E & \begin{matrix}0&\cdots&0\\1&&\\&\ddots&\\&&1\end{matrix} & & \leftarrow \\ \cline{2-3} E & & \begin{matrix}-1&&\\&\ddots&\\&&-1\end{matrix} & \\ \cline{2-3} \end{array}$$

So $M \underset{\sigma\tau}{\approx} \sum\Gamma \oplus \langle +1\rangle$, and we have

$$\{6a\} \qquad \begin{array}{c|c|} \cline{2-2} \sigma & \sum\Gamma \oplus \langle -1\rangle \\ \tau & \sum\Gamma \oplus \langle -1\rangle \\ \sigma\tau & \sum\Gamma \oplus \langle +1\rangle \\ \cline{2-2} \end{array}$$

$$\{6b\}\quad D = \begin{array}{|c|c|}\hline \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} \\ \hline \begin{matrix}0&1&&\\\vdots&&\ddots&\\0&&&1\end{matrix} & \begin{matrix}1&&&0\\&\ddots&&\vdots\\&&1&0\end{matrix} \\ \hline \end{array} \qquad A = \begin{array}{c|c|c|} & \overset{\downarrow}{c_1} & \\ \cline{2-3} E & & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} \\ \cline{2-3} -E & \begin{matrix}0&1&&\\\vdots&&\ddots&\\0&&&1\end{matrix} & \\ \cline{2-3} & E & -E \end{array}$$

$M \approx \sum\Gamma \oplus \langle +1\rangle$, $\langle +1\rangle = \langle c_1\rangle$

$$B = \begin{array}{r|c|c|} & & \overset{\displaystyle d_n}{\downarrow} \\ \hline E & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \\ \hline -E & & \begin{matrix}1&&&0\\&\ddots&&\vdots\\&&1&0\end{matrix} \\ \hline & -E & E \end{array}$$

$$M \underset{\tau}{\approx} \sum\Gamma \oplus \langle +1\rangle, \quad \langle +1\rangle = \langle d_{n_4} \rangle$$

$$AB = \begin{array}{r|c|c|} \hline E & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} \\ \hline E & \begin{matrix}0&-1&&\\\vdots&&\ddots&\\0&&&-1\end{matrix} & \begin{matrix}-1&&&0\\&\ddots&&\vdots\\&&-1&0\end{matrix} \\ \hline & -E & -E \end{array}$$

This can be transformed by row and column operations to AB of the module {4b}. Therefore

$$AB \sim \begin{array}{r|c|c|} & \hfill\downarrow & \\ \hline E & \begin{matrix}1&&&0\\&\ddots&&\vdots\\&&1&0\end{matrix} & \\ \hline E & & \begin{matrix}-1&&\\&\ddots&\\&&-1\end{matrix} \\ \hline & -E & -E \end{array}$$

So $M \underset{\sigma\tau}{\approx} \sum\Gamma \oplus \langle -1\rangle$, and we have

$$\{6b\} \qquad \begin{array}{r|l|} \hline \sigma & \sum\Gamma \oplus \langle +1\rangle \\ \tau & \sum\Gamma \oplus \langle +1\rangle \\ \sigma\tau & \sum\Gamma \oplus \langle -1\rangle \\ \hline \end{array} \, .$$

{7a} $D = \left[\begin{array}{c|c} \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & \begin{matrix}1 & & \\ & \ddots & \\ & & 1 \\ 0 & \cdots & 0\end{matrix} \\ \hline \begin{matrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1\end{matrix} & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\end{array}\right]$ $\quad A = \begin{array}{cc|c|c|c} & \overset{c_1}{\downarrow} & & & \\ \hline E & & \begin{matrix}1 & & \\ & \ddots & \\ & & 1 \\ 0 & \cdots & 0\end{matrix} & & \leftarrow a_{n_1} \\ \hline -E & \begin{matrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1\end{matrix} & & & \\ \hline & E & -E & & \end{array}$

$M \underset{\sigma}{\approx} \sum\Gamma \oplus <+1> \oplus <+1>$

$$B = \begin{array}{c|c|c|} \hline E & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & \\ \hline -E & & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} \\ \hline & -E & E \end{array}$$

$M \underset{\tau}{\approx} \sum\Gamma$

$$AB = \begin{array}{c|c|c|} \hline E & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & \begin{matrix}1 & & \\ & \ddots & \\ & & 1 \\ 0 & \cdots & 0\end{matrix} \\ \hline E & \begin{matrix} 0 & -1 & & \\ \vdots & & \ddots & \\ 0 & & & -1\end{matrix} & \begin{matrix}-1 & & \\ & \ddots & \\ & & -1\end{matrix} \\ \hline & -E & -E \end{array}$$

This can be transformed by row and column operations to AB of module {3a}.

Hence $AB \sim \begin{array}{c|c|c|} \hline E & & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} \\ \hline E & \begin{matrix}-1 & & \\ & \ddots & \\ & & -1\end{matrix} & \\ \hline & -E & -E \end{array}$

So $M \underset{\sigma\tau}{\simeq} \sum\Gamma$, and we have

$$\{7a\} \qquad \begin{array}{r|l|} \cline{2-2} \sigma & \sum\Gamma \oplus \langle +1\rangle \oplus \langle +1\rangle \\ \tau & \sum\Gamma \\ \sigma\tau & \sum\Gamma \\ \cline{2-2} \end{array} .$$

$$\{7b\} \quad D = \left[\begin{array}{c|c} \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}0&1&&\\ \vdots&&\ddots&\\ 0&&&1\end{matrix} \\ \hline \begin{matrix}1&&\\&\ddots&\\&&1\\0&\cdots&0\end{matrix} & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} \end{array}\right] \qquad A = \begin{array}{r|c|c|l} \multicolumn{1}{r}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\overset{\downarrow}{d_1}} & \\ \cline{2-3} E & & \begin{matrix}0&1&&\\ \vdots&&\ddots&\\ 0&&&1\end{matrix} & \\ \cline{2-3} -E & \begin{matrix}1&&\\&\ddots&\\&&1\\0&\cdots&0\end{matrix} & & \leftarrow b_n \\ \cline{2-3} \multicolumn{1}{r}{} & \multicolumn{1}{c}{E} & \multicolumn{1}{c}{-E} & \end{array}$$

$M \underset{\sigma}{\simeq} \sum\Gamma \oplus \langle -1\rangle \oplus \langle -1\rangle$

$$B = \begin{array}{r|c|c|} \cline{2-3} E & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \\ \cline{2-3} -E & & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} \\ \cline{2-3} \multicolumn{1}{r}{} & \multicolumn{1}{c}{-E} & \multicolumn{1}{c}{E} \end{array}$$

$M \underset{\tau}{\simeq} \sum\Gamma$

AB can be transformed into the AB matrix for {3b}. Hence

$$AB \sim \begin{array}{r|c|c|} \cline{2-3} E & & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} \\ \cline{2-3} E & \begin{matrix}-1&&\\&\ddots&\\&&-1\end{matrix} & \\ \cline{2-3} \multicolumn{1}{r}{} & \multicolumn{1}{c}{-E} & \multicolumn{1}{c}{-E} \end{array}$$

So $M \underset{\sigma\tau}{\simeq} \sum\Gamma$

and we have

{7b} $$\begin{array}{r|l|} \cline{2-2} \sigma & \sum\Gamma \oplus \langle -1\rangle \oplus \langle -1\rangle \\ \tau & \sum\Gamma \\ \sigma\tau & \sum\Gamma \\ \cline{2-2} \end{array}$$.

{8} $$\left[\begin{array}{c|c} \begin{smallmatrix}1&&\\&\ddots&\\&&1\end{smallmatrix} & \begin{smallmatrix}1&&\\&\ddots&\\&&1\end{smallmatrix} \\ \hline \begin{smallmatrix}1&&\\&\ddots&\\&&1\end{smallmatrix} & \begin{smallmatrix}0&1&&\\&\ddots&\ddots&\\&&\ddots&1\\&&&0\end{smallmatrix} \end{array}\right] \qquad A = \begin{array}{c|c|c|} \cline{2-3} E & & \begin{smallmatrix}1&&\\&\ddots&\\&&1\end{smallmatrix} \\ \cline{2-3} -E & \begin{smallmatrix}1&&\\&\ddots&\\&&1\end{smallmatrix} & \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c}{E} & \multicolumn{1}{c}{-E} \end{array}$$

$M \underset{\sigma}{\simeq} \sum\Gamma$

$$B = \begin{array}{c|c|c|l} \cline{2-3} E & \begin{smallmatrix}1&&\\&\ddots&\\&&1\end{smallmatrix} & & \\ \cline{2-3} -E & & \begin{smallmatrix}0&1&&\\&\ddots&\ddots&\\&&\ddots&1\\&&&0\end{smallmatrix} & \leftarrow b_{n_2} \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c}{-E} & \multicolumn{1}{c}{E} & \\ \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{l}{\begin{smallmatrix}d_1\\ \uparrow\end{smallmatrix}} & \end{array}$$

$M \underset{\tau}{\simeq} \sum\Gamma \oplus \langle +1\rangle \oplus \langle -1\rangle$ with $\langle +1\rangle = \langle d_1\rangle$, $\langle -1\rangle = \langle b_{n_2}\rangle$.

AB can be transformed into AB for module {1}. Hence

$$AB \sim \begin{array}{c|c|c|} \cline{2-3} E & \begin{smallmatrix}1&&\\&\ddots&\\&&1\end{smallmatrix} & \\ \cline{2-3} E & & \begin{smallmatrix}1&&\\&\ddots&\\&&1\end{smallmatrix} \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c}{-E} & \multicolumn{1}{c}{-E} \end{array}$$

So $M \underset{\sigma\tau}{\simeq} \sum\Gamma$

and we have

$$\{8\} \qquad \begin{array}{c|l|} \hline \sigma & \sum\Gamma \\ \tau & \sum\Gamma \oplus \langle +1\rangle \oplus \langle -1\rangle \\ \sigma\tau & \sum\Gamma \\ \hline \end{array}\ .$$

$$\{9a\} \quad A = \begin{array}{c|c|c|} \hline E & & \begin{matrix} \overset{\overset{d_1}{\downarrow}}{0} & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{matrix} \\ \hline -E & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \\ \hline & E & -E \end{array}$$

$M \underset{\sigma}{\cong} \sum\Gamma \oplus \langle -1\rangle$ with $\langle -1\rangle = \langle d_1\rangle$.

$$B = \begin{array}{c|c|c|} \hline E & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \\ \hline -E & & \begin{matrix} 1 & & & \overset{\overset{\downarrow}{d_n}}{0} \\ & \ddots & & \vdots \\ & & 1 & 0 \end{matrix} \\ \hline & -E & E \end{array}$$

$M \underset{\tau}{\cong} \sum\Gamma \oplus \langle +1\rangle$ with $\langle +1\rangle = \langle d_{n_4}\rangle$. AB can be transformed into AB for module {2a}.

So $M \underset{\sigma\tau}{\cong} \sum\Gamma \oplus \langle -1\rangle$, and we have

$$\{9a\} \qquad \begin{array}{c|l|} \hline \sigma & \sum\Gamma \oplus \langle -1\rangle \\ \tau & \sum\Gamma \oplus \langle +1\rangle \\ \sigma\tau & \sum\Gamma \oplus \langle -1\rangle \\ \hline \end{array}\ .$$

{9b} $$D = \begin{array}{|c|c|}\hline \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}1&&\\&\ddots&\\&&1\\0&\cdots&0\end{matrix} \\ \hline \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}0&\cdots&0\\1&&\\&\ddots&\\&&1\end{matrix} \\ \hline\end{array} \qquad A = \begin{array}{c|c|c|l} \cline{2-3} E & & \begin{matrix}1&&\\&\ddots&\\&&1\\0&\cdots&0\end{matrix} & \leftarrow a_n \\ \cline{2-3} -E & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & & \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c}{E} & \multicolumn{1}{c}{-E} & \end{array}$$

$M \underset{\sigma}{\cong} \sum\Gamma \oplus \langle +1\rangle$ with $\langle +1\rangle = \langle a_{n_1}\rangle$.

$$B = \begin{array}{c|c|c|l} \cline{2-3} E & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & & \\ \cline{2-3} -E & & \begin{matrix}0&\cdots&0\\1&&\\&\ddots&\\&&1\end{matrix} & \leftarrow b_1 \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c}{-E} & \multicolumn{1}{c}{E} & \end{array}$$

$M \underset{\tau}{\cong} \sum\Gamma \oplus \langle -1\rangle$, $\langle -1\rangle = \langle b_1\rangle$. AB is equivelent to AB in module {2b}. So $M \underset{\sigma\tau}{\cong} \sum\Gamma \oplus \langle +1\rangle$, and we have

{9b} $$\begin{array}{c|c|} \cline{2-2} \sigma & \sum\Gamma \oplus \langle +1\rangle \\ \tau & \sum\Gamma \oplus \langle -1\rangle \\ \sigma\tau & \sum\Gamma \oplus \langle +1\rangle \\ \cline{2-2} \end{array} .$$

{10} $$D = \begin{array}{|c|c|}\hline \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}0&&&\\1&\ddots&&\\&\ddots&\ddots&\\&&1&0\end{matrix} \\ \hline \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} \\ \hline\end{array}$$

$$A = \begin{array}{r|c|c|l}
 & & \hfill \overset{d_n}{\downarrow} & \\
\cline{2-3}
E & & \begin{matrix}0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0\end{matrix} & \leftarrow a_1 \\
\cline{2-3}
-E & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & & \\
\cline{2-3}
\multicolumn{1}{r}{} & \multicolumn{1}{c}{E} & \multicolumn{1}{c}{-E} &
\end{array}$$

$M \underset{\sigma}{\cong} \sum\Gamma \oplus <+1> \oplus <-1>$. $<+1> = <a_1>$, $<-1> = <d_{n_4}>$

$$B = \begin{array}{r|c|c|}
\cline{2-3}
E & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & \\
\cline{2-3}
-E & & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} \\
\cline{2-3}
\multicolumn{1}{r}{} & \multicolumn{1}{c}{-E} & \multicolumn{1}{c}{E}
\end{array}$$

$M \underset{\tau}{\cong} \sum\Gamma$. AB is equivelent to AB of module {5}. So $M \underset{\sigma\tau}{\cong} \sum\Gamma$,

and we have

$$\{10\} \qquad \begin{array}{r|l|}
\cline{2-2}
\sigma & \sum\Gamma \oplus <+1> \oplus <-1> \\
\tau & \sum\Gamma \\
\sigma\tau & \sum\Gamma \\
\cline{2-2}
\end{array} .$$

We now consider the modules with representations of the form

$$D = \begin{array}{|c|c|}
\hline
\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & D_4 \\
\hline
\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} \\
\hline
\end{array}$$

where D_4 is an indecomposeable matrix over Z_2. Obviously module {10} is included in this case.

Since D_4 may be varied up to conjugation, we may assume D_4 is of the form

$$D_4 = \begin{pmatrix} 0 & \cdots & \cdots & 0 & \alpha_0 \\ 1 & \ddots & & \vdots & \vdots \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & 0 & \alpha_{m-2} \\ & & & 1 & \alpha_{m-1} \end{pmatrix}$$

where $p(x) = X^m + \alpha_{m-1}X^{m-1} + \ldots + \alpha_0$ is the minimal polynomial for D_4 over $\mathbb{Z}_2$.

We consider two cases:

<u>Case 1</u>. D_4 is not invertible over $\mathbb{Z}_2$.

This is equivelent to $\alpha_0 = 0$, since for the above matrix the minimal and characteristic polynomials are the same and α_0 is thus the determinant of D_4. Moreover, since D_4 is indecomposeable, the polynomial $p(x)$ is irreducible or is a power of an irreducible polynomial, and therefore since $\alpha_0 = 0$, we must have $p(x) = x^m$. So

$$D_4 = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 1 & \ddots & & \vdots \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{pmatrix}$$

So if D_4 is not invertible over $\mathbb{Z}_2$, the representation for this type of module is given by

$$D = \left[\begin{array}{c|c} \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & \begin{matrix}0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0\end{matrix} \\ \hline \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} \end{array}\right]$$

and M is a module of type {10}.

Case 2. D_4 is invertible over $\mathbb{Z}_2$.

We must have $\alpha_0 = 1$.

$$A = \left(\begin{array}{cc|cc} E & & & D_4 \\ & -E & E & \\ \hline & & E & \\ & & & -E \end{array}\right).$$

If we conjugate by a matrix of the form

$$\left(\begin{array}{cc|cc} X & & & \\ & E & & \\ \hline & & E & \\ & & & Y \end{array}\right).$$

A is transformed into

$$\left(\begin{array}{cc|cc} E & & & D_4' \\ & -E & E & \\ \hline & & E & \\ & & & -E \end{array}\right)$$

where $D_4' = XD_4Y^{-1}$. Hence D_4 may be replaced by any matrix gotten from D_4 by applying elementary row operations and elementary column operations independently.

Since

$$D_4 = \left(\begin{array}{cccccc} 0 & \cdots & \cdots & \cdots & 0 & 1 \\ 1 & \ddots & & & \cdot & \alpha_2 \\ & \ddots & \ddots & & \cdot & \cdot \\ & & \ddots & \ddots & \cdot & \cdot \\ & & & \ddots & 0 & \alpha_{m-2} \\ & & & & 1 & \alpha_{m-1} \end{array}\right)$$

if $\alpha_2 \neq 0$ we can add a multiple of the first column to the last column so that the new entry in that position is zero. Continuing the same way, D_4 may be reduced to the form

$$\left(\begin{array}{cccccc} 0 & \cdots & \cdots & \cdots & 0 & 1 \\ 1 & \ddots & & & \cdot & 0 \\ & \ddots & \ddots & & \cdot & \cdot \\ & & \ddots & \ddots & 0 & 0 \\ & & & \ddots & 1 & 0 \end{array}\right)$$

and then a column permutation brings this into the form of the identity matrix. Hence

$$A \sim \left(\begin{array}{cc|cc} E & & & E \\ & -E & E & \\ \hline & & E & \\ & & & -E \end{array}\right)$$

and therefore $M \underset{\sigma}{\cong} \sum\Gamma$.

Since

$$B = \left(\begin{array}{cc|cc} E & & E & \\ & -E & & E \\ \hline & & -E & \\ & & & E \end{array}\right)$$

we see $M \underset{\tau}{\approx} \sum\Gamma$ also.

Finally we consider the structure of M with respect to $\sigma\tau$

$$AB = \left(\begin{array}{cc|cc} E & & E & D_4 \\ & E & -E & -E \\ \hline & & -E & \\ & & & -E \end{array}\right).$$

Recalling that we can perform elementary row and column operations on D, i.e., conjugate by $\left(\begin{array}{c|c} X & \\ \hline & Y \end{array}\right)$

$$AB \sim \left(\begin{array}{cc|cc} E & & & D_4-E \\ & E & E & \\ \hline & & -E & \\ & & & -E \end{array}\right)$$

$$D_4-E = D_4+E \quad = \begin{pmatrix} 1 & 0 \cdots & 0 & \alpha_0 \\ 1 & & \vdots & \vdots \\ & \ddots \ddots & 0 & \vdots \\ & & 1 & \alpha_{m-2} \\ & & 1 & \alpha_{m-1}+1 \end{pmatrix}$$

Adding the first row to the second (mod 2) and then the second to the third and so on, this matrix becomes:

$$\begin{pmatrix} 1 & 0 & & & 0 & \alpha_0 \\ 0 & 1 & 0 & \cdots & 0 & \alpha_0+\alpha_1 \\ 0 & 0 & 1 & \cdots & 0 & \alpha_0+\alpha_1+\alpha_2 \\ \vdots & & & \ddots & \vdots & \\ \vdots & & & & 1 & \alpha_0+\ldots+\alpha_{m-2} \\ 0 & \cdots & \cdots & \cdots & 0 & \alpha_0+\ldots+\alpha_{m-1}+1 \end{pmatrix}$$

Now by adding multiples of the $1^{\underline{st}},\ldots,m-1^{\underline{st}}$ columns to the last column, the matrix becomes:

$$\begin{pmatrix} 1 & 0 \cdots & 0 & 0 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & & 0 & \vdots \\ \vdots & & 1 & 0 \\ 0 & \cdots & 0 & \beta \end{pmatrix}$$

where $\beta = \alpha_0 + \alpha_1 + \ldots + \alpha_{m-1} + 1$.

Recall that $p(x)$ = minimal polynomial for D_4 = the characteristic polynomial for $D_4 = \det(D_4 - xE)$. But

$$\beta = \alpha_0+\alpha_1+\ldots+\alpha_{m-1}+1 = p(x)\Big|_{x=1} = \det(D_4-E).$$

Note that we have been assuming, in this case, that D_4 was invertible.

We now must consider two subcases:

Case 2a. (D_4-E) is not invertible.

Case 2b. (D_4-E) is invertible.

If (D_4-E) is not invertible, then $\beta = 0$ and hence

$$AB \sim \left(\begin{array}{cc|c|c} E & & & \begin{smallmatrix} & & & \downarrow & \\ 1 & & & 0 & \\ & \ddots & 1 & \vdots & \\ 0 & \cdots & & 0 & \leftarrow \end{smallmatrix} \\ \cline{3-4} & E & E & \\ \hline & & \multicolumn{1}{c}{-E} & \\ & & \multicolumn{1}{c}{} & -E \end{array}\right)$$

and $M \underset{\sigma\tau}{\cong} \sum\Gamma \oplus \langle +1\rangle \oplus \langle -1\rangle$.

On the other hand if (D_4-E) is invertible, then

$$AB \sim \left(\begin{array}{cc|cc} E & & & E \\ & E & E & \\ \hline & & -E & \\ & & & -E \end{array}\right)$$

and hence $M \underset{\sigma\tau}{\cong} \sum\Gamma$.

So for these two cases, we have that the induced Z_2 structures are:

<u>D_4 invertible and (D_4-E) not invertible</u>

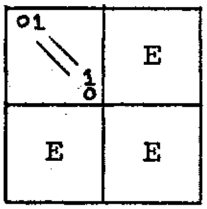
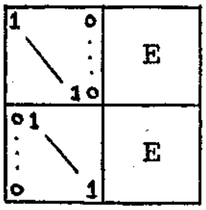
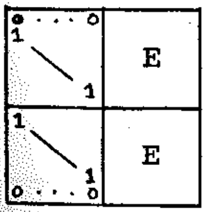
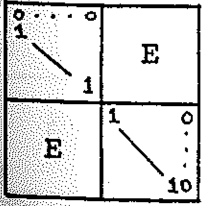
σ	$\sum\Gamma$
τ	$\sum\Gamma$
$\sigma\tau$	$\sum\Gamma \oplus \langle +1\rangle \oplus \langle -1\rangle$

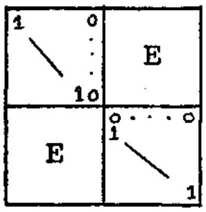
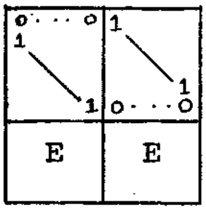
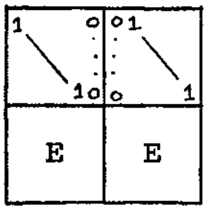
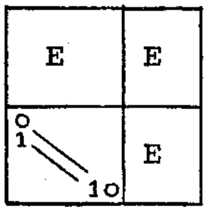
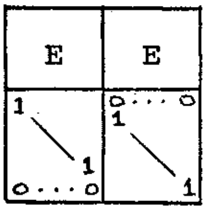
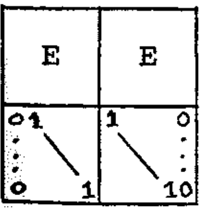
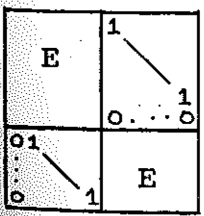
D_4 invertible and (D_4-E) also invertible

σ	$\sum\Gamma$
τ	$\sum\Gamma$
$\sigma\tau$	$\sum\Gamma$

Summary of Z_2-structure and cohomology.

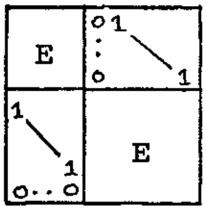
For convenience we list the representations along with their Z_2 structures and the corresponding first and second Z_2-cohomology groups.

	D		Z_2-structure	$H^1(Z_2,M)$	$H^2(Z_2,M)$
{1}		σ	$\sum\Gamma$	0	0
		τ	$\sum\Gamma\oplus\langle+1\rangle\oplus\langle-1\rangle$	Z_2	Z_2
		$\sigma\tau$	$\sum\Gamma$	0	0
{2a}		σ	$\sum\Gamma\ \oplus\ \langle+1\rangle$	0	Z_2
		τ	$\sum\Gamma\ \oplus\ \langle-1\rangle$	Z_2	0
		$\sigma\tau$	$\sum\Gamma\ \oplus\ \langle-1\rangle$	Z_2	0
{2b}		σ	$\sum\Gamma\ \oplus\ \langle-1\rangle$	Z_2	0
		τ	$\sum\Gamma\ \oplus\ \langle+1\rangle$	0	Z_2
		$\sigma\tau$	$\sum\Gamma\ \oplus\ \langle+1\rangle$	0	Z_2
{3a}		σ	$\sum\Gamma$	0	0
		τ	$\sum\Gamma\oplus\langle+1\rangle\oplus\langle+1\rangle$	0	$Z_2\oplus Z_2$
		$\sigma\tau$	$\sum\Gamma$	0	0

	D		Z_2-structure	$H^1(Z_2,M)$	$H^2(Z_2,M)$
{3b}		σ	$\sum\Gamma$	0	0
		τ	$\sum\Gamma\oplus\langle-1\rangle\oplus\langle-1\rangle$	$Z_2\oplus Z_2$	0
		$\sigma\tau$	$\sum\Gamma$	0	0
{4a}		σ	$\sum\Gamma \oplus \langle+1\rangle$	0	Z_2
		τ	$\sum\Gamma \oplus \langle+1\rangle$	0	Z_2
		$\sigma\tau$	$\sum\Gamma \oplus \langle+1\rangle$	0	Z_2
{4b}		σ	$\sum\Gamma \oplus \langle-1\rangle$	Z_2	0
		τ	$\sum\Gamma \oplus \langle-1\rangle$	Z_2	0
		$\sigma\tau$	$\sum\Gamma \oplus \langle-1\rangle$	Z_2	0
{5}		σ	$\sum\Gamma\oplus\langle+1\rangle\oplus\langle-1\rangle$	Z_2	Z_2
		τ	$\sum\Gamma$	0	0
		$\sigma\tau$	$\sum\Gamma$	0	0
{6a}		σ	$\sum\Gamma \oplus \langle-1\rangle$	Z_2	0
		τ	$\sum\Gamma \oplus \langle-1\rangle$	Z_2	0
		$\sigma\tau$	$\sum\Gamma \oplus \langle+1\rangle$	0	Z_2
{6b}		α	$\sum\Gamma \oplus \langle+1\rangle$	0	Z_2
		τ	$\sum\Gamma \oplus \langle+1\rangle$	0	Z_2
		$\sigma\tau$	$\sum\Gamma \oplus \langle-1\rangle$	Z_2	0
{7a}		σ	$\sum\Gamma\oplus\langle+1\rangle\oplus\langle+1\rangle$	0	$Z_2\oplus Z_2$
		τ	$\sum\Gamma$	0	0
		$\sigma\tau$	$\sum\Gamma$	0	0

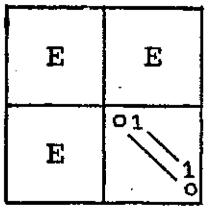
D

{7b}



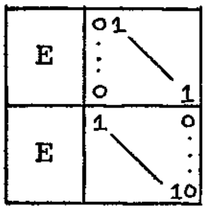
	Z_2-structure	$H^1(Z_2,M)$	$H^2(Z_2,M)$
σ	$\sum\Gamma\oplus\langle-1\rangle\oplus\langle-1\rangle$	$Z_2\oplus Z_2$	0
τ	$\sum\Gamma$	0	0
$\sigma\tau$	$\sum\Gamma$	0	0

{8}



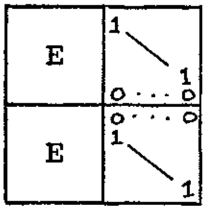
σ	$\sum\Gamma$	0	0
τ	$\sum\Gamma\oplus\langle+1\rangle\oplus\langle-1\rangle$	Z_2	Z_2
$\sigma\tau$	$\sum\Gamma$	0	0

{9a}



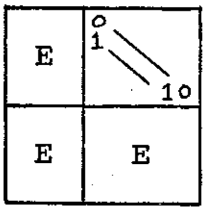
σ	$\sum\Gamma \oplus \langle-1\rangle$	Z_2	0
τ	$\sum\Gamma \oplus \langle+1\rangle$	0	Z_2
$\sigma\tau$	$\sum\Gamma \oplus \langle-1\rangle$	Z_2	0

{9b}



σ	$\sum\Gamma \oplus \langle+1\rangle$	0	Z_2
τ	$\sum\Gamma \oplus \langle-1\rangle$	Z_2	0
$\sigma\tau$	$\sum\Gamma \oplus \langle+1\rangle$	0	Z_2

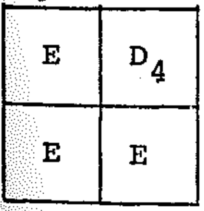
{10}



σ	$\sum\Gamma\oplus\langle+1\rangle\oplus\langle-1\rangle$	Z_2	Z_2
τ	$\sum\Gamma$	0	0
$\sigma\tau$	$\sum\Gamma$	0	0

D_4 invertible a) $(D_4 - E)$ not invertible:

{∞}



σ	$\sum\Gamma$	0	0
τ	$\sum\Gamma$	0	0
$\sigma\tau$	$\sum\Gamma\oplus\langle+1\rangle\oplus\langle-1\rangle$	Z_2	Z_2

b) $(D_4 - E)$ invertible:

	Z_2-structure	$H^1(Z_2,M)$	$H^2(Z_2,M)$
σ	$\sum\Gamma$	0	0
τ	$\sum\Gamma$	0	0
$\sigma\tau$	$\sum\Gamma$	0	0

§4. APPLICATION OF HOCHSCHILD-SERRE SPECTRAL SEQUENCE

For the above list of modules we use the Hochschild Serre spectral sequence to get information about the cohomology of these modules over $Z_2 \times Z_2$.

The Hochschild Serre spectral sequence applies to modules, A, over a group G where K is a normal subgroup of G. This spectral sequence yields an exact sequence under certain circumstances.

I. Let $m \geq 1$. If $H^n(K,A) = 0$ for $0 < n < m$ then there is an exact sequence

$$0 \longrightarrow H^m(G/K,A^K) \longrightarrow H^m(G,A) \xrightarrow{r_K} H^m(K,A)^G \longrightarrow$$
$$\longrightarrow H^{m+1}(G/K,A^K) \longrightarrow H^{m+1}(G,A)$$

where r_K is the restriction map induced by the inclusion $K \hookrightarrow G$.

II. Let $m \geq 1$. If $H^n(K,A) = 0$ for $1 < n < m$ then for $0 < n < m$ there is an exact sequence

$$\ldots \longrightarrow H^n(G/K,A^K) \longrightarrow H^n(G,A) \longrightarrow H^{n-1}(G/K,H^1(K,A)) \longrightarrow$$
$$\longrightarrow H^{n+1}(G/K,A^K) \longrightarrow H^{n+1}(G,A) \longrightarrow$$

In particular if we take $m = 2$ in I, we get;

Ia. If $H^1(K,A) = 0$ then $H^1(G,A) \cong H^1(G/K,A^K)$ and we have an exact sequence

$$0 \longrightarrow H^2(G/K,A^K) \longrightarrow H^2(G,A) \xrightarrow{r_K} H^2(K,A)^G$$
$$\longrightarrow H^3(G/K,A^K) \longrightarrow H^3(G,A)$$

Taking m = 3 gives:

Ib. If $H^1(K,A) = H^2(K,A) = 0$ then $H^2(G,A) \simeq H^2(G/K,A^K)$.

And, taking m = 3 in the sequence II yields:

IIa. If $H^2(K,A) = 0$ then the following is exact

$$\longrightarrow H^1(G/K,A^K) \longrightarrow H^1(G,A) \longrightarrow H^0(G/K,H^1(K,A)) \longrightarrow$$
$$\longrightarrow H^2(G/K,A^K) \longrightarrow H^2(G,A) \longrightarrow H^1(G/K,H^1(K,A)) \longrightarrow$$
$$\longrightarrow H^3(G/K,A^K) \longrightarrow H^3(G,A) \longrightarrow$$

Now let M be a module over the group $\Phi = Z_2 \times Z_2$, then the subgroups $\langle\sigma\rangle$, $\langle\tau\rangle$ and $\langle\sigma\tau\rangle$ are normal since Φ is abelian. We write M^σ for $M^{\langle\sigma\rangle}$ etc. Since

$$M^\sigma = \ker\{(\sigma-1): M \longrightarrow M\},$$

we can readily calculate M^σ, M^τ and $M^{\sigma\tau}$ in terms of our representation

$$D = \boxed{\begin{array}{c|c} D_1 & D_4 \\ \hline D_2 & D_3 \end{array}} \qquad \text{as follows.}$$

Let (W,X,Y,Z) be the coordinates of an arbitrary element of M in terms of the basis for

$$M = V_1 \oplus V_2 \oplus V_3 \oplus V_4$$

as free abelian group.

Lemma 4.1. $M^{\sigma} = \{(W,X,Y,0) \mid D_2Y = 2X\}$

$M^{\tau} = \{(W,X,0,Z) \mid D_3Z = 2X\}$

$M^{\sigma\tau} = \{(W,X,0,0)\}$

Proof. This is a straightforward verification. We will just calculate M^{σ}.

The action of σ-1 on M with respect to the basis is given by

$$(1-\sigma)M = (E-A)\begin{pmatrix} W \\ X \\ Y \\ Z \end{pmatrix} = \left(\begin{array}{cc|cc} 0 & & & -D_4 \\ & 2E & -D_2 & \\ \hline & & 0 & \\ & & & 2E \end{array}\right)\begin{pmatrix} W \\ X \\ Y \\ Z \end{pmatrix} =$$

$$= \begin{pmatrix} -D_4Z \\ 2X-D_2Y \\ 0 \\ 2Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{iff} \quad Z = 0 \text{ and } D_2Y = 2X$$

Proposition 4.2.

{a} If $D_2 = E$ then

$$H^1(\Phi/\langle\sigma\rangle, M^{\sigma}) \simeq Z_2^{n_2}$$

$$H^2(\Phi/\langle\sigma\rangle, M^{\sigma}) \simeq Z_2^{n_1}$$

{b} If $D_3 = E$ then

$$H^1(\Phi/<\tau>,M^\tau) \approx Z_2^{n_2}$$

$$H^2(\Phi/<\tau>,M^\tau) \approx Z_2^{n_1}$$

{c} With no hypothesis on D_2 or D_3

$$H^1(\Phi/<\sigma\tau>,M^{\sigma\tau}) \approx Z_2^{n_2}$$

$$H^2(\Phi/<\sigma\tau>,M^{\sigma\tau}) \approx Z_2^{n_1}$$

<u>Proof</u>.

{a} If $D_2 = E$ then $M^\sigma = \{(W,X,2X,0)\}$, so we can choose as a basis for M^σ the set

$$\{a_1,\ldots,a_{n_1}\} \cup \bigcup_{j=1}^{n_2} \{b_j + 2c_j\}.$$

Then the action of $\Phi/<\sigma>$ on M^σ is that of $<\tau> \approx \Phi/<\sigma>$ on M^σ. The effect of τ on this chosen basis for M^σ can be seen by conjugating the matrix B by the matrix

$$C = \left(\begin{array}{cc|cc} E & & & \\ & E & & \\ \hline & 2E & E & \\ & & & E \end{array}\right).$$

The first two sets of columns of C are the coordinates of the chosen basis vectors for M^σ. C is non singular and if we conjugate by this change of basis matrix, we get

$$\left(\begin{array}{cc|cc} E & 2D_1 & D_1 & 0 \\ 0 & -E & 0 & D_3 \\ \hline 0 & 0 & -E & -2D_3 \\ 0 & 0 & 0 & E \end{array}\right)$$

The upper left block $\begin{pmatrix} E & 2D_1 \\ 0 & -E \end{pmatrix}$ provides a representation of the action of τ on M^σ. Furthermore, such a matrix is always congruent to $\begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}$;

$$\begin{pmatrix} E & D_1 \\ 0 & E \end{pmatrix}\begin{pmatrix} E & 2D_1 \\ 0 & -E \end{pmatrix}\begin{pmatrix} E & -D_1 \\ 0 & E \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}.$$

But this representation tells us that M^σ is isomorphic to a direct sum of <+1> and <-1> summands, and keeping track of the ranks of the E and -E blocks, we see that E has rank n_1 (corresponding to the basis subset $\{a_1,\ldots,a_{n_1}\}$), and -E has rank n_2.

So $M^\sigma \underset{\tau}{\cong} \langle +1\rangle^{n_1} \oplus \langle -1\rangle^{n_2}$ and hence

$$H^1(\langle\tau\rangle, M^\sigma) \simeq H^1(\Phi/\langle\sigma\rangle, M^\sigma) \simeq Z_2^{n_2}$$

and

$$H^2(\langle\tau\rangle, M^\sigma) \simeq H^2(\Phi/\langle\sigma\rangle, M^\sigma) \simeq Z_2^{n_1}$$

Proof of {b}:

$M^{\tau} = \{(W,X,0,2X)\}$ in this case. Analogously to the proof of {a} we would conjugate the matrix A by the change of basis matrix

$$\left(\begin{array}{cc|cc} E & 0 & & \\ & E & & \\ \hline & 0 & E & \\ & 2E & & E \end{array}\right)$$

and get the representation

$$\begin{pmatrix} E & 2D_4 \\ & -E \end{pmatrix}$$

which is congruent to

$$\begin{pmatrix} E & \\ & -E \end{pmatrix}$$

also.

Finally for {c}, $M^{\sigma\tau} = \{(W,X,0,0)\}$ in this case; and the upper left block of AB itself provides a representation

$$\begin{pmatrix} E & \\ & -E \end{pmatrix}$$

for the action of either σ or τ on $M^{\sigma\tau}$.

There is an analogous result for the modules on the second list. We include it here but do not give a proof since we are not going to study the cohomology of these modules.

Proposition 4.3. If $D_2 = E$ then

$$H^1(\Phi/\langle\sigma\rangle, M^\sigma) \simeq Z_2^{n_2} \oplus U_1$$

and

$$H^2(\Phi/\langle\sigma\rangle, M^\sigma) \simeq Z_2^{n_1} \oplus U_2$$

where

$$U_1 = \begin{cases} Z_2 & \text{if there is a } \Gamma\text{-summand in coker}(\sigma-\tau) \\ 0 & \text{otherwise} \end{cases}$$

and

$$U_2 = \begin{cases} Z_2 & \text{if there is a } \Gamma\text{-summand in ker}(\sigma-\tau) \\ 0 & \text{otherwise} \end{cases}$$

We now return to the application of the Hochschild-Serre sequence to the modules in the family we are considering.

{1} The representation for this module is given by

$$D = \left[\begin{array}{c|c} \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{matrix} & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \\ \hline \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{array}\right]$$

From earlier calculations $H^1(\langle\sigma\rangle, M) = H^2(\langle\sigma\rangle, M) = 0$. Therefore, by Ia and the above corollary

$$H^1(\Phi, M) \simeq H^1(\Phi/\langle\sigma\rangle, M^\sigma) \simeq Z_2^{n_2}.$$

Also, by IB and the corollary

$$H^2(\Phi, M) \simeq H^2(\Phi/\langle\sigma\rangle, M^\sigma) \simeq Z_2^{n_1}.$$

{2a} $$D = \left[\begin{array}{c|c} \begin{matrix}1 & & & 0\\ & \ddots & & \vdots\\ & & 1 & 0\end{matrix} & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} \\ \hline \begin{matrix}0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1\end{matrix} & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\end{array}\right]$$

Here we have $H^1(\langle\sigma\rangle, M) = 0$. So by Ia

$$H^1(\Phi, M) \simeq H^1(\Phi/\langle\sigma\rangle, M^\sigma)$$

and we have an exact sequence

$$0 \longrightarrow H^2(\Phi/\langle\sigma\rangle, M^\sigma) \longrightarrow H^2(\Phi, M) \longrightarrow H^2(\langle\sigma\rangle, M)^\Phi \longrightarrow$$
$$\longrightarrow H^3(\Phi/\langle\sigma\rangle, M^\sigma) \longrightarrow H^2(\Phi, M)$$

To determine $H^i(\Phi/\langle\sigma\rangle, M^\sigma) = H^i(\langle\tau\rangle, M^\sigma)$ we must go through some direct calculations.

First we calculate $M^\sigma = \ker(A-E)$.

$$(A-E)\begin{pmatrix}W\\X\\Y\\Z\end{pmatrix} = \left(\begin{array}{cc|cc} 0 & & & E\\ & -2E & D_2 & \\ \hline & & 0 & \\ & & & -2E\end{array}\right)\begin{pmatrix}W\\X\\Y\\Z\end{pmatrix} = \begin{pmatrix}Z\\-2X+D_2Y\\0\\-2Z\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

if and only if $Z = 0$ and $D_2Y = 2X$.

We can calculate D_2Y

$$\begin{pmatrix}0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1\end{pmatrix}\begin{pmatrix}y_1\\ \vdots\\ y_r\end{pmatrix} = \begin{pmatrix}y_2\\ \vdots\\ y_r\end{pmatrix} = 2X = \begin{pmatrix}2x_1\\ \vdots\\ 2x_{r-1}\end{pmatrix}$$

So $Y = [y_1,\ldots,y_r] = [y_1, 2x_1, 2x_2, \ldots, 2x_{r-1}]$.

So $M^\sigma = \{(W', X', Y', o) \mid X' = [x_1,\ldots,x_{r-1}], Y' = [y_1, 2x_1, \ldots, 2x_{r-1}]\}$

$M^\sigma \simeq Z^{n_1+n_2+1}$.

The columns of the following $n \times (n_1 + n_2 + 1)$ matrix, then form a Z-basis for M^σ.

$$\left(\begin{array}{c|c|c} 1 & & 0 \\ \ddots & & \vdots \\ 1 & & 0 \\ \hline & 1 & 0 \\ & \ddots & \vdots \\ & 1 & 0 \\ \cline{2-3} & 0 \cdots 0 & 1 \\ & 2 & 0 \\ & \ddots & \vdots \\ & 2 & 0 \\ \cline{2-3} & & 0 \\ & & \vdots \\ & & 0 \end{array}\right)$$

This matrix constitutes the first $n_1 + n_2 + 1$ columns of the non-singular (change of basis) matrix.

n_1+n_2+1 columns

$$C = \left(\begin{array}{c|c|c|c} E & & & \\ \cline{2-4} & E & & \\ \hline & T & E & \\ \hline & 0 & & E \end{array}\right) \text{ with inverse } = \left(\begin{array}{cc|cc} E & & & \\ & E & & \\ \hline & -T & E & \\ & & & E \end{array}\right)$$

where $T = \begin{pmatrix} 0 \cdots 0 \\ 2 \ddots \\ \quad 2 \end{pmatrix}$.

So to get the corresponding representation of $\langle\tau\rangle$ on M^{σ} we can conjugate B by C and consider the upper left $(n_1+n_2+1) \times (n_1+n_2+1)$ block.

$$C^{-1}BC = \left(\begin{array}{cc|cc} E & D_1T & D_1 & \\ & -E & & E \\ \hline & & -E & -T \\ & & & E \end{array}\right) .$$

So $<\tau>$ has a representation in M^{σ} of the form

$$\left(\begin{array}{cc|c} E & D_1T & \begin{smallmatrix}1\\0\\\vdots\\0\end{smallmatrix} \\ & -E & \begin{smallmatrix}0\\\vdots\\0\end{smallmatrix} \\ \hline & & -1 \end{array}\right) = \left(\begin{array}{ccc|cccc|c} 1 & & & 0 & \cdots & & 0 & 1 \\ & \ddots & & 2 & & & \vdots & 0 \\ & & & & \ddots & & \vdots & \vdots \\ & & 1 & & & 2 & 0 & 0 \\ \hline & & & -1 & & & & \\ & & & & \ddots & & & \\ & & & & & & -1 & \\ \hline & & & & & & & -1 \end{array}\right)$$

This matrix is congruent to the matrix

$$\left(\begin{array}{ccc|ccc|cc} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ \hline & & & -1 & & & & \\ & & & & \ddots & & & \\ & & & & & -1 & & \\ \hline & & & & & & 1 & 1 \\ & & & & & & 0 & -1 \end{array}\right)$$

The $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ summand is congruent to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Hence $M^{\sigma} \underset{\tau}{\approx} <+1>^{n_1-1} \oplus <-1>^{n_1} \oplus \Gamma$. So

$$H^1(\Phi/<\sigma>, M^{\sigma}) = H^1(<\tau>, M^{\sigma}) \simeq Z_2^{n_2}$$

and

$$H^2(\Phi/<\sigma>, M^{\sigma}) \simeq Z_2^{n_1-1}.$$

Also to compute the term $H^2(<\sigma>, M)^{\Phi}$, note $H^2(<\sigma>, M) \simeq Z_2$ and the only way Φ can act on Z_2 is trivially, hence $H^2(<\sigma>, M)^{\Phi} \simeq Z_2$. So from the Hochschild-Serre sequence we get

$$H^1(\Phi,M) \simeq H^1(\Phi/\langle\sigma\rangle,M^\sigma) \simeq Z_2^{n_2}$$

and we have the exact sequence

$$0 \longrightarrow Z_2^{n_1-1} \longrightarrow H^2(\Phi,M) \xrightarrow{r_\sigma} Z_2 \longrightarrow Z_2^{n_2} \longrightarrow H^3(\Phi,M).$$

In order for this sequence to yield any further information about $H^2(\Phi,M)$ we need to know whether r_σ: $H^2(\Phi,M) \longrightarrow Z_2$ is onto and whether $H^2(\Phi,M)$ has any elements of order four. In order to do this systematically for all the modules, we delay this until later and continue to calculate $H^2(\Phi,M)$ for the other modules, "up to exact sequences."

Turning now to the module type {2b} with

$$D = \left[\begin{array}{c|c} \begin{matrix} 0 & \cdots & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \\ \hline \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & \cdots & 0 \end{matrix} & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{array}\right]$$

For these modules, $H^1(\langle\tau\rangle,M) = 0$. So

$$H^1(\Phi,M) \simeq H^1(\Phi/\langle\tau\rangle,M^\tau) \simeq Z_2^{n_2}$$

by the corollary, since $D_3 = E$. We also have the exact sequence

$$0 \longrightarrow H^2(\Phi/\langle\tau\rangle, M^\tau) \longrightarrow H^2(\Phi, M) \xrightarrow{r_\tau} H^2(\langle\tau\rangle, M)^\Phi \longrightarrow$$

$$\longrightarrow H^3(\Phi/\langle\tau\rangle, M^\tau) \longrightarrow H^3(\Phi, M)$$

From the corollary, and because $H^2(\langle\tau\rangle,M)^\Phi = Z_2^\Phi = Z_2$, this sequence becomes

$$0 \longrightarrow Z_2^{n_1} \longrightarrow H^2(\Phi,M) \xrightarrow{r_\tau} Z_2 \longrightarrow Z_2^{n_2} \longrightarrow H^3(\Phi,M)$$

{3a}

$$D = \left[\begin{array}{ccc|ccc} 0 & \cdots & 0 & 1 & & \\ 1 & & & & \ddots & \\ & \ddots & & & & 1 \\ & & 1 & & & \\ \hline 1 & & & 1 & & 0 \\ & \ddots & & & \ddots & \vdots \\ & & 1 & & 1 & 0 \end{array}\right]$$

Here $H^1(\langle\sigma\rangle,M) = H^2(\langle\sigma\rangle,M) = 0$. So

$$H^1(\Phi,M) \simeq H^1(\Phi/\langle\sigma\rangle, M^\sigma)$$

and

$$H^2(\Phi,M) \simeq H^2(\Phi/\langle\sigma\rangle, M^\sigma).$$

Since $D_2 = E$ in this case, by the corollary we have

$$H^1(\Phi,M) \simeq Z_2^{n_2}$$

$$H^2(\Phi,M) \simeq Z_2^{n_1}$$

{3b}

$$\left[\begin{array}{ccc|ccc} 1 & & 0 & 1 & & \\ & \ddots & \vdots & & \ddots & \\ & 1 & 0 & & & 1 \\ \hline 1 & & & 0 & \cdots & 0 \\ & \ddots & & 1 & & \\ & & & & \ddots & \\ & & 1 & & & 1 \end{array}\right]$$

Again $H^1(<\sigma>,M) = H^2(<\sigma>,M) = 0$ and $D_2 = E$, so as before

$$H^1(\Phi,M) \simeq Z_2^{n_2}$$

$$H^2(\Phi,M) \simeq Z_2^{n_1}$$

{4a}
$$\left[\begin{array}{c|c} \begin{matrix} 0 & \cdots & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & \cdots & 0 \end{matrix} \\ \hline \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{array}\right]$$

$H^1(<\sigma>,M) = 0$, $H^2(<\sigma>,M) = Z_2$ and $D_2 = E$. Thus, we have

$$H^1(\Phi,M) \simeq H^1(\Phi/<\sigma>,M) \simeq Z_2^{n_2}$$

and the exact sequence

$$0 \longrightarrow H^2(\Phi/<\sigma>,M^\sigma) \longrightarrow H^2(\Phi,M) \longrightarrow H^2(<\sigma>,M)^\Phi \longrightarrow$$
$$H^3(\Phi/<\sigma>,M^\sigma) \longrightarrow H^3(\Phi,M)$$

which becomes

$$0 \longrightarrow Z_2^{n_1} \longrightarrow H^2(\Phi,M) \longrightarrow Z_2 \longrightarrow Z_2^{n_2} \longrightarrow H^3(\Phi,M)$$

{4b}
$$D = \left[\begin{array}{c|c} \begin{matrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{matrix} & \begin{matrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{matrix} \\ \hline \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{array}\right]$$

Here $H^2(\langle\sigma\rangle,M) = H^2(\langle\tau\rangle,M) = H^2(\langle\sigma\tau\rangle,M) = 0$.

By IIa we have an exact sequence

$$\longrightarrow H^1(\Phi/\langle\sigma\rangle,M^\sigma) \longrightarrow H^1(\Phi,M) \longrightarrow H^0(\Phi/\langle\sigma\rangle,H^1(\langle\sigma\rangle,M)) \longrightarrow$$
$$\longrightarrow H^2(\Phi/\langle\sigma\rangle,M^\sigma) \longrightarrow H^2(\Phi,M) \longrightarrow H^1(\Phi/\langle\sigma\rangle,H^1(\langle\sigma\rangle,M)) \longrightarrow$$
$$H^3(\Phi/\langle\sigma\rangle,M^\sigma) \longrightarrow H^3(\Phi,M).$$

We also have the usual exact sequence

$$0 \longrightarrow H^1(\Phi/\langle\sigma\rangle,M^\sigma) \longrightarrow H^1(\Phi,M) \longrightarrow H^1(\langle\sigma\rangle,M)^\Phi \longrightarrow$$
$$\longrightarrow H^2(\Phi/\langle\sigma\rangle,M^\sigma) \longrightarrow H^2(\Phi,M).$$

Since $D_2 = E$, $H^1(\Phi/\langle\sigma\rangle,M^\sigma) \simeq Z_2^{n_2}$ and $H^2(\Phi/\langle\sigma\rangle,M^\sigma) \simeq Z_2^{n_1}$. Also $H^1(\langle\sigma\rangle,M) \simeq Z_2$, so $H^0(\Phi/\langle\sigma\rangle,H^1(\langle\sigma\rangle,M)) \simeq Z_2^\Phi \simeq Z_2$ and $H^1(\Phi/\langle\sigma\rangle,H^1(\langle\sigma\rangle,M)) \simeq \dfrac{\ker\{(\tau+1):Z_2 \longrightarrow Z_2\}}{\mathrm{Im}\{(\tau-1):Z_2 \longrightarrow Z_2\}}$.

Since τ must act trivially on Z_2, then $(\tau+1)Z_2 = (1+1)Z_2 = 0\cdot Z_2$. So $\ker\{(\tau+1):Z_2 \longrightarrow Z_2\} = Z_2$. Also $(\tau-1)Z_2 = (1-1)Z_2 = 0$. So $\mathrm{Im}(\tau-1) = 0$. Hence $H^1(\Phi/\langle\sigma\rangle,H^1(\langle\sigma\rangle,M) \simeq Z_2$. So the above two exact sequences become

$$0 \longrightarrow Z_2^{n_2} \longrightarrow H^1(\Phi,M) \longrightarrow Z_2 \longrightarrow Z_2^{n_1} \longrightarrow H^2(\phi,M) \longrightarrow Z_2 \longrightarrow$$
$$\longrightarrow Z_2^{n_2} \longrightarrow H^3(\Phi,M)$$

and

$$0 \longrightarrow Z_2^{n_2} \longrightarrow H^1(\Phi,M) \longrightarrow Z_2 \longrightarrow Z_2^{n_1} \longrightarrow H^2(\Phi,M).$$

<u>Remark:</u> This module will require special treatment later on, in order to eliminate the indeteriminacy of the exact sequences.

{5}

$$D = \left[\begin{array}{c|c} \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} \\ \hline \begin{matrix}0&&&\\1&\ddots&&\\&\ddots&\ddots&\\&&1&0\end{matrix} & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} \end{array}\right]$$

For these modules $H^1(<\tau>,M) = H^2(<\tau>,M) = 0$ and $D_3 = E$, so we have

$$H^1(\Phi,M) \simeq H^1(\Phi/<\tau>,M^\tau) \simeq Z_2^{n_2}$$

and

$$H^2(\Phi,M) \simeq H^2(\Phi/<\tau>,M^\tau) \simeq Z_2^{n_1}$$

{6a}

$$D = \left[\begin{array}{c|c} \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} \\ \hline \begin{matrix}1&&\\&\ddots&\\&&1\\0&\cdots&0\end{matrix} & \begin{matrix}0&\cdots&0\\1&&\\&\ddots&\\&&1\end{matrix} \end{array}\right]$$

$H^1(<\sigma\tau>,M) = 0$ and $H^2(<\sigma\tau>,M) = Z_2$. So we have $H^1(\Phi,M) \simeq H^1(\Phi/<\sigma\tau>,M^{\sigma\tau})$. By the corollary, $H^1(\Phi/<\sigma\tau>,M^{\sigma\tau}) \simeq Z_2^{n_2}$. Hence $H^1(\Phi,M) \simeq Z_2^{n_2}$. We also have the exact sequence

$$0 \longrightarrow H^2(\Phi/\langle\sigma\tau\rangle, M^{\sigma\tau}) \longrightarrow H^2(\Phi, M) \longrightarrow H^2(\langle\sigma\tau\rangle, M)^{\Phi} \longrightarrow$$
$$\longrightarrow H^3(\Phi/\langle\sigma\tau\rangle, M^{\sigma\tau}) \longrightarrow H^3(\Phi, M)$$

which becomes

$$0 \longrightarrow Z_2^{n_1} \longrightarrow H^2(\Phi, M) \xrightarrow{r_{\sigma\tau}} Z_2 \longrightarrow Z_2^{n_2} \longrightarrow H^3(\Phi, M)$$

{6b} $$D = \begin{array}{|c|c|}\hline \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} \\ \hline \begin{matrix}0&1&&\\ \vdots&&\ddots&\\ 0&&&1\end{matrix} & \begin{matrix}1&&&0\\&\ddots&&\vdots\\&&1&0\end{matrix} \\ \hline \end{array}$$

$H^1(\langle\sigma\rangle, M) = 0$. So we have $H^1(\Phi/\langle\sigma\rangle, M) \simeq H^1(\Phi, M)$ and the exact sequence

$$0 \longrightarrow H^2(\Phi/\langle\sigma\rangle, M^{\sigma}) \longrightarrow H^2(\Phi, M) \longrightarrow H^2(\langle\tau\rangle, M)^{\Phi} \longrightarrow$$
$$\longrightarrow H^3(\Phi/\langle\sigma\rangle, M^{\sigma}) \longrightarrow H^3(\Phi, M).$$

Since the corollary does not apply here, we compute directly $H^i(\Phi/\langle\sigma\rangle, M^{\sigma})$.

$$M^{\sigma} = \{(W', X', Y', 0) \mid D_2 Y' = 2X'\}$$

$$D_2 Y' = \begin{pmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{n_3} \end{pmatrix} = \begin{pmatrix} y_2 \\ \vdots \\ y_{n_3} \end{pmatrix} = 2X' = \begin{pmatrix} 2x_1 \\ \vdots \\ 2x_{n_3-1} \end{pmatrix}$$

So $M^{\sigma} = \left\{(W', [x_1, \ldots, x_{n_3-1}], [y_1, 2x_1, 2x_2, \ldots, 2x_{n_3-1}], 0)\right\}$

$M^{\sigma} \simeq Z^{n_1+n_2+1}$ as a free abelian group and the first n_1+n_2+1 columns of the nonsingular matrix

$$C = \left(\begin{array}{c|c|c|c} E & & & \\ \hline & E & & \\ \hline & \begin{matrix}0 & \cdots & 0\\ 2 & & \\ & \ddots & \\ & & 2\end{matrix} & E & \\ \hline & & & E \end{array}\right) \text{ generate } M^{\sigma}$$

Conjugating the representation for $\langle\tau\rangle \simeq \Phi/\langle\sigma\rangle$ gives,

(letting $T = \begin{pmatrix}0 & \cdots & 0\\ 2 & & \\ & \ddots & \\ & & 2\end{pmatrix}$

$$C^{-1}BC = \left(\begin{array}{cc|cc} E & T & E & \\ & -E & & D_3 \\ \hline & & -E & -TD_3 \\ & & & E \end{array}\right)$$

The upper left $(n_1+n_2+1) \times (n_1+n_2+1)$ block gives a representation of $\langle\tau\rangle$ on M^{σ} of the form

$$\left(\begin{array}{c|c|c} \begin{matrix}1 & 0 & \cdots & 0\\ & \ddots & & \vdots\\ & & & 0\\ & & & 1\end{matrix} & \begin{matrix}0 & \cdots & 0\\ 2 & & \\ & \ddots & \\ & & 2\end{matrix} & \begin{matrix}1\\ 0\\ \vdots\\ 0\end{matrix} \\ \hline & \begin{matrix}-1 & & \\ & \ddots & \\ & & -1\end{matrix} & \\ \hline & & -1 \end{array}\right)$$

This representation is equivalent to

$$\left(\begin{array}{c|c|c} \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & & \begin{matrix}1\\0\\\vdots\\0\end{matrix} \\ \hline & \begin{matrix}-1&&\\&\ddots&\\&&-1\end{matrix} & \\ \hline & & -1 \end{array}\right)$$

which is in turn equivalent to

$$\left(\begin{array}{c|c|c} \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & & \\ \hline & \begin{matrix}-1&&\\&\ddots&\\&&-1\end{matrix} & \\ \hline & & \begin{matrix}0&1\\1&0\end{matrix} \end{array}\right)$$

Hence $M^{\sigma} \underset{\tau}{\approx} \langle +1 \rangle^{n_1 - 1} \oplus \langle -1 \rangle^{n_2} \oplus \Gamma$. Therefore by the corollary we have

$$H^1(\Phi,M) \approx H^1(\Phi/\langle\sigma\rangle, M^{\sigma}) \approx Z_2^{n_2}$$

and the exact sequence becomes

$$0 \longrightarrow Z_2^{n_1-1} \longrightarrow H^2(\Phi,M) \xrightarrow{r_\tau} Z_2 \longrightarrow Z_2^{n_2} \longrightarrow H^3(\Phi,M)$$

{7a} $$D = \left[\begin{array}{c|c} \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}0&1&&\\ \vdots&&\ddots&\\0&&&1\end{matrix} \\ \hline \begin{matrix}1&&\\&\ddots&\\&&1\\0&\cdots&0\end{matrix} & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix}\end{array}\right]$$

Here $H^1(<\tau>,M) = H^2(<\tau>,M) = 0$. So $H^i(\Phi,M) \simeq H^i(\Phi/<\tau>,M^\tau)$. $i = 1,2$. Since $D_3 = E$

$$H^1(\Phi,M) \simeq Z_2^{n_2}$$

$$H^2(\Phi,M) \simeq Z_2^{n_1}$$

{7b} $$D = \left[\begin{array}{c|c} \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}1&&\\&\ddots&\\&&1\\0&\cdots&0\end{matrix} \\ \hline \begin{matrix}0&1&&\\ \vdots&&\ddots&\\0&&&1\end{matrix} & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix}\end{array}\right]$$

Again $H^i(<\tau>,M) = 0$, $i = 1,2$, and, as in {7a}

$$H^1(\Phi,M) \simeq Z_2^{n_2}$$

$$H^2(\Phi,M) \simeq Z_2^{n_1}$$

{8} $$D = \left[\begin{array}{c|c} \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} \\ \hline \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}0&1&&\\&\ddots&\ddots&\\&&&1\\&&&0\end{matrix}\end{array}\right]$$

$H^1(<\sigma>,M) = H^2(<\sigma>,M) = 0$ and $D_2 = E$. So

$$H^1(\Phi,M) \simeq H^1(\Phi/<\sigma>,M) \simeq Z_2^{n_2}$$

and

$$H^2(\Phi,M) \simeq H^2(\Phi/<\sigma>,M) \simeq Z_2^{n_1}$$

{9a}
$$D = \left[\begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \begin{matrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{matrix} \\ \hline \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \begin{matrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{matrix} \end{array}\right]$$

$H^1(<\tau>,M) = 0$ and $H^2(<\tau>,M) \simeq Z_2$. We have $H^1(\Phi,M) \simeq H^1(\Phi/<\tau>,M^\tau)$ and the exact sequence

$$0 \longrightarrow H^2(\Phi/<\tau>,M) \longrightarrow H^2(\Phi,M) \longrightarrow H^2(\Phi<\tau>,M)^{\Phi} \longrightarrow$$
$$\longrightarrow H^3(\Phi/<\tau>,M) \longrightarrow H^3(\Phi,M).$$

Here we must compute $H^i(\Phi/<\tau>,M^\tau)$ $i = 1,2$, directly.

$$M^\tau = \{(W',X',0,Z') \mid D_3Z' = 2X\}$$

$$D_3Z' = \begin{pmatrix} 1 & & 0 \\ & \ddots & \vdots \\ & & 1\,0 \end{pmatrix}\begin{pmatrix} z_1 \\ \vdots \\ z_{n_4} \end{pmatrix} = \begin{pmatrix} z_1 \\ \vdots \\ z_{n_4-1} \end{pmatrix} = 2X = \begin{pmatrix} 2x_1 \\ \vdots \\ 2x_{n_4-1} \end{pmatrix}.$$

So $M^\tau = \left\{ (W',[x_1,\ldots,x_{n_4-1}],\ 0,\ [2x_1,\ldots,2x_{n_4-1},z_{n_4}] \right\}$

M^{τ} is spanned by the first n_1+n_2 and the last column of the nonsingular matrix

$$C = \left(\begin{array}{c|c|c|c} E & & & \\ \hline & E & & \\ \hline & 0 & E & \\ \hline & T & & E \end{array}\right)$$

where T is the matrix

$$\begin{pmatrix} 2 & & \\ & \ddots & \\ & & 2 \\ 0 & \cdots & 0 \end{pmatrix} .$$

Conjugating A by C gives

$$C^{-1}AC = \left(\begin{array}{cc|cc} E & D_4T & & D_4 \\ & -E & E & \\ \hline & & E & \\ & & -T & -E \end{array}\right) .$$

We get a representation of $<\sigma>$ on M^{τ} by considering the $(n_1+n_2+1) \times (n_1+n_2+1)$ submatrix of those elements in rows and columns 1 through n_2 and column n_4, which can be shown is congruent to the matrix

$$\left(\begin{array}{ccc|ccc|cc} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ \hline & & & -1 & & & & \\ & & & & \ddots & & & \\ & & & & & -1 & & \\ \hline & & & & & & 0 & 1 \\ & & & & & & 1 & 0 \end{array}\right)$$

So $M^{\sigma} \underset{\tau}{\cong} \langle +1\rangle^{n_1-1} \oplus \langle -1\rangle^{n_2} \oplus \Gamma$. Hence

$$H^1(\Phi/\langle\sigma\rangle, M^{\sigma}) \simeq Z_2^{n_2}$$

and

$$H^2(\Phi/\langle\sigma\rangle, M^{\sigma}) \simeq Z_2^{n_1-1}.$$

So $H^1(\Phi,M) \simeq Z_2^{n_2}$ and the exact sequence we had becomes

$$0 \longrightarrow Z_2^{n_1-1} \longrightarrow H^2(\Phi,M) \xrightarrow{r_\tau} Z_2 \longrightarrow Z_2^{n_2} \longrightarrow H^3(\Phi,M).$$

{9b}

$$\begin{array}{|c|c|}\hline \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}1&&\\&\ddots&\\&&1\\0&\cdots&0\end{matrix} \\ \hline \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}0&\cdots&0\\1&&\\&\ddots&\\&&1\end{matrix} \\ \hline\end{array}$$

$H^1(\langle\sigma\rangle,M) = 0$, $H^2(\langle\sigma\rangle,M) = Z_2$. Since $D_2 = E$,

$$H^1(\Phi,M) \simeq H^1(\Phi/\langle\sigma\rangle, M^{\sigma}) \simeq Z_2^{n_2}$$

and the exact sequence in this case

$$0 \longrightarrow H^2(\Phi/\langle\sigma\rangle, M^{\sigma}) \longrightarrow H^2(\Phi,M) \longrightarrow H^2(\langle\sigma\rangle,M)^{\Phi}$$
$$\longrightarrow H^3(\Phi/\langle\sigma\rangle, M^{\sigma}) \longrightarrow H^3(\Phi,M)$$

becomes

$$0 \longrightarrow Z_2^{n_1} \longrightarrow H^2(\Phi,M) \longrightarrow Z_2 \longrightarrow Z_2^{n_2} \longrightarrow H^3(\Phi,M).$$

$$\{10\} \qquad D = \left[\begin{array}{c|c} \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}0&&&\\1&\ddots&&\\&\ddots&\ddots&\\&&1&0\end{matrix} \\ \hline \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} \end{array}\right]$$

Here $H^1(<\tau>,M) = H^2(<\tau>,M) = 0$ and $D_3 = E$. So

$$H^1(\Phi,M) \simeq H^1(\Phi/<\tau>,M^{\tau}) \simeq Z_2^{n_2}$$

and

$$H^2(\Phi,M) \simeq H^2(\Phi/<\tau>,M^{\tau}) \simeq Z_2^{n_2}.$$

Turning now to the modules of the form

$$\{\infty\} \qquad D = \left[\begin{array}{c|c} E & D_4 \\ \hline E & E \end{array}\right],$$

where D_4 is invertible and is indecomposeable as a matrix over Z_2, there were two cases, depending on whether or not D_4-E was indecomposeable.

For both of these cases, by a previous calculation

$$H^1(<\tau>,M) = H^2(<\tau>,M) = 0$$

and since $D_3 = E$, we have

$$H^1(\Phi,M) \simeq H^1(\Phi/\langle\tau\rangle,M^\tau) \simeq Z_2^{n_2}$$

and

$$H^2(\Phi,M) \simeq H^2(\Phi/\langle\tau\rangle,M^\tau) \simeq Z_2^{n_1}.$$

Remark. Although one may feel that $H^2(\Phi,M)$ will always be isomorphic to $Z_2^{n_1}$, this turns out not to be the case, as subsequent work will show.

Summary of Cohomology and Exact Sequences

Type	$H^1(\Phi,M)$	$H^2(\Phi,M)$
{1}	$Z_2^{n_2}$	$Z_2^{n_1}$
{2a}	$Z_2^{n_2}$	$0 \longrightarrow Z_2^{n_1-1} \longrightarrow H^2(\Phi,M) \xrightarrow{r_\sigma} Z_2 \longrightarrow Z_2^{n_2}$
{2b}	$Z_2^{n_2}$	$0 \longrightarrow Z_2^{n_1} \longrightarrow H^2(\Phi,M) \xrightarrow{r_\tau} Z_2 \longrightarrow Z_2^{n_2}$
{3a}	$Z_2^{n_2}$	$Z_2^{n_1}$
{3b}	$Z_2^{n_2}$	$Z_2^{n_1}$
{4a}	$Z_2^{n_2}$	$0 \longrightarrow Z_2^{n_1} \longrightarrow H^2(\Phi,M) \xrightarrow{r_\sigma} Z_2 \longrightarrow Z_2^{n_2}$
{4b}	For $H^1(\Phi,M)$	$0 \longrightarrow Z_2^{n_2} \longrightarrow H^1(\Phi,M) \longrightarrow Z_2 \longrightarrow$
	and $H^2(\Phi,M)$	$\longrightarrow Z_2^{n_1} \longrightarrow H^2(\Phi,M) \longrightarrow Z_2 \longrightarrow Z_2^{n_2}$

Type	$H^1(\Phi,M)$	$H^2(\Phi,M)$
{5}	$Z_2^{n_2}$	$Z_2^{n_1}$
{6a}	$Z_2^{n_2}$	$0 \longrightarrow Z_2^{n_1} \longrightarrow H^2(\Phi,M) \xrightarrow{r_{\sigma\tau}} Z_2 \longrightarrow Z_2^{n_2} \longrightarrow$
{6b}	$Z_2^{n_2}$	$0 \longrightarrow Z_2^{n_1-1} \longrightarrow H^2(\Phi,M) \xrightarrow{r_{\tau}} Z_2 \longrightarrow Z_2^{n_2}$
{7a}	$Z_2^{n_2}$	$Z_2^{n_1}$
{7b}	$Z_2^{n_2}$	$Z_2^{n_1}$
{8}	$Z_2^{n_2}$	$Z_2^{n_1}$
{9a}	$Z_2^{n_2}$	$0 \longrightarrow Z_2^{n_1-1} \longrightarrow H^2(\Phi,M) \xrightarrow{r_{\tau}} Z_2 \longrightarrow Z_2^{n_2}$
{9b}	$Z_2^{n_2}$	$0 \longrightarrow Z_2^{n_1} \longrightarrow H^2(\Phi,M) \xrightarrow{r_{\sigma}} Z_2 \longrightarrow Z_n^{n_2}$
{10}	$Z_2^{n_2}$	$Z_2^{n_1}$
{∞}	$Z_2^{n_2}$	$Z_2^{n_1}$

§5. SUPERNORMALIZED COCYCLES

The next stage in unravelling the second cohomology groups of these modules, is to analyze the restriction maps:

$$H^2(\Phi,M) \longrightarrow H^2(\langle\gamma\rangle,M)$$

where γ is a generator of Φ.

Suppose we are in the category of G modules with K a normal subgroup of G.

In cohomology, it is convenient to work with normalized cochains. In the situation of $K \triangleleft G$, we give a definition, suggested by H. Sah, of a more restrictive type of 2-cochain which can be used to define the second cohomology group and which has some useful properties.

Definition. Let $\pi: G \to G/K$ be the natural propection map. Let $F \in C^2(G,M)$ be a normalized cochain. Then F is a supernormalized cochain if there exists a section $\tau: G/K \to K$ (not necessarily a homomorphism) of the natural projection map π such that

$$F(k,\tau(\overline{g})) = 0,$$

for all $k \in K$, $\overline{g} \in G/K$. ($F(k,\tau(\overline{g})) = 1$ in multiplicative notation).

We will say that in the above situation, F is supernormalized relative to τ.

Notation. Let $C_0^2(G,M)$, $B_0^2(G,M)$ and $Z_0^2(G,M)$ be the supernormalized cochains, coboundaries and cocycles.

Clearly $C_0^2(G,M)$, $B_0^2(G,M)$ and $Z_0^2(G,M)$ are all subgroups of $C^2(G,M)$.

Remark. Supernormalized cocycles have the following useful property.

If F is a supernormalized cocycle relative to $\tau: G/K \to G$, we can evaluate $F(k,g)$ for all $g \in G$ as follows.

g has a unique expression (relative to τ) of the form $g = k_g \cdot \tau(\bar{g})$ where $k_g \in K$.

Now apply the cocycle condition to F.

$$0 = kF(k_g, \tau(\bar{g})) - F(kk_g, \tau(\bar{g})) + F(k, k_g\tau(\bar{g})) - F(k, k_g)$$

Since F is supernormalized, the first two terms are zero. We have

$$0 = F(k, k_g\tau(\bar{g})) - F(k, k_g) = F(k,g) - F(k,k_g)$$

that is,

$$F(k,g) = F(k,k_g).$$

Hence $F|_{K\times G}$ depends only on $F|_{K\times K}$.

In fact, if $\rho_\tau : G \to K$ is the function given by $\rho_\tau(g) = g_k$ then we have a commutative diagram:

$$\begin{tikzcd} G \times G \arrow[r, "F"] & M \\ K \times G \arrow[u] \arrow[r, "1\times\rho_\tau"'] & K \times K \arrow[u, "F"'] \end{tikzcd}$$

Now if $Z_0^2(G,M) \longrightarrow Z^2(G,M)$ is the inclusion map, then we have a natural induced map

$$Z_0^2(G,M) \longrightarrow \frac{Z^2(G,M)}{B^2(G,M)} = H^2(G,M)$$

with kernal, $B_0^2(G,M)$.

<u>Lemma 5.1</u>. $H^2(G,M) \simeq \dfrac{Z_0^2(G,M)}{B_0^2(G,M)}$.

<u>Proof</u>: We show that $Z_0^2(G,M) \longrightarrow \dfrac{Z^2(G,M)}{B^2(G,M)}$ is onto by proving the following slightly stronger result:

Let $\tau : G/K \to G$ be a section such that $\tau(\overline{1}) = 1$, then any cocycle $F \in Z^2(G,M)$ is cohomologous to a supernormalized cocycle relative to τ.

Let F be represented by any extension $1 \to M \to E \to G \to 1$ with corresponding set of representatives $\{r_g\}_{g\in G}$. (We use multiplicative notation in M, regarded as a subgroup of E).

We assume that $r_1 = 1 \varepsilon E$ (i.e., that F is normalized in the usual sense).

The elements of E can be expressed uniquely in the form xr_g, with $x \varepsilon M$, $g \varepsilon G$. Then the group structure in E follows from the equations

$$r_g \cdot x = x^g \cdot r_g$$

where x^g indicates the module action $G \times M \to M$ and

$$r_g r_{g'} = F(g,g') r_{gg'}$$

Let $\tau : G/K \to K$ be any section of π such that $\tau(\bar{1}) = 1$. We modify the family of representatives as follows.

For each $g \varepsilon G$, g can be written uniquely as $g = k_g \cdot \tau(\bar{g})$, $k_g \varepsilon K$, $\bar{g} \varepsilon G/K$. Now define

$$r'_g = r_{k_g} \cdot r_{\tau(\bar{g})}$$

Note that $r'_k = r_k$ for all $k \varepsilon K$, since $k = k \cdot 1 = k \cdot \tau(\bar{1})$ and therefore $r'_k = r_k \cdot r_1 = r_k$. Also $r'_{\tau(\bar{g})} = r_{\tau(\bar{g})}$ by a similar argument.

Modifying the set of representatives $\{r_g\}_{g \varepsilon G}$ results in changing F to a cohomologous cocycle F' where

$$F'(g,g') = r'_g r'_{g'} (r'_{gg'})^{-1}$$

and

$$F'(k,\tau(\bar{g})) = r'_k \cdot r_{\tau(\bar{g})} \cdot (r'_{k\cdot\tau(\bar{g})})^{-1}$$

$$= r_k \cdot r_{\tau(\bar{g})} \cdot (r_k \cdot r_{\tau(\bar{g})})^{-1} = 1.$$

So the new cocycle F' is supernormalized relative to the section τ. (Note however that $F'|_K = F_K$ since $r'_k = r_k$.)

<u>Proposition 5.2</u>. Let $G = K \times L$ and let $z \in H^2(G,M)$ be represented by a supernormalized cocycle F relative to the canonical section $\tau: L \to K \times L$. Let $F|_K = f$ and $F|_L = g$, then F is given explicitly by the formula

$$F(k_1\ell_1, k_2\ell_2) = k_1 m_{k_2}^{\ell_1} + f(k_1,k_2) + k_1k_2 g(\ell_1,\ell_2)$$

where $m_{k_2}^{\ell_1} = F(\ell_1,k_2)$. (So F is expressible in terms of its restrictions and a mixed term $F(\ell_1,k_2)$).

<u>Proof</u>: The ideas and notation for much of this proof are from MacLane-<u>Homology</u>. Hence, additive notation is used although the group extensions considered are not assumed abelian.

Let E be the extension of M by G of the form $E = \{(x,g) | x \in M, g \in G\}$ with addition given by

$$(x,g) + (x',g') = (x+gx'+F(g,g'),gg')$$

Then $E_K = \{(x,k) | k \in K\}$ forms a normal subgroup of E as it is the kernal of the composition

$$E \longrightarrow G \longrightarrow G/K.$$

E_K can then be regarded as the extension of M by K corresponding to $F|_K = f$.

E can also be regarded as an extension of the non-abelian group E_K by the group $G/K = L$. We have then, a commutative diagram of groups with exact rows and columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 1 \arrow[d] & \\
0 \arrow[r] & E_K \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & K \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

Let $\theta: E \longrightarrow \text{Aut } E_K$ be the homomorphism that associates to each element of E, the operation of conjugation by that element in E_K. Then $\theta(E_K) \subset \text{Inn}E_K$, the inner automorphisms of E_K. Hence, there is an induced homomorphism $\psi: L \rightarrow \dfrac{\text{Aut } E_K}{\text{Inn}E_K} = \text{Out } E_K$, called the "conjugation class" of the extension, making the following diagram commute.

$$\begin{tikzcd}
0 \arrow[r] & E_K \arrow[r] \arrow[d, "\mu"] & E \arrow[r] \arrow[d, "\theta"] & L \arrow[r] \arrow[d, "\psi"] & 1 \\
1 \arrow[r] & \text{Inn}\, E_K \arrow[r] & \text{Aut } E_K \arrow[r] & \text{Out } E_K \arrow[r] & 1
\end{tikzcd}$$

So in MacLane's terminology, E is an extension of the abstract kernal (L, E_K, ψ).

Now according to MacLane [7], Lemma 8.2, p. 125, if we choose $\phi(\ell) \in \psi(\ell)$ for all $\ell \in L$, (with $\phi(1) = 1$), then the extension

$$0 \longrightarrow E_K \xrightarrow{\alpha} E \xrightarrow{\beta} L \longrightarrow 1$$

is congruent to a "crossed product extension" $[E_K, \phi, h, L]$. Moreover, MacLane shows in his proof how to construct the proper crossed product extension and the congruence. First we choose a family of representatives $\{u_\ell\}_{\ell \in L}$, (i.e., a section of β). In our case, we define u_ℓ by letting $u_\ell = (0, \ell)$ for all $\ell \in L$. Then we select a $\phi \in \psi$ by the rule

$$\phi(\ell)x = u_\ell + x - u_\ell \text{ for all } \ell \in L,\ x \in E_K,$$

then $\phi(1) = 1$. Now define a function $h: L \times L \to E_K$ by

$$h(\ell_1, \ell_2) = u_{\ell_1} + u_{\ell_2} - u_{\ell_1 \ell_2}.$$

$(h(\ell_1, \ell_2) \in E_K$ since $\beta h(\ell_1, \ell_2) = \beta u_{\ell_1} \cdot \beta u_{\ell_2} \cdot (\beta u_{\ell_1 \ell_2})^{-1}$

$$= (\ell_1 \cdot \ell_2)(\ell_1 \cdot \ell_2)^{-1} = 1)$$

In fact,

$$\begin{aligned} h(\ell_1,\ell_2) &= (0,\ell_1) + (0,\ell_2) - (0,\ell_1\ell_2) \\ &= (g(\ell_1,\ell_2),\ell_1\ell_2) + (-(\ell_1\ell_2)^{-1}g(\ell_1\ell_2,(\ell_1\ell_2)^{-1}),(\ell_1\ell_2)^{-1}) \\ &= (g(\ell_1,\ell_2),1). \end{aligned}$$

Then, as MacLane mentions, the following two identities hold:

$$[\phi(\ell_1)h(\ell_2,\ell_3)] + h(\ell_1,\ell_2\ell_3) = h(\ell_1,\ell_2) + h(\ell_1\ell_2,\ell_3)$$

(from associativity on $u_{\ell_1} + u_{\ell_2} + u_{\ell_3}$), and

$$\phi(\ell_1)\phi(\ell_2) = \mu[h(\ell_1,\ell_2)]\phi(\ell_1\ell_2)$$

where $\mu = \theta\Big|_{E_K}$.

Also h: $L \times L \to E_K$ satisfies the normalization conditions:

$$h(\ell,1) = 0 = h(1,\ell),$$

since $u_\ell = 1$. Hence $[E_K,\phi,h,L]$ defines a crossed product extension of E_K by L given by

$$B_0 = [E_K,\phi,h,L] = \{(x,\ell)|x \in E_K, \ \ell \in L\}$$

with addition defined by

$$(x,\ell) + (x',\ell') = (x+\phi(\ell)x' + h(\ell,\ell'),\ell\ell').$$

Moreover, by MacLane's [7] lemma 8.2, our original extension of the abstract kernal (L,E_K,ψ) is congruent to the crossed product extension $[E_K,\phi,h,L]$.

$$\begin{tikzcd}
0 \arrow[r] & E_K \arrow[r] \arrow[d, "1"] & E \arrow[r] \arrow[d, "\Gamma"] & L \arrow[r] \arrow[d, "1"] & 1 \\
0 \arrow[r] & E_K \arrow[r] & B_0 \arrow[r] & L \arrow[r] & 1
\end{tikzcd}$$

Γ is defined as follows; each element of E is uniquely expressible in the form $\nu + u_\ell$ with $\nu \in E_K$, $\ell \in L$. Then if we let $\Gamma(\nu + u_\ell) = (\nu,\ell)$, Γ is a congruence.

Now we calculate Γ in our extension situation. An arbitrary element of E is of the form $(x,k\ell)$. Since F is supernormalized we have

$$(x,k\ell) = (x,k) + (0,\ell) = (x,k) + \mu_\ell$$

(Note $(x,k) \in E_K$ so this is the required decomposition). Then $\Gamma(x,k\ell) = \Gamma\{(x,k) + \mu_\ell\} = [(x,k),\ell]$.

Now since Γ is a homomorphism we have on the one hand, that

$$\begin{aligned} &\Gamma\{(x,k\ell) + (y,k'\ell')\} \\ (1)\qquad &=\Gamma\{(x+k\ell y+F(k\ell,k'\ell'),kk'\ell\ell')\} \\ &=[(x+k\ell y+F(k\ell,k'\ell'),kk'),\ell\ell'] \end{aligned}$$

On the other hand we have

$$\begin{aligned} &\Gamma(x,k\ell) + \Gamma(y,k'\ell') \\ (2)\qquad &= [(x,k),\ell] + [(y,k'),\ell'] \\ &= [(x,k) + \phi(\ell)(y,k') + h(\ell,\ell'),\ell\ell']. \end{aligned}$$

To evaluate the last expression recall that

$$\begin{aligned}\phi(\ell)(y,k') &= u_\ell + (y,k') - u_\ell \\ &= (0,\ell) + (y,k') - (0,\ell) \\ &= (\ell + F(\ell,k'),k'\ell) + (-\ell^{-1}F(\ell,\ell^{-1}),\ell^{-1}) \\ &= (\ell y + F(\ell,k') - k'F(\ell,\ell^{-1}) + (F(k'\ell,\ell^{-1}),k').\end{aligned}$$

Since F is a cocycle we have

$$-k'F(\ell,\ell^{-1}) + F(k'\ell,\ell^{-1}) = F(k',\ell\ell^{-1}) - F(k',\ell).$$

Since F is normalized, and in fact supernormalized, the right side above is zero. Hence

$$\phi(\ell)(y,k') = (\ell y+F(\ell,k'),k').$$

Now define $m^{\ell}_{k'} = F(\ell,k')$ and then (2) becomes

$$[(x,k) + (\ell y+m^{\ell}_{k'},k') + (g(\ell,\ell'),1),\ell\ell']$$

which becomes

$$\begin{aligned}&[(x+k\ell y+km^{\ell}_{k'}+F(k,k'),kk') + (g(\ell,\ell'),1),\ell\ell'] \\ &= [x+k\ell y+km^{\ell}_{k'}+F(k,k') + kk'g(\ell,\ell'),kk'),\ell\ell'].\end{aligned}$$

Finally, equating (1) and (2) and using $F|_K = f$ we have

$$F(k\ell,k'\ell') = km^{\ell}_{k'} + f(k,k') + kk'g(\ell,\ell'),$$

which is the desired formula.

Remark. The proposition generalizes for the assumption that $K \triangleleft G$, but the correspondingly more complicated formula and proof do not seem particularly useful.

Let $r_K\colon H^2(K \times L, M) \longrightarrow H^2(K, M)$ and

$r_L\colon H^2(K \times L, M) \longrightarrow H^2(L, M)$ be the restriction maps induced by the inclusions $K \hookrightarrow K \times L$, $L \hookrightarrow K \times L$.

Corollary 5.3. Let $z \in H^2(K \times L, M)$, and let $r_K(z) = [f]$ and $r_L(z) = [g]$ where f and g are cocycles on M. Then z has a cocycle representative F such that

$$F(k\ell, k'\ell') = km^{\ell}_{k'} + f(k,k') + kk'g(\ell,\ell')$$

where $m^{\ell}_{k'} = F(\ell, k')$

Proof. Let F' be an arbitrary representative of z then

$$\left\{\begin{array}{l} F'|_K = f + \delta_K s \\ F'|_L = g + \delta_L t \end{array}\right\} \text{ where } s \in C^1(K,M) \text{ and } t \in C^1(L,M).$$

Let $w \in C^1(G,M)$ be given by

$$\begin{cases} w|_K = s \\ w|_L = t \\ w(g) = 0 \text{ if } g \notin K \text{ and } g \notin L. \end{cases}$$

Then $F'' = F' - \delta_G w \in Z^2(G,M)$ and F'' has the property that

$F''|_K = f$ and $F''|_L = g$. Finally, by the proof of Lemma 5.1, F'' is cohomologous to a supernormalized cocycle, F, relative to the natural section $L \longrightarrow G = K \times L$ with $F|_K = F'|_K$ and $F|_L = F'|_L$. Now, applying the previous proposition gives us the required formula for F in terms of its restrictions.

At this stage, we would like to "turn this formula around" and use it to investigate the cocycle F in terms of its restrictions and to find some condition on these restrictions which would allow us to lift to $H^2(G,M)$ in the case where $G \simeq K \times L$. (As might be expected, the key lies entirely in the term m^{ℓ}_{k}, [or, in fact, the element

$m \in C^0(L \times K,M)$ defined by m^{ℓ}_{k},]).

§6. LIFTING THE RESTRICTION MAP

Notation. In this section, Φ will stand for a group of the form $\Phi = A \times B$ where B is cyclic of finite order n, generated by the element τ. M is a module over the group ring of Φ.

The main result of this section is that given a possible value of the restriction map, we can associate to it, a family of obstruction cocycles, such that the vanishing of the cohomology class of one of these cocycles allows us to construct a lifting of the restriction map. [L. Charlap and H. Sah have communicated that the lifting of res: $H^2(G,M) \longrightarrow H^2(K,M)$, for the more general situation of $K \triangleleft G$, involves two cohomology class obstructions].

Let r_A: $H^2(\Phi,M) \longrightarrow H^2(A,M)$ be the restriction map induced by $A \subset \Phi$. Since A is normal in Φ, $H^2(A,M)$ is a Φ module with action defined (on the cocycle level) by $(gf)t = g(f(g^{-1}t))$, $g \varepsilon \Gamma(\Phi)$, $t \varepsilon B(A)$, ($B(A)$ stands for the standard bar resolution) and $f \varepsilon \mathrm{Hom}_A(B(A),M)$. Now if $[f] \varepsilon \mathrm{Im}(r_A)$, then $f = F|_A + \delta_A s$ where $F \varepsilon Z^2(\Phi,M)$ and $s \varepsilon C^1(A,M)$. In particular $F \varepsilon \mathrm{Hom}_\Phi(B(\Phi),M)$. Therefore $g(F(g^{-1}t)) = F(t)$ and so

$$gf = gF|_A + g\delta_A s = F|_A + \delta_A gs \sim F|_A \sim f$$

So $[gf] = [f]$ and hence the image of r_A: $H^2(\Phi,M) \longrightarrow H^2(A,M)$ for $A \subset \Phi$, lies always in $H^2(A,M)^{\Phi}$. Moreover, the elements of $H^2(A,M)$ are fixed under the action of A itself (by an analogous argument to the above, using the fact that cocycles are in particular, A-homomorphisms). Hence $H^2(A,M)^{\Phi} = H^2(A,M)^B$.

So it appears that a necessary condition for there to exist a lift of $\alpha \varepsilon H^2(A,M)$ to $H^2(\Phi,M)$ is that $\alpha \varepsilon H^2(A,M)^B$.

Let $\alpha \varepsilon H^2(A,M)^B$ and suppose $f \varepsilon \alpha$. Since α is invariant under B,

$$[(\tau-1)f] = (\tau-1)[f] = (\tau-1)\alpha = 0$$

and therefore there exists a $\mu \varepsilon C^1(A,M)$ such that

$$(\tau-1)f = \delta_A\mu.$$

Now consider $N_\tau(\mu) = (1+\tau+\ldots+\tau^{n-1})\mu \varepsilon C^1(A,M^B)$. Applying δ_A we get

$$\delta_A N_\tau(\mu) = N_\tau \delta_A \mu$$

(since the elements of A and B commute with one another). But $N_\tau \delta_A \mu = N_\tau(\tau-1)f = 0$. So $N_\tau(\mu) \varepsilon Z^1(A,M^B)$.

Definition: Let $\alpha \varepsilon H^2(A,M)^B$. An obstruction cocycle for α is an element of the form $N_\tau(\mu)$, where $\mu \varepsilon C^1(A,M)$, and such that $\delta_A\mu = (\tau-1)f$ for some $f \varepsilon \alpha$. In this situation we will sometimes refer to $N_\tau(\mu)$ as an obstruction cocycle for α corresponding to f. Obviously, there will generally be a whole family of such objects corresponding to a given $\alpha \in H^2(A,M)^B$.

The following theorem gives a constructive proof, on the cocycle level, that the vanishing of the cohomology class of an obstruction cocycle for α is equivalent to the existence of a lifting of α under the restriction map.

Theorem 6.1. Let $\alpha \in H^2(A,M)^B$. Then

$$\alpha \in \mathrm{Im}\{r_A\colon H^2(\Phi,M) \longrightarrow H^2(A,M)^B\}$$

if and only if there exists an obstruction cocycle $N_\tau\mu$ for α such that $[N_\tau\mu] = 0 \in H^1(A,M^B)$.

Proof: Suppose $\alpha = r_A([F])$ where $[F] \in H^2(\Phi,M)$. Let $f = F|_A$, then $f \in \alpha$. Let E_A be the extension of M by A corresponding to f and let E be the extension of M by Φ corresponding to F. Then as in the previous section, E can be regarded as an extension of E_A by $\Phi/A \simeq B$ and we have the commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 1 \arrow[d] & \\
(f) & 0 \arrow[r] & M \arrow[r] \arrow[d] & E_A \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 1 \\
(F) & 0 \arrow[r] & M \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \Phi \arrow[r] \arrow[d] & 1 \\
 & & 1 \arrow[r] & B \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 1 \\
 & & & 1 \arrow[r] & 1 &
\end{tikzcd}$$

As before, we use additive notation in our group extensions. We also remark, as before, that the elements of E operating by conjugation, act as automorphisms of the normal subgroup E_A. We now use the fact that B is cyclic to define a particularly nice element, $\phi: \Phi/A \to \text{Aut } E_A$ of the conjugation class $\psi: \Phi/A \to \text{Out}(E_A) = \text{Aut } E_A/\text{Inn } E_A$ of the extension

$$(E_A) \qquad 0 \longrightarrow E_A \longrightarrow E \longrightarrow B \longrightarrow 1$$

For $k = 0,1,\ldots,n-1$ define $\phi(\tau^k)$ to be conjugation of E_A by the element $k\cdot(0,\tau)$ of E. Note, ϕ is not a homomorphism from $\Phi/A \simeq B \longrightarrow \text{Aut } E_A$, but the extent to which it fails is "captured" in the term $\{\Phi(\tau)\}^n$, in the sense that $\phi(\tau)^k = \phi(\tau^k)$, $k = 0,\ldots,n-1$ but

$$\phi(\tau)^n \neq \phi(\tau^n) = \phi(1) = 1.$$

If $(x,\sigma) \varepsilon E_A$ then clearly $\phi(\tau)(x,\sigma) = (0,\tau) + (x,\sigma) - (0,\tau)$ is of the form $(\tau x+\mu_\sigma,\tau\sigma\tau^{-1}) = (\tau x+\mu_\sigma,\sigma)$ where μ_σ is some element of M. The function $\sigma \longmapsto \mu_\sigma$ defines an element $\mu \varepsilon C^1(A,M)$. If we let

$$\mu_\sigma^{\tau^k} = (1+\tau+\ldots+\tau^{k-1})\mu_\sigma, \text{ for } k = 1,\ldots,n-1$$

then $\mu_\sigma^\tau = \mu_\sigma$ and it is easily seen inductively that

$$\phi(\tau^k)(x,\sigma) = (\tau^k x + (1+\tau+\dots+\tau^{k-1})\mu_\sigma,\sigma)$$
$$= (\tau^k x+\mu_\sigma^{\tau^k},\sigma) \text{ for } k = 1,\dots,n-1.$$

Moreover,

$$\phi(\tau^n)(x,\sigma) = \phi(\tau)\cdot\phi(\tau^{n-1})(x,\sigma) = \phi(\tau)(\tau^{n-1}x+\mu_\sigma^{\tau^{n-1}},\sigma) = (x+N_\tau\mu_\sigma,\sigma).$$

At this point in the proof some technical lemmas are needed.

Lemma 6.2. $k\cdot(0,\tau) = (F(\tau^{k-1},\tau)+\dots+F(\tau^2,\tau)+F(\tau,\tau),\tau^k) =$ $(F(\tau,\tau^{k-1})+\tau F(\tau,\tau^{k-2})+\dots+\tau^{k-3}F(\tau,\tau^2)+\tau^{k-2}F(\tau,\tau),\tau^k)$ for $k \geq 1$.

Proof. This is true for $k = 1$ because $F(1,\tau) = 0 = F(\tau,1)$. Assuming true for $k = m$, with $m > 1$ then $(m+1)\cdot(0,\tau) =$

$$m\cdot(0,\tau) + (0,\tau) = (\sum_{\ell=1}^{m-1} F(\tau^\ell,\tau),\tau^m) + (0,\tau)$$
$$= (\sum_{\ell=1}^{m} F(\tau^\ell,\tau),\tau^{m+1}).$$

Also $(m+1)\cdot(0,\tau) = (0,\tau) + m(0,\tau)$

$$= (0,\tau) + (\sum_{i+j=m-1} \tau^i F(\tau,\tau^j),\tau^m)$$
$$= (\sum_{i+j=m-1} \tau^{i+1}F(\tau,\tau^j) + F(\tau,\tau^m),\tau^{m+1})$$
$$= (\sum_{i+j=m} \tau^i F(\tau,\tau^j),\tau^{m+1}).$$

Definition: Let $F_k = \sum_{i=1}^{k-1} F(\tau^i,\tau) = \sum_{i+j=k-1} \tau^i F(\tau,\tau^j)$, then Lemma 6.2 becomes $k\cdot(0,\tau) = (F_k,\tau^k)$, $k > 1$. In particular we have $n\cdot(0,\tau) = (F_n,1)$.

Lemma 6.3. $F_n \varepsilon M^B$.

Proof. We show $\tau F_n = F_n$ by applying τ to the right hand side in the above definition and comparing the result with the left hand side. Since $F(\tau,\tau^n) = 0$ we can write

$$\tau F_n = F(\tau,\tau^n)+\tau F_n = F(\tau,\tau^n)+\tau F(\tau,\tau^{n-1})+\ldots+\tau^{n-1}F(\tau,\tau)$$

$$= \sum_{i+j=n} \tau^i F(\tau,\tau^j) = F_{n+1} = F(\tau^n,\tau)+F(\tau^{n-1},\tau)+\ldots+F(\tau,\tau)$$

$$= F(\tau^{n-1},\tau)+\ldots+F(\tau,\tau) = F_n.$$

Now returning to the proof, we have $(\phi(\tau))^n(x,\sigma) = (x+N_\tau\mu_\sigma,\sigma) = n\cdot(0,\tau)+(x,\sigma)-n\cdot(0,\tau) = (F_n,1)+(x,\sigma)-(F_n,1) = (x+(1-\sigma)F_n,\sigma)$ for all $(x,\sigma) \varepsilon E_A$. So $N_\tau\mu_\sigma = (1-\sigma)F_n$ for all $\sigma \varepsilon A$ and since $F_n \varepsilon M^B$, if we regard $F_n \varepsilon C^0(A,M^B)$, we have $N_\tau\mu = \delta_A F_n \varepsilon B^1(A,M^B)$. Hence, $[N_\tau\mu] = 0 \varepsilon H^1(A,M^B)$.

It remains to show that $N_\tau\mu$ is an obstruction cocycle for α. Applying the automorphism $\phi(\tau)$ to $(x,\sigma)+(y,\sigma)$ gives

$$(1) \quad \phi(\tau)\{(x,\sigma)+(y,\sigma')\} = \phi(\tau)(x+\sigma y+f(\sigma,\sigma'),\sigma\sigma')$$

$$= (\tau x+\tau\sigma y+\tau f(\sigma,\sigma')+\mu_{\sigma\sigma'},\sigma\sigma')$$

(2) and $\phi(\tau)(x,\sigma)+\phi(\tau)(y,\sigma') = (\tau x+\mu_\sigma,\sigma)+(\tau y+\mu_{\sigma'},\sigma')$
$=(\tau x+\mu_\sigma+\sigma\tau y+\sigma\mu_{\sigma'}+f(\sigma,\sigma'),\sigma\sigma')$

Equating (1) and (2) gives

$$(\tau-1)f(\sigma,\sigma') = \sigma\mu_{\sigma'}-\mu_{\sigma\sigma'}+\mu_\sigma = (\delta_A\mu)(\sigma,\sigma')$$

i.e., $(\tau-1)f = \delta_A\mu$. Since $f \in \alpha$, $N_\tau\mu$ is consequently an obstruction cocycle for α corresponding to f and $[N_\tau\mu] = 0$.

To prove the converse, we assume $N_\tau\mu$ is an obstruction cocycle for α such that $[N_\tau\mu] = 0 \in H^1(A,M^B)$. Then there exists an $f \in \alpha$ such that $(\tau-1)f = \delta_A\mu$ and there exists an $\ell \in C^0(A,M^B)$ such that $N_\tau\mu = \delta_A\ell$. Since $\ell \in M^B$, using the usual resolution for modules over a cyclic group, ℓ can be viewed as an element of $Z^2(B,M)$ and hence defines a cohomology class $[\ell] \in H^2(B,M)$. We can realize this class by a 2-cocycle g, relative to the bar resolution as follows.

<u>Definition</u>. Let $g\colon B\times B \longrightarrow M$ be defined by

$$g(\tau^i,\tau^j) = \begin{cases} 0 \text{ if } i+j<n \\ \ell \text{ if } i+j\geq n \end{cases} \quad \text{with } i,j<n.$$

<u>Lemma 6.4</u>. For the cyclic group B, let $[B_i]$ be the standard bar resolution, and let $[\Gamma]$ be the simpler, cyclic group resolution. Then with g defined as above,

a) $g \in Z^2([B_i],M)$

b) There exists a chain homomorphism $\theta:[\Gamma] \to [B_i]$ such that $H^j(\theta): H^j([B_i],M) \longrightarrow H^j([\Gamma],M)$ is an isomorphism for all j, and $H^2(\theta)([g]) = [\ell]$.

c) $g(\tau^{n-1},\tau)+\ldots+g(\tau,\tau) = \ell$.

d) If g' is any other cocycle satisfying c) then $g' \sim g$.

The above lemma is probably well known, hence we omit its somewhat technical proof.

Now for each $\tau^k \in B$ $k = 0,\ldots,n-1$ define an element $\mu^{\tau^k} \in C^1(A,M)$ (i.e., an element of $C^1(B,C^1(A,M))$, as follows. Let $\mu_\sigma^1 = 0$, $\mu_\sigma^\tau = \mu_\sigma$ for any $\sigma \in A$, (so $\mu^\tau = \mu$) and then let $\mu_\sigma^{\tau^k} = (1+\tau+\ldots+\tau^{k-1})\mu_\sigma^\tau$. $k = 2,\ldots,n-1$.

Lemma 6.5. a) $$\delta_B\mu(\tau^i,\tau^j) = \delta_A g(\tau^i,\tau^j) = \begin{cases} 0 & \text{if } i+j<n \\ N_\tau\mu & \text{if } i+j\geq n \end{cases}$$

b) $$\delta_A\mu^{\tau^k} = (\tau^k-1)f \text{ for all } \tau^k \in B$$

Proof. a) We must show that for all $i,j<n$

$$\tau^i\mu^{\tau^j}-\mu^{\tau^i\tau^j}+\mu^{\tau^i} = \delta_A g(\tau^i,\tau^j) = \begin{cases} 0 & \text{if } i+j<n \\ N_\tau\mu & \text{if } i+j\geq n. \end{cases}$$

(Here $g(\tau^i,\tau^j)$ is regarded as an element of $C^0(A,M)$). If $i+j<n$ then the middle expression is zero. The left hand expression is then

$$\tau^i\mu^{\tau^j} - \mu^{\tau^{(i+j)}} + \mu^{\tau^i}$$

$$= \tau^i(1+\tau+\ldots+\tau^{j-1})\mu + (1+\tau+\ldots+\tau^{i+j-1}) + (1+\tau+\ldots+\tau^{i-1})\mu$$

$$= \{(1+\tau+\ldots+\tau^{i-1}) + (\tau^i+\ldots+\tau^{i+j-1})\}\mu - (1+\tau+\ldots+\tau^{i+j-1})\mu$$

$$= 0.$$

If $i+j\geq n$, the first expression is

$$\tau^i\mu^{\tau^j} - \mu^{\tau^{i+j-n}} + \mu^{\tau^i}$$

$$= (\tau^i+\ldots+\tau^{i+j-1})\mu - (1+\ldots+\tau^{i+j-n-1})\mu + (1+\ldots+\tau^{i-1})\mu$$

$$= (1+\tau+\ldots+\tau^{i+j-1})\mu - (1+\tau+\ldots+\tau^{i+j-n-1})\mu$$

$$= (\tau^{i+j-n}+\ldots+\tau^{i+j-1})\mu.$$

The coeficient of μ is a sum of n successive powers of τ, and since the order of τ is n, the coefficient must be $N_\tau\mu = (1+\tau+\ldots+\tau^{n-1})\mu$. So if $i+j>n$, the first expression is $N_\tau\mu$. But $N_\tau\mu = \delta_A\ell = \delta_A g(\tau^i,\tau^j)$, for in this case, by definition of g, $g(\tau^i,\tau^j) = \ell$.

<u>Proof of b)</u>. $\delta_A\mu^{\tau^k} = \delta_A(1+\tau+\ldots+\tau^{k-1})\mu^\tau = (1+\tau+\ldots+\tau^{k-1})\delta_A\mu$ $= (1+\tau+\ldots+\tau^{k-1})(\tau-1)f = (\tau^k-1)f$. (Note, this says that

$\delta_A\mu = \delta_B f$ with f regarded as an element of $C^0(B,Z^2(A,M))$.

Finally, we can define a cocycle $F \varepsilon Z^2(\Phi,M)$ lifting f (and therefore [F] lifts [f] = α) in terms of f,g and μ^{τ^k} as follows. If $ab,a'b' \varepsilon A \times B = \Phi$, then let

$$F(ab,a'b') = a\mu^{b}_{a'} + f(a,a') + aa'g(b,b').$$

It is apparent that $F|_{A\times A} = f$ and also, in fact, that $F|_{B\times B} = g$ because $\mu^1_a = 0 = \mu^b_1$ for all a ε A, b ε B. So once F is seen to be a cocycle, the theorem will be proved. The following lemma establishes this.

<u>Lemma 6.6</u>. With F defined as above,

$$\begin{aligned} &a_1b_1F(a_2b_2,a_3b_3) - F(a_1a_2b_1b_2,a_3b_3) \\ &+ F(a_1b_1,a_2a_3b_2b_3) - F(a_1b_1,a_2b_2) \\ &= 0. \end{aligned}$$

<u>Proof</u>. Evaluating, we have that the above expression

$$\begin{aligned} &= a_1b_1[a_2\mu^{b_2}_{a_3} + f(a_2,a_3) + a_2a_3g(b_2,b_3)] + \\ &- [a_1a_2\mu^{b_1b_2}_{a_3} + f(a_1a_2,a_3) + a_1a_2a_3g(b_1b_2,b_3)] + \\ &+ [a_1\mu^{b_1}_{a_2a_3} + f(a_1,a_2a_3) + a_1a_2a_3g(b_1,b_2b_3)] + \\ &- [a_1\mu^{b_1}_{a_2} + f(a_1,a_2) + a_1a_2g(b_1,b_2)] = \end{aligned}$$

$= $(upon adding and subtracting $a_1a_2\mu_{a_3}^{b_1}$),

$$(3) \quad a_1a_2(b_1\mu_{a_3}^{b_2}-\mu_{a_3}^{b_1b_2}+\mu_{a_3}^{b_1}) - a_1(a_2\mu_{a_3}^{b_1}-\mu_{a_2a_3}^{b_1}+\mu_{a_2}^{b_1}) +$$

$$(4) \quad a_1b_1f(a_2,a_3) + [-f(a_1a_2,a_3) + f(a_1,a_2a_3) - f(a_1,a_2)] +$$

$$(5) \quad a_1a_2a_3[b_1g(b_2,b_3) - g(b_1b_2,b_3) + g(b_1,b_2b_3)] - a_1a_2g(b_1,b_2).$$

After applying Lemma 6.5 to line (3) above, and the facts that f and g are cocycles to lines (4) and (5), the above is equal to

$$\begin{aligned}
&a_1a_2[(1-a_3)g(b_1,b_2)] - a_1[(b_1-1)f(a_2,a_3)] + \\
&+ a_1b_1f(a_2,a_3) - a_1f(a_2,a_3) + \\
&+ a_1a_2a_3g(b_1,b_2) - a_1a_2g(b_1,b_2) \\
&= [a_1a_2(1-a_3) + a_1a_2a_3 - a_1a_2]g(b_1,b_2) + \\
&+ [-a_1(b_1-1) + a_1b_1 - a_1]f(a_2,a_3) \\
&= 0.
\end{aligned}$$

Q.E.D.

Corollary 6.7. Let $\alpha \in H^2(A,M)^B$. If there exists an $f \in \alpha$, such that $f \in C^2(A,M^B)$ then

$$\alpha \in \mathrm{Im}\{\mathrm{res}_A\colon H^2(\Phi,M) \longrightarrow H^2(A,M)^B\}.$$

Proof. Let $f \in \alpha$ with $f \in C^2(A,M^B)$. Then $(\tau-1)f = 0 = \delta_A 0$. So $N_\tau(0) = 0$ is an obstruction cocycle for α, and of course $[N_\tau(0)] = 0$.

The above corollary will be very useful in the calculations of the last section.

Now consider the map $N_\tau^*: H^1(A,M) \longrightarrow H^1(A,M^B)$ induced by the norm homomorphism $N_\tau: M \longrightarrow M^B$. The conditions needed to assure a lift of $\alpha \in H^2(A,M)^B$ can be weakened somewhat and we can say something more about where the family of obstruction cocycles for α lie by using the following proposition.

Proposition 6.8. $\alpha \in H^2(A,M)^B$ and let $N_\tau\mu$ be an obstruction cocycle for α, then there exists a lift of α under $r_A: H^2(\Phi,M) \longrightarrow H^2(A,M)^B$ if and only if

$$[N_\tau\mu] \in \text{Im}\{N_\tau^*: H^1(A,M) \longrightarrow H^1(A,M^B)\}.$$

Proof. Suppose there exists an extension to α, then by the previous theorem, there exists an $f \in \alpha$ and an obstruction cocycle $N_\tau\mu$ for α such that $(\tau-1)f = \delta_A\mu$ where $[N_\tau\mu] = 0 \in H^1(A,M^B)$. If $N_\tau\mu'$ is another obstruction cocycle for α, then there exists an $f' \in \alpha$, $\mu' \in C^1(A,M)$ such that $(\tau-1)f' = \delta_A\mu'$. Since $f \sim f'$, $f' = f+\delta_A h$ for some $h \in C^1(A,M)$. Then

$$(\tau-1)\delta_A h = (\tau-1)(f'-f) = \delta_A(\mu'-\mu).$$

So

$$\delta_A\{(\tau-1)h + \mu - \mu'\} = 0.$$

Hence $\{(\tau-1)h + \mu - \mu'\} \varepsilon Z^1(A,M)$. Let $\nu = \{(\tau-1)h + \mu - \mu'\}$, then $\mu' = \{\mu + (1-\tau)h - \nu\}$. Applying N_τ we get $N_\tau\mu' = N_\tau\mu + N_\tau(1-\tau)h - N_\tau\nu = N_\tau\mu - N_\tau\nu$. So in $H^1(A,M^B)$ we have $[N_\tau\mu'] = [N_\tau\mu] - [N_\tau\nu] = -[N_\tau\nu]$. However, since $\nu \varepsilon Z^1(A,M)$, we have that $[N_\tau\mu'] = -[N_\tau\nu] = N^*_\tau[-\nu] \varepsilon \mathrm{Im}\{N^*_\tau\colon H^1(A,M) \longrightarrow H^1(A,M^B)\}$.

For the converse, suppose we are given an obstruction cocycle $N_\tau\mu$ for α corresponding to some $f \varepsilon \alpha$, such that $[N_\tau\mu] \varepsilon \mathrm{Im}N^*_\tau$. Then we have $(\tau-1)f = \delta_A\mu$ and $N_\tau\mu = N_\tau\nu + \delta_A r$ for some $\nu \varepsilon Z^1(A,M)$, $r \varepsilon C^0(A,M^B)$. Since $N_\tau(\mu-\nu) = \delta_A r$, $[N_\tau(\mu-\nu)] = 0 \varepsilon H^1(A,M^B)$. Furthermore, $\delta_A(\mu-\nu) = \delta_A\mu - \delta_A\nu = \delta_A\mu = (\tau-1)f$. So $N_\tau(\mu-\nu)$ is an obstruction cocycle for α which becomes 0 in $H^1(A,M^B)$. Therefore, α has an extension.

Corollary 6.9. If $H^1(A,M) = 0$, then α has an extension in $H^2(\Phi,M)$ if and only if $[N_\tau\mu] = 0$, for all obstruction cocycles $N_\tau\mu$ for α.

To further simplify the problem of lifting the restriction map we have the following proposition.

Proposition 6.10. If $H^2(B,M) = 0$, then $\alpha \varepsilon H^2(A,M)^B$ extends if and only if 0 is an obstruction cocycle for α. (So instead of $N_\tau\mu$ being just a coboundary we can assume it is identically zero.)

__Proof__. Clearly if $N_\tau \mu = 0 \in Z^1(A,M^B)$ then $[N_\tau \mu] = 0$. On the other hand, if $H^2(B,M) = 0$, then the diagram

$$\begin{tikzcd}
 & & & {} \arrow[d] & \\
0 \arrow[r] & Z \arrow[r, hook, "j"] & M \arrow[r, "\tau"] \arrow[dr, "N_\tau"] & N_\tau(M) \arrow[r] \arrow[d, hook, "i"] & 0 \\
 & & & M^B \arrow[d] & \\
 & & & \frac{M^B}{N_\tau M} \simeq H^2(B,M) \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

induces a diagram

$$\begin{tikzcd}
 & & {} \arrow[d] & & \\
 & & H^0(A,H^2(B,M)) \arrow[d] & & \\
{} \arrow[r] & H^1(A,M) \arrow[r, "\theta"] \arrow[dr, "N_\tau^*"'] & H^1(A,N_\tau M) \arrow[r, "\overline{\partial}"] \arrow[d, "i^*"] & H^2(A,Z) \arrow[r] & {} \\
 & & H^1(A,M^B) \arrow[d] & & \\
 & & H^1(A,H^2(B,M)) \arrow[d] & & \\
 & & {} & &
\end{tikzcd}$$

which reduces to

$$\begin{tikzcd}
{} \arrow[r] & H^1(A,M) \arrow[r, "\theta"] \arrow[dr, "N^*_\tau"'] & H^1(A,N_\tau\mu) \arrow[r] \arrow[d, "\simeq"', "i^*"] & H^2(A,Z) \arrow[r] & {} \\
& & H^1(A,M^B) & &
\end{tikzcd}$$

where $Z = \ker\{N_\tau\colon M \longrightarrow N_\tau(M)\}$, and $\bar{\partial}$ is the connecting homomorphism.

Suppose α extends. Then let $N_\tau\mu$ be an obstruction cocycle for α corresponding to some $f \in \alpha$ such that $[N_\tau\mu] = 0 \in H^1(A,M^B)$. Then $\bar{\partial} \circ i^{*-1}\{[N_\tau(\mu)]\} = 0 \in H^2(A,Z)$. But $\bar{\partial} \circ i^{*-1}[N_\tau(\mu)] = \bar{\partial}[N_\tau(\mu)] = [\delta_A\mu]$. So $\delta_A\mu = \delta_A\mu'$ for some $\mu' \in C^1(A,Z)$. But then $(\tau-1)f = \delta_A\mu = \delta_A\mu'$. So $N_\tau\mu'$ is also an obstruction cocycle for α. But $\mu' \in Z = \ker N_\tau$. So $N_\tau\mu' = 0$, and hence, α has 0 as an obstruction cocycle.

Lemma 6.11. Let $N_\tau\mu$ be an obstruction cocycle for $\alpha \in H^2(A,M)^B$, then $N_\tau\mu$ corresponds to f for all $f \in \alpha$.

Proof. By definition, there exists an $f \in \alpha$ such that $(\tau-1)f = \delta_A\mu$. If $f' \in \alpha$ then $f' = f + \delta_A\mu'$ for some $\mu' \in C^1(A,M)$. Then $(\tau-1)f' = (\tau-1)f + (\tau-1)\delta_A\mu' = \delta_A\mu + (\tau-1)\mu' = \delta_A(\mu+(\tau-1)\mu')$. So $N_\tau(\mu+(\tau-1)\mu') = N_\tau\mu + N_\tau(\tau-1)\mu' = N_\tau\mu$ which is therefore an obstruction cocycle corresponding also to f'.

Corollary 6.12. Let $f \in \alpha \in H^2(A,M)^B$. If $H^2(B,M) = 0$ then α extends to $H^2(\Phi,M)$ if and only if $(\tau-1)f = \delta_A\mu$ for some $\mu \in \ker N_\tau$.

Proof. Assume α extends, then by Proposition 6.10 we can assume there exists an $f' \in \alpha$ such that $(\tau-1)f' = \delta_A\mu$ with $\mu \in \ker N_\tau$. By the lemma, $0 = N_\tau\mu$, then, is an obstruction cocycle corresponding to all $f \in \alpha$.

The converse is clear from the proposition.

So by the above, if we want to see whether α lifts in the case where $H^2(B,M) = 0$, we are free to choose any cocycle f belonging to α; then a lift exists if and only if we can, simultaneously solve the equations

$$(6) \qquad \begin{cases} (\tau-1)f = \delta_A\mu \\ N_\tau\mu \quad = 0 \end{cases}$$

In fact, this is the situation for practically all the modules on Nazarova's list.

§7. ANTISPECIAL CLASSES

We would like to apply the results of the previous section to the exact sequences of the type

$$0 \longrightarrow Z_2^t \longrightarrow H^2(\Phi,M) \xrightarrow{r_K} H^2(K,M)^\Phi \longrightarrow Z_2^s \longrightarrow$$

that recurred so frequently in section 4. If it turns out, for a particular module that $r_K = 0$, then we know $H^2(\Phi,M)$. However, usually it happens that $r_K \neq 0$ and we get an exact sequence of the form

$$0 \longrightarrow Z_2^t \longrightarrow H^2(\Phi,M) \longrightarrow Z_2 \longrightarrow 0.$$

In this situation, there are two possibilities; either

$$H^2(\Phi,M) \simeq Z_2^{t+1}$$

or

$$H^2(\Phi,M) \simeq Z_2^{t-1} \oplus Z_4.$$

To distinguish between these two possibilities, consider the case where $H^2(\Phi,M)$ has an element z of order 4. Since for all our modules, $H^2(K,M)$ has exponent 2 for all $K \subsetneq G$, we must have z

$$r_K(2z) = 2r_K(z) = 0$$

where r_K: $H^2(\Phi,M) \longrightarrow H^2(K,M)$. Hence the element $2Z \neq 0$ of $H^2(\Phi,M)$ has the property that $2Z \in \bigcap_{K\in\Lambda} \ker r_K$ where Λ is the set of cyclic subgroups of Φ. Such an element of $H^2(\Phi,M)$ is called an "antispecial cohomology class." This terminology is suggested by L. Charlap's [1] definition of "special classes." So a necessary condition for $H^2(\Phi,M)$ to have a Z_4 summand is the existence of a non-zero antispecial class in the subgroup $2H^2(\Phi,M)$ of $H^2(\Phi,M)$.

In this section, we use some results of section 4, and a method suggested by H. Sah to study the antispecial classes in $H^2(\Phi,M)$, for Φ, the Klein group.

Let γ stand for an arbitrary element of

$$\Phi = \{\sigma,\tau,\rho,1 \mid \rho = \sigma\tau = \tau\sigma, \quad \delta^2 = \tau^2 = 1\}.$$

Now suppose Z is an antispecial cohomology class. Then, taking $K = \langle\sigma\rangle$, $L = \langle\tau\rangle$, and $f = 0 = g$ in the corollary on page 78, there exists a cocycle representative F for z such that $F(k\ell,k'\ell') = km^{\ell}_{k'} = kF(\ell,k')$. In particular, $F(k,k') = kF(1,k') = 0$, $F(\ell,\ell') = F(\ell,1) = 0$ and $F(k,\ell) = kF(1,1) = 0$, so F is supernormalized relative to the canonical section τ: $L \to K \times L$ (assuming a given isomorphism $\Phi \simeq K \times L$).

<u>Notation</u>. Let $\mathfrak{A}$ be the subgroup of $H^2(\Phi,M)$ of antispecial

classes, let $M^{-\gamma} = \ker\{(1+\gamma): M \to M\}$ for all $\gamma \in \Phi$ and $N_\gamma M = (1+\gamma)M$.

Theorem 7.1. $\mathfrak{A} \simeq \dfrac{M^{-\tau} \cap N_\rho M}{[(1-\sigma)M^{-\tau}+(\tau-1)M^{-\sigma}]}$

Proof. Using MacLane's [7] terminology, we define an additive relation $\Gamma: \mathfrak{A} \longrightarrow M$ with image;

$$\operatorname{Im} \Gamma = M^{-\tau} \cap N_\rho M,$$

indeterminancy;

$$\operatorname{Ind} \Gamma = [(1-\sigma)M^{-\tau} + (\tau-1)M^{-\sigma}]$$

and domain of definition; Def $\Gamma = \mathfrak{A}$.

Let $z \in \mathfrak{A}$, then by the preceeding discussion, z has a cocycle representative F, such that $F|_{K\times K} = 0 = F|_{L\times L}$ and $F(k,\ell) = 0$, for all $k \in \langle\sigma\rangle$, $\ell \in \langle\tau\rangle$. It is more convenient, in this proof, to change to multiplicative notation. There exists an extension

$$1 \longrightarrow M \longrightarrow E \longrightarrow \Phi \longrightarrow 1$$

and a set $\{r_\gamma\}_{\gamma\in\Phi}$ of representatives such that $r_{\gamma_1}r_{\gamma_2} = F(\gamma_1,\gamma_2)r_{\gamma_1\gamma_2}$. Because $F(k,k) = 1 = F(\ell,\ell) = F(k,\ell)$ for all $(k,\ell) \in \langle\sigma\rangle \times \langle\tau\rangle$, we have that

$$(r_\sigma)^2 = r_{\sigma^2} = r_1 = 1 = r_{\tau^2} = (r_\tau)^2$$

and

$$r_\sigma r_\tau = r_{\sigma\tau} = r_\rho .$$

Define $\Gamma\colon \mathfrak{A} \longrightarrow M$ by

$$\Gamma(z) = F(\sigma\tau,\sigma\tau).$$

The relation $\Gamma \subset \mathfrak{A} \times M$ is closed under addition (multiplication) because if $\Gamma(z_1) = F_1(\sigma\tau,\sigma\tau)$ and $\Gamma(z_2) = F_2(\sigma\tau,\sigma\tau)$, then the cocycle F_1F_2 has the requisite properties so that $F_1(\sigma\tau,\sigma\tau)F_2(\sigma\tau,\sigma\tau)$ could be chosen as a value of $\Gamma(z_1+z_2)$. That is,

$$(z_1,F_1(\sigma\tau,\sigma\tau)) + (z_2,F_2(\sigma\tau,\sigma\tau)) =$$
$$= (z_1+z_2,F_1(\sigma\tau,\sigma\tau)F_2(\sigma\tau,\sigma\tau)) \in \Gamma .$$

<u>Claim</u>. $\operatorname{Im}\Gamma = M^{-\tau} \cap N_\rho M$.

Let $\ell \in \operatorname{Im}\Gamma$. Then $\ell = F(\sigma\tau,\sigma\tau) = F(\rho,\rho)$ where $z = [F] \in \mathfrak{A}$. Then

$$\ell = F(\rho,\rho) = F(\rho,\rho)r_{\rho^2} = r_1 = r_\rho r_\rho = (r_\sigma r_\tau)(r_\sigma r_\tau)$$

So $r_\tau \ell r_\tau^{-1} = r_\tau r_\sigma r_\tau r_\sigma$. I.e.,

$$\ell^\tau = r_\tau r_\sigma r_\tau r_\sigma .$$

Then

$$\ell \cdot \ell^{\tau} = (r_{\sigma}r_{\tau}r_{\sigma}r_{\tau})(r_{\tau}r_{\sigma}r_{\tau}r_{\sigma}) = 1,$$

since $(r_{\sigma})^2 = (r_{\tau})^2 = 1$. So $\ell^{\tau} = \ell^{-1}$ and hence $\ell \in M^{-\tau}$.

Now consider $\text{res}_{\rho} : H^2(\Phi,M) \longrightarrow H^2(<\rho>,M)$. We have, $\text{res}_{\rho}(z) = 0$, but $\text{res}_{\rho}(z) = [F|_{<\rho>\times<\rho>}]$. So $F|_{<\rho>} \in B_2(<\rho>,M)$ and therefore $F(\rho,\rho) = \ell \in N_{\rho}M$. $(F(\rho,\rho) = (\delta G)(\rho,\rho) = \rho G(\rho) + G(\rho))$.

To prove the opposite inclusion, let $\ell \in M^{-\tau} \cap N_{\rho}M$. Then define $F: \Phi \times \Phi \longrightarrow M$ according to the table

	σ	τ	ρ
σ	1	1	1
τ	ℓ^{-1}	1	ℓ^{-1}
ρ	ℓ	1	ℓ

(where, for example, the second line says $F(\tau,\sigma) = \ell^{-1}$, $F(\tau,\tau) = 1$, and $F(\tau,\rho) = \ell^{-1}$). Then it can be verified that F is a cocycle. Moreover F has the properties that

$$F(k,k) = F(\ell,\ell) = F(k,\ell) = 1$$

for all $(k,\ell) \in <\sigma> \times <\tau>$ and

$$\Gamma([F] = F(\rho,\rho) = \ell.$$

Claim: $\text{Ind}\ \Gamma = [(\sigma-1)M^{-\tau} + (1-\tau)M^{-\sigma}]$.

By definition, $\text{Ind}\ \Gamma = \{m \in M | (0,m) \in \Gamma\}$.

Proof. Let $m \in \operatorname{Ind}\mathfrak{M}$, then there exists a coboundary $F \in B^2(\Phi,M)$ such that

$$F\big|_{\langle\sigma\rangle\times\langle\sigma\rangle} = F\big|_{\langle\tau\rangle\times\langle\tau\rangle} = F\big|_{\langle\sigma\rangle\times\langle\tau\rangle} = 1 \text{ and } F(\rho,\rho) = M.$$

Let F correspond to the extension

$$1 \longrightarrow M \longrightarrow E \longrightarrow \Phi \longrightarrow 1$$

with associated set of representatives $\{r_\gamma\}_{\gamma\varepsilon\Phi}$. Since F is a coboundary, this extension splits. Therefore, there is another set of representatives $\{r'_\gamma\}_{\gamma\varepsilon\Phi}$, such that the section $r'\colon \Phi \to E$ given by $r'(\gamma) = r'_\gamma$ is a homomorphism. So with respect to this new set of representatives, the extension corresponds to the trivial cocycle.

Now define $c \;\varepsilon\; C^1(\Phi,M)$, via the change of representatives, by $r'_\gamma = c_\gamma r_\gamma$ for all $\gamma \;\varepsilon\; \Phi$. Then for $\gamma \;\varepsilon\; \Phi$.

$$1 = r'_{\gamma 2} = (r'_\gamma)^2 = c_\gamma r_\gamma c_\gamma r_\gamma = c_\gamma c_\gamma^\gamma (r_\gamma)^2.$$

For $\gamma = \sigma$ or τ, we have $F(\gamma,\gamma) = 1$, so that $(r_\gamma)^2 = 1$, when $\gamma = \sigma,\tau$. Hence we have that $1 = c_\sigma c_\sigma^\sigma = c_\tau c_\tau^\tau$. Therefore $c_\sigma \;\varepsilon\; M^{-\sigma}$ and $c_\tau \;\varepsilon\; M^{-\tau}$. We also must have

$$1 = r'_{\rho^2} = (r'_\rho)^2 = (r'_\sigma r'_\tau)^2$$

$$= (c_\sigma r_\sigma c_\tau r_\tau)^2 = (c_\sigma c_\tau^\sigma r_\sigma r_\tau)^2$$

$$= (c_\sigma c_\tau^\sigma r_\rho)(c_\sigma c_\tau^\sigma r_\rho) = c_\sigma c_\tau^\sigma (c_\sigma c_\tau^\sigma)^\rho r_\rho^2$$

$$= c_\sigma c_\tau^\sigma c_\sigma^\rho c_\tau^\tau m.$$

Since $c_\sigma^\rho = c_\sigma^{\sigma\tau} = (c_\sigma^{-1})^\tau = c_\sigma^{-\tau}$, and $c_\tau^\tau = c^{-1}$ and since the c_γ commute, this becomes

$$1 = c_\sigma c_\tau^\sigma c_\sigma^{-\tau} c_\tau^{-1} m = c_\sigma^{(1-\tau)} \cdot c_\tau^{(\sigma-1)} \cdot m.$$

So $m = c_\sigma^{(\tau-1)} \cdot c_\tau^{(1-\sigma)}$. Since $c_\sigma \in M^{-\tau}$ and $c_\tau \in M^{-\sigma}$, additively we have $m \in [(\tau-1)M^{-\sigma} + (1-\sigma)M^{-\tau}]$.

Now we show $[(\tau-1)M^{-\sigma} + (1-\sigma)M^{-\tau}] \subseteq \text{Ind } \Gamma$. Let $m \in (\tau-1)M^{-\sigma} + (1-\sigma)M^{-\tau}$, then $m = c_\sigma^{(\tau-1)} \cdot c_\tau^{(1-\sigma)}$ for some $c_\sigma \in M^{-\sigma}$, $c_\tau \in M^{-\tau}$. Analogously to the proof that $\text{Im } \Gamma = M^{-\tau} \cap N_\rho M$ we define a cocycle $F: \Phi \times \Phi \to M$ by the table

	σ	τ	$\sigma\tau$
σ	1	1	1
$\tau\tau$	m^{-1}	1	m^{-1}
$\sigma\tau$	m	1	m

Then F has the usual properties and $F(\rho,\rho) = m$. We will prove that F is a coboundary by showing that F is cohomologous to the trivial cocycle.

Suppose F corresponds to an extension of M by Φ with respect to a set of representatives $\{r_\gamma\}_{\gamma \varepsilon \Phi}$. We change representatives by defining

$$r'_\sigma = c_\sigma r_\sigma, \; r'_\tau = c_\tau r_\tau \text{ and } r'_\rho = r'_\sigma \, r'_\tau.$$

To verify that r' is a homomorphism we must show that $(r'_\sigma)^2 = (r'_\tau)^2 = (r'_\rho)^2 = 1$ and that $r'_\rho = r'_\tau r'_\sigma$, then the other multiplicative properties easily follow.

$$(r'_\sigma)^2 = c_\sigma r_\sigma c_\sigma r_\sigma = c_\sigma c_\sigma^\sigma (r_\sigma)^2 = c_\sigma c_\sigma^{-1} \cdot 1 = 1.$$

Similarly, $(r'_\tau)^2 = 1$.

$$\begin{aligned} (r'_\tau r'_\sigma) &= c_\tau r_\tau c_\sigma r_\sigma \\ &= c_\tau c_\sigma^\tau r_\tau r_\sigma = c_\tau c_\sigma^\tau F(\tau,\sigma) r_\rho \\ &= c_\tau c_\sigma^\tau \mathfrak{m}^{-1} r_\rho = c_\tau c_\sigma^\tau c_\sigma^{(1-\tau)} \cdot c_\tau^{(\sigma-1)} r_\rho \\ &= c_\tau^\sigma c_\sigma r_\rho = c_\sigma c_\tau^\sigma r_\rho = c_\sigma c_\tau^\sigma r_\sigma r_\tau \\ &= c_\sigma r_\sigma c_\tau r_\tau = r'_\sigma r'_\tau = r'_\rho. \end{aligned}$$

And finally we have

$$\begin{aligned} (r'_\rho)^2 &= (r'_\sigma r'_\tau)^2 = r'_\sigma (r'_\tau r'_\sigma) r'_\tau \\ &= r'_\sigma (r'_\sigma r'_\tau) r'_\tau = (r'_\sigma)^2 \cdot (r'_\tau)^2 \\ &= 1. \end{aligned}$$

Since changing to the set of representatives r' results in the section r' being a homomorphism, the resulting cocycle is the trivial one, and this new cocycle is cohomologous to the original cocycle F.

Clearly Def $\Gamma = \mathfrak{A}$, so Γ induces a homomorphism

$$\Gamma^0 : \mathfrak{A} \longrightarrow \frac{M^{-\tau} \cap N_\rho M}{[(\tau-1)M^{-\sigma}+(1-\sigma)M^{-\tau}]}$$

Moreover, the preceeding discussion showed that if $\Gamma(z) \in [(\tau-1)M^{-\sigma} + (1-\sigma)M^{-\tau}]$ then $z = 0$. Hence $\ker \Gamma^0 = 0$. We therefore have an isomorphism

$$\Gamma^0 : \mathfrak{A} \longrightarrow \frac{M^{-\tau} \quad N_\rho M}{[(\tau-1)M^{-\delta}+ 1-\sigma)M^{-\tau}]}$$

Q.E.D.

Remark. $M^{-\tau} \cap N_\rho M = M^{-\sigma} \cap M^{-\tau} \cap N_\rho M$ because if $m \in M^{-\tau} \cap N_\rho M$ then $m \in M^\rho$, and then $m = \rho m = \sigma\tau m = \sigma(-m) = -\sigma m$. So $\sigma m = -m$. Also, $[(\tau-1)M^{-\sigma} + (1-\sigma)M^{-\tau}] \subset N_\rho M$ because if $c_\sigma \in M^{-\sigma}$ and $c_\tau \in M^{-\tau}$ then $c_\sigma^{(\tau-1)} \cdot c_\tau^{(1-\sigma)} = (c_\tau c_\sigma^{-1})^{1+\rho}$.

§8. APPLICATION OF ANTISPECIAL CLASSES

In view of the isomorphism Γ^0, we shall refer to elements of $M^{-\tau}\quad N_\rho M$ as *antispecial* *cocycles* and to elements of $[(\tau-1)M^{-\sigma} + (1-\sigma(M^{-\tau}]$ as *antispecial* *coboundaries*, where σ,τ, and ρ are the three generators of $\Phi = Z_2 \times Z_2$.

It will be useful to describe the antispecial cocycles and coboundaries in terms of Nazarova's representations. We assume the generators σ,τ and $\rho=\sigma\tau$ correspond to the matrices A, B and AB respectively. First we compute $M^{-\tau} \cap N_\rho M = M^{-\sigma} \cap M^{-\tau} \cap N_\rho M$.

The action of σ on M corresponds to the matrix A acting on $\mathbb{Z}^n$. So $M^{-\sigma} = \ker(E+A)$.

$$(E+A)\begin{pmatrix} W \\ X \\ Y \\ Z \end{pmatrix} = \left(\begin{array}{cc|cc} 2E & & 0 & D_4 \\ & 0 & D_2 & 0 \\ \hline & & 2E & \\ & & & 0 \end{array}\right)\begin{pmatrix} W \\ X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 2W + D_4Z \\ D_2Y \\ 2Y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

if and only if $Y = 0$ and $D_4Z = -2W$.

So $M^{-\sigma} = \{(W,X,0,Z) \mid D_4Z = -2W\}$. $M^{-\tau} = \ker(E+B)$.

$$(E+B)\begin{pmatrix} W \\ X \\ Y \\ Z \end{pmatrix} = \left(\begin{array}{cc|cc} 2E & & D_1 & 0 \\ & 0 & 0 & D_3 \\ \hline & & 0 & \\ & & & 2E \end{array}\right)\begin{pmatrix} W \\ X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 2W + D_1Y \\ D_3Z \\ 0 \\ 2Z \end{pmatrix}$$

So $M^{-\tau} = \{(W,X,Y,0)\,|\,D_1Y = -2W\}$. Therefore $M^{-\sigma} \cap M^{-\tau} =$ $\{(0,X,0,0)\}$.

$$N_\rho\begin{pmatrix} W' \\ X' \\ Y' \\ Z' \end{pmatrix} = \left(\begin{array}{cc|cc} 2E & & D_1 & D_4 \\ & 2E & -D_2 & -D_3 \\ \hline & & 0 & \\ & & & 2E \end{array}\right)\begin{pmatrix} W' \\ X' \\ Y' \\ Z' \end{pmatrix} = \begin{pmatrix} 2W' + D_1Y' + D_4Z' \\ 2X' - D_2Y' - D_3Z' \\ 0 \\ 0 \end{pmatrix}$$

Setting

$$\begin{pmatrix} 2W' + D_1Y' + D_4Z' \\ 2X' - D_2Y' - D_3Z' \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ X \\ 0 \\ 0 \end{pmatrix},$$

we get $X = 2X' - D_2Y' - D_3Z'$ where $-D_1Y' - D_4Z'$ is of the form $2W'$ for some W'. Or, replacing Y' and Z' by $-Y'$, and $-Z'$, we have

$$M^{-\tau} \cap N_\rho M = \left\{\begin{pmatrix} 0 \\ X \\ 0 \\ 0 \end{pmatrix}\right\}$$

where $X = [2X' + D_2Y' + D_3Z']$ and $[D_1Y' + D_4Z' = 2W']$.

We now calculate $[(\sigma-1)M^{-\tau} + (\tau-1)M^{-\sigma}]$.

Let $(W'',X'',Y'',Z'') \varepsilon M^{-\tau}$ then $D_1Y'' = -2W''$ and $Z'' = 0$.

$$(\sigma-1)(W'',X'',Y'',0) = \left(\begin{array}{cc|cc} 0 & & & D_4 \\ & -2E & D_2 & \\ \hline & & 0 & \\ & & & -2E \end{array}\right)\begin{pmatrix} W'' \\ X'' \\ Y'' \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2X''+D_2Y'' \\ 0 \\ 0 \end{pmatrix}$$

So $(\sigma-1)M^{-\tau} = \{(0,X,0,0)\ X = (-2X'' + D_2Y'')$ and $D_1Y'' = -2W''\}$

For $(W''',X''',Y''',Z''') \varepsilon M^{-\sigma}$ we have $Y''' = 0$ and $D_4Z''' = -2W'''$. Then $(\tau-1)(W''',X''',0,Z''') =$

$$\left(\begin{array}{cc|cc} 0 & & D_1 & \\ & -2E & & D_3 \\ \hline & & -2E & \\ & & & 0 \end{array}\right)\begin{pmatrix} W''' \\ X''' \\ 0 \\ Z''' \end{pmatrix} = \begin{pmatrix} 0 \\ -2X'''+D_3Z''' \\ 0 \\ 0 \end{pmatrix}$$

So $(\tau-1)M^{-\sigma} = \{(0,(-2X'''+D_3Z'''),0,0)\,|\,D_4Z''' = -2W'''$ for some $W'''\}$. Therefore

$$[(\tau-1)M^{-\sigma} + (\sigma-1)M^{-\tau}] = \left\{\begin{pmatrix} 0 \\ X \\ 0 \\ 0 \end{pmatrix}\right\}$$

where

$$X = [-2X'' - 2X''' + D_2Y'' + D_3Z'']$$

where

$$D_1Y'' = -2W''$$

and

$$D_4Z''' = -2W'''.$$

Since X'' and X''' are free variables we can rewrite these defining conditions as

$$\left\{\begin{pmatrix}0\\X\\0\\0\end{pmatrix}\right\} \text{ such that } X = \left[2X' + D_2Y' + D_3Z\right]$$

where

$$D_1Y' = 2W'$$

and

$$D_4Z' = 2W''.$$

Summarizing we have:

(1) <u>Antispecial cocycles</u>: $\begin{pmatrix}0\\\overline{X}\\0\\0\end{pmatrix}$ such that $\overline{X} = \left[2X' + D_2Y + D_3Z\right]$

where Y and Z must also satisfy a) $\left[D_1Y + D_4Z\right] = 2W.$

(2) <u>Antispecial coboundaries</u>: $\begin{pmatrix}0\\\overline{X}\\0\\0\end{pmatrix}$ such that $\overline{X} = \left[2X' + D_2Y + D_3Z\right]$

where Y and Z satisfy

$$\begin{array}{ll} \text{b)} & D_1Y = 2W' \\ \text{c)} & D_4Z = 2W'' \end{array} \quad \text{with } X',W,W',+W'' \text{ free.}$$

<u>Note:</u> Let $Z^2_{\mathfrak{A}}$ be the group of antispecial cocycles and let $B^2_{\mathfrak{A}}$ be the group of antispecial coboundaries for a given module under discussion.

We give as an example, the calculation of $\mathfrak{A}$ for modules of type {2b}.

$$D = \begin{array}{c|c|c|}
 & n_1-1 & n_1 \\ \hline
n_1 & \begin{matrix} 0 & \cdots & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \\ \hline
n_1 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & \cdots & 0 \end{matrix} & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \\ \hline
\end{array}$$

The ranks of V_1, V_2, V_3 and V_4 are n_1, n_1, n_1-1, and n_1 respectively.

Z^2_q: condition a), $D_1Y + D_4Z = 2W \quad D_1Y + Z = 2W$ which implies that $Z = 2W - D_1Y$. Then evaluating $\overline{X}$, we have

$$\overline{X} = [2X' + D_2Y + D_3Z] = [2X' + D_2Y + Z]$$
$$= [2X' + D_2Y + (2W-D_1Y)] = 2(X'+W) + (D_2-D_1)Y.$$

Now $(D_2-D_1) = \underset{n_1x(n_1-1)}{\begin{pmatrix} -1 & & \\ 1 & \ddots & \\ & \ddots & -1 \\ & & 1 \end{pmatrix}}$, which has maximal column rank (n_1-1), so it spans a free abelian subgroup V' of V_2 of rank n_1. Moreover (D_2-D_1) is a submatrix of the unimodular matrix

$$\begin{pmatrix} 1 & -1 & & \\ 0 & 1 & \ddots & \\ \vdots & & \ddots & -1 \\ 0 & & & 1 \end{pmatrix}$$

and therefore V' is a rank (n_1-1) direct summand of V_2. Hence $\{\overline{X}\} \simeq 2Z \oplus Z^{n_1-1} \simeq Z^2_{\alpha}$.

Finding B^2_{α} is easier: the conditions

$$\left.\begin{array}{ll} a) & D_1Y = 2W' \\ b) & D_4Z = 2W'' \end{array}\right\} \text{imply} \begin{pmatrix} 0 & \cdots & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix} = 2W' \Rightarrow Y = 2S$$

also $D_4Z = Z = 2W''$. Then

$$\overline{X} = [2X' + D_2Y + D_3Z] = [2X' + D_2Y + Z]$$

$$= [2X' + D_2(2S) + 2W''] = 2(X'+D_2S+W'').$$

Since X' is a free variable here, $\{\overline{X}\} \simeq 2Z^{n_1}$, and therefore $B^2_{\alpha} \simeq 2Z^{n_1}$. Furthermore $\{\overline{X}\}$ can be spanned by the columns of the matrix $\begin{pmatrix} 2 & 2 & 0 & \cdots & 0 \\ 0 & 2 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & \ddots & 2 \\ 0 & & & & 2 \end{pmatrix}$ while Z^2_{α} is spanned by the columns of $\begin{pmatrix} 2 & -1 & & \\ 0 & 1 & \ddots & \\ \vdots & & \ddots & -1 \\ 0 & & & 1 \end{pmatrix}$. So in evaluating

$\frac{Z^2_{\mathfrak{A}}}{B^2_{\mathfrak{A}}} \simeq \frac{2Z \oplus Z^{n_1-1}}{2Z^{n_1}}$ we can mod out within each "column summand'

and get $\mathfrak{A} \simeq Z_2^{n_1-1}$ (i.e., if $L_i \subseteq K_i$ $i = 1,\ldots,n$ then

$$\frac{\overset{n}{\underset{i=1}{\oplus}} K_i}{\overset{n}{\underset{i=1}{\oplus}} L_i} \simeq \overset{n}{\underset{i=1}{\oplus}} K_i/L_i).$$

Remark: In general, the evaluation of $\mathfrak{A}$ will not be necessary for our purposes, however, the characterization of just the antispecial coboundaries will usually suffice.

Suppose that $z \in H^2(\Phi,M)$, then if K is a cyclic order 2 subgroup, $H^2(K,M)$ has exponent 2. Therefore $\text{res}_K(2z) = 0$, for all such K. Hence $2z \in \mathfrak{A}$ for all $z \in H^2(\Phi,M)$.

We now give a procedure, which will be useful later, for constructing an antispecial cocycle representative for 2z and the corresponding value of $\Gamma(2z)$, and $\Gamma^0(2z)$.

Let $K = \langle\sigma\rangle$ and $L = \langle\tau\rangle$ be two of the order 2 subgroups of Φ then $\Phi \simeq K \times L$. Let f, and g be arbitrary cocycle representatives of $\text{res}_K(Z) + \text{res}_L(Z)$ respectively. Then by the corollary of page 78, z has a cocycle repretative F with the property that

$$F(k\ell,k'\ell') = kF(\ell,k') + f(k,k') + kk'g(\ell,\ell').$$

In particular $F(\sigma,\sigma) = f(\sigma,\sigma)$, $F(\tau,\tau) = g(\tau,\tau)$ and $F(\sigma,\tau) = 0$. We note that f and g are determined by the elements $f_1 = f(\sigma,\sigma)$ and $g_1 = g(\tau,\tau)$ and the cocycle conditions imply that $f_1 \varepsilon M^{<\sigma>}$ and $g_1 \varepsilon M^{<\tau>}$ (e.g., $\sigma f(\sigma,\sigma) - f(1,\sigma) + f(\sigma,1) - f(\sigma,\sigma) = 0$). If we let $F(\tau,\sigma) = m$, then $F(\sigma\tau,\sigma\tau) = \sigma m + f_1 + g_1$.

We would like to find a value for $\Gamma^0(2z)$, and to that end, we must modify the cocycle $2F \varepsilon 2z$ so that it has certain properties. Define $h \varepsilon C^1(\Phi,M)$ by $h_\sigma = -f_1$, $h_\tau = -g_1$ and $h_{\sigma\tau} = \sigma h_\tau + h_\sigma = -f_1 - \sigma g_1$. Then the cocycle $2F + \delta_\Phi h \varepsilon 2z$ has the properties that

$$(2F+\delta_\Phi h)(\sigma,\sigma) = 2F(\sigma,\sigma) + (1+\sigma)h_\sigma$$

$$= 2f_1 + (1+\sigma)(-f_1) = 2f_1 - 2f_1$$

$$= 0.$$

Similarly $(2F+\delta_\Phi h)\Big|_{<\tau>\times<\tau>} = 0$, and finally

$$(2F+\delta_\Phi h(\sigma,\tau)$$

$$= 2F(\sigma,\tau) + \sigma h_\tau - h_{\sigma\tau} + h_\sigma$$

$$= 0.$$

So we can associate a value to $\Gamma(2z)$ via this cocycle.

We get that

$$\begin{aligned}\Gamma(2z) &= (2F+\delta_\Phi h)(\sigma\tau,\sigma\tau) = 2F(\sigma\tau,\sigma\tau) + (1+\sigma\tau)h_{\sigma\tau} \\ &= 2(\sigma m+f_1+g_1) + (1+\sigma\tau)(-f_1-\sigma g_1) \\ &= 2\sigma m + (1-\tau)f_1 + (1-\sigma)g_1\end{aligned}$$

so that

$$\Gamma^0(2z) = [2\sigma m+(1-\tau)f_1 + (1-\sigma)g_1]$$

<u>Remark.</u> Suppose $B^2_{\mathfrak{A}} = [(1-\sigma)M^{-\tau} + (1-\tau)M^{-\sigma}] = 2V_2$ and f_1 can be chosen in either M^τ or $M^{-\tau}$ and that g_1 can be chosen in M^σ or $M^{-\sigma}$. Then $[2\sigma m + (1-\tau)f_1 + (1-\sigma)g_1] \epsilon\ 2V_2$. Therefore $\Gamma^0(2z) = 0$, and thus $2z = 0 \ \epsilon\ \mathfrak{A} \subseteq H^2(\Phi,M)$. We have therefore proven the following lemma (which is applicable in many of our cases).

<u>Lemma 8.1</u>. Suppose Φ has two order 2 subgroups $<\sigma>$, and $<\tau>$ such that the generators of $H^2(<\sigma>,M)^\Phi$ and $H^2(<\tau>,M)^\Phi$ have representatives $\{f_i\}_{i\epsilon\Lambda}$, $\{g_j\}_{j\epsilon\Lambda'}$, with $f_i \ \epsilon\ M^\tau$ or $M^{-\tau}$, and $g_j \ \epsilon\ M^\sigma$ or $M^{-\sigma}$ for all $i \ \epsilon\ \Lambda$, $j \ \epsilon\ \Lambda'$. Also suppose $B^2 = 2V_2$ where $B^2 = \{(1-\sigma)M^{-\tau} + (1-\tau)M^{-\sigma}\}$. Then $H^2(\Phi,M)$ has no elements of order 4.

As an application of the above lemma, we resolve the question of the existence of elements of order 4 for modules of type {2b}.

We calculated on page 109 that $[(1-\sigma)M^{-\tau} + (1-\tau)M^{-\sigma}] = 2V_2$. Also for these modules, the discussion on page showed that $H^2(<\sigma>,M) = 0$, and $H^2(<\tau>,M) = Z_2$ $(=H^2(<\tau>,m)^{\Phi})$ has generator $[a_1]$. Since $a_1 \varepsilon M^{\tau}$, (and of course $0 \varepsilon M^{\sigma}$), $H^2(\Phi,M)$ has exponent 2 for modules of this type.

We will show in the next section that in fact, there are no cohomology elements of order 4 for the other modules as well.

§9. FINAL CALCULATIONS

In this chapter, we systematically take each module type whose cohomology is in doubt, and apply the results of section 6 on lifting the restriction map. If the restriction map in question is non-trivial we then find $B^2_{\mathfrak{A}}$ and use Lemma 8.1 where it is applicable. We take these modules up in the order: {2a}, {9a}, {2b}, {9b}, {6a}, {6b}, {4a} and {4b}.

Notation. $\Phi = \{\sigma,\tau,\sigma\tau,1 \mid \sigma^2 = \tau^2 = 1,\ \sigma\tau = \tau\sigma\}$.
$A = \langle\sigma\rangle$, $B = \langle\tau\rangle$, $AB = \langle\sigma\tau\rangle$.

For convenience we list here the forms of $\delta_A M$, $\delta_B M$, $\delta_{AB} M$, $\ker N_\sigma$, $\ker N_\tau$, and $\ker N_{\sigma\tau}$, which can easily be calculated from the matrix representations

$$\delta_A M = N_\sigma M = \{(2W+D_4Z, D_2Y, 2Y, 0)\}$$

$$\ker N_\sigma = \{(W,X,0,Z) \mid D_4Z = -2W\}$$

$$\delta_B M = N_\tau M = \{(2W+D_1Y, D_3Z, 0, 2Z)\}$$

$$\ker N_\tau = \{(W,X,Y,0) \mid D_1Y = -2W\}$$

$$\delta_{AB} M = N_{\sigma\tau} = \{(2W+D_1Y+D_4Z, 2X-D_2Y-D_3Z, 0, 0)\}$$

$$\ker N_{\sigma\tau} = \{(W,X,Y,Z) \mid D_1Y+D_4Z = -2W, \quad D_2Y+D_2Z = 2X\}$$

Modules of type {2a}. $D = \left[\begin{array}{ccc|ccc} 1 & & 0 & 1 & & \\ & \ddots & \vdots & & \ddots & \\ & 1 & 0 & & & 1 \\ \hline 0 & 1 & & 1 & & \\ \vdots & & \ddots & & \ddots & \\ 0 & & 1 & & & 1 \end{array}\right]$

We have an exact sequence

$$0 \longrightarrow Z_2^{n_1-1} \longrightarrow H^2(\Phi,M) \xrightarrow{r_\sigma} Z_2 \longrightarrow Z_2^{n_2} \longrightarrow$$

and must decide whether or not r_σ: $H^2(\Phi,M) \longrightarrow H^2(A,M)^B \simeq Z_2$ is onto. For these modules $H^2(B,M) = 0$, and Corollary 6.12 is applicable. So we may choose any cocycle representative f for the generator α of $H^2(A,M)^B$, and then α lifts (i.e., $r_\sigma \neq 0$) if and only if we can solve the equations

$$\begin{cases} (\tau-1)f = \delta_A m \\ N_\tau m = 0 \end{cases} .$$

(Since every $\mu \in C^1(A,M)$ is completely determined by its only possible non-zero value $\mu_\sigma = m$, we write m for μ in the equations (6) on page 95.)

For modules {2a} the generator α of $H^2(A,M)^B$ has the basis element c_1 (see page 15), as a cocycle representative. Then we claim that there is no solution to the equations

$$\begin{cases} (\tau-1)c_1 = \delta_A m. \\ N_\tau m = 0. \end{cases}$$

For suppose $N_\tau m = 0$ then $m = (W,X,Y,0)$ where $D_1Y = -2W$. Then $\delta_A m = (2W,D_2Y,2Y,0)$. But

$$(\tau-1)c_1 = \left(\begin{array}{cc|cc} 0 & & D_1 & \\ & -2E & & D_3 \\ \hline & & -2E & \\ & & & 0 \end{array}\right) \begin{pmatrix} 0 \\ 0 \\ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ 0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ 0 \\ \begin{pmatrix} -2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ 0 \end{pmatrix}$$

So for a solution we must have $(2W,D_2Y,2Y,0) = ([1,0,\dots,0],0,[-2,0,\dots,0],0)$ which is impossible since $2W \neq [1,0,\dots,0]$. Hence $r_\sigma\colon H^2(\Phi,M) \longrightarrow H^2(A,M)^B$ is the zero map and the exact sequence becomes

$$0 \longrightarrow Z_2^{n_1-1} \xrightarrow[\simeq]{} H^2(\Phi,M) \longrightarrow 0.$$

{9a}. We carry out the analogous calculations for modules of type {9a} at this point, because the calculations are very similar to {2a}.

Here we have the exact sequence

$$0 \longrightarrow Z_2^{n_1-1} \longrightarrow H^2(\Phi,M) \xrightarrow{r_\tau} Z_2 \longrightarrow Z_2^{n_2} \longrightarrow$$

and we examine $r_\tau\colon H^2(\Phi,M) \longrightarrow H^2(B,M)^A \simeq Z_2$. $H^2(B,M)^A$ has a single generator for which we can select the basis element d_{n_4} as a cocycle representative. Moreover, $H^2(A,M) = 0$ here so we can apply the same corollary as before. Therefore $r_\tau \neq 0$ if and only if there exists a solution to

$$\begin{cases} (\sigma-1)d_{n_4} = \delta_B M \\ \text{with } N_\sigma(m) = 0 \end{cases}$$

Here too, there is no solution, for if $N_\sigma(m) = 0$, then $m = (W,X,0,Z)$ with $D_4 Z = -2W$. Then $\delta_B M = (2W, D_3 Z, 0, 2Z)$, but $(\sigma-1)d_{n_4} =$

$$\left(\begin{array}{cc|cc} 0 & & & D_4 \\ & -2E & D_2 & \\ \hline & & 0 & \\ & & & -2E \end{array}\right)\begin{pmatrix}0\\0\\0\\ \begin{pmatrix}0\\ \vdots\\0\\1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}\begin{pmatrix}0\\ \vdots\\0\\1\end{pmatrix}\\0\\0\\ \begin{pmatrix}0\\ \vdots\\0\\-2\end{pmatrix}\end{pmatrix}$$

Since there does not exist a W such that $2W = [0,\ldots,0,1]$, there is no solution. So, as before we have

$$0 \longrightarrow Z_2^{n_1-1} \xrightarrow[\cong]{} H^2(\Phi,M) \longrightarrow 0$$

{2b}. $D = \left[\begin{array}{c|c} \begin{matrix}0 & \cdots & 0\\ 1 & & \\ & \ddots & \\ & & 1\end{matrix} & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} \\ \hline \begin{matrix}1 & & \\ & \ddots & \\ & & 1\\ 0 & \cdots & 0\end{matrix} & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\end{array}\right]$

Turning to modules of type {2b} we have the exact sequence

$$0 \longrightarrow Z_2^{n_1} \longrightarrow H^2(\Phi,M) \xrightarrow{r_\tau} Z_2 \longrightarrow Z_2^{n_2} \longrightarrow$$

Here $r_\tau\colon H^2(\Phi,M) \longrightarrow H^2(B,M)^A \simeq Z_2$. The basis element a_1 (see page 18) is a cocycle representative for the generator of $H^2(B,M)^A$. Since $a_1 \in M^A \subseteq C^2(B,M^A)$, Corollary 6.7 implies that $\alpha \in I_m\{r : H^2(\Phi,M) \longrightarrow H^2(B,M)^A\}$. So r_τ is onto and the exact sequence is

$$0 \longrightarrow Z_2^{n_1} \longrightarrow H^2(\Phi,M) \xrightarrow{r_\tau} Z_2 \longrightarrow 0.$$

Since we have shown previously that $H^2(\Phi,M)$ has no elements of order 4 for modules of this type,

$$H^2(\Phi,M) \simeq Z_2^{n_1+1}.$$

The situation for modules of type {9b} is similar. We have

$$0 \longrightarrow Z_2^{n_1} \longrightarrow H^2(\Phi,M) \xrightarrow{r_\tau} Z_2 \longrightarrow Z_2^{n_2} \longrightarrow$$

Again, the element a_1 serves as cocycle representative for the generator of $H^2(A,M)^B$ (see page 31). Since $a_1 \in M^A$, the generator lifts and therefore we have the exact sequence

$$0 \longrightarrow Z_2^{n_1} \longrightarrow H^2(\Phi,M) \longrightarrow Z_2 \longrightarrow 0.$$

A similar argument and calculation to the one for case {2b}, shows that $H^2(\Phi,M)$ has no order 4 elements. Hence

$$H^2(\Phi,M) \simeq Z_2^{n_1+1}.$$

{6a}. $$D = \begin{array}{|c|c|} \hline \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \\ \hline \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & \cdots & 0 \end{matrix} & \begin{matrix} 0 & \cdots & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \\ \hline \end{array}$$

Because of the form of the representations, {6a} presents a special problem. We have the exact sequence

$$0 \longrightarrow Z_2^{n_1} \longrightarrow H^2(\Phi,M) \xrightarrow{r_{\sigma\tau}} Z_2 \longrightarrow Z_2^{n_2} \longrightarrow$$

and would like to find a representative for the generator of $H^2(\langle\sigma\tau\rangle,M)^A$. The basis element $a_1 = \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is clearly a cocycle. We must show that it is not a coboundary. $N_{\sigma\tau}M = \delta_{\sigma\tau}M = \{(2W+D_1Y+D_4Z,2X-D_2Y-D_3Z,0,0\}$. In this case $\delta_{\sigma\tau}M = \{2W+Y+Z,2X-D_2Y-D_3Z,0,0\}$. This submodule is spanned by the columns of

$$\left(\begin{array}{c|c|c|c} 2E & 0 & E & E \\ \hline 0 & 2E & -D_2 & -D_3 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

By column operations this reduces to

$$\left(\begin{array}{c|c|c|c} 2E & & E & 0 \\ \hline & 2E & -D_2 & -D_2+D_3 \\ \hline & & & \end{array}\right) = \left(\begin{array}{c|c|c|c} 2E & & E & 0 \\ \hline & 2E & \begin{smallmatrix} -1 & & \\ & \ddots & \\ & & -1 \\ 0 & \cdots & 0 \end{smallmatrix} & \begin{smallmatrix} -1 & & \\ 1 & \ddots & \\ & \ddots & -1 \\ & & 1 \end{smallmatrix} \\ \hline & & & \end{array}\right)$$

from which it is clear that $a_1 \notin \delta_{\sigma\tau} M$.

So a_1 is a cocycle representative for the generator of $H^2(\langle\sigma\tau\rangle, M)^A$. Moreover $a_1 \in M^A$, so Corollary 6.7 shows that the generator can be lifted.

So we have a short exact sequence

$$0 \longrightarrow Z_2^{n_1} \longrightarrow H^2(\Phi, M) \xrightarrow{\gamma_{\sigma\tau}} Z_2 \longrightarrow 0.$$

To determine whether $H^2(\Phi, M)$ has any elements of order 4, we compute $B^2_{\sigma\tau}$.

The coboundary conditions

a) $D_1 Y = 2W'$

b) $D_4 Z = 2W''$

are $Y=2W'$ and $Z=2W''$. Then $B^2_{\sigma\tau} = \left\{ \begin{pmatrix} 0 \\ \bar{X} \\ 0 \\ 0 \end{pmatrix} \middle| \bar{X} = 2X+D_2Y+D_3Z \right\} =$

$$= \left\{ \begin{pmatrix}0\\X\\0\\0\end{pmatrix} \middle| \; X = 2X + 2D_2W' + 2D_3W'' \right\} \quad \text{where } X \in \mathbb{Z}^{n_2}.$$

Hence $B^2_\alpha = 2V_2$. To apply Lemma 8.1, note that we can choose as representative for both $H^2(A,M)$ and $H^2(B.M)$ the cocycle $0 \in M^\sigma \cap M^\tau$. Therefore $H^2(\Phi,M)$ has exponent 2 and we have

$$H^2(\Phi,M) \simeq Z_2^{n_1+1} .$$

Turning to modules of type $\{6b\}$ where

$$D = \left[\begin{array}{c|c} \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} \\ \hline \begin{matrix}0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1\end{matrix} & \begin{matrix}1 & & & 0\\ & \ddots & & \vdots \\ & & 1 & 0\end{matrix}\end{array}\right]$$

From section 5 we had the exact sequence

$$0 \longrightarrow Z_2^{n_1-1} \longrightarrow H^2(\Phi,M) \xrightarrow{r_\tau} Z_2 \longrightarrow Z_2^{n_2}.$$

For these modules $H^2(AB,M) = 0$ and as in cases $\{2a\}$ and $\{9a\}$, we can apply Corollary 6.12 to the representative d_{n_4} of the generator of $H^2(B,M)^\Phi \simeq Z_2$. So the generator of $H^2(B,M)$ lifts if and only if we can solve the equations:

$$\begin{cases}(\sigma\tau - 1)\, d_{n_4} = \delta_B m \\ N_{\sigma\tau} m = 0\end{cases}$$

By a calculation similar to that for case $\{2a\}$, one finds there is no solution possible for the above equations. Hence $r_\tau = 0$ and therefore

$$H^2(\Phi,M) \simeq Z_2^{n_1-1}.$$

So, we recall that for {6a} we had

$$H^2(\Phi,M) \simeq Z_2^{n_1+1},$$

and for {6b} we have

$$H^2(\Phi,M) \simeq Z_2^{n_1-1}.$$

{4a} $D = \left[\begin{array}{c|c} \begin{matrix} 0 & \cdots & 0 \\ 1 & \ddots & \\ & & 1 \end{matrix} & \begin{matrix} 1 & \ddots & \\ & & 1 \\ 0 & \cdots & 0 \end{matrix} \\ \hline \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{array}\right]$

Coming to the next to last case, that of modules {4a}, we have the exact sequence

$$0 \longrightarrow Z_2^{n_1} \longrightarrow H^2(\Phi,M) \xrightarrow{r_\sigma} Z_2 \longrightarrow Z_2^{n_2}.$$

We can choose $a_{n_1} \in M^B$ as representative for the generator of $H^2(A,M)^B$. So as before, the generator lifts, and we have a short sequence

$$0 \longrightarrow Z_2^{n_1} \longrightarrow H^2(\Phi,M) \xrightarrow{r_\tau} Z_2 \longrightarrow 0$$

We calculate $B^2_{\mathfrak{A}} = [(1-\sigma)M^{-\tau} + (1-\tau)M^{-\sigma}]$.

The antispecial coboundary conditions

$$\left.\begin{array}{ll} \text{a)} & D_1Y = 2W' \\ \text{b)} & D_4Z = 2W'' \end{array}\right\} \text{ imply } \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 1 & & & \\ & \ddots & & \\ & & & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_m \end{pmatrix} = \begin{pmatrix} 0 \\ y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_m \end{pmatrix} = 2W'$$

and $\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & & 1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_m \end{pmatrix} = \begin{pmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_m \\ 0 \end{pmatrix} = 2W''$. Both imply

that $Y = 2Y'$, and $Z = 2Z'$ for some $Y', Z' \in \mathbb{Z}^m$. Then

$B^2_{\mathfrak{a}} = \left\{ \begin{pmatrix} 0 \\ \overline{X} \\ 0 \\ 0 \end{pmatrix} \,\middle|\, \overline{X} = 2X+D_2Y+D_3Z \text{ with } X \text{ free} \right\}$. Here

$\overline{X} = 2X+D_2(2Y')+D_3(2Z') = 2(X+Y'+Z')$. Since X is free, we see that $B^2_{\mathfrak{a}} = 2V_2$. We apply Lemma 8.1, noting that as representatives of $H^2(A,M)^B$ and $H^2(B,M)^A$ we can choose the elements $a_{n_1} \in M^B$ and $a_1 \in M^A$ respectively. Hence $H^2(\Phi,M)$ has exponent 2 and therefore $H^2(\Phi,M) \simeq Z_2^{n_1+1}$.

Finally, we come to modules of type {4b}. The Hochschild-Serre sequence yielded an exact sequence

$$\longrightarrow H^1(\Phi/\langle\sigma\rangle, M^\sigma) \xrightarrow{\text{inf}} H^1(\Phi, M) \longrightarrow H^0(\Phi/\langle\sigma\rangle, H^1(\langle\sigma\rangle, M)) \longrightarrow$$

$$\longrightarrow H^2(\Phi/\langle\sigma\rangle, M^\sigma) \xrightarrow{\text{inf}} H^2(\Phi, M) \longrightarrow H^1(\Phi/\langle\sigma\rangle, H^1(\langle\sigma\rangle, M)) \longrightarrow$$

where inf is the inflation map.

To find $H^2(\Phi,M)$, we take advantage of the fact that $H^2(K,M) = 0$ for $K = \langle\sigma\rangle, \langle\tau\rangle$ and $\langle\sigma\tau\rangle$. So in this case $H^2(\Phi,M) \approx \mathfrak{A}$. We calculate $\mathfrak{A}$ for this case.

$$D = \left[\begin{array}{ccc|ccc} 1 & & 0 & 0 & 1 & \\ & \ddots & \vdots & \vdots & & \ddots \\ & 1 & 0 & 0 & & 1 \\ \hline 1 & & & 1 & & \\ & \ddots & & & \ddots & \\ & & 1 & & & 1 \end{array}\right]$$

The antispecial cocycle conditions

$$D_1Y+D_4Z = 2W' \text{ implies } \begin{pmatrix} 1 & & 0 \\ & \ddots & \vdots \\ & 1 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} + \begin{pmatrix} 0 & 1 & \\ \vdots & & \ddots \\ 0 & & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} =$$

$$= \begin{pmatrix} y_1 \\ \vdots \\ y_{m-1} \end{pmatrix} + \begin{pmatrix} z_2 \\ \vdots \\ z_m \end{pmatrix} = 2W', \qquad \text{which implies that}$$

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_{m-1} \\ y_m \end{pmatrix} = \begin{pmatrix} -z_2 \\ -z_3 \\ \vdots \\ -z_m \\ y_m \end{pmatrix} + \underbrace{\begin{pmatrix} 2w_1 \\ \vdots \\ 2w_m \\ 0 \end{pmatrix}}_{\substack{\parallel \\ 2W''}}.$$

Then, recall

$$z^2_{\mathfrak{a}} = \left\{ \begin{pmatrix} 0 \\ \overline{X} \\ 0 \\ 0 \end{pmatrix} \middle| \; \overline{X} = 2X+D_2Y+D_3Z \right\} \subseteq Z^m.$$ Here

$$\overline{X} = 2X+Y+Z = 2X + \left\{ \begin{pmatrix} -z_2 \\ \vdots \\ -z_m \\ y_m \end{pmatrix} + 2W'' \right\} + \begin{pmatrix} z_1 \\ \vdots \\ z_{m-1} \\ z_m \end{pmatrix}$$

$$= 2(X+W'') + \begin{pmatrix} z_1-z_2 \\ \vdots \\ z_{m-1}-z_m \\ z_m \quad +y_m \end{pmatrix}$$

$$= 2(X+W'') + \begin{pmatrix} 1 & -1 & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & -1 & 0 \\ & & & 1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \\ y_m \end{pmatrix}$$

Since the first m columns of the m × (m+1) matrix

$\begin{pmatrix} 1 & -1 & & 0 \\ & \ddots & \ddots & \vdots \\ & & -1 & 0 \\ & & 1 & 1 \end{pmatrix}$ form a unimodular submatrix, the elements of

the form $\begin{pmatrix} z_1-z_2 \\ \vdots \\ z_{m-1}-z_m \\ z_m + y_m \end{pmatrix}$ span all of $\mathbb{Z}^M$ and therefore, the elements

$$2(X+W'') + \begin{pmatrix} z_1-z_2 \\ \vdots \\ z_{m-1}+z_m \\ z_m + y_m \end{pmatrix} \text{ generate } \mathbb{Z}^M. \quad \text{Hence } \{\overline{X}\} \simeq \mathbb{Z}^M.$$

Since $m = \text{rank } V_3 = \text{rank } V_2 = \text{rank } V_1+1 = n_1+1$, we have $Z^2_{\mathfrak{A}} = V_2 \simeq Z^{n_1+1}$.

Now to evaluate $B^2_{\mathfrak{A}}$; the coboundary conditions

$$\left.\begin{array}{l} D_1Y = 2W' \\ D_4Z = 2W'' \end{array}\right\} \text{ imply } \begin{pmatrix} y_1 \\ \vdots \\ y_{m-1} \end{pmatrix} = 2W' \text{ and } \begin{pmatrix} z_2 \\ \vdots \\ z_m \end{pmatrix} = 2W''.$$

So $Y = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ y_m \end{pmatrix} + 2S$ and $Z = \begin{pmatrix} z_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + 2T$ for some $S,T \in Z^M$.

Then

$$\overline{X} = 2X+D_2Y+D_3Z = 2X+Y+Z = 2X + \begin{pmatrix} y_1 \\ 0 \\ \vdots \\ 0 \\ z_m \end{pmatrix}.$$

So $\{\overline{X}\} \simeq Z^2 \oplus 2Z^{m-2} = Z^2 \oplus Z^{n_1-1}$. To evaluate $\mathfrak{A}$, then, we note that $B^2_{\mathfrak{A}}$ is the submodule of $Z^2 = V_2 = \{(0,[x_1,\ldots,x_{n_1+1}],0,0\}$ consisting of all elements of the form $(0,[x'_1,2x'_2,\ldots,2x'_{n_1},x'_{n+1}],0,0)$. So

$$\mathfrak{A} \simeq Z_2^{n_1-1} \simeq H^2(\Phi,M).$$

Hence the exact sequence becomes

$$\longrightarrow Z_2^{n_2} \xrightarrow{\text{inf}} H^1(\Phi,M) \xrightarrow{\gamma_1} Z_2 \xrightarrow{\gamma_1} Z_2^{n_1} \xrightarrow{\gamma_3} Z_2^{n_1-1} \longrightarrow Z_2 \longrightarrow$$

From this it follows that γ_3 is not injective, thus Im $\gamma_2 \neq 0$, but then γ_2 must be injective so that $\gamma_1 = 0$, and inf is surjective.

$$\text{So } H^1(\Phi,M) \simeq \frac{Z_2^{n_2}}{\text{ker inf}:\{H^1(<\tau>,M^\sigma) \to H^1(\Phi,M)\}}.$$

But from the usual exact sequence

$$0 \longrightarrow H^1(\Phi/<\sigma>,M^\sigma) \xrightarrow{\text{inf}} H^1(\Phi,M) \longrightarrow H^1(<\sigma>,M)^\Phi \longrightarrow$$

we have that ker(inf) = 0. Hence $H^1(\Phi,M) \simeq Z_2^{n_2}$

$$H^1(\Phi,M) \simeq Z_2^{n_2}.$$

We can summarize $H^1(\Phi,M)$ and $H^2(\Phi,M)$ for these modules as follows. If n_1 = rank V_1, n_2 = rank V_2 then in all cases

$$\boxed{H^1(\Phi,M) \simeq Z_2^{n_2}}.$$

$H^2(\Phi,M)$ for each case is as follows.

Module type	$H^2(\Phi,M)$
1	$Z_2^{n_1}$
2a	$Z_2^{n_1-1}$
2b	$Z_2^{n_1+1}$
3a	$Z_2^{n_1}$
3b	$Z_2^{n_1}$
4a	$Z_2^{n_1+1}$
4b	$Z_2^{n_1-1}$
5	$Z_2^{n_1}$
6a	$Z_2^{n_1+1}$
6b	$Z_2^{n_1-1}$
7a	$Z_2^{n_1}$
7b	$Z_2^{n_1}$
8	$Z_2^{n_1}$

Module type	$H^2(\Phi,M)$
9a	$Z_2^{n_1-1}$
9b	$Z_2^{n_1+1}$
10	$Z_2^{n_1}$
(∞)	$Z_2^{n_1}$

§10. CONCLUSION

The computations of cohomology groups in this paper have applications to compact flat Riemannian manifolds.

Charlap has shown [1] that given a group Φ and a module M over $Z[\Phi]$, there is a one-one correspondence between "semi-linear equivalence classes" of "special classes" in $H^2(\Phi,M)$ and diffeomorphism classes of flat manifolds with Φ as holonomy group and M maximal abelian in the homotopy group of the manifold. In this paper we have shown how to calculate $H^2(\Phi,M)$ for the modules on Nazarova's first list, and have provided explicit formulas for lifting cohomology classes under the restriction map. So given a module M, it is possible, modulo combinatorial difficulties to ascertain what are the special cohomology classes, and the corresponding flat manifolds.

The techniques developed here can be applied to the modules on Nazarova's second list. However, since there appears to be an error in that second list, we have not studied those modules at this time.

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