ANTI-HOLOMORPHIC INVOLUTIONS OF ANALYTIC FAMILIES OF ABELIAN VARIETIES

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Allan Russell Adler

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Abstract of the Dissertation

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In this paper, we investigate anti-holomorphic involutions of certain analytic families of abelian varieties parametrized by compact local Hermitian symmetric spaces. One first observes that such an involution must be fibre preserving and therefore induce an anti-holomorphic involution of the parameter space, which is an invariant of the original involution. By means of this observation, one can reduce the problem to that of describing anti-holomorphic maps between complex tori, which is easily solved. Namely, every anti-holomorphic map between complex tori is an anti-holomorphic homomorphism followed by a translation. As one varies over the parameter space, one obtains an "analytic family of anti-holomorphic homomorphisms" and an "analytic family of translation parts". These are classified by certain cohomological invariants of

the data defining the fibre variety which, together with the involution induced on the parameter space, constitute a complete set of invariants. In Chapter II, we obtain a necessary and sufficient condition for the existence of an anti-holomorphic involution of a fibre variety with prescribed invariants (Theorem 1). In Chapter IV, we specialize our results to the case of analytic families of abelian varieties belonging to totally indefinite quaternion division algebras over totally real number fields. In this case, the invariants have a natural description in terms of the division algebra, and the existence theorem specializes to arithmetic criterion for the existence of involutions with prescribed invariants (Theorems 2 and 3).

In the course of determining some of these invariants, we have had to compute the group of holomorphic sections of these fibre varieties. This is done at the end of Chapter I and Chapter IV and yields an interesting class of algebraic cycles on the fibre-variety.

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CHAPTER 0.

INTRODUCTION

A quotient space $U = \Gamma\backslash X$ of a symmetric domain X by a discontinuous group Γ is a projective algebraic variety. Moreover, U is often the parameter space of a family V of abelian varieties.

In that case, V is also a projective algebraic variety. It is to this fact that the arithmetic theory of the arithmetic group Γ owes its success. For an arithmetic discontinuous group Γ_o acting on a non-Hermitian symmetric space X_o, this powerful resource is not available. However, as John Milson has suggested, if we could realize $U_o = \Gamma_o\backslash X_o$ as a real algebraic variety in $P^N(R)$, we could attempt to study the deeper arithmetic theory of Γ_o. According to this point of view, one could, for example, study the arithmetic of the orthogonal group of a quadratic form over a number field by studying the unitary groups of the corresponding Hermitian form over all quadratic extensions of that number field.

With this aim in mind, there have been several investigations in recent years of the possibility of realizing such a manifold $U_o = \Gamma_o\backslash X_o$ as a connected component of a real cross-section $P^N(R) \cap U$ of a local Hermitian symmetric space $U = \Gamma\backslash X$ embedded in $P^N(C)$ and defined over R. This problem is reduced, using Weil's results on the field of definition of an algebraic variety, to the investigation of anti-holomorphic involutions

σ of $U = \Gamma\backslash X$.

Harris Jaffee [5] has classified the anti-holomorphic involutions of Hermitian symmetric spaces X and Steve Kudla [4] has investigated anti-holomorphic involutions of $U = \Gamma\backslash X$ for the case of an arithmetic group Γ belonging to a quaternion algebra and acting on a product of copies of the upper half-plane.

Our purpose in this paper is to investigate anti-holomorphic involutions σ of certain families $V \xrightarrow{\pi} U$ of abelian varieties parametrized by a local Hermitian symmetric space $U = \Gamma\backslash X$.

We begin by observing that such an involution σ must be fibre preserving, i.e. there must exist an anti-holomorphic involution σ_o of U such that $\pi\circ\sigma = \sigma_o\circ\pi$. Therefore, one is reduced to 1) investigating conditions on an anti-holomorphic involution σ_o on U which will guarantee that it is induced in this manner, and 2) given a σ_o satisfying these conditions, to classify all anti-holomorphic involutions σ of V which induce σ_o on U. The first of these problems is solved by Theorem 1 which gives a necessary and sufficient condition for σ_o to lift to V.

Since σ must be fibre preserving, we can view it as an "analytic family of anti-holomorphic maps between complex tori". Such maps are well-known to have a particularly simple form, namely, they can be uniquely written as an anti-holomorphic homomorphism followed by a translation.

As one varies over the parameter space U, one obtains an "analytic family of anti-holomorphic homomorphisms", i.e. an anti-holomorphic mapping C : V → V which induces a homomorphism on each fibre, and an "analytic family of translations", i.e. an anti-holomorphic mapping b : U → V such that $\pi \circ b = \sigma_o$.

In Chapter II, we show that the translation part b is classified by an element of $H^1(\Gamma, L)$, where Γ, L are data defining V (namely Γ is a discontinuous group acting on X and L is a lattice on which Γ acts). In Chapter III, we define cohomology with non-abelian coefficients and use this to classify the homomorphism part C.

In Chapter IV, we specialize our results to the case of an analytic family of abelian varieties belonging to a totally indefinite quaternion division algebra D over a totally real number field k. In this case, the invariants of σ, described above, have a natural interpretation in terms of the arithmetic of the division algebra. In this case, we can prove that if σ_o has a fixed point, then it lifts (Theorem 2). Moreover, if we specialize Theorem 1 to this situation, we obtain an arithmetic criterion for the existence of lifts in case σ_o does not have a fixed point. Finally, at the end of Chapter IV, we apply our results on the group Ξ of sections of V at the end of Chapter I to the exact determination of that group in this special setting. This provides us with an interesting class of algebraic cycles on V.

It is quite likely that we can extend our arithmetic results to the general case by considering fibre varieties belonging to a semi-simple algebra with involution over a number field. We have already made some preliminary investigations of this possibility, the results of which will appear in a subsequent paper.

CHAPTER I

KUGA'S FIBRE VARIETIES

1. Let W be a vector-space of dimension $2n$ over the field $\mathbb{R}$ of real numbers, let $W_{\mathbb{C}} = W \otimes_R \mathbb{C}$ denote its complexification, and let $Gr(W_{\mathbb{C}})$ denote the Grassman manifold of complex n-planes P in $W_{\mathbb{C}}$.

If X is any complex manifold and if φ is a holomorphic mapping of X into $Gr(W_{\mathbb{C}})$, then φ determines a holomorphic vector-bundle $\beta(\varphi)$ over X in the following way. The total space E_φ of $\beta(\varphi)$ consists of all pairs (x,v) belonging to $X \times W_{\mathbb{C}}$ for which the vector v belongs to the n-plane $\varphi(x)$, and the projection mapping $E_\varphi \xrightarrow{\pi_\varphi} X$ is given by $\pi_\varphi(x,v) = x$. The addition and scalar multiplication are defined by the rules $(x,v_1) + (x,v_2) = (x,v_1+v_2)$, $c\cdot(x,v) = (x,cv)$.

2. By a complex structure on the real vector space W, we mean a linear endomorphism of W such that $J^2 = -1_W$. If J is given, we can define on W the structure of a complex vector space by the rule $(a+bi)\cdot v = a\cdot v + bJ(v)$, for $a,b \in R$ and $v \in W$. We will denote by (W,J) the complex vector-space so obtained.

If J is a complex structure on W, then its $\mathbb{C}$-linear extension to $W_{\mathbb{C}}$ satisfies $J^2 = -1_{W_{\mathbb{C}}}$, and we can write $W_{\mathbb{C}} = W_J^+ \oplus W_J^-$, where W_J^+ is the i-eigenspace of J and W_J^- is the $-i$-eigenspace of J. W_J^+ is a complex n-plane in $W_{\mathbb{C}}$ canonically associated to the complex structure J, and we say W_J^+ belongs to J.

It is easy to see that the mapping $w \to w-iJ(w)$ defines an isomorphism of $x_J(W,J)$ onto W_J^+.

3. Suppose we are given a non-degenerate skew-symmetric bilinear form $B : W \times W \mapsto \mathbb{R}$. Denote by $\mathcal{H}_B$ the set of all complex structures J on W for which the bilinear form $S_J : W \times W \to \mathbb{R}$ defined by $S_J(x,y) = B(x,Jy)$ is symmetric and positive definite.

We have a canonical injection $\varkappa : \mathcal{H}_B \to Gr(W_{\mathbb{C}})$ given by $J \mapsto \varkappa(J) = W_J^+$ which maps $\mathcal{H}_B$ onto an open subset of the sets of n-planes totally isotropic for B. Thus, $\mathcal{H}_B$ has a natural structure of complex manifolds for which the mapping $\varkappa$ is holomorphic.

4. Let X be a bounded symmetric domain and let Γ be a group acting holomorphically and properly discontinuously on X. Suppose that $\Gamma\backslash X$ is compact.

Let $\varphi_o : X \to \mathcal{H}_B$ be a holomorphic mapping, and let $\varphi = \varkappa \circ \varphi_o : X \to Gr(W_{\mathbb{C}})$. By the results of §1, φ determines a holomorphic vector-bundle $\beta(\varphi) = (E_\varphi, \pi_\varphi)$ over X. The fibre $\pi_\varphi^{-1}(x)$ is just $\{x\} \times \varphi(x)$.

Using the remark at the end of §2, we define, for every $x \in X$, a complex linear isomorphism $\lambda_x : (W,\varphi_o(x)) \to \varphi(x)$ by the rule $\lambda_x(v) = v-i\varphi_o(x)(v)$.

Let L be a lattice in W such that B is integer valued on $L \times L$. Let Aut(W,B,L) denote the group of all linear automorphisms g of W such that

1) $g(L) = L$.

2) $B(g(x),g(y)) = B(x,y)$ for all $x,y \in W$.

Let $\rho : \Gamma \to \mathrm{Aut}(W,B,L)$ be a representation. Let $\tilde{\Gamma}$ denote the semi-direct product of Γ and L via ρ. Explicitly, $\tilde{\Gamma} = \Gamma \times L$ with the group law defined by $(\gamma_1,\ell_1)\cdot(\gamma_2,\ell_2) = (\gamma_1 * \gamma_2, \ell_1+\rho(\gamma_1)(\ell_2))$.

We require that for $x \in X$ and $\gamma \in \Gamma$, we have $\varphi_o(\gamma\cdot x) = \rho(\gamma)\circ\varphi_o(x)\circ\rho(\gamma)^{-1}$. It follows that $\varphi(\gamma\cdot x) = \rho(\gamma)(\varphi(x))$.

We define an action of $\tilde{\Gamma}$ on E_φ by the rule $(\gamma,\ell)\cdot(x,v) = (\gamma\cdot x, \lambda_{\gamma\cdot x}(\ell) + \rho(\gamma)(v))$. It is straightforward to verify that in this way $\tilde{\Gamma}$ acts holomorphically and properly discontinuously on E_φ.

Denote by V the complex manifold $\tilde{\Gamma}\backslash E_\varphi$ and U the complex manifold $\Gamma\backslash X$.

For $\tilde{\gamma} = (\gamma,\ell) \in \tilde{\Gamma}$, the diagram

$$\begin{array}{ccc} E_\varphi & \xrightarrow{\tilde{\gamma}} & E_\varphi \\ \big\downarrow \pi_\varphi & & \big\downarrow \pi_\varphi \\ X & \longrightarrow & X \end{array}$$

is commutative.

It follows that π_φ induces a holomorphic mapping $V \xrightarrow{\pi} U$.

Denote by $V \coprod V$ the set of all ordered pairs (v_1,v_2) such that $\pi(v_1) = \pi(v_2)$. For $(v_1,v_2) \in V \coprod V$, we can unambiguously define $\pi(v_1, v_2)$ to be $\pi(v_1) = \pi(v_2)$.

The addition in E_φ determines a mapping $+ : V$ which is holomorphic, and the 0-section of E_φ determines a holomorphic section $\eta : U \to V$. Finally, the map $(x,v) \mapsto (x,-v)$ of E_φ determines a holomorphic map $\theta : V \to V$.

In this way, the sextuple $(V, \pi, U, +, \eta, \theta)$ can be viewed as an analytic family of compact complex Lie groups. Moreover, for each $x \in U$, $\pi^{-1}(x)$ has the structure of a polarized abelian variety.

Henceforth we will denote by F_x the fibre $\pi^{-1}(x)$.

5. We recall now, for future reference, the following fact which is proved in [1]. $\tilde{\Gamma}$ acts on $X \times W$ by the rule $(\gamma, \ell)(x,u) = (\gamma \cdot x, \ell + \rho(\gamma) \cdot u)$.

Lemma: There is a unique structure of holomorphic vector-bundle on $X \times W$ such that $\tilde{\Gamma}$ acts holomorphically and such that for all $x \in X$, the complex structure induced on the vector-space $\{x\} \times W$ is $\varphi_o(x)$.

Actually, we have essentially already proved the existence, for the mapping $\lambda : X \times W \to E_{\varphi_o}$ given by $(x,w) \to (x, \lambda_x(w))$ is a bijection which induces on $X \times W$ the desired structure. For the uniqueness, see [1].

6. As we noted in the construction of V in §4, the group $\tilde{\Gamma}$

acts holomorphically and properly discontinuously on the space E_φ. Therefore, the subgroup Γ of $\tilde{\Gamma}$ acts holomorphically and properly discontinuously on E_φ, and we denote by $V^\#$ the quotient space $\Gamma\backslash E_\varphi$. The projection mapping $\pi_\varphi : E_\varphi \to X$ commutes with the action of Γ on these spaces, and therefore induces a mapping $\pi^\# : V^\# \to U$. The pair $(V^\#, \pi^\#)$ is a holomorphic vector-bundle over U. It is proved in [1] that for all q, $H^q(U,V^\#)$ is canonically isomorphic to $H^q(\Gamma,W)$.

7. The mapping $\lambda : X \times W \to E_\varphi$ constructed in §5 maps $X \times L$ onto a subspace $\tilde{\Lambda}$ of E_φ. It is evident that $\tilde{\Lambda}$ is a complex submanifold of E_φ invariant under the action of Γ. Denote by Λ the quotient space $\Gamma\backslash\tilde{\Lambda}$, which we view as a subspace of $V^\#$. Λ is a sheaf over U. It is proved in [1] that for all q, $H^q(U,\Lambda)$ is canonically isomorphic to $H^q(\Gamma,L)$ and that the diagram

$$\begin{array}{ccc} H^q(U,\Lambda) & \longrightarrow & H^q(U,V^\#) \\ \big\downarrow & & \big\downarrow \\ H^q(\Gamma,L) & \longrightarrow & H^q(\Gamma,W) \end{array}$$

is commutative,

where the horizontal arrows are coefficient homomorphisms and the vertical arrows are canonical isomorphisms.

In particular, we have $H^q(U,V^\#) \cong H^q(U,\Lambda) \otimes \mathbb{R}$.

8. In this section, we compute the group Ξ of sections of the fibre variety $\pi : V \to U$.

In the category of analytic families of complex Lie groups over U, the sequence

$$(*)\qquad 0 \to \Lambda \to V^{\#} \to V \to 0 \qquad \text{is exact,}$$

where $V^{\#} \to V$ is the natural mapping $\Gamma\backslash E_{\varphi} \to \tilde{\Gamma}\backslash E_{\varphi}$.

Denote by $\mathfrak{v}^{\#}$ the sheaf of germs of holomorphic sections of $V^{\#}$, and by $\mathfrak{v}$ the sheaf of germs of holomorphic sections of V.

As we noted in §7, Λ is already a sheaf. Then the sequence (*) determines an exact sequence

$$(**)\qquad 0 \to \Lambda \to \mathfrak{v}^{\#} \to \mathfrak{v} \to 0$$

of sheaves over U.

We therefore have an exact cohomology sequence

$$0 \to H^{0}(U,\Lambda) \to H^{0}(U,\mathfrak{v}^{\#}) \to H^{0}(U,\mathfrak{v}) \xrightarrow{\delta} H^{1}(U,\Lambda) \to H^{1}(U,\mathfrak{v}^{\#}).$$

Of course, $H^{0}(U,\mathfrak{v}) = \Xi$.

Using the isomorphisms described in §§6-7, we have that

$$0 \to H^{0}(\Gamma,L) \to H^{0}(\Gamma,W) \to \Xi \xrightarrow{\delta} H^{1}(\Gamma,L) \to H^{1}(\Gamma,W)$$

is exact.

The kernel of $H^{1}(\Gamma,L) \to H^{1}(\Gamma,W)$ is just the torsion subgroup $H^{1}(\Gamma,L)^{\text{tors.}}$ of $H^{1}(\Gamma,L)$ so that we obtain the following exact sequence

$$0 \to \frac{H^{0}(\Gamma,W)}{H^{0}(\Gamma,L)} \to \Xi \to H^{1}(\Gamma,L)^{\text{tors.}} \to 0.$$

This sequence splits since $\frac{H^{0}(\Gamma,W)}{H^{0}(\Gamma,L)}$ is a divisible

$$\therefore\quad \Xi \cong \frac{H^{0}(\Gamma,W)}{H^{0}(\Gamma,L)} \times H^{1}(\Gamma,L)^{\underline{\text{tors}}.}$$

In all the cases we will consider, $H^{0}(\Gamma,W) = 0$, so that

$\Xi \cong H^1(\Gamma, L)^{\underline{\text{tors.}}}$. Moreover, since the group Γ is finitely generated, so is the group $H^1(\Gamma, L)^{\text{tors.}}$. Therefore, for suitable N, $H^1(\Gamma, L)^{\text{tors.}}$ consists entirely of N-torsion.

From the exact sequence of Γ-modules $0 \to L \xrightarrow{N} L \to L/NL \to 0$, we have the coefficient sequence $H^0(\Gamma, L) \to H^0(\Gamma, L/NL) \xrightarrow{\delta} H^1(\Gamma, L) \xrightarrow{N} H^1(\Gamma, L)$.

Since we have $H^0(\Gamma, L) = 0$, and the image of δ is the N-torsion in $H^1(\Gamma, L)$, we conclude that

$$(***) \qquad \Xi \cong H^1(\Gamma, L)^{\underline{\text{tors.}}} \cong H^0(\Gamma, L/NL).$$

9. The elements of Ξ determine algebraic cycles in V. It would be interesting to study the homological properties of these cycles. For example, when do two elements of Ξ determine homologous cycles?

We can describe these cycles explicitly in a way which sheds more light on the isomorphism $\Xi \cong H^0(\Gamma, L/NL)$. Let $c \in L$ be an element which represents an element of $H^0(\Gamma, L/NL)$, so that $\rho(\gamma)\cdot(c) \equiv c \pmod{NL}$ for every $\gamma \in \Gamma$. Then $\rho(\gamma)(\frac{1}{N}c) \equiv \frac{1}{N}c \pmod{L}$ for every $\gamma \in \Gamma$.

It follows that there is a uniquely determined section ω of V such that the diagram

$$\begin{array}{ccccccc} X & \longrightarrow & X \times \{\frac{1}{N}c\} & \subseteq & X \times W & \xrightarrow{\lambda} & E_\varphi \\ \downarrow & & & & & & \downarrow \\ U & & & \xrightarrow{\omega} & & & V \end{array}$$

is commutative.

Then ω and c correspond to each other under the isomorphism (***).

10. We say that a holomorphic mapping $f : V \to V$ is an endomorphism of V if $\pi \circ f = \pi$ and if for every $x \in U$, f induces on the fibre F_x an endomorphism of that complex Lie group. If f is bijective, we say it is an automorphism of V. The set of all endomorphisms of V forms a ring $End(V)$, which we will always view as operating on V from the right. The group of units of $End(V)$ consists of the automorphisms of C, and is denoted by $Aut(V)$.

CHAPTER II

EXISTENCE OF COMPLEX CONJUGATIONS

1. Let M and N be complex manifolds, and let $f : M \to N$ be a differentiable function. We say f is <u>anti-holomorphic</u> if $df : T(M) \to T(N)$ induces conjugate-linear maps on the tangent spaces to M.

The following proposition is well-known, but for the sake of completeness, we will give a proof.

<u>Proposition 1</u>: Let M and N be complex tori and let $f : M \to N$ be an anti-holomorphic mapping. Then there is a unique element $b \in N$ and a unique anti-holomorphic homomorphism $C : M \to N$ such that for all $x \in M$ we have $f(x) = C(x)+b$.

<u>Proof</u>. We may as well write $M = \widetilde{M}/L_1$, $N = \widetilde{N}/L_2$, where $\widetilde{M},\widetilde{N}$ are complex vector spaces and L_1 and L_2 are lattices in $\widetilde{M}$ and $\widetilde{N}$ respectively. We can identify $\widetilde{M}$ and $\widetilde{N}$ with the universal covering spaces of M and N respectively. We can therefore cover f with an anti-holomorphic mapping $\widetilde{f} : \widetilde{M} \to \widetilde{N}$. For each $\lambda \in L$, and each $z \in \widetilde{M}$, $\widetilde{f}(z+\lambda)$ and $\widetilde{f}(z)$ represent the same point of N. Therefore, there is a point $\varphi(\lambda,z) \in L$ such that $\widetilde{f}(z+\lambda) = \widetilde{f}(z) + \varphi(\lambda,z)$. Since φ is continuous with respect to z and L_2 is discrete, φ must actually be independent of z, so we can write $\varphi(\lambda,z) = \varphi(\lambda)$.

Then for all $\lambda \in L_1$ and $z \in \widetilde{M}$, we have $d\widetilde{f}(z+\lambda) = d\widetilde{f}(z)$, so that $d\widetilde{f} : \widetilde{M} \to \mathrm{Hom}(\widetilde{M},\widetilde{N})$ is a periodic anti-holomorphic function with period L_1. Therefore $d\tilde{f}$ is constant. This proves that $\tilde{f}$ is of the form $\tilde{C}+\tilde{b}$, where $\tilde{b} \in \tilde{N}$ and where $\tilde{C} : \tilde{M} \to \tilde{N}$ is a conjugate-linear mapping. The proposition follows as once.

Q.E.D.

By a <u>complex conjugation</u> of a complex manifold M, we mean an anti-holomorphic involution of M, i.e., an anti-holomorphic mapping $f : M \to M$ such that $f \circ f = 1_M$.

2. Let $V \xrightarrow{\pi} U$ a Kuga fibre variety, that is to say, a fibre system of abelian varieties over U of the type discussed in §4 of Chapter I. Our purpose is to study the complex conjugations of V.

Let us begin by remarking that a complex conjugation σ of V must preserve the fibres of π. To see this, let $x \in U$. Let $f : F_x \to U$ be the restriction of $\pi \circ \sigma$ to F_x. f is anti-holomorphic and is induced by an anti-holomorphic mapping $\tilde{f}$ of the universal covering space of F_x, which is a complex vector space, into the universal covering space of U, which is X. Let $g : X \to X$ be an anti-holomorphic isometry. Then $g \circ \tilde{f}$ is a bounded holomorphic function on a complex vector-space and is therefore constant. It follows that f must be a constant map, say, $f(u) = y \in U$ for all $u \in F_x$. Then f maps F_x into F_y, which proves our assertion.

Let σ be a complex conjugation of V. Since this fibres of π are in 1-1 correspondence with the points of U, σ induces a function $\sigma_0 : U \to U$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\sigma} & V \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ U & \xrightarrow{\sigma_o} & U \end{array}$$ is commutative.

Thus, $\sigma(F_x) = F_{\sigma_o(x)}$ for all $x \in U$. We can write $\sigma_o \circ \pi \circ \sigma_o \circ \eta$, so that σ_o is anti-holomorphic.

For every $x \in U$, F_x is a complex torus, and σ induces on F_x an anti-holomorphic mapping of F_x into $F_{\sigma_o(x)}$. Applying the proposition of the preceding section, we conclude that for each $x \in U$, there is a unique element b_x of $F_{\sigma_o(x)}$ and a unique anti-holomorphic homomorphism $C_x : F_x \to F_{\sigma_o(x)}$ such that for all $u \in F_x$, we have $\sigma(u) = C_x(u) + b_x$.

We can therefore define a function $b : U \to V$ and a function $C : V \to V$ by the rules

$$b(x) = b_x = \sigma \circ \eta(x)$$

and $C(u) = C_x(u) = +\circ(\sigma(u), \theta \circ b \circ \pi(u))$, if $\pi(u) = x$.

It is evident from the descriptions that b and C are anti-holomorphic, and we have $\sigma = C + (b \circ \pi)$.

For $u \in V$, if we put $x = \pi(u)$, we have

$$u = \sigma^2(u) = C^2(u) + C(b_x) + b_{\sigma_o(x)}.$$

In particular, if $u = \eta(x)$, we get

$$\eta(x) = C(b_x) + b_{\sigma_o(x)}$$

which we can rewrite as

$$(1) \qquad C \circ b = -b \circ \sigma_o.$$

For arbitrary u, therefore, we have

$$u = C^2(u) + \eta(x) = C^2(u),$$

so that C is also fibre preserving complex conjugation of V.

We call C the homomorphism part of σ and b the translation part of σ.

In order to classify complex conjugations of V, it is therefore sufficient to classify all possible C's (i.e. all those which leave the image of η invariant), and then to find all b's which, for a given C satisfy (1).

3. The first invariant of the complex conjugation σ is the complex conjugation σ_o of U. As we noted above, the homomorphism part C of σ is also a complex conjugation of V and it induces the same complex conjugation on U. It is natural to try to determine those complex conjugations of U which are obtained in this way.

Let σ_o be any complex conjugation of U, and let $\tilde{\sigma}_o$ be a lifing of σ_o to an anti-holomorphic mapping from X to itself. Then $\tilde{\sigma}_o^2$ is a holomorphic covering transformation of X over U, and therefore can be identified with an element γ_o of Γ. Moreover, the mapping $\gamma \mapsto \tilde{\sigma}_o \gamma \tilde{\sigma}_o^{-1}$ defines an automorphism of Γ.

<u>Theorem 1</u>: A necessary and sufficient condition for the complex conjugation σ_o of U to be induced on U by a complex conjugation of V is that there exist a linear transformation $A : W \to W$ with the following four properties:

1) $A^2 = \rho(\gamma_o)$

2) A normalizes $\rho(\Gamma)$ and induces on Γ the automorphism

$\gamma \to \tilde{\sigma}_o \gamma \tilde{\sigma}_o^{-1}$.

3) $A(L) = L$

4) for every $x \in X$, we have $A \circ \varphi_o(x) = -\varphi(\tilde{\sigma}_o(x)) \circ A$.

Proof. By the results of §2 of this chapter, we may as well require that σ_o be induced by a complex conjugation preserving the image of η.

As we noted in §5 of Chapter I, there is a unique structure of holomorphic vector-bundle on $X \times W \to X$ such that $\tilde{\Gamma}$ acts holomorphically on $X \times W$ and such that the complex structure induced on the vector-space W is $\varphi_o(x)$.

The mapping λ defined in that section determines a holomorphic isomorphism of that vector-bundle onto E_φ which is compatible with the action of $\tilde{\Gamma}$. Therefore, we may view $X \times W$, with this $\tilde{\Gamma}$-action and complex structure, as the universal covering manifold of V.

Now suppose that C is a complex conjugation of V preserving η and inducing σ_o on U. Then there is a unique lifing of C to an anti-holomorphic mapping $\tilde{C} : X \times W \to X \times W$ such that the diagram

$$\begin{array}{ccc} X \times W & \xrightarrow{\tilde{C}} & X \times W \\ \downarrow & & \downarrow \\ X & \xrightarrow{\tilde{\sigma}_o} & X \end{array}$$

is commutative

and such that for every $x \in X$, $\tilde{C}$ induces a real-linear transformation of the vector-space $\{x\} \times W$ onto $\{\tilde{\sigma}_o(x)\} \times W$. Therefore, we can write $\widetilde{C}$ in the form $\widetilde{C}(x,u) = (\widetilde{\sigma}_o(x), A_x u)$, where $A_x : W \to W$ is R-linear. If $\widetilde{\gamma} \in \widetilde{\Gamma}$, then $\widetilde{C}\widetilde{\gamma}\widetilde{C}$ is a holomorphic covering transformation of $X \times W$ over V, and therefore belongs to $\widetilde{\Gamma}$. In particular, $\widetilde{C}^2$ belongs to $\widetilde{\Gamma}$, and for every $\widetilde{\gamma} \in \widetilde{\Gamma}$, $\widetilde{C}\widetilde{\gamma}\widetilde{C}^{-1} = (\widetilde{C}\widetilde{\gamma}\widetilde{C})\widetilde{C}^{-2}$ belongs to $\widetilde{\Gamma}$. Actually, if we write $\widetilde{C}^2 = (\gamma, \ell)$, we must have $\gamma = \gamma_o$ and $\ell_o = 0$, since $\widetilde{C}$ covers $\widetilde{\sigma}_o$ and preserves the o-section.

Therefore, $\widetilde{C}^2 = (\gamma_o, 0)$ and $\widetilde{C}$ normalizes $\widetilde{\Gamma}$. Since $\widetilde{C}$ normalizes $\widetilde{\Gamma}$, it follows that for any $\ell_o \in L$, we can find $(\gamma, \ell) \in \widetilde{\Gamma}$ such that $\widetilde{C}(0, \ell_o) = (\gamma, \ell)\widetilde{C}$.

For any $x \in X$, we have

$$\widetilde{C}\,(0,\ell_o)(x,0) = \widetilde{C}(x,\ell) = (\widetilde{\sigma}_o x, A_x(\ell_o))$$

$$(\gamma,\ell)\widetilde{C}(x,0) = (\gamma,\ell)(\widetilde{\sigma}_o x, 0) = (\gamma\widetilde{\sigma}_o, \ell),$$

so that $(\widetilde{\sigma}_o x, A_x(\ell_o)) = (\gamma\widetilde{\sigma}_o, \ell)$.

Therefore $\gamma = 1$ and $A_x(\ell_o) = \ell$, independently of x. Since L contains a basis for W over R, it follows that the transformation A_x is independent of x. Therefore, we can write $A = A_x$. This is the linear transformation we are looking for.

What we have shown is that $A(L) \subseteq L$.

Since $\widetilde{C}^2 = (\gamma_o, 0)$, for every $(x,u) \in X \times W$ we have $(\gamma_o x, \rho(\gamma_o)(u)) = \widetilde{C}^2(x,u) = \widetilde{C}(\widetilde{\sigma}_o x, A(u)) = (\widetilde{\sigma}_o^2 x, A^2(u)) = (\gamma_o x, A^2(u))$, so that $A^2 = \rho(\gamma_o)$, which proves 1).

Since $A(L) \subseteq L$, we have $L = \rho(\gamma_o)(L) = A^2(L) \subseteq A(L) \subseteq L$, so that $A(L) = L$, which proves 3).

For $\gamma \in \Gamma$, we can find $(\gamma_1, \ell_1) \in \tilde{\Gamma} \ni \tilde{C}(\gamma, 0) = (\gamma_1, \ell_1)\tilde{C}$, since $\tilde{C}$ normalizes $\tilde{\Gamma}$. Then for $(x,u) \in X \times W$, we have $\tilde{C}(\gamma, 0)(x,u) = \tilde{C}(\gamma x, \rho(\gamma)u) = (\tilde{\sigma}_o \gamma x, A\rho(\gamma)(u))$ and $(\gamma_1, \ell_1)\tilde{C}(x,u) = (\gamma_1, \ell_1)(\tilde{\sigma}_o x, A(u)) = (\gamma_1 \tilde{\sigma}_o x, \ell_1 + \rho(\gamma_1)A(u))$. Therefore, we have $(\tilde{\sigma}_o \gamma x, A \circ \rho(\gamma)(u)) = (\gamma_1 \tilde{\sigma}_o x, \ell_1 + \rho(\sim_1) \circ A(u))$ for all $(x,u) \in X \times W$.

Thus we have

a) $\ell_1 = 0$

b) $\gamma_1 = \tilde{\sigma}_o \gamma \tilde{\sigma}_o^{-1}$

c) $A \circ \rho(\gamma) \circ A^{-1} = \rho(\tilde{\sigma}_o \gamma \tilde{\sigma}_o^{-1})$

which proves 2).

Finally, since $\tilde{C}$ is anti-holomorphic, the linear mapping $(W, \varphi_o(x)) \to (W, \varphi_o(\tilde{\sigma}_o(x)))$, given by $u \mapsto A_u$, must be anti-holomorphic as well, i.e. it must be conjugate linear. Therefore, we must have $A \circ \varphi_o(x) = -\varphi_o(\tilde{\sigma}_o(x)) \circ A$ for all $x \in X$, which proves 4).

Conversely, suppose we are given a linear mapping $A : W \to W$ satisfying conditions 1) - 4). Define the mapping $\tilde{C} : X \times W \to X \times W$ by the rule $\tilde{C}(x,u) = (\tilde{\sigma}_o x, A(u))$. Then $\tilde{C}^{-1}$ is given by $\tilde{C}^{-1}(x,u) = (\tilde{\sigma}_o^{-1} x, A^{-1}(u))$.

By 1), $\tilde{C}^2 = (\gamma_o, 0) \in \tilde{\Gamma}$. If $(\gamma, \ell) \in \tilde{\Gamma}$, then for all $(x,u) \in X \times W$ we have $\tilde{C}(\gamma, \ell)\tilde{C}^{-1}(x,u) = (\tilde{\sigma}_o \gamma \tilde{\sigma}_o^{-1} x, A(\ell) + A \circ \rho(\gamma) \circ A^{-1}(u))$.

By 2), we have $A \circ \rho(\gamma) \circ A^{-1} = \rho(\tilde{\sigma}_o \gamma \tilde{\sigma}_o^{-1})$, and by 3), $A(\ell) \in L$, so that the right hand side of the above equation becomes

$$(\tilde{\sigma}_o \gamma \tilde{\sigma}_o^{-1} x, A(\ell) + \rho(\tilde{\sigma}_o \gamma \tilde{\sigma}_o^{-1})(u)) = (\tilde{\sigma}_o \gamma \tilde{\sigma}_o^{-1}, A(\ell))(x,u)$$

$\therefore \widetilde{C} \circ (\gamma, \ell) \circ \widetilde{C}^{-1} \in \widetilde{\Gamma}$, so $\widetilde{C}$ normalizes $\widetilde{\Gamma}$.

Since $\widetilde{C}^2 \in \widetilde{\Gamma}$, we conclude that $\widetilde{C}$ induces on V a differentiable mapping $C : V \to V$ such that $C^2 = 1_V$. Obviously, C preserves the fibres of π and the image of η. Moreover, C must induce the complex conjugation σ_o on U. It only remains to prove that C is anti-holomorphic.

Denote by N the manifold $X \times W$ with the complex structure described in §5 of Chapter 1, and denote by M the manifold $X \times W$ with the unique structure of complex manifold for which the mapping $\widetilde{C} : M \to N$ is anti-holomorphic. We only have to show that $M = N$, i.e. that these complex structures coincide. Let $\mathcal{J}$ denote the almost complex structure on $X \times W$ determined by the complex manifold N, let $\mathcal{J}'$ denote the almost complex structure on $X \times W$ determined by the complex manifold M, and let $\mathcal{J}_o$ denote the almost complex structure on X.

It is easy to see that $\mathcal{J}' = -d\widetilde{C}^{-1} \circ \mathcal{J} \circ d\widetilde{C}$. In view of the uniqueness part of Lemma 1 of §5, Chapter I, in order to prove that $M = N$, or what is the same, that $\mathcal{J}' = \mathcal{J}$, we only have to verify that the almost complex structure $\mathcal{J}'$ has the other properties described in that lemma.

We begin by proving that the mapping $\widetilde{\pi} : X \times W \to X$, given by $\widetilde{\pi}(x,u) = x$, is holomorphic with respect to $\mathcal{J}'$. We already know that it is holomorphic with respect to $\mathcal{J}$, so that $d\widetilde{\pi} \circ \mathcal{J} = \mathcal{J}_o \circ d\widetilde{\pi}$. Then we have $d\widetilde{\pi} \circ \mathcal{J}' = -d\widetilde{\pi} \circ d\widetilde{C}^{-1} \circ \mathcal{J} \circ d\widetilde{C} = -d\widetilde{\sigma}_o^{-1} \circ d\widetilde{\pi} \circ \mathcal{J} \circ dC$
$= -d\widetilde{\sigma}_o^{-1} \circ \mathcal{J}_o \circ d\widetilde{\pi} \circ d\widetilde{C} = -d\widetilde{\sigma}_o \circ \mathcal{J}_o \circ d\widetilde{\sigma}_o \circ d\widetilde{\pi} = \mathcal{J}_o \circ d\widetilde{\pi}$ as required, since

$\tilde{\sigma}_o$ is anti-holomorphic.

Now we will prove that addition is holomorphic. Denote by $\mathcal{J} \times \mathcal{J}$ and $\mathcal{J}' \times \mathcal{J}'$ the product almost complex structures on $N \times N$ and $M \times M$ respectively.

Denote by $\mathcal{J} \| \mathcal{J}$ the almost complex structure on $X \times W \times W$ for which the mapping $(x,w_1,w_2) \mapsto ((x,w_1),(x,w_2))$ of $X \times W \times W$ into $N \times N$ is holomorphic, and denote by $\mathcal{J}' \| \mathcal{J}'$ the almost complex structure on $X \times W \times W$ into $M \times M$ is holomorphic. Call the resulting complex manifolds $N \| N$ and $M \| M$ respectively.

The mapping $\tilde{C} \times \tilde{C} : M \times M \to N \times N$ maps the image of $M \| M$ in $M \times M$ onto the image of $N \| N$ in $N \times N$, and induces a mapping $\tilde{C} \| \tilde{C} : M \| M \to N \| N$ which is anti-holomorphic.

Denote by $\tilde{+}$ the mapping $X \times W \times W \to X \times W$ given by $\tilde{+}(x,w_1,w_2) = (x,w_1+w_2)$.

We want to prove that $\mathcal{J}' \circ d\tilde{+} = d\tilde{+} \circ (\mathcal{J}' \| \mathcal{J}')$. We know that $\mathcal{J} \circ d\tilde{+} = d\tilde{+} \circ (\mathcal{J} \| \mathcal{J})$.

We have

$$d\tilde{+} \circ (\mathcal{J}' \| \mathcal{J}') = -d\tilde{+} \circ d(\tilde{C} \| \tilde{C})^{-1} \circ (\mathcal{J} \| \mathcal{J}) \circ d(\tilde{C} \| \tilde{C})$$
$$= -(d\tilde{C})^{-1} \circ d\tilde{+} \circ (\mathcal{J} \| \mathcal{J}) \circ d(\tilde{C} \| \tilde{C})$$

since $\tilde{C}^{-1}$ is a homomorphism on each fibre

$$= -(d\tilde{C})^{-1} \circ \mathcal{J} \circ d\tilde{+} \circ d(\tilde{C} \| \tilde{C}) = -(d\tilde{C})^{-1} \circ \mathcal{J} \circ d\tilde{C} \circ d\tilde{+}$$
$$= \mathcal{J}' \circ d\tilde{+}.$$

Next we check that the scalar multiplication mapping is holomorphic. Denote by J the almost complex structure on $\mathbb{C}$, and by $J \otimes \mathcal{J}'$ the product almost complex structure on $\mathbb{C} \times M$. Similarly, $J \otimes \mathcal{J}$ is the product almost complex structure on $\mathbb{C} \times N$.

Define $\mu : \mathbb{C} \times X \times W \to X \times W$ by $\mu(z,x,w) = (x,z,w)$. We want to prove $d\mu \circ (J \otimes \mathcal{J}') = d\mathcal{J}' \circ d\mu$. We know that $d\mu \circ (J\otimes\mathcal{J}) = \mathcal{J} \circ d\mu$. We have

$$d\mu \circ (J\otimes\mathcal{J}') = d\mu \circ (J \otimes (-d\tilde{C}^{-1} \circ \mathcal{J} \circ d\tilde{C}))$$

$$= d\mu \circ (1_{T(C)} \otimes (-d\tilde{C}^{-1})) \circ (J\otimes\mathcal{J}) \circ (1_{T(C)} \oplus d\tilde{C})$$

$$= -d\tilde{C}^{-1} \circ d\mu \circ (J\otimes\mathcal{J}) \circ (1_{T(C)} \otimes d\tilde{C})$$

since $\tilde{C}$ is linear in every fibre

$$= -d\tilde{C}^{-1} \circ \mathcal{J} \circ d\mu \circ (1_{T(C)} \otimes d\tilde{C})$$

$$= -d\tilde{C}^{-1} \circ \mathcal{J} \circ dC \circ d\mu = \mathcal{J}' \circ d\tilde{C}.$$

Denote by $\tilde{\eta} : X \to X \times W$ the mapping $x \mapsto (x,0)$. Then we have $d\tilde{\eta} \circ \mathcal{J}_0 = \mathcal{J}_0 d\tilde{\eta}$, so that

$$\mathcal{J}' \circ d\tilde{\eta} = -d\tilde{C}^{-1} \circ \mathcal{J} \circ d\tilde{C} \circ d\tilde{\eta}$$

$$= -d\tilde{C}^{-1} \circ \mathcal{J} \circ d\tilde{\eta} \circ d\tilde{\sigma}_0 = -d\tilde{C}^{-1} \circ d\tilde{\eta} \circ \mathcal{J}_0 \circ d\tilde{\sigma}_0$$

$$= -d\tilde{\eta} \circ d\tilde{\sigma}_0^{-1} \circ \mathcal{J}_0 \circ d\tilde{\sigma}_0 = d\tilde{\eta} \circ \mathcal{J}_0,$$

so $\tilde{\eta}$ holomorphic with respect to $\mathcal{J}'$.

This proves that $\tilde{\pi} : M \to X$ is a holomorphic vector-bundle.

Next we prove that $\tilde{\Gamma}$ operates holomorphically on M. Let $\tilde{\gamma} \in \tilde{\Gamma}$. Then $d\tilde{\gamma} \circ \mathcal{J} = \mathcal{J} \circ d\tilde{\gamma}$. We have

$$d\tilde{\gamma} \circ \mathcal{J}' = -d\tilde{\gamma} \circ d\tilde{C}^{-1} \circ \mathcal{J} \circ d\tilde{C}$$

$$= -d\tilde{C}^{-1} \circ d(\tilde{C} \circ \tilde{\gamma} \circ \tilde{C}^{-1}) \circ \mathcal{J} \circ d\tilde{C}$$

$$= -d\tilde{C}^{-1} \circ \mathcal{J} \circ d(\tilde{C} \circ \tilde{\gamma} \circ \tilde{C}^{-1}) \circ d\tilde{C}$$

since $\tilde{C} \circ \tilde{\gamma} \circ \tilde{C}^{-1} \in \tilde{\Gamma}$

$$= -d\tilde{C} \circ \mathcal{J} \circ d\tilde{C} \circ d\tilde{\gamma} = \mathcal{J}' \circ d\tilde{\gamma}.$$

Finally, we check that the complex structure induced on the vector-space $\{x\} \times W$ by the vector-bundle $\tilde{\pi} : M \to X$ is $\varphi_o(x)$ for every $x \in X$.

Denote by i_x the mapping $X \to X \times W$ given by $u \mapsto (x,u)$. For every $x \in X$, i_x is an $\mathbb{R}$-linear isomorphism of W onto the fibre $\tilde{\pi}^{-1}(x)$.

We have to prove the i_x maps $(W, \varphi_o(x))$ complex linearly into the fibre $\tilde{\pi}^{-1}(x)$, or, what amounts to the same thing, that di_x maps the tangent space to 0 in $(W, \varphi_o(x))$ complex linearly into the tangent space to $i_x(0)$ in M.

Since the complex structure on the tangent space to 0 in $(W, \varphi_o(x))$ is $d\varphi_o(x)|_o$, it suffices to prove that for all $x \in X$, we have $di_x \circ d\varphi_o(x) = \mathcal{J}' \circ di_x$.

Since we know that the complex structure induced on the vector-space $\{x\} \times W$ by the vector-bundle $\tilde{\pi} : N \to X$ is $\varphi_o(x)$, we have $di_x \circ d\varphi_o(x) = \mathcal{J} \circ di_x$.

$$\begin{aligned} \text{Then } \mathcal{J}' \circ di_x &= -d\tilde{C}^{-1} \circ \mathcal{J} \circ d\tilde{C} \circ di_x \\ &= -d\tilde{C}^{-1} \circ \mathcal{J} \circ di_{\tilde{\sigma}_o(x)} \circ dA \\ &= -d\tilde{C}^{-1} \circ di_{\tilde{\sigma}_o(x)} \circ d\varphi_o(\tilde{\sigma}_o(x)) \circ dA \\ &= -di_x \circ dA^{-1} \circ d\varphi_o(\tilde{\sigma}_o(x)) \circ dA \\ &= di_x \circ d(-A^{-1} \varphi_o(\tilde{\sigma}_o(x)) \circ A) \\ &= di_x \circ d\varphi_o(x) \end{aligned}$$

because A satisfies property 4).

This shows that $\mathcal{J}'$ has the properties described in Lemma 1 of §5, Chapter I, so that $\mathcal{J}' = \mathcal{J}$, $M = N$, and the mapping $\tilde{C} : X \times W \to X \times W$ is anti-holomorphic. This completes the proof.

Q.E.D.

4. Let C be a complex conjugation of V which preserves the fibres of π, the image of η, and which induces the complex conjugation σ_o on U. The purpose of this section is to describe the set of translation parts b which are compatible with C.

In order that $b : U \to V$ be a translation part, it is necessary and sufficient that b be anti-holomorphic, that $\pi \circ b = \sigma_o$, and that $C \circ b = -b \circ \sigma_o$. It is clear that the set of translation parts for C forms a group under pointwise operations. Denote this group by Trans(C). If $b \in$ Trans(C), then $b \circ \sigma_o$ is a holomorphic section of V, which we call ω_b.

The mapping $b \mapsto \omega_b$ defines an isomorphism of Trans(C) onto the group of all sections $\omega \in \Xi$ such that $C \circ \omega \circ \sigma_o = -\omega$.

In Chapter I, §8, it was shown that Ξ is isomorphic to $H^0(\Gamma, L/NL)$ for suitable N. Let $\tilde{\sigma}_o$ be a lift of σ_o to an anti-holomorphic mapping of X onto itself. By the results of §3, we can find a linear operator A on W which satisfies the conditions 1) - 4) of Theorem 1, and such that the mapping $\tilde{C} : X \times W \to X \times W$, given by $\tilde{C}(x,u) = (\tilde{\sigma}_o x, Au)$, induces C on V.

Let ω be a section of V, and let $\tilde{\omega}$ be a mapping $\tilde{\omega} : X \to X \times W$ which induces the section ω on U. By the results of §9, Chapter I, we can write $\tilde{\omega}$ in the form $\tilde{\omega}(x) = (x, \frac{1}{N}c)$, where $c \in L$ satisfies the condition $\rho(\gamma)(c) \equiv c \pmod{NL}$. Then the section $C \circ \omega \circ \sigma_o$ is induced on U by the mapping $\tilde{C} \circ \tilde{\omega} \circ \tilde{\sigma}_o^{-1}$ (since σ_o has order two), which is given by $\tilde{C} \circ \tilde{\omega} \circ \tilde{\sigma}_o^{-1}(x) = (x, \frac{1}{N}A(c))$.

Therefore, the mapping $\omega \mapsto C \circ \omega \circ \sigma_0$ of Ξ into itself has the following description in terms of $H^0(\Gamma, L/NL)$. By condition 3) of Theorem 1, $A(L) = L$, so $A(NL) = NL$. Therefore A acts on L/NL. Moreover, since $A^2 \in \rho(\Gamma)$, A^2 must act trivially on $H^0(\Gamma, L/NL)$. Thus A is an involution of $H^0(\Gamma, L/NL)$, and the group Trans(C) is isomorphic to the subgroup of $H^0(\Gamma, L/NL)$ consisting of elements $\ell \in H^0(\Gamma, L/NL)$ such that $A(\ell) = -\ell$.

In Chapter IV, §14, we will determine the group $H^1(\Gamma, L)_{\underline{\text{tors}}}$. for special choices of Γ and L, so that the problem of determining Trans(C) is reduced to a reasonable computation.

If C' is a fibre preserving complex conjugation of V which is conjugate to C by an element $\alpha \in \mathrm{Aut}(V)$, say $C' = \alpha C \alpha^{-1}$, then $\mathrm{Trans}(C') = \alpha \mathrm{Trans}(C)$, so that the determination of Trans(C) depends only on the conjugacy class of C. In Chapter III, §8, and Chapter IV, §§11-12, we will discuss how to classify C up to conjugacy.

CHAPTER III

COHOMOLOGY WITH NON-ABELIAN COEFFICIENTS

1. Let G be a group. By a G-module, we mean a group A together with a homomorphism of G into the group of automorphisms of A. For $g \in G$, we denote the action of g on an element $a \in A$ by $a \mapsto {}^g a$.

By a 1-cocycle of G valued in A, we mean a mapping $\alpha : G \to A$ such that for all $g,h \in G$ we have $\alpha(gh) = \alpha(g)\,{}^g\alpha(h)$. We denote by $Z^1(G,A)$ the set of all these 1-cocycles. This is a pointed set whose basepoint is the cocycle which maps G onto the identity element of A. Given elements α, α' of $Z^1(G,A)$, we say that α and α' are cohomologous if there exists an element $b \in A$ such that the equation $\alpha'(g) = b^{-1}\alpha(g)\,{}^g b$ holds for every $g \in G$. This defines an equivalence relation on $Z^1(G,A)$. We denote by $H^1(G,A)$ the set of all the equivalence classes. This is a pointed set whose basepoint is the cohomology class of the basepoint of $Z^1(G,A)$. We call $H^1(G,A)$ the first cohomology set of G with coefficients in A.

2. Let A and B be G-modules. By a morphism from A to B, we mean a homomorphism $\phi : A \to B$ such that for all $g \in G$ and $a \in A$ we have $\phi({}^g a) = {}^g\phi(a)$.

Let $\phi : A \to B$ be a morphism. If $\alpha : G \to A$ is a 1-cocycle, then $\phi \circ \alpha : G \to B$ is easily seen to be a 1-cocycle valued in B. If $\alpha' : G \to A$ is a 1-cocycle cohomologous to α, then $\phi \circ \alpha'$ and

$\phi\circ\alpha$ are cohomologous as well.

In this way, ϕ determines a commutative diagram

$$\begin{array}{ccc} Z^1(G,A) & \xrightarrow{Z^1(\phi)} & Z^1(G,B) \\ \downarrow & & \downarrow \\ H^1(G,A) & \xrightarrow{H^1(\phi)} & H^1(G,B) \end{array}$$

in the category of pointed sets.

In this way, Z^1 and H^1 are seen to be functors from the category of G-modules to the category of pointed sets.

3. For the rest of this chapter, we will consider only the case where G has two elements, say $G = \{1,g\}$. For any G-module A, we denote the action of g on A by $a \mapsto \bar{a}$ for all $a \in A$.

The mapping $\alpha \mapsto \alpha(g)$ defines a bijection between $Z^1(G,A)$ and the set of all elements $a \in A \ni \bar{a} = a^{-1}$. In view of this bijection, we will often refer to such elements of A as cocycles. If $a,b \in A$ are such, they determine cohomologous cocycles if and only if there exists an element $c \in A$ such that $a = c^{-1}b\bar{c}$.

We will devote the rest of this chapter to examining some interesting examples of these cohomology sets.

4. Let $\widetilde{M}$ be a differentiable manifold with a connection $\widetilde{\nabla}$, and suppose that given any two points $x,y \in \widetilde{M}$, there is a unique geodesic joining x to y. Let A be a group of connection preserving diffeomorphisms which act properly discontinuously

on $\widetilde{M}$. Denote by M the quotient manifold and by ∇ the connection induced on M by $\widetilde{\nabla}$. Let σ be a diffeomorphism of M onto itself, other than the identity, such that $\sigma^2 = 1_M$, and suppose that σ preserves the connection ∇.

Suppose further that σ has a fixed point x, fixed throughout this paragraph and the next. Let $\widetilde{x}$ be a point of $\widetilde{M}$ representing x. We can find a lifting of σ to a connection preserving mapping $\widetilde{\sigma} : \widetilde{M} \to \widetilde{M}$ such that $\widetilde{\sigma}(\widetilde{x}) = \widetilde{x}$. (<u>Proof</u>: Let $\widetilde{\sigma}'$ be any lifting to $\widetilde{M}$. It automatically preserves $\widetilde{\nabla}$. Since $\widetilde{x}$ represents a fixed point of σ, we can find $a \in A$ such that $\widetilde{\sigma}'\widetilde{x} = a\widetilde{x}$. Then we can take $\widetilde{\sigma} = a^{-1}\widetilde{\sigma}'$.)

Then $\widetilde{\sigma}^2 \in A$, anf fixes $\widetilde{x}$, so $\widetilde{\sigma}^2 = 1_{\widetilde{M}}$. Moreover, $\widetilde{\sigma}$ normalizes A, so we can view A as a G-module, $G = \{1_{\widetilde{M}}, \widetilde{\sigma}\}$. As in §3, we denote $\widetilde{\sigma}a\widetilde{\sigma}$ by $\bar{a}$ for all $a \in A$.

5. We will now define, for the situation described in §4, a bijection between $H^1(G,A)$ and the set of path components of the set of fixed points of σ.

Let x_1 be a fixed point of σ, and let $\widetilde{x}_1$ be a representative of x_1 in $\widetilde{M}$. Let $\widetilde{\sigma}_1$ be the unique lift of σ to $\widetilde{M}$ such that $\widetilde{\sigma}_1(\widetilde{x}_1) = \widetilde{x}_1$. Then we can write $\widetilde{\sigma}_1 = a\widetilde{\sigma}$ with $a \in A$. The element a is uniquely determined and satisfies
$a\bar{a} = a(\widetilde{\sigma}a\widetilde{\sigma}) = \widetilde{\sigma}_1^2 = 1_M$. Thus, a is a 1-cocycle, determed by $\widetilde{x}_1$, of G valued in A. If $\widetilde{x}_1'$ is another representative of x_1, we can write $\widetilde{x}_1' = b\widetilde{x}_1$ with $b \in A$. If $\widetilde{\sigma}_1'$ is the lift of σ to $\widetilde{M}$ fixing $\widetilde{x}_1'$, then we have $b^{-1}\widetilde{\sigma}_1'b(\widetilde{x}_1) = \widetilde{x}_1 = \widetilde{\sigma}_1(\widetilde{x}_1)$, which implies

that $b^{-1}\tilde{\sigma}_1' b = \tilde{\sigma}_1$. If a' is the 1-cocycle determined by $\tilde{\sigma}_1'$, then we have $\tilde{\sigma}_1' = a'\tilde{\sigma}$ so that $a\tilde{\sigma} = b^{-1}a'\tilde{\sigma}b$, i.e. $a = b^{-1}a'\tilde{\sigma}b\tilde{\sigma} = b^{-1}a\bar{b}$. Therefore the cohomology class we obtain from x_1 is independent of the representative $\tilde{x}_1$.

We claim that the cohomology class actually depends only on the path component of x_1 in the fixpoint set of σ. For suppose x_2 belongs to that component. Let p be a path joining x_1 to x_2 such that $\sigma \circ p = p$. Let $\tilde{x}_1$ be a representative of x_1 in $\tilde{M}$, and let $\tilde{p}$ be the unique lift of p to a path in $\tilde{M}$ beginning at $\tilde{x}_1$. Let $\tilde{x}_2 = \tilde{p}(1)$. Let $\tilde{\sigma}_1, \tilde{\sigma}_2$ be the lifts of σ which fix $\tilde{x}_1, \tilde{x}_2$ respectively, and write $\tilde{\sigma}_1 = a\tilde{\sigma}$. Then since σ fixes p, $\tilde{\sigma}_1\tilde{p}$ is a lift of p to a path beginning at $\tilde{\sigma}_1(\tilde{x}_1) = \tilde{x}_1$, and so $\tilde{\sigma}_1\tilde{p} = \tilde{p}$. In particular, $\tilde{\sigma}_1(\tilde{x}_2) = \tilde{\sigma}_1(\tilde{p}(1)) = \tilde{p}(1) = \tilde{x}_2 = \tilde{\sigma}_2(\tilde{x}_2)$. So $\tilde{\sigma}_1 = \tilde{\sigma}_2$. Therefore x_1 and x_2 determine the same cohomology class.

This defines a mapping from the set of path components of the fixpoint set into $H^1(G,A)$. It remains to prove it is a bijection.

First we prove it is injective. Let x_1, x_2 be fixed points of σ and suppose they determine the same cohomology class. Let $\tilde{x}_1, \tilde{x}_2$ be representatives of x_1, x_2 respectively, $\tilde{\sigma}_1, \tilde{\sigma}_2$ the corresponding lifts of σ, and a and b the corresponding cocycles. By hypothesis, these are cohomologous, so we can find $c \in A$ such that $a = c^{-1}b\bar{c}$.

Then $c\tilde{\sigma}_1 = ca\tilde{\sigma} = b\bar{c}\tilde{\sigma} = b\tilde{\sigma}c = \tilde{\sigma}_2 c$, so that $\tilde{\sigma}_2(c\tilde{x}_1) = c\tilde{x}_1$.

Let $\widetilde{p}$ denote the unique geodesic joining $c\widetilde{x}_1$ to $\widetilde{x}_2$. Since $\widetilde{\sigma}_2$ preserves $\widetilde{\nabla}$, $\widetilde{\sigma}\circ\widetilde{p}$ is a geodesic and it joints $\widetilde{\sigma}_2(c\widetilde{x}_1) = c\widetilde{x}_1$ to $\widetilde{\sigma}_2(\widetilde{x}_2) = \widetilde{x}_2$. Therefore $\widetilde{\sigma}\widetilde{p} = \widetilde{p}$, so that $\widetilde{p}$ lies over a path p in M which is fixed by σ and joins x_1 to x_2.

Finally, we prove the mapping is surjective. Let $a \in A$ be a 1-cocycle and let $\widetilde{\sigma}_1 = a\widetilde{\sigma}$. We will be done if we can prove $\widetilde{\sigma}_1$ has a fixed point. Let $\widetilde{m} \in \widetilde{M}$. If $\widetilde{m}$ is fixed by $\widetilde{\sigma}_1$, we are done. If not, let $\widetilde{p}$ be the unique geodesic joining $\widetilde{m}$ to $\widetilde{\sigma}_1(\widetilde{m})$. Then $\widetilde{\sigma}_1\circ\widetilde{p}$ is the unique geodesic joining $\widetilde{\sigma}_1(\widetilde{m})$ to $\widetilde{m}$, so we must have $\widetilde{\sigma}_1\circ\widetilde{p}(t) = \widetilde{p}(1-t)$ for all $t \in [0,1]$. Then $\widetilde{\sigma}_1(\widetilde{p}(\frac{1}{2})) = \widetilde{p}(\frac{1}{2})$ and we are done.

6. a) If we take $\widetilde{M} = X$, $A = \Gamma$, $\widetilde{\nabla}$ to be the Riemannian connection for the natural metric on X, and take σ to be a complex conjugation σ_0 of U, then we conclude that the components of the fixpoint set of σ_0 are classified by $H'(G,\Gamma)$.

b) If we take $\widetilde{M} = W$, $A = L$, $\widetilde{\nabla}$ to be the covariant constant connection on W, and σ to be a complex conjugation on $M = W/L$ with respect to some complex structure on M, then the components of the fixpoint set of σ are classified by $H'(G,L)$. In this case, the components form a principal homogeneous space for the group $H^1(G,L)$.

c) If we take $\widetilde{M} = X \times W$, $A = \widetilde{\Gamma}$, $\widetilde{\nabla}$ = product of the two connections given in a) and b), and σ to be a complex conjugation of V, then the components of the fixpoint set of σ are classified by $H^1(G,\tilde{\Gamma})$.

7. Let R be a ring with unity and let a be a unit of R contained in the center of R. Let S denote the set of all elements $x \in R$ such that $x^2 = a$, and suppose S is nonempty. Let x_o be a non-central element of S fixed throughout this discussion. Then x_o is a unit, and the mapping $r \mapsto x^{-1}rx$ determines an automorphism ψ of R of order 2 which we denote by $r \mapsto \bar{r}$. If we take $G = \{1,\psi\}$, then the group R^x of units of R is a G-module.

On the other hand, R^x acts by conjugation on S since a is in the center of R.

We will construct a bijection from the set $R^x\backslash S$ of R^x-conjugacy classes in S onto $H^1(G,R^x)$. Let $x \in S$. We can write $x = x_o\alpha$ with $\alpha \in R^x$. Then $a = x^2 = x_o\alpha x_o\alpha = a(x_o^{-1}\alpha x)\alpha = a\bar{\alpha}\alpha$, so that $\bar{\alpha}\alpha = 1$ and α is a 1-cocycle valued in R^x. Conversely, if α is a 1-cocycle, then $x = x_o\alpha$ is clearly in S, so every cocycle is obtained in this way. Finally, two elements x,y of S are conjugate by an element r of R^x if and only if

$r^{-1}x_o(x_o^{-1}x)r = x_o(x_o^{-1}y)$,

i.e. iff $\bar{r}(x_o^{-1}y)r^{-1} = x_o^{-1}x$

i.e. iff the cocycles $x_o^{-1}x$ and $x_o^{-1}y$ are cohomologous.

8. Let σ_o be a complex conjugation of U. Denote by $h(\sigma_o)$ the set of all complex conjugations of V which preserve the image of η, and which induce on U the complex conjugation σ_o.

Suppose that $h(\sigma_o)$ is non-empty. Let $C_o \in h(\sigma_o)$ be an element fixed throughout this discussion. Let $G = \{1_V, C_o\}$. If $\alpha \in \mathrm{Aut}(V)$, then $C_o\alpha C_o \in \mathrm{Aut}(V)$ as well. In this way, Aut(V) is a G-module.

The group Aut(V) acts on the set $h(\sigma_o)$ by conjugation. We will define a bijection of the set $\mathrm{Aut}(V)\backslash h(\sigma_o)$ of Aut(V) conjugacy classes in $h(\sigma_o)$ onto $H'(G, \mathrm{Aut}(V))$.

Given $C \in h(\sigma_o)$, we can write $C = \alpha C_o$ uniquely with $\alpha \in \mathrm{Aut}(V)$. Then $1_V = C^2 = \alpha(C_o\alpha C_o) = \alpha\bar{\alpha}$, so α is a 1-cocycle. Conversely, it is clear that every 1-cocycle is obtained in this manner.

If $C, C' \in h(\sigma_o)$, and $C = \alpha C_o$ and $C' = \beta C_o$, then α and β are cohomologous iff $\exists\ \gamma \in \mathrm{Aut}(V)$ such that $\beta = \gamma^{-1}\alpha\bar{\gamma}$. This is equivalent to saying $C' = \beta C_o = \gamma^{-1}\alpha\bar{\gamma}C_o = \gamma^{-1}\alpha C_o\gamma = \gamma^{-1}C\gamma$, i.e. that C' and C are conjugate.

CHAPTER IV

SOME SPECIAL CASES

1. Let D be a totally indefinite quaternion division algebra over a totally real number field K, and let $n = [K:\mathbb{Q}]$.

Let $\mathfrak{O}$ be an order in D, let $\mathfrak{O}^\times$ be the group of units in $\mathfrak{O}$, let $\mathfrak{O}^1$ be the subgroup of $\mathfrak{O}^\times$ consisting of units of reduced norm 1, and let $\Gamma \subseteq \mathfrak{O}^1$ be a torsion-free subgroup of finite index in $\mathfrak{O}^1$.

We identify $D_{\mathbb{R}} = D \otimes_{\mathbb{Q}} \mathbb{R}$ with $M_2(\mathbb{R}) \times \ldots \times M_2(\mathbb{R})$, where the number of factors is n. Let X equal the product $\mathfrak{H} \times \ldots \times \mathfrak{H}$ of n copies of the upper half-plane.

Denote by $D_{\mathbb{R}}^\times$ the group of units $GL(2,\mathbb{R}) \times \ldots \times GL(2,\mathbb{R})$ of $D_{\mathbb{R}}$, and by $D_{\mathbb{R}}^1$ the subgroup $SL(2,\mathbb{R}) \times \ldots \times SL(2,\mathbb{R})$ of $D_{\mathbb{R}}^\times$.

We define an action of $D_{\mathbb{R}}^\times$ on X as follows. Given $g = (g_1, \ldots, g_n) \in D_{\mathbb{R}}^\times$, where $g_\nu = \begin{pmatrix} a_\nu & b_\nu \\ c_\nu & d_\nu \end{pmatrix}$ for $\nu = 1, \ldots, n$, and $x = (x_1, \ldots, x_n)$, we put $g \cdot x = x' = (x_1', \ldots, x_n')$ where

$$x_\nu' = \begin{cases} \dfrac{a_\nu x_\nu + b_\nu}{c_\nu x_\nu + d_\nu} & \text{if } a_\nu d_\nu - b_\nu c_\nu > 0 \\[2ex] \dfrac{a_\nu \bar{x}_\nu + b_\nu}{c_\nu \bar{x}_\nu + d_\nu} & \text{if } a_\nu d_\nu - b_\nu c_\nu < 0. \end{cases}$$

Then $D_{\mathbb{R}}^\times$ acts as isometries of X for the Bergmann metric.

Denote by Σ_n the group of permutations on the set $\{1,2,\ldots,n\}$. Σ_n acts as a group of automorphisms of $D_{\mathbb{R}}$ and as a group of isometries of X. In both cases, the action is given by the rule

$$\sigma\cdot(x_1,\ldots,x_n) = (x_{\sigma(1)},\ldots,x_{\sigma(n)}) \text{ for } \sigma \in \Sigma_n.$$

Σ_n also acts as automorphisms of the product $PGL(2,\mathbb{R}) \times\ldots\times PGL(2,\mathbb{R})$ of n copies of $PGL(2,\mathbb{R})$ in the same way. Denote by $G^{\#}$ the semi-direct product $\Sigma_n \circ PGL(2,\mathbb{R})^n$.

2. If ψ is an automorphism of $D_{\mathbb{R}}$, it induces an automorphism on the center $K_{\mathbb{R}} = K \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R} \times\ldots\times \mathbb{R}$. Since $\mathbb{R}$ has only the identity automorphism, that automorphism of $K_{\mathbb{R}}$ must belong to Σ_n. Call that permutation σ. Then $\sigma^{-1}\circ\psi$ is an automorphism of $D_{\mathbb{R}}$ which is the identity on the center $K_{\mathbb{R}}$, and therefore induces an automorphism on each of the factors $M_2(\mathbb{R})$ of $D_{\mathbb{R}}$.

By the Skolem-Noether theorem, it follows that the automorphism $\sigma^{-1}\circ\psi$ is an inner-automorphism $x \mapsto axa^{-1}$ with $a \in D_{\mathbb{R}}^{\times}$. It follows that the group $\mathrm{Aut}(D_{\mathbb{R}})$ is the semidirect product of Σ_n with the group of inner-automorphisms of $D_{\mathbb{R}}$. The group of inner-automorphisms is just $D_{\mathbb{R}}^{\times}$ modulo its center, which is $GL(2,\mathbb{R}) \times\ldots\times GL(2,\mathbb{R})/\mathbb{R}^{\times} \times\ldots\times \mathbb{R}^{\times} = PGL(2,\mathbb{R}) \times\ldots\times PGL(2,\mathbb{R})$.

Thus $\mathrm{Aut}(D_{\mathbb{R}})$ is canonically isomorphic to $G^{\#}$.

Let ψ be an automorphism of $D_{\mathbb{R}}$ and let ω be the corresponding isometry of X. ψ induces an automorphism on $D_{\mathbb{R}}^{\times}$, and therefore on the group $D_{\mathbb{R}}^{\times}$ modulo its center, which is $PGL(2,\mathbb{R}) \times\ldots\times PGL(2,\mathbb{R})$. On the other hand, the inner automorphism of $G^{\#}$ determined by ω leaves $PGL(2,\mathbb{R}) \times\ldots\times PGL(2,\mathbb{R})$ invariant and induces an automorphism on that group which coincides with the automorphism determined by ψ. This is easily seen to be true. If $\psi \in \Sigma_n$, it is obvious. If $\psi \in PGL(2,\mathbb{R}) \times \ldots \times PGL(2,\mathbb{R})$, ψ acts by conjugation on $D_{\mathbb{R}}$, and therefore also acts by conjugation on $G^{\#}$, so it obviously coincides with the conjugation by ω. We will denote by Γ' the image of Γ in $Isom(X)$. Our hypotheses imply that Γ is mapped isomorphically onto Γ'.

3. The elements of $D_{\mathbb{R}}^{\times}$ act as isometries of X. The kernel of this action is again the center of $D_{\mathbb{R}}^{\times}$, so that the action factors through $PGL(2,\mathbb{R}) \times \ldots \times PGL(2,\mathbb{R})$. The subgroup $PSL(2,\mathbb{R}) \times \ldots \times PSL(2,\mathbb{R})$ acts transitively on X. If σ is an isometry of X, it can be written uniquely as the composition of an element $g \in PGL(2,\mathbb{R}) \times \ldots \times PGL(2,\mathbb{R})$ and a permutation $\omega \in \Sigma_n$. In this way, the group $Isom(X)$ of isometries of X is canonically isomorphic to $G^{\#}$.

It follows that $Aut(D_{\mathbb{R}})$ and $Isom(X)$ are canonically isomorphic.

4. Denote by $x \to \bar{x}$ the canonical involution of D given by $\bar{x} = tr(x) - x$, where $tr : D \to K$ is the reduced trace. Let $S \in \mathfrak{O}$ be a non-zero element such that $\bar{S} = -S$. Define a skew-symmetric $\mathbb{Q}$-bilinear form $B_o : D \times D \to \mathbb{Q}$ by the rule $B_o(x,y) = tr_{D/\mathbb{Q}}(S\bar{x}y)$. B_o is easily seen to be integer-valued on $\mathfrak{O} \times \mathfrak{O}$ and non-degenerate.

Put $W = D_{\mathbb{R}}$, $B = \mathbb{R}$-bilinear extension of B_o to $D_{\mathbb{R}}$, and take $L = $ 2-sided $\mathfrak{O}$-ideal in $\mathfrak{O}$.

For the action of $D_{\mathbb{R}}^{x}$ on X defined in §1 of this chapter, Γ acts holomorphically and properly discontinuously on X. Denote by ρ the left-regular representation of $D_{\mathbb{R}}$ on itself. We will also denote by ρ the restrictions of the left-regular representation to $D_{\mathbb{R}}^{x}$ and to Γ.

For $\gamma \in \Gamma \subseteq \mathfrak{O}^{1}$, we have $\bar{\gamma}\gamma = 1$. Therefore, for any $x,y \in D$ we have $B_{o}(\gamma x, \gamma y) = tr_{D/\mathbb{Q}}(S(\overline{\gamma x})\gamma y) = tr_{D/\mathbb{Q}}(S\bar{x}\bar{\gamma}\gamma xy)$ $= tr_{D/\mathbb{Q}}(S\bar{x}y) = B_{o}(x,y)$, so $\rho(\gamma)$ preserves B_{o} and therefore B as well.

Moreover, $\rho(\gamma)L = \gamma\mathfrak{A} = \mathfrak{A} = L$ and since $\mathfrak{A} \subseteq \mathfrak{O}$, B is integer-valued on $L \times L$. $\therefore \rho$ maps Γ into Aut(W,B,L).

We define $\varphi_{o} : X \to \mathcal{H}_{B}$ as follows. Denote by J the element $(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \in D_{\mathbb{R}}$, and denote by z_{o} the point $(i,i,\ldots,i)$ of X. Fox $z \in X$, we can find $g \in D_{\mathbb{R}}^{1}$ such that $g \circ z_{o} = z$. We then define $\varphi_{o}(z) = \rho(gJg^{-1}) = \rho(J\,{}^{t}g^{-1}g^{-1})$. It is easy to see that this is well-defined.

For $\gamma \in \Gamma$, we have $\varphi_{o}(\gamma z) = \rho(\gamma gJg^{-1}\gamma^{-1}) = \rho(\gamma)\varphi_{o}(z)\rho(\gamma)^{-1}$.

5. We will prove that φ_{o} is holomorphic. In view of the definition in Chapter I, §3 of the complex structure on $\mathcal{H}_{B}$, it suffices to prove that the mapping $\varphi = \varkappa \circ \varphi_{o} : X \to G_{r}(W_{\mathbb{C}})$, which assigns to each $x \in X$ the i-eigenspace of $\varphi_{o}(x)$ in $W_{\mathbb{C}}$, is holomorphic. For $i = 1,\ldots,n$, let $\chi_{i} : M_{2}(\mathbb{R}) \to D_{\mathbb{R}}$ be the inclusion into the i-th factor of $D_{\mathbb{R}}$. For every point $z \in \mathcal{H}$, let $\mu(z)$ be the subspace $\mathbb{C}\begin{bmatrix} z \\ 1 \end{bmatrix}$ of $\mathbb{C}^{2}$, where $\mathbb{C}^{2}$ is viewed as a space of column vectors. Then, using the identification of

$\mathbb{P}^1(\mathbb{C})$ with the extended complex plane, the mapping μ can be identified with the inclusion $\mathcal{H} \to \mathbb{P}^1(\mathbb{C})$, and is therefore holomorphic. We can identify $\mathbb{C}^2 \oplus \mathbb{C}^2$ with $M_2(\mathbb{R}) \otimes \mathbb{C}$, and we denote by $G_{2,4}$ the complex manifold of complex 2-planes in $M_2(\mathbb{R}) \otimes \mathbb{C}$. We have a natural mapping $i : \mathbb{P}^1(\mathbb{C}) \to G_{2,4}$ given by $P \mapsto P \oplus P$, which is clearly holomorphic.

If we identify $D_{\mathbb{R}} \otimes \mathbb{C}$ with $(M_2(\mathbb{R})\otimes\mathbb{C}) \oplus \ldots \oplus (M_2(\mathbb{R})\otimes\mathbb{C})$, we have a natural mapping $j : G_{2,4} \times \ldots \times G_{2,4} \to G_r(W_{\mathbb{C}})$ given by $(P_1, \ldots, P_n) \overset{j}{\to} P_1 \oplus \ldots \oplus P_n$ which is also holomorphic. Then for $x \in X$, $x = (x_1, \ldots, x_n)$, we have that $\varphi(x) = j(i \circ \mu(x_1), \ldots, i \circ \mu(x_n))$, so φ is holomorphic.

It follows that the data $(X, \Gamma, W, B, L, \varphi_o, \rho)$ satisfy the conditions of Chapter I, §§1-4, and, as was shown in those sections, determine a Kuga fibre variety V. For the remainder of this chapter, we will deal only with such V.

6. In this section, we prove that Γ contains a basis for D over $\mathbb{Q}$. Let $D_o = \mathbb{Q}[\Gamma]$ denote the $\mathbb{Q}$-linear span of Γ. Then D_o is a division algebra over $\mathbb{Q}$. We would like to prove that $D = D_o$. Let ρ_o denote the left-regular representation of Γ on D_o and on $D_{o\mathbb{R}} = D_o \otimes_{\mathbb{Q}} \mathbb{R}$. We can view D as a left-vector-space of dimension m over D_o, so that $D_{\mathbb{R}} \cong D_{o\mathbb{R}} \times \ldots \times D_{o\mathbb{R}}$ (m copies). Therefore $D_{\mathbb{R}}^{\times} \cong D_{o\mathbb{R}}^{\times} \times \ldots \times D_{o\mathbb{R}}^{\times}$, and if we denote by Z the center of $D_{o\mathbb{R}}^{\times}$, we have that $PSL(2,\mathbb{R}) \times \ldots \times PSL(2,\mathbb{R})$ (n copies) is isomorphic to the connected component of $(D_{o\mathbb{R}}^{\times}/Z) \times \ldots \times (D_{o\mathbb{R}}^{\times}/Z)$. Using a basis for D over D_o, we can view

ρ as the direct sum of m copies of ρ_o. Now, Γ acts on $PSL(2,\mathbb{R}) \times \ldots \times PSL(2,\mathbb{R})$ by conjugation, and it is not difficult to see that this corresponds, under the above isomorphism, to the action of Γ on $(D^X_{o\mathbb{R}}/Z) \times \ldots \times (D^X_{o\mathbb{R}}/Z)$ given by

$$\gamma \cdot (\delta_1 Z, \delta_2 Z, \ldots, \delta_m Z) = (\rho_o(\gamma)\delta_1 Z, \ldots, \rho_o(\gamma)\delta_m Z).$$

Let π_i, $i = 1,\ldots,m$, denote the projection of $(D^X_{o\mathbb{R}}/Z) \times \ldots \times (D^X_{o\mathbb{R}}/Z)$ onto its i-th factor. Let $H = PSO(2) \times \ldots \times PSO(2) \subseteq PSL(2,\mathbb{R}) \times \ldots \times PSL(2,\mathbb{R})$. H is a maximal compact subgroup, and its image in $(D^X_{o\mathbb{R}}/Z) \times \ldots \times (D^X_{o\mathbb{R}})$ which we call H', must be a maximal compact subgroup. Therefore, $H' = \pi_1(H') \times \ldots \times \pi_n(H')$.

Let $X_i = (D^X_{o\mathbb{R}}/Z)/\pi_i(H')$. Then we have $X \cong X_1 \times \ldots \times X_m$, and this isomorphism is compatible with the action of Γ.

Now suppose that m is greater than 1.

Then $\Gamma\backslash X$ can be viewed as a fibre bundle over $\Gamma\backslash X_1$ whose fibre is $X_2 \times \ldots \times X_m$. Since $\Gamma\backslash X$ is compact, $\Gamma\backslash X_1$ and $X_2 \times \ldots \times X_m$ must also be compact. However, since each $\pi_i(H')$ is compact, $X_2 \times \ldots \times X_m$ is compact if and only if $D^X_{o\mathbb{R}}/Z$ is. But that would imply that $PSL(2,\mathbb{R}) \times \ldots \times PSL(2,\mathbb{R}) \cong (D^X_{o\mathbb{R}}/Z) \times \ldots \ldots \times (D^X_{o\mathbb{R}}/Z)$ is compact, which is a contradiction. Therefore $m = 1$, $D_o = D$, and we are done.

7. Let σ_o be an anti-holomorphic involution of U. Let $\tilde{\sigma}_o$ be a lifting of σ_o to an anti-holomorphic mapping of X into itself. It is not hard to prove that $\tilde{\sigma}_o$ must be an isometry of X for the Bergmann metric. Moreover, we can choose $\tilde{\sigma}_o$ to

be an involution of X if and only if σ_o has a fixed point. Let ψ be the automorphism of $D_{\mathbb{R}}$ corresponding to $\tilde{\sigma}_o$ under the canonical isomorphism of §3. Then ψ must leave Γ invariant.

Lemma: $\psi \circ \varphi_o(x) = -\varphi_o(\tilde{\sigma}_o(x)) \circ \psi$ for every $x \in X$.

Proof. By §3, we can write $\psi = @ \circ T_h$, where $h \in D_{\mathbb{R}}^{\times}$, and T_h denotes the inner-automorphism determined by h. Since $\tilde{\sigma}_o$ is anti-holomorphic, we can choose $h = (h_1, \ldots, h_n) \in D_{\mathbb{R}}^{\times}$ so that $\det(h_i) = -1$ for $i = 1, \ldots, n$.

Let $\alpha = ((\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}), \ldots, (\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix})) \in D_{\mathbb{R}}$. Let $x \in X$. We can write $x = g \cdot z_o$, where $g = (g_1, \ldots, g_n) \in D_{\mathbb{R}}^1$, and $z_o = (i, i, \ldots, i)$ as in §4.

Then it is easy to check that $h \cdot z = (hg\alpha) \circ z_o$, since $\alpha \circ z_o = z_o$, and the element $hg\alpha$ belongs to $D_{\mathbb{R}}^1$. Therefore $\varphi_o(h \circ z) = \rho(hg\alpha J \alpha g^{-1} h^{-1}) = -\rho(hgJg^{-1}h^{-1})$. For $y \in D_{\mathbb{R}}$, we have then

$$\begin{aligned}\varphi_o(h \cdot z)(y) &= -hgJg^{-1}h^{-1}y \\ &= -h((gJg^{-1})(h^{-1}yh))h^{-1} = -T_h \circ \varphi_o(z) \circ T_h^{-1}(y),\end{aligned}$$

so that $\varphi_o(h \cdot z) T_h = -T_h \circ \varphi_o(z)$. Finally

$$\begin{aligned}\varphi_o(\tilde{\sigma}_o \circ z) &= \varphi_o(@ \circ (T_h g\alpha) \circ z_o) \\ &= \rho(@(hg\alpha)J@(hg\alpha)^{-1} \\ &= @ \circ \rho((hg\alpha)J(hg\alpha)^{-1}) \circ @^{-1} \\ &= @ \circ \varphi_o(h \cdot z) \circ @^{-1} = -@ \circ T_h \circ \varphi_o(z) \circ T_h^{-1} \circ @^{-1}\end{aligned}$$

so $\varphi_o(\tilde{\sigma}_o \circ z) \circ \psi = -\psi \circ \varphi_o(z)$, as desired.

8. Let σ_0 be as in §7. Let V be the Kuga fibre variety defined in §§4-5 of this chapter, where we take $\mathfrak{A} = \mathfrak{G}$. Our object in this section is to prove the following result.

<u>Theorem 2</u>: Suppose that σ_0 has a fixpoint. Let $\tilde{\sigma}_0$ be a complex conjugation of X covering σ_0, and let ψ be the corresponding automorphism of $D_{\mathbb{R}}$. Then σ_0 is induced by a complex conjugation of V if and only if $\psi(\mathfrak{G}) = \mathfrak{G}$.

<u>Proof</u>. ($\Rightarrow$) Suppose σ_0 is so induced. Then by Theorem 1 of Chapter II, §3, and the results of §2 of this chapter, we can find an operator A on W such that

1) $A^2 = 1_W$

2) A normalizes $\rho(\Gamma)$ and induces the automorphism ψ on Γ.

3) $A(\mathfrak{G}) = \mathfrak{G}$.

4) $A \circ \varphi_0(x) = -\varphi_0(\tilde{\sigma}_0 x) \circ A$ for all $x \in X$.

Let $B = A \circ \psi$. Since $A^2 = \psi^2 = 1_W$, for any $\gamma \in \Gamma$ and any $y \in W$, we have

$$\begin{aligned} B \circ \rho(\gamma) \circ B^{-1}(y) &= A \circ \psi \circ \rho(\gamma) \circ \psi \circ A(y) \\ &= A(\psi(\gamma) A(y)) = A \circ \rho(\psi(\gamma)) \circ A(y) \\ &= \rho(\psi^2 \gamma)(y) = \rho(\gamma)(y). \end{aligned}$$

Thus, B commutes with the elements of $\rho(\Gamma)$. By the results of §6 of this chapter, Γ contains a basis for $D_{\mathbb{R}}$ over $\mathbb{R}$, so B commutes with the elements of $\rho(D_{\mathbb{R}})$. Therefore B has the form $x \mapsto xb$ with $b \in D_{\mathbb{R}}$. Therefore, A is given by $A(x) = \psi(x)b$ for all $x \in D_R$. By 3), we have $\mathfrak{O} = A(\mathfrak{O}) = \psi(\mathfrak{O})b = \psi(\mathfrak{O})\psi(\mathfrak{O})b = \psi(\mathfrak{O})\mathfrak{O}$, since $\psi(\mathfrak{O})$ is an order, and therefore $\psi(\mathfrak{O}) \subseteq \mathfrak{O}$. Applying ψ, we get $\mathfrak{O} = \psi^2(\mathfrak{O}) \subseteq \psi(\mathfrak{O})$, so $\psi(\mathfrak{O}) = \mathfrak{O}$.

($\Leftarrow$) Suppose $\psi(\mathfrak{O}) = \mathfrak{O}$. Let $A = \psi$. Then we claim 1) - 4) are satisfied. For 1) and 2), it is obvious, 3) holds by hypothesis, and 4) holds by the lemma of §7.

9. In the situation described in §§4-5, we now take $\mathfrak{O}$ to be a maximal order. Let $T \in D^{\times}$, and suppose that $T\Gamma T^{-1} = \Gamma$. Then the action of T on X induces a mapping on $U = \Gamma\backslash X$. Moreover, if we require that $T^2 \in K^{\times}T$ and that the reduced norm $\nu(T)$ of T be a totally negative element of K, then that induced mapping will be a complex conjugation of U, which we call σ_o.

<u>Theorem 3</u>: In order that σ_o be induced by a complex conjugation of V, it is necessary that $T\mathfrak{O} = \mathfrak{O}T$ and that $K(\sqrt{-\nu(T)})$ be embeddable in D over K.

If the ideal class group of K is generated by the primes of K at which D is ramified, then these conditions are also sufficient.

<u>Proof</u>. Let c be an element of $K \otimes_{\mathbb{Q}} \mathbb{R}$ whose square is $-\nu(T)$, and let $\tilde{\sigma}_o = c^{-1}T$. If we write $\tilde{\sigma}_o = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n)$, then $\det(\tilde{\sigma}_i) = -1$, for $i = 1, \ldots, n$, and $\tilde{\sigma}_o$ determines an anti-holomorphic mapping of X into itself which covers σ_o. Suppose σ_o is induced by a complex conjugation of V_o. Then by the

results of Chapter II, §3, we can find an operator A on W such that

1) $A^2 = \rho(\tilde{\sigma}_o)^2$.

2) A normalizes $\rho(\Gamma)$ and induces on Γ the same automorphism that $\tilde{\sigma}_o$ does.

3) $A(\mathfrak{A}) = \mathfrak{A}$.

4) $A\ \varphi_o(x) = -\varphi_o(\tilde{\sigma}_o \cdot z)A$.

Let $A_o = \rho(\sigma_o)$ and let $B = A_o^{-1}A$. Since A and A_o induce the same automorphism on $\rho(\Gamma)$, it follows that B commutes with the elements of $\rho(\Gamma)$. Therefore, B commutes with the elements of $\rho(D_R)$ and is of the form $x \mapsto xb$, with $b \in D_R$. Since B commutes with the elements of $\rho(D_R)$, in particular, B commutes with A_o, so A commutes with A_o. Therefore, since $A^2 = A_o^2$, by 1) we have $B^2 = 1_W$, so $b^2 = 1$.

By 3), we have $\mathfrak{A} = A(\mathfrak{A}) = A_oB(\mathfrak{A}) = \tilde{\sigma}_o\mathfrak{A}b$. Let $\beta = c^{-1}b$. Then $\mathfrak{A} = \tilde{\sigma}_o\mathfrak{A}b = T\mathfrak{A}\beta$, so that $\beta \in D^x$, $\beta^{-2} = c^2b^{-2} = -\nu(T)$, so $K(\sqrt{-\nu(T)})$ embeds in D.

Since $\mathfrak{O}$ is maximal, $\mathfrak{O}$ is the left order of $\mathfrak{A}$, and $T\mathfrak{O}T^{-1}$ is maximal. Then $\mathfrak{A} = T\mathfrak{A}\beta = T\mathfrak{O}\mathfrak{A}\beta = T\mathfrak{O}T^{-1}T\mathfrak{A}\beta = T\mathfrak{O}T^{-1}\mathfrak{A}$ so that $T\mathfrak{O}T^{-1} \subseteq \mathfrak{O}$, which implies $T\mathfrak{O} = \mathfrak{O}T$.

Conversely, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the distinct primes of K where D is ramified, and suppose that they generate the ideal class group of K. We will prove that the conditions $T\mathfrak{O} = \mathfrak{O}T$ and $K(\sqrt{-\nu(T)})$ embeddable in D imply that σ_o is induced on U by a complex conjugation σ of V.

We can, by multiplying T by an element of K^x if necessary, assume without loss of generality that $T \in \mathfrak{O}$ and there is an element $\delta \in \mathfrak{O}$ such that $\delta^2 = -\nu(T) = c^2$. Put $b = c^{-1}\delta$, $\beta = c^{-2}\delta$, and let $B \in GL(D_R)$ be defined by $B(x) = xb$ for all $x \in D_R$. Then B commutes with $\rho(D_R)$. Let $A_o = \rho(\tilde{\sigma}_o)$ and let $A = A_oB$, so that for $x \in D_R$, we have $A(x) = \tilde{\sigma}_o xb$. We clearly have $B^2 = 1_{D_R}$.

We will prove that A satisfies the condition 1) - 4) given in the direct part of this proof.

Proof of 1): Since $A_o \in \rho(D_R)$, A_o and B commute. Therefore, $A^2 = (A_oB)^2 = A_o^2B^2 = A_o^2 = \rho(\tilde{\sigma}_o)^2$.

Proof of 2): Since B commutes with $\rho(\Gamma)$, this is obvious.

Proof of 3): For each prime ideal $\mathfrak{P}$ of K, there is a unique maximal ideal $\mathfrak{m}_{\mathfrak{p}}$ of $\mathfrak{O}$ containing $\mathfrak{p}\mathfrak{O}$. We have

$$\mathfrak{p}\mathfrak{O} = \begin{cases} \mathfrak{m}_{\mathfrak{p}} & \text{if D is unramified at } \mathfrak{p}. \\ \mathfrak{m}_{\mathfrak{p}}^2 & \text{if D is ramified at } \mathfrak{p}. \end{cases}$$

It is well-known that the finitely generated two-sided submodules of D (other than (0)) form an abelian group under ideal multiplication, which is freely generated by the $\mathfrak{m}_{\mathfrak{p}}$'s.

By hypothesis, $T\mathfrak{O} = \mathfrak{O}T$ is a two-sided ideal in $\mathfrak{O}$. Therefore, we can write

$$T\mathfrak{O} = \prod_{\mathfrak{p}} \mathfrak{m}_{\mathfrak{p}}^{n_{\mathfrak{p}}}$$

uniquely with $n_{\mathfrak{p}} \geq 0$ for all $\mathfrak{p}$.

If $\mathfrak{p}$ is a prime of K, then $\bar{\mathfrak{p}} = \mathfrak{p}$, so that $\overline{\mathfrak{m}_{\mathfrak{p}}}$ is a maximal ideal containing $\mathfrak{p}\mathfrak{O}$, and therefore $\overline{\mathfrak{m}_{\mathfrak{p}}} = \mathfrak{m}_{\mathfrak{p}}$. Here, of course,

$\bar{x}$ denotes, for all x, the image of x under the canonical involution of D over K. It follows that $\bar{\mathfrak{B}} = \mathfrak{B}$ for any two-sided $\mathfrak{O}$-module P. Let $P = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. Taking reduced norms, we now see that

$$\nu(T)\mathfrak{O}_K = \prod_{\mathfrak{p} \in P} \mathfrak{p}^{n_\mathfrak{p}} \cdot \prod_{\mathfrak{p} \notin P} \mathfrak{p}^{2n_\mathfrak{p}}.$$

Since the primes in P generate the ideal class group of K, we can find an element $\xi \in K^x$, and integers $m_1, \ldots, m_r$ such that

$$\prod_{\mathfrak{p} \notin P} \mathfrak{p}^{n_\mathfrak{p}} = \eta^{-1} \prod_{i=1}^{r} \mathfrak{p}_i^{m_i}.$$

Then we have

$$\nu(\eta T)\mathfrak{O}_K = \prod_{i=1}^{r} \mathfrak{p}_i^{n_{\mathfrak{p}_i} + 2m_i}.$$

It is obvious that T and ηT induce the same automorphism on Γ, that $\eta T\mathfrak{O} = \mathfrak{O}\eta T$, and that $K(\sqrt{-\nu(T)}) = K(\sqrt{-\nu(\eta T)})$. Therefore, replacing T by ηT if necessary, we may assume without loss of generality that for $\mathfrak{p} \notin P$, $n_\mathfrak{p} = 0$.

Since $\delta^2 = -\nu(T)$, it follows that for $\mathfrak{p} \notin P$, δ is a unit in $\mathfrak{O}_\mathfrak{p}$, so that $\delta\mathfrak{O}_\mathfrak{p} = \mathfrak{O}_\mathfrak{p}\delta = \mathfrak{O}_\mathfrak{p}$. For $\mathfrak{p} \in P$, $D_\mathfrak{p}$ is a division algebra, so $\delta\mathfrak{O}_\mathfrak{p} = \mathfrak{O}_\mathfrak{p}\delta$ automatically.

Therefore $\delta\mathfrak{O} = \mathfrak{O}\delta$. Since $\delta^2 = -\nu(T)$, we have

$$\begin{aligned}(\delta\mathfrak{O})^2 &= \delta\mathfrak{O}\delta\mathfrak{O} = \delta^2\mathfrak{O} = \nu(T)\mathfrak{O} \\ &= T\bar{T}\mathfrak{O} = T\mathfrak{O}\bar{T}\mathfrak{O} = (T\mathfrak{O})(\overline{T\mathfrak{O}}) \\ &= (T\mathfrak{O})^2,\end{aligned}$$

so that $\delta\mathfrak{O} = T\mathfrak{O}$ and so $T\delta^{-1} \in \mathfrak{O}^x$.

Therefore, $T\beta = T\delta c^{-2} = T\delta(-\nu(T))^{-1} = T\delta\delta^{-2} = T\delta^{-1} \in \mathfrak{O}^{\times}$, so that

$$A(\mathfrak{A}) = T\mathfrak{A}\beta = T\mathfrak{A}T^{-1}T\beta = T\mathfrak{O}\mathfrak{A}\mathfrak{O}T^{-1}T\beta$$
$$= (T\mathfrak{O})\mathfrak{A}(T^{-1}\mathfrak{O})T\beta = \mathfrak{A}T\beta = \mathfrak{A}.$$

<u>Proof of 4)</u>: It follows immediately from the result of §7 that $A_o \circ \varphi_o(x) = \varphi_o(\tilde{\sigma}_o x) \circ A_o$. Since B commutes with the elements of $\rho(D_R)$, and since $\varphi_o(x) \in \rho(D_R)$ for all x, we have

$$A \circ \varphi_o(x) = B \circ A_o \circ \varphi_o(x) = B \circ \varphi_o(\tilde{\sigma}_o x) \circ A_o$$
$$= -\varphi_o(\tilde{\sigma}_o x) \circ B \circ A_o = -\varphi_o(\tilde{\sigma}_o x) A,$$

as required. Q.E.D.

10. In this section, we will describe the endomorphism ring of V. Let $\alpha \in \mathrm{End}(V)$, and let $\tilde{\alpha}$ be a holomorphic mapping of $X \times W$ onto itself which induces α on U. We can take $\tilde{\alpha}$ to be of the form $(x,u) \mapsto (x,\mathfrak{a}(u))$, where $\mathfrak{a}$ is a linear operator on W. It follows that $\mathfrak{a}$ must commute with the elements of $\rho(\Gamma)$ and therefore $\mathfrak{a}$ is of the form $u \mapsto u \circ a$, with $a \in \mathbb{D}_R$. Since $\mathfrak{a}$ must preserve $\mathfrak{A}$, $a \in D$ and belongs to the right order $\mathfrak{O}'$ of $\mathfrak{A}$. Conversely, if $a \in \mathfrak{O}'$, then one can show, using a proof similar to theproof of Theorem 2, that $(x,u) \to (x,ua)$ induces an endomorphism of V. We omit the details. For a different approach, see Shimura [2].

11. Let σ_o, $\tilde{\sigma}_o$, ψ, V, and in particular $\mathfrak{A}$, be as in §8 of this chapter. Then $\mathfrak{A} = \mathfrak{O}$, and so the right order of $\mathfrak{A}$ is $\mathfrak{O}$. In this way, we have $\mathrm{End}(V) \cong \mathfrak{O}$. Suppose that $\psi(\mathfrak{O}) = \mathfrak{O}$.

Let $\widetilde{C}$ be the complex conjugation of $X \times W$ given by $\widetilde{C}(x,u) = (\widetilde{\sigma}_o x, \psi(u))$, and let C be the complex conjugation induced on V by $\widetilde{C}$. Let $G = \{1, \psi\}$. We can view $\mathfrak{G}$ as a G-module in two different ways, as follows. For $\alpha \in \mathrm{End}(V)$, $C \circ \alpha \circ C$ also belongs to End(V). Thus, letting G act on End(V), by conjugation, via $\{1, C\}$, End(V) becomes a G-module.

Using theidentification of End(V) with $\mathfrak{G}$, $\mathfrak{G}$ becomes a G-module, which we call M_1.

On the other hand, by hypothesis, $\psi(\mathfrak{G}) = \mathfrak{G}$, so that by letting G act on $\mathfrak{G}$ via $\{1, \psi\}$, $\mathfrak{G}$ becomes a G-module which we call M_2.

We will prove that $M_1 = M_2$, in other words, that these two G-actions are the same.

Let $\alpha \in \mathfrak{G}$. Then α acts on $X \times W$ by the rule $(x,u)\alpha = (x, u\alpha)$. Call that mapping $\widetilde{\alpha}$. The endomorphism $C \circ \alpha \circ C$ is induced on V by the mapping $\widetilde{C} \circ \widetilde{\alpha} \circ \widetilde{C} : X \times W \to X \times W$ given by $\widetilde{C} \circ \widetilde{\alpha} \circ \widetilde{C}(x,u) = \widetilde{C} \circ \widetilde{\alpha}(\widetilde{\sigma}_o x, \psi(u)) = \widetilde{C}(\widetilde{\sigma}_o x, \psi(u)\alpha) = (x, u\ \psi(\alpha))$. Thus $C \circ \alpha \circ C$ is the endomorphism corresponding to $\psi(\alpha)$, which proves our assertion.

12. By the results of Chapter III, §8, we can identify the set of Aut(V)-conjugacy classes in $\omega(\sigma_o)$ with the cohomology set $H^1(G, \mathrm{Aut}(V))$. By the results of §11, we can identify this set with $H^1(\{1, \psi\}, \mathfrak{G}^x)$. Now suppose that ψ is inner. Let $T \in \mathfrak{G}$ such that $\psi(x) = TxT^{-1}$ for all $x \in D$. Since $\psi^2 = 1$, $T^2 \in K$, and since $\psi \neq 1$, $T \notin K$. Therefore $\bar{T} = -T$, so $T^2 = -\nu(T)$.

By the results of Chapter III, §7, we can identify $H^1(\{1,\psi\},\mathfrak{O}^x)$ with the set of $\mathfrak{O}^x$-conjugacy classes of elements $\delta \in \mathfrak{O}$ such that $\delta^2 = -\nu(T)$. This cohomology set was computed in [4],[

13. We continue with the assumptions and notations of §11. By the results of that section, the action of $\{1,\psi\}$ on $\mathfrak{O}$ via $\{1,c\}$ coincide with the usual action of $\{1,\psi\}$ on $\mathfrak{O}$.

On the other hand, the automorphism induced by $\tilde{\sigma}_o$ on Γ coincides with ψ. Therefore, we can say unambiguously that the composition $\Gamma \hookrightarrow \mathfrak{O}^x \tilde{\to} \mathrm{Aut}(V)$ is a morphism of G-modules.

By Chapter III, §2, there is an induced mapping $H^1(G,\Gamma) \to H^1(G,\mathrm{Aut}(V))$. By the results of Chapter III, §§5-6, we can identify $H^1(G,\Gamma)$ with the set of path components of the fixpoint set of σ_o.

It follows now that every component of the fixed point set of σ_o determines an Aut(V)-conjugacy class of liftings of σ_o to V.

14. If N is a positive integer, we denote by Γ_N the group of all $\gamma \in \mathfrak{O}^1$ such that $\gamma - 1 \in N\mathfrak{O}$. For sufficiently large N, Γ_N is torsion-free, and for all N, Γ_N has finite index in $\mathfrak{O}^1$.

In this section, we will compute $H^1(\Gamma_N,\mathfrak{O})^{\underline{\text{tors.}}}$. Our result is that $H^1(\Gamma_N,\mathfrak{O})^{\text{tors.}}$ is isomorphic to $\mathfrak{O}/N\mathfrak{O}$.

For every prime $\mathfrak{p}$ of K, denote by $K_\mathfrak{p}$ the completion of K at $\mathfrak{p}$ and by $\mathfrak{O}_\mathfrak{p}$ the closure of $\mathfrak{O}$ in $D_\mathfrak{p} = D \otimes_K K_\mathfrak{p}$. Let $\mathfrak{O}_\mathfrak{p}^x$ denote the group of units of $\mathfrak{O}_\mathfrak{p}$, and write $\Gamma_N^\mathfrak{p}$ for the subgroup of $\mathfrak{O}_{\mathfrak{p}}^{x}$ consisting of all $\gamma \in \mathfrak{O}_{\mathfrak{p}}^{x}$ whose reduced norm is 1, and such that $\gamma - 1 \in N\mathfrak{O}_{\mathfrak{p}}$. By the approximation theorem, Γ_N is dense in $\Gamma_N^{\mathfrak{p}}$ for every $\mathfrak{p}$.

Let $n_{\mathfrak{p}} = \text{ord}_{\mathfrak{p}} N$ for every $\mathfrak{p}$. Then Γ_N and $\Gamma_N^{\mathfrak{p}}$ have the same image in $\mathfrak{O}_{\mathfrak{p}}/\mathfrak{p}^{n+\ell}\mathfrak{O}_{\mathfrak{p}}$ for every $\ell > 0$, namely they both map onto

$$\{x \in \mathfrak{O}_{\mathfrak{p}}/\mathfrak{p}^{n_{\mathfrak{p}}+\ell}\mathfrak{O}_{\mathfrak{p}} \mid \nu(x) = 1 \text{ and } x-1 \in \mathfrak{p}^{n_{\mathfrak{p}}}\mathfrak{O}_{\mathfrak{p}}/\mathfrak{p}^{n_{\mathfrak{p}}+\ell}\mathfrak{O}_{\mathfrak{p}}\}.$$

Here, of course, ν denotes the mapping $\mathfrak{O}_{\mathfrak{p}}/\mathfrak{p}^{n_{\mathfrak{p}}+\ell}\mathfrak{O}_{\mathfrak{p}} \to \mathfrak{O}_{K_{\mathfrak{p}}}/\mathfrak{p}^{n_{\mathfrak{p}}+\ell}\mathfrak{O}_{K_{\mathfrak{p}}}$ induced by the reduced norm mapping $D \to K$.

It is clear that $\mathfrak{p}^{\ell}\mathfrak{O}_{\mathfrak{p}}/\mathfrak{p}^{n_{\mathfrak{p}}+\ell}\mathfrak{O}_{\mathfrak{p}} \subseteq H^0(\Gamma_N^{\mathfrak{p}}, \mathfrak{O}_{\mathfrak{p}}/\mathfrak{p}^{n_{\mathfrak{p}}+\ell}\mathfrak{O}_{\mathfrak{p}})$. We will prove the inclusion is actually equality.

<u>Case I:</u> <u>D is ramified at $\mathfrak{p}$</u>

Then $D_{\mathfrak{p}}$ is a division algebra over $K_{\mathfrak{p}}$ and $\mathfrak{O}_{\mathfrak{p}}$ is a valuation ring with maximal ideal $\mathfrak{p}$. Therefore $\mathfrak{O}_{\mathfrak{p}}/\mathfrak{p}^{n_{\mathfrak{p}}+\ell}\mathfrak{O}_{\mathfrak{p}}$ is a valuation ring whose maximal ideal is $\mathfrak{p}\mathfrak{O}_{\mathfrak{p}}/\mathfrak{p}^{n_{\mathfrak{p}}+\ell}\mathfrak{O}_{\mathfrak{p}}$. Let $x \in H^0(\Gamma_N^{\mathfrak{p}}, \mathfrak{O}_{\mathfrak{p}}/\mathfrak{p}^{n_{\mathfrak{p}}+\ell}\mathfrak{O}_{\mathfrak{p}})$. Let $\gamma \in \Gamma_N^{\mathfrak{p}}$ be an element such that its image in $\mathfrak{O}_{\mathfrak{p}}/\mathfrak{p}^{n_{\mathfrak{p}}+\ell}\mathfrak{O}_{\mathfrak{p}}$ does not belong to $(1+\mathfrak{p}^{n_{\mathfrak{p}}+1}\mathfrak{O}_P)/\mathfrak{p}^{n_{\mathfrak{p}}+\ell}\mathfrak{O}_{\mathfrak{p}}$.

This is possible in view of our determination of the image of $\Gamma_N^{\mathfrak{p}}$ in $\mathfrak{O}_{\mathfrak{p}}/\mathfrak{p}^{n_{\mathfrak{p}}+\ell}\mathfrak{O}_{\mathfrak{p}}$. We have $\text{ord}_{\mathfrak{p}}(\bar{\gamma}-1) \leq n_{\mathfrak{p}}$.

Since $x = \bar{\gamma}x = x+(\bar{\gamma}-1)x$, we have $(\bar{\gamma}-1)x = 0$. Therefore $\text{ord}_{\mathfrak{p}}x \geq (n_{\mathfrak{p}}+\ell) - \text{ord}_{\mathfrak{p}}(\bar{\gamma}-1) \geq \ell$, which proves

$$x \in \mathfrak{p}^{\ell}\mathfrak{O}_{\mathfrak{p}}/\mathfrak{p}^{n_{\mathfrak{p}}+\ell}\mathfrak{O}_{\mathfrak{p}} \therefore \mathfrak{p}^{\ell}\mathfrak{O}_{\mathfrak{p}}|\mathfrak{p}^{n_{\mathfrak{p}}+\ell}\mathfrak{O}_{\mathfrak{p}} = H^0(\Gamma_N^{\mathfrak{p}}, \mathfrak{O}_{\mathfrak{p}}/\mathfrak{p}^{n_{\mathfrak{p}}+\ell}\mathfrak{O}_{\mathfrak{p}}).$$

<u>Case II:</u> <u>D is unramified at $\mathfrak{p}$</u>

Then $D_{\mathfrak{p}} \cong M_2(K_{\mathfrak{p}})$ and $\mathfrak{O}_{\mathfrak{p}} \cong M_2(\mathfrak{O}_{K_{\mathfrak{p}}})$.

Denote by $R_\mathfrak{p}^\ell$ the residence class ring $\mathfrak{o}_{K_\mathfrak{p}}/\mathfrak{p}^{n_\mathfrak{p}+\ell}\mathfrak{o}_{K_\mathfrak{p}}$. Then $\mathfrak{o}_\mathfrak{p}/\mathfrak{p}^{n_\mathfrak{p}+\ell}\mathfrak{o}_\mathfrak{p}$ is isomorphic to $M_2(R_\mathfrak{p}^\ell)$ in such a way that the image of $\Gamma_N^\mathfrak{p}$ corresponds to

$$\{x \in M_2(R_\mathfrak{p}^\ell) \mid \det x = 1 \text{ and } x \equiv \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} (\mathrm{mod}\ \mathfrak{p}^{n_\mathfrak{p}} M_2(R_\mathfrak{p}^\ell))\}.$$

Let $\pi \in \mathfrak{o}_{K_\mathfrak{p}}$ be a prime element, and denote by $\bar{\pi}$ its image in $R_\mathfrak{p}^\ell$.

Then

$$\begin{pmatrix}1 & \bar{\pi}^{n_\mathfrak{p}}\\ 0 & 1\end{pmatrix} \text{ and } \begin{pmatrix}1 & 0\\ \bar{\pi}^{n_\mathfrak{p}} & 1\end{pmatrix}$$

both belong to the image of $\Gamma_N^\mathfrak{p}$ in $M_2(R_\mathfrak{p}^\ell)$.

Let

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in H^0(\Gamma_N^\mathfrak{p}, \mathfrak{o}_\mathfrak{p}/\mathfrak{p}^{n_\mathfrak{p}+\ell}\mathfrak{o}_\mathfrak{p}).$$

Then

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}1 & \bar{\pi}^{n_\mathfrak{p}}\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a+\bar{\pi}^{n_\mathfrak{p}}c & b+\bar{\pi}^{n_\mathfrak{p}}d\\ c & d\end{pmatrix}$$

and

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}1 & 0\\ \bar{\pi}^{n_\mathfrak{p}} & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a & b\\ \bar{\pi}^{n_\mathfrak{p}}a+c & \bar{\pi}^{n_\mathfrak{p}}b+d\end{pmatrix}$$

so that $\bar{\pi}^{\mathfrak{p}P}a + \bar{\pi}^{n_\mathfrak{p}}b = \bar{\pi}^{n_\mathfrak{p}}c = \bar{\pi}^{n_\mathfrak{p}}d = 0$. Therefore, $a,b,c,d \in \mathfrak{p}^\ell R_\mathfrak{p}^\ell$, so that $\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathfrak{p}^\ell M_2(R_\mathfrak{p}^\ell)$. This proves that

$$\mathfrak{p}^\ell\mathfrak{o}_\mathfrak{p}/\mathfrak{p}^{n_\mathfrak{p}+\ell}\mathfrak{o}_\mathfrak{p} = H^0(\Gamma_N^\mathfrak{p}, \mathfrak{o}_\mathfrak{p}/\mathfrak{p}^{n_\mathfrak{p}+\ell}\mathfrak{o}_\mathfrak{p}).$$

Since the inclusion $\mathfrak{o}/\mathfrak{p}^{n_\mathfrak{p}+\ell}\mathfrak{o} \hookrightarrow \mathfrak{o}_\mathfrak{p}/\mathfrak{p}^{n_\mathfrak{p}+\ell}\mathfrak{o}_\mathfrak{p}$ is an isomorphism and since Γ_N and $\Gamma_N^\mathfrak{p}$ have the same image in $\mathfrak{o}_\mathfrak{p}/\mathfrak{p}^{n_\mathfrak{p}+\ell}\mathfrak{o}_\mathfrak{p}$, we conclude that

$$\mathfrak{p}^\ell\mathfrak{o}/\mathfrak{p}^{n_\mathfrak{p}+\ell}\mathfrak{o} = H^0(\Gamma_N, \mathfrak{o}/\mathfrak{p}^{n_\mathfrak{p}+\ell}\mathfrak{o}).$$

As we remarked in Chapter I, §8, for any positive integer M, $H^0(\Gamma_N, \mathfrak{O}/_{MN\mathfrak{O}})$ is naturally isomorphic to the MN-torsion in $H^1(\Gamma_N, \mathfrak{O})$. Moreover, we observed that $H^1(\Gamma_N, \mathfrak{O})^{\text{tors.}}$ is a finite group. Therefore, for suitable M, we have $H^0(\Gamma_N, \mathfrak{O}/MN\mathfrak{O}) \cong H^1(\Gamma_N, \mathfrak{O})^{\underline{\text{tors.}}}$.

For every prime $\mathfrak{p}$ of K, let $\ell_{\mathfrak{p}} = \text{ord}_{\mathfrak{p}} M$. Then $\mathfrak{O}/MN\mathfrak{O} \underset{\Gamma_N}{\cong} \mathfrak{O}/\mathfrak{p}^{n_{\mathfrak{p}}+\ell_{\mathfrak{p}}}\mathfrak{O}$, so that by what we have proved above, we have

$$\begin{aligned} H^0(\Gamma_N, \mathfrak{O}/MN\mathfrak{O}) &\cong \bigoplus_{\mathfrak{p}} H^0(\Gamma_N, \mathfrak{O}/\mathfrak{p}^{n_{\mathfrak{p}}+\ell_{\mathfrak{p}}}\mathfrak{O}) \\ &\cong \bigoplus_{\mathfrak{p}} \mathfrak{p}^{\ell_{\mathfrak{p}}}\mathfrak{O}/\mathfrak{p}^{n_{\mathfrak{p}}+\ell_{\mathfrak{p}}}\mathfrak{O} \\ &\cong M\mathfrak{O}/MN\mathfrak{O} \cong \mathfrak{O}/N\mathfrak{O}. \end{aligned}$$

Therefore, $H^1(\Gamma_N, \mathfrak{O})^{\text{tors.}} \cong \mathfrak{O}/N\mathfrak{O}$.

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