A Class of Compact Manifolds With Positive Ricci Curvature

A Dissertation presented

by

Horacio Hernandez

to

The Graduate School

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in

Mathematics

College of Arts and Sciences

State University of New York

at

Stony Brook

December, 1972

STATE UNIVERSITY OF NEW YORK

AT STONY BROOK

THE GRADUATE SCHOOL

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Jan 15, 1973

Abstract of the Dissertation

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In this thesis we prove the existence of a large class of compact Riemannian manifolds (most of which are not locally homogeneous) which admit a metric such that the Ricci curvature is strictly positive. They are certain Brieskorn varieties with a metric induced naturally from some modified imbedding into euclidean space.

In II we develop the general theory, in III we apply it to the modified Brieskorn manifolds and we derive our main results. In IV we discuss the scope of the results proven in III, and give further applications.

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I. Introduction

The problem we are dealing with is part of an important question in global Riemannian geometry: To study compact manifolds of strictly positive curvature. So far, one of the main difficulties has been the lack of enough examples. In fact, all known examples of compact manifolds with positive sectional curvature are diffeomorphic to locally homogeneous spaces. It is somewhat surprising that in the much more general case of positive Ricci curvature, again no examples other than locally homogeneous spaces are known (up to diffeomorphism). In particular, it is an interesting question whether or not exotic spheres (which can never be homogeneous) admit metrics of positive sectional curvature, or at least positive Ricci curvature. Only this year, Cheeger [2] has found the first non-homogeneous manifolds of positive Ricci curvature (including the Kervaire spheres), by means though of a very isolated construction, which is essentially intrinsic.

In this paper we shall give a very large new class of (in general not locally homogeneous) compact manifolds with positive Ricci curvature, including a very rich class of exotic spheres; compare sections III, IV. These examples are Brieskorn varieties endowed with a metric induced from euclidean space after a modification of the standard embedding. The methods involved seems to be quite promising to produce even more new examples of positively curved manifolds among certain algebraic varieties.

To determine properties of the curvature tensor explicitly for a given Riemannian manifold, is in general very difficult. Basically our approach is very classical, namely, to study Riemannian submanifolds of euclidean spaces which are globally defined by equations. Our key observation is that there are many interesting examples of such manifolds which are orthogonal intersections of level surfaces of simple functions.

I would like to take the opportunity to thank my advisor, Professor D. Gromoll, for his kindness and essential help in many ways, to Professors W. Meyer and J. Cheeger for help in the very early stage of the work, to Virginia LaLumia for doing an excellent typing of my dissertation, and to the State University of New York for financially supporting almost all of my stay at Stony Brook.

II. The Ricci Curvature of a Variety

In this section we first review the fundamental curvature quantities of a Riemannian manifold M. Our main objective is to estimate the curvature of M in computable terms. This problem seems to be least difficult for Riemannian submanifolds which are globally defined by equations and which for our purposes, we call varieties. Even though our approach is very classical and familiar in the case of hypersurfaces, hardly any results have been known for higher codimension. We develop some new ideas how to obtain estimates for the curvature of certain varieties in terms of the defining ideal of functions. We finally prepare all formulas for application in our study of Brieskorn varieties in Chapter III. For basic definitions and facts in Riemannian geometry, we refer to [5], [8].

We consider an n-dimensional Riemannian manifold M with metric $\langle\ ,\ \rangle$ and Levi-Civita connection ∇. All data will always be of sufficiently high differentiability class, say C^∞ for convenience (however it is not necessary to suppose that). For $p \in M$, let M_p denote the tangent space of M at p. The curvature tensor R of M assigns to vector fields X, Y, Z a new vector field.

(1) $$R(X,Y)Z = \nabla_x(\nabla_y Z) - \nabla_y(\nabla_x Z) - \nabla_{[x,y]}Z$$

One may consider R as a skew symmetric 2-form which assigns to $u,v \in M_p$ a skew adjoint endomorphism $R(u,v)$ of M_p.

Let $\delta \subset M_p$ be a 2-dimensional linear subspace of M_p and $u,v \in \delta$ independent vectors. Then the <u>sectional curvature</u> K_δ of M with respect to the plane δ is the real number defined by:

(2) $$K_\delta = K(u,v) = \frac{\langle R(u,v)v,u\rangle}{\|u\|^2\|v\|^2-\langle u,v\rangle^2}$$

K_δ depends only on the plane δ.

The <u>Ricci tensor</u> S is the 2-form on M defined by:

(3) $$s(u,v) = \text{trace}(w \to R(w,u)v)$$

where $u,v,w \in M_p$. So, if $e_1,\ldots,e_n$ is any orthonormal basis of M_p,

(3') $$s(u,v) = \sum_{j=1}^{n} \langle R(e_j,u)v,e_j\rangle$$

In particular, S is symmetric. The <u>Ricci curvature</u> Ric(u) is a real number associated with the 1-dimensional linear subspace of M_p generated by a vector $u \neq 0$,

(4) $$\text{Ric}(u) = \frac{1}{\|u\|^2}\, s(u,u)$$

If $\|u\| = 1$, we obtain using (2) and (3'), (4') $$\mathrm{Ric}(u) = \sum_{j=1}^{n-1} K(e_j, u)$$

for any orthonormal basis $e_1, \ldots, e_{n-1}$, u of M_p .

Remark: Some authors call $\frac{1}{n-1}$ Ric(u) the Ricci curvature, the reason being that this number is exactly the average of all sectional curvature with respect to planes δ which contain u. In our context, such a normalization is irrelevant.

By definition, M has positive (negative) Ricci curvature everywhere if and only if the Ricci tensor S is positive (negative) definite at any point $p \in M$.

The scalar curvature of M is a real valued function s on M defined by

(5) $$s = \text{trace } S$$

where the trace is taken pointwise with respect to the inner product $\langle\ ,\ \rangle$. Hence,

(5') $$s(p) = \sum_{j=1}^{n} \mathrm{Ric}(e_j) = 2 \sum_{1 \le i < j \le n} K(e_i, e_j)$$

for any orthonormal basis $e_1, \ldots, e_n$ of M_p .

Now we turn to the situation where M is a Riemannian submanifold of some Riemannian manifold $\widetilde{M}$ of dimension $n + k$. So $M \subset \widetilde{M}$, and the inclusion map i is an isometric imbedding. We identify the tangent space M_p with

$i_* M_p \subset \tilde{M}_p$. The normal space of M at p is the orthogonal complement $M_p^\perp$ of M_p in $\tilde{M}_p$. For $w \in \tilde{M}_p$, we write $w^T \in M_p$, $w^\perp \in M_p^\perp$ for the tangent and normal components of $w = w^T + w^\perp$.

Let $\tilde{\nabla}$ denote the Levi-Civita connection of $\tilde{M}$. One has the following relation

$$(6) \qquad \nabla_X Y = (\tilde{\nabla}_X Y)^T ,$$

for vector fields X, Y on M, which we may also consider as vector fields in $\tilde{M}$ along M, i.e., along the inclusion i.

By applying (6) to (1), one can express the curvature tensor R of M in terms of the curvature tensor $\tilde{R}$ of $\tilde{M}$ and in terms of derivates involving only $\tilde{\nabla}$,

$$(7) \quad R(X,Y)Z = [\tilde{R}(X,Y)Z]^T - (\tilde{\nabla}_X(\tilde{\nabla}_Y Z)^\perp)^T + (\tilde{\nabla}_Y(\tilde{\nabla}_X Z)^\perp)^T$$

Now one introduces the second fundamental tensor of M at p as a 1-form A on M_p with values in linear transformations $M_p^\perp \to M_p$ by

$$(8) \qquad A(u)N_p = (\tilde{\nabla}_u N)^T ,$$

where $u \in M_p$, $N_p \in M_p^\perp$, and N is any normal vector field along M with $N(p) = N_p$. The right hand side of (8) depends only on N_p . Let A^* denote the 1-form on M_p with values in linear transformations $M_p \to M_p^\perp$ defined by $A^*(u) = A(u)^*$, the adjoint of $A(u)$. One has

(9) $$A^*(u)v = -(\tilde{\nabla}_u Y)^\perp ,$$

where $v \in M_p$, and Y is any tangential field along M with $Y(p) = v$. Using (8) and (9) in (7) we obtain the <u>Gauss equations</u>.

(10) $$R(u,v) = \tilde{R}(u,v)^T + A(u) \circ A^*(v) - A(v) \circ A^*(u).$$

Of course, the "product" $A(u) \circ A^*(v)$ is the composition of linear transformations. Let us consider $\Delta = R - \tilde{R}^T$ as the curvature difference of M and $\tilde{M}$ at p, and let $A \wedge A^*$ denote the skew 2-form on M_p, with values in endomorphisms $M_p \to M_p$ given by:

$$A \wedge A^*(u,v) = A(u) \circ A^*(v) - A(v) \circ A^*(u)$$

then the Gauss equations (10) take the more suggestive form

(11) $$\Delta = R - \tilde{R}^T = A \wedge A^* .$$

For to work with the curvature difference Δ explicitly, one often chooses a basis $N_1,\ldots,N_k$ for $M_p^\perp$ and consider the self-adjoint endomorphisms A_λ of M_p with $A_\lambda u = A(u)N_\lambda$ for $1 \le \lambda \le k$. In many cases, it need not necessarily be advantageous to assume that the fixed basis is orthonormal, for example, when M is a variety. For any $a \in M_p^{\perp}$, we have the identity

$$a = \sum_{\lambda,\mu=1}^{k} \phi_{\lambda\mu}\langle a,N_\lambda\rangle N_\mu ,$$

where $\phi_{\lambda\mu}$ is the inverse matrix of $\langle N_\mu,N_\lambda\rangle$.

Since $\langle A^*(u)w,N_\lambda\rangle = \langle w,A(u)N_\lambda\rangle = \langle w,A_\lambda u\rangle$,

$$A^*(u)w = \sum_{\lambda,\mu=1}^{k} \phi_{\lambda\mu} \langle A_\lambda u,w\rangle N_\mu .$$

Therefore, (10) becomes

$$(12) \qquad \Delta(u,v)w = \sum_{\lambda,\mu=1}^{k} \phi_{\lambda\mu}(\langle A_\lambda v,w\rangle A_\mu u - \langle A_\lambda u,w\rangle A_\mu v).$$

If $N_1,\ldots,N_k$ are <u>orthonormal</u>,

$$(12') \qquad \Delta(u,v)w = \sum_{\lambda=1}^{k} (\langle A_\lambda v,w\rangle A_\lambda u - \langle A_\lambda u,w\rangle A_\lambda v)$$

To deal with the curvature tensor R in many applications, the Gauss equations can only be used successfully if, at least locally, k in dependent normal fields of M can be found such that the transformations A_λ in (12) are computable, provided in addition, that $\tilde{R}$ is known. Moreover, the codimension k should be finally compared to n = dim M. The most favorable situation seems to be when M is defined by fairly simple "equations", this was studied to some extent by Dombrowski, whose paper [3] inspired our investigation.

Recall that the gradient of a function f defined in some open subset U of $\widetilde{M}$ is the vector field ∇f on U such that $\langle \nabla f, X \rangle = Xf$ for all vector fields X on U. At $p \in \widetilde{M}$ one has

$$\nabla f/p = \sum_{i=1}^{n+k} (e_i f) e_i$$

for any orthonormal basis $e_1, \ldots, e_{n+k}$ of $\widetilde{M}_p$.

We say that a subset M of $\widetilde{M}$ is a variety in $\widetilde{M}$ if there is an open neighborhood U of M in $\widetilde{M}$ and an ideal I_M of functions in the ring $C^\infty(u)$ of real valued C^∞-functions on U such that:

a) $M = \{p/f(p) = 0 \text{ for all } f \in I_M\}$

b) I_M is generated by $f_1, \ldots, f_k$ and $\nabla f_1/p, \ldots, \nabla f_k/p$ are linearly independent for all $p \in M$.

Clearly, $k \leq \dim \widetilde{M}$, and necessarily, M is a closed submanifold of U, of codimension k, since 0 is a regular value of $F : U \to R^k$ by b), $F = (f_1, \ldots, f_k)$, and $M = F^{-1}(0)$ by a).

<u>Proposition II.1</u>

The submanifold M of $\widetilde{M}$ is a variety if and only if the normal bundle, ν_M, of M is trivial.

Proof: We only sketch the straightforward argument. Clearly, if M is a variety, the vector fields $\nabla f_1,\ldots,\nabla f_k$ give a parallelization of ν_M. Conversely, assume that ν_M is trivial. Then we can find $k = \operatorname{co\,dim} M$ orthonormal vector fields $N_1,\ldots,N_k$ along M.

The exponential map exp of $\widetilde{M}$ maps some open neighborhood V of the zero section in ν_M diffeomorphically onto a neighborhood W of M in $\widetilde{M}$. Let ν_i denote the sub-bundle of ν_M whose fibers $\nu_{i/p}$ over $p \in M$ is the linear hyperplane in $M_p^{\perp}$ perpendicular to $N_{i/p}$, $1 \leq i \leq k$. Let $V_i = V \cap \nu_i$. Then exp maps V_i diffeomorphically onto a hypersurface W_i such that $M = W_1 \cap \ldots \cap W_k$. Moreover, M is an "orthogonal intersection of hypersurfaces", i.e., the normal vectors $N_{i/p}$ of W_i at $p \in M$ are orthogonal. Let $\hat{V}_i$ denote the trivial normal line bundle of W_i in $\widetilde{M}$ and consider the section $\hat{N}_i$ over W_i such that $\|\hat{N}_i\| = 1$ and $\hat{N}_i/M = N_i$. We have the function $\varphi_i \hat{V}_i \to R$ with $\varphi_i(w) = \langle w, \hat{N}_i \rangle$. Again the exponential map exp of $\widetilde{M}$ maps an open neighborhood $\hat{V}_i$ of the zero section in $\hat{V}_i$ diffeomorphically onto a neighborhood U_i of W_i in $\widetilde{M}$. Let f_i be the restriction of $\varphi_i \circ (\exp/\overline{V}_i)^{-1}$ to $U = U_1 \cap \ldots \cap U_k$. Then $\nabla f_{i/p} = N_{i/p}$ for $p \in M$ and $F_i^{-1}(0) = W_i$. Hence M is a variety with defining ideal I_M generated by $f_1,\ldots,f_k$ in $C^{\infty}(U)$.

We have two immediate consequences.

Corollary II.2.

1) Any submanifold M of $\widetilde{M}$ is locally a variety, i.e., every point $p \in M$ has an open neighborhood in M which is a variety in $\widetilde{M}$.

2) The defining ideal I_M of any variety M in $\widetilde{M}$ has an "orthonormal generator system", i.e., I_M can be generated by $f_1,\ldots,f_k$ such that $\nabla f_1,\ldots,\nabla f_k$ are orthonormal along M.

We want to emphasize that the whole point for introducing varieties is that they are globally defined by functions. Global curvature estimates can be obtained just by studying first and second derivates of k generating functions, as discussed next. This is usually a less difficult problem than working in the general case, in particular, when the defining functions and the ambient space are fairly "simple". We should mention that the second part of Corollary II.2 is mainly of theoretical interest. In examples, generating functions form hardly ever an orthonormal system, and explicit orthonormalization is in general too complicated to be of any practical use. However, one of our crucial observations is that sometimes the defining functions of a variety M in $\widetilde{M}$ can be slightly modified to yield at least an orthogonal generator system for a new ideal $I_{M'}$ which defines a variety M' diffeomorphic to M.

For a real valued function f defined on some open subset, U, of $\widetilde{M}$ and $p \in U$, the <u>hessian tensor</u> is the self adjoint <u>endomorphism</u> $H_f : \widetilde{M}_p \to \widetilde{M}_p$ with

$$(13) \qquad H_f v = \nabla_v \nabla f$$

H_f is given by the matrix of second partial derivates of f at p in the case $\widetilde{M} = R^k$.

Let M be a variety in $\widetilde{M}$ defined by the functions $f_1, \ldots, f_k$ on $U \supset M$. Setting $H_\lambda = H_{f_\lambda}$ and $N_\lambda = \nabla f_\lambda / M$, we obtain from (12) and (13) for the curvature difference tensor Δ of M in $\widetilde{M}$,

$$(14) \qquad \Delta(u,v)w = \sum_{\lambda=1}^{k} \frac{1}{\|\nabla f_\lambda\|^2} (\langle H_\lambda v, w\rangle H_\lambda^T u - \langle H_\lambda u, w\rangle H_\lambda^T v)$$

Here, $H_\lambda^T : M_p \to M_p$ is the tangential projection of H_λ, i.e., $H_\lambda^T u = (H_\lambda u)^T$. From now on we restrict attention to the case where M is a variety in flat euclidean space $\widetilde{M} = R^{n+k}$, so $\Delta = R$. Furthermore, we always assume that $\nabla f_1, \ldots, \nabla f_k$ are mutually orthogonal along M. Then for the sectional curvature of M, we get by using (14') in (2),

(15) $$K_\delta = K(u,v) = \sum_{\lambda=1}^{k} \frac{1}{\|\nabla f_\lambda\|^2} (\langle H_\lambda u,u\rangle \cdot \langle H_\lambda v,v\rangle - \langle H_\lambda u,v\rangle^2)$$

where $u,v \in M_p$ orthonormal. Similarly, using (14') in (3') and (5') yields for Ricci and scalar curvature:

(16) $$s(u,u) = \sum_{\lambda=1}^{k} \frac{1}{\|\nabla f_\lambda\|^2} (\langle H_\lambda u,u\rangle \operatorname{tr} H_\lambda^T - \|H_\lambda^T u\|^2),$$

(17) $$s(p) = \sum_{\lambda=1}^{k} \frac{1}{\|\nabla f_\lambda\|^2} ((\operatorname{tr} H_\lambda^T)^2 - \operatorname{tr}(H_\lambda^T)^2)$$

Now let $\tilde{M} = C^n = R^n \oplus iR^n \cong R^{2n}$ complex n-space, where we identify $z = (z_1,\ldots,z_n) \in C^n$ with $(x_1,\ldots,x_n; y_1,\ldots,y_n) \in R^{2n}$, $z_k = x_k + iy_k$. Let $\langle u,v\rangle = \sum_{k=1}^{n} u_k \cdot \bar{v}_k$ denote the canonical hermitian inner product for C^n. Then the real part $\operatorname{Re}\langle u,v\rangle$ is the canonical inner product for $R^{2n} \cong C^n$. Suppose $f : U \to C$ is a holomorphic function defined on some open subset U of C^n. Setting $\varphi = \operatorname{Re} f$ and $\psi = \operatorname{Im} f$, $f = \varphi + i\psi$ satisfies the Cauchy-Riemann equations.

(18) $$\nabla\varphi = -i\nabla\psi$$

Hence,

(19) $$\operatorname{Re}\langle\nabla\varphi,\nabla\psi\rangle = 0,$$

so $\nabla\varphi$, $\nabla\psi$ are always <u>orthogonal</u> in R^{2n}. The <u>complex</u>

<u>gradient</u> of f is the vector field $\nabla f = (\overline{\frac{\partial f}{\partial z_1}}, \ldots, \overline{\frac{\partial f}{\partial z_n}})$ on U. We have

(20) $\nabla f = \frac{1}{2}(\nabla\varphi - i\,\nabla\psi) = \nabla\varphi = -i\nabla\psi$

The <u>complex hessian</u> H of f at $p \in U$ is a C-linear endomorphism of $C^n \cong \tilde{M}_p$ given by the matrix $(\frac{\partial^2 f}{\partial z_j \partial z_k}|p)$ with respect to the canonical basis of C^n, $1 \leq j, k \leq n$. Now suppose, we are also given a real valued function g on U, such that the ideal generated by $\varphi = \text{Re}f$, $\psi = \text{Im}f$, g defines a variety M in C^n. So M is a real submanifold of codimension 3 in C^n which is the intersection of the complex hypersurface $f^{-1}(0)$ and the real hypersurface $g^{-1}(0)$. Now let us assume in addition that the complex gradient ∇f and the real gradient ∇g are complex orthogonal, $\langle \nabla f, \overline{\nabla g} \rangle = 0$. Then we obtain the following result for the curvature of M. Using the Cauchy-Riemann equations, the complex hessian H of f and the real hessians H_φ, H_ψ are easily seen to be related as follows:

(20) $\langle Hu, \bar{v} \rangle = \text{Re}\langle H_\varphi u, v \rangle + i\text{Re}\langle H_\psi u, v \rangle$

<u>Lemma II.3</u>

Let u be a tangent vector of M at p. Then

$$(21)\quad s(u,u) = \frac{1}{\|\nabla f\|^2}\left(-\mathrm{Re}\langle Hu,\bar{u}\rangle\langle H\frac{\nabla g}{\|\nabla g\|},\frac{\overline{\nabla g}}{\|\nabla g\|}\rangle - 2\|Hu\|^2\right.$$

$$\left. + 2\left|\langle Hu,\frac{\overline{\nabla f}}{\|\nabla f\|}\rangle\right|^2 + \left|\langle Hu,\frac{\overline{\nabla g}}{\|\nabla g\|}\rangle\right|^2\right)$$

$$+ \frac{1}{\|\nabla g\|^2}\left(\langle H_g u,u\rangle \operatorname{tr} H_g^T - \|H_g^T u\|^2\right),$$

$$(22)\quad s(p) = \frac{4}{\|\nabla f\|^2}\left(\|H\frac{\nabla g}{\|\nabla g\|}\|^2 + 2\|H\frac{\nabla f}{\|\nabla f\|}\|^2 - \left|\langle H\frac{\nabla f}{\|\nabla f\|},\frac{\overline{\nabla f}}{\|\nabla f\|}\rangle\right|^2\right.$$

$$\left. - \left|\langle H\frac{\nabla f}{\|\nabla f\|},\frac{\overline{\nabla g}}{\|\nabla g\|}\rangle\right|^2 - \operatorname{tr} H\bar{H}\right)$$

$$+ \frac{1}{\|\nabla g\|^2}\left((\operatorname{tr} H_g^T)^2 - \operatorname{tr}(H_g^T)^2\right).$$

<u>Proof</u>: The above formulas follow straightforward from (16), (17), and (20), using a complex orthonormal basis $e_1,\ldots,e_{n-2}, \frac{\nabla g}{\|\nabla g\|}|p, \frac{\nabla f}{\|\nabla f\|}|p$ for C^n. Note that $e_1, ie_1,\ldots,e_{n-2}, ie_{n-2}, i\frac{\nabla g}{\|\nabla g\|}|p$ is a real orthonormal basis of the tangent space of M at p.

In applications, formulas (21) and (22) can be used advantageously in particular when the function g is very simple, as we will study now in the next section.

III. Results for Brieskorn Varieties

Let $n \geq 3$ and $a_1 \geq a_2 \geq \ldots \geq a_n \geq 2$ integers. Consider the polynomial $f : C^n \to C$, $f(z) = \sum_{k=1}^{n} z_k^{a_k}$, and the function $g : C^n \to R$, $g(z) = \sum_{k=1}^{n} z_k \cdot \overline{z_k} - 1$. The intersection of the complex hypersurface $f^{-1}(0)$, which is singular only at $z = 0$, and the euclidean sphere $g^{-1}(0)$ is a compact submanifold of C^n with real codimension 3, the <u>Brieskorn variety</u> $V(a_1,\ldots,a_n)$. These manifolds have been studied extensively in recent years from various viewpoints and turned out to be extremely important and interesting. (For a general account, compare [9]). In differential geometry, of course, one is interested in their curvature behavior. Clearly $V(a_1,\ldots,a_n)$ is a variety in our sense defined by the ideal generated by (Ref, Imf, g). It is very easy to see, and has been observed by many people that $V(a_1,\ldots,a_n)$ with its induced metric always has sectional curvature of either sign. Our starting point was to study the Ricci Tensor of V which in some very special cases turned out to be positive, for example, when $a_1=\ldots=a_n=2$. However, already in the fairly simple case $a_1=3$, $a_2=\ldots=a_n=2$ (which contain exotic spheres), the formulas became too complicated to work with. In our attempt to make computations more accessible, we discovered that there is a simple modification of the imbedding of V in C^n which is an orthogonal intersection of a perturbed complex hypersurface and an ellipsoid. Then the fairly simple formula (21) is applicable, and that makes it possible to obtain estimates for the Ricci curvature.

Let $\alpha_1,\ldots,\alpha_n > 0$ and $\omega_1,\ldots,\omega_n > 0$ real numbers. Then we consider the generalized Brieskorn variety $V'(a_1,\ldots,a_n) = f^{-1}(0) \cap g^{-1}(0)$, where $f(z) = \sum_{k=1}^{n} \alpha_k z_k^{a_k}$, $g(z) = \sum_{k=1}^{n} \omega_k z_k \cdot \overline{z_k} - 1$; Note that on V', the complex gradient ∇f and the real gradient ∇g are independent over C : For $z \neq 0$, $\nabla f_z = (\alpha_1 a_1 \bar{z}_1^{a_1-1},\ldots,\alpha_n a_n \bar{z}_n^{a_n-1}) \neq 0$, $\nabla g_z = 2(\omega_1 z_1,\ldots,\omega_n z_n) \neq 0$. Now if for some $0 \neq \mu \in C$ and $z \in V'$, $\nabla f|_z = \mu \nabla g|_z$, we would have $\alpha_k a_k \bar{z}_k^{a_k-1} = 2\mu\omega_k z_k$, so $\alpha_k \bar{z}_k^{a_k} = \frac{2\mu\omega_k}{a_k} z_k \cdot \overline{z_k}$. Summing over k yields $0 = 2\mu \sum_{k=1}^{n} \frac{\omega_k}{a_k} z_k \cdot \overline{z_k}$, which is impossible. Moreover, we have the following fact.

Proposition III.1.

$V'(a_1,\ldots,a_n)$ and $V(a_1,\ldots,a_n)$ are diffeomorphic and isotopic in C^n.

Proof: Let $H : C^n \times [0,1] \to C \times R$ defined by

$$H(z,t) = \left(\sum_{k=1}^{n} \alpha_k(t) z_k^{a_k}, \sum_{k=1}^{n} \omega_k(t) z_k \cdot \overline{z_k}\right), \text{ where}$$

$\alpha_k(t) = 1 - t + t\alpha_k > 0$, $\omega_k(t) = 1 - t + t\omega_k > 0$. By the above argument, 0 is a regular value of H. So, $Q = H^{-1}(0)$ is a compact submanifold with boundary $V \times 0 \cup V' \times 1$. The restriction λ of the function $(z,t) \to t$ to Q has no critical points, $\lambda^{-1}(0) = V \times 0$, $\lambda^{-1}(1) = V' \times 1$. Hence, Q is diffeomorphic to $V \times [0,1]$, and this completes the argument.

Now once and for all, given $a_1 \geq a_2 \geq \ldots \geq a_n \geq 2$ we choose $\alpha_k = \frac{1}{a_k(a_k-1)}$ and $\omega_k = \frac{1}{a_k}$ and consider always the Brieskorn variety,

$V'(a_1,\ldots,a_n) = f^{-1}(0) \cap g^{-1}(0)$, where

$$f(z) = \sum_{k=1}^{n} \frac{z_k^{a_k}}{a_k(a_k-1)}, \quad g(z) = \sum_{k=1}^{n} \frac{z_k \cdot \bar{z}_k}{a_k} - 1, \tag{23}$$

endowed with the induced Riemannian structure as a submanifold of C^n. We have

$$\nabla f\,|_z = \left(\frac{\bar{z}_1^{a_1-1}}{a_1-1}, \ldots, \frac{\bar{z}_n^{a_n-1}}{a_n-1}\right) \tag{24}$$

$$\nabla g\,|_z = 2\left(\frac{z_1}{a_1}, \ldots, \frac{z_n}{a_n}\right) \tag{25}$$

Hence, if $z \in V'$ it follows using (23) that

$$\langle \nabla f, \nabla g \rangle\,|_z = 0. \tag{26}$$

We are now in a situation where the techniques developed at the end of Section II can be applied. In particular, we want to find lower bounds for the Ricci curvature of V' using the formula (21). For this purpose, we need the following quantities:

(27) $$\|\nabla f\|^2_{1z} = \sum_{k=1}^{n} \frac{|z_k|^{2(a_k-1)}}{(a_k-1)^2}$$

(28) $$\|\nabla g\|^2_{1z} = 4 \sum_{k=1}^{n} \frac{|z_k|^2}{a_k^2} ,$$

(29) $$H = \begin{pmatrix} z_1^{a_1-2} & & 0 \\ & \ddots & \\ 0 & & z_n^{a_n-2} \end{pmatrix}$$

(30) $$\mathrm{Re}\langle H_g u, v\rangle = 2\mathrm{Re} \sum_{k=1}^{n} \frac{u_k \bar{v}_k}{a_k}$$

the norm of H at z is the number

(31) $$\|H\| = \sup_{\substack{C^n 3 \\ u\neq 0}} \frac{\|Hu\|}{\|u\|} = \max_k |z_k|^{a_k-2}$$

<u>Lemma III.2.</u>

A lower bound for the Ricci curvature of V' is given by

(32) $$\mathrm{Ric}(u) \geq -3 \sup_{z\in V'} \frac{\|H\|^2}{\|\nabla f\|^2} + \frac{a_n}{a_1^2} (2n-4)$$

In the case of equal exponents $a_1 = \ldots = a_n = p$ we have a better bound

(32') $$\mathrm{Ric}(u) \geq -2 \sup_{z\in V'} \frac{\|H\|^2}{\|\nabla f\|^2} + \frac{2n-4}{p}$$

Proof: We may assume, $\|u\|=1$. Thus, from (21) we obtain

(33) $$s(u,u) \geq -\frac{1}{\|\nabla f\|^2}\left(|\langle Hu,\bar{u}\rangle|\,|\langle H\frac{\nabla g}{\|\nabla g\|}, \frac{\overline{\nabla g}}{\|\nabla g\|}\rangle| + 2\|Hu\|\right) + \frac{1}{\|\nabla g\|^2}\left(\langle H_g u,u\rangle \operatorname{tr} H_g^T - \|H_g^T u\|^2\right).$$

Now

$$|\langle Hu,\bar{u}\rangle|\,|\langle H\frac{\nabla g}{\|\nabla g\|}, \frac{\overline{\nabla g}}{\|\nabla g\|}\rangle| \leq \|H\|^2 .$$

Note that $\langle H\frac{\nabla g}{\|\nabla g\|}, \frac{\overline{\nabla g}}{\|\nabla g\|}\rangle = 0$ in the case of equal exponents.

For the second term in (33) we get using (15),

$$\frac{1}{\|\nabla g\|^2}\left(\langle H_g u,u\rangle \operatorname{tr} H_g^T - \|H_g^T u\|^2\right) \geq (2n-4)K\min,$$

where Kmin is the minimal sectional curvature of the ellipsoid $g^{-1}(0)$ in $R^{2n} \cong C^n$. Then the second term in the lower bound for Ric(u) in (32) follow from

(34) $$K\min = \frac{a_n}{a_1^2}$$

To see that let u,v be orthonormal tangent vectors of the ellipsoid at $z \in g^{-1}(0)$. Then

$$K(u,v) = \frac{\langle H_g u,u\rangle\langle H_g v,v\rangle - \langle H_g u,v\rangle^2}{\|\nabla g\|^2}$$

From (28) we obtain

$$\|\nabla g\|^2 = 4 \sum_{k=1}^{n} \frac{|z_k|^2}{a_k^2} \le \frac{4}{a_n} \sum_{k=1}^{n} \frac{|z_k|^2}{a_k} = \frac{4}{a_n}$$

So

(35) $$\|\nabla g\|^2 \le \frac{4}{a_n},$$

with equality holding for $z_0 = (0,\ldots,0,\sqrt{a_n})$. On the other hand, setting $u'_k = \alpha_k u_k$, $v'_k = \alpha_k v_k$ for $1 \le k \le 2n$, $\alpha_k = \sqrt{\frac{2}{a_k}}$ for $1 \le k \le n$, $\alpha_{k+n} = \alpha_k$,

$$\langle H_g u,u\rangle\langle H_g v,v\rangle - \langle H_g u,v\rangle^2 =$$

$$\langle u',u'\rangle\langle v',v'\rangle - \langle u',v'\rangle^2 = \sum_{1\le k<\ell\le 2n} (u'_k v'_\ell - u'_\ell \cdot v'_k)^2 =$$

$$\sum_{1\le k<\ell\le 2n} \alpha_k^2\alpha_\ell^2(u_k v_\ell - u_\ell\ v_k)^2 \ge \min_{k<\ell} \alpha_k^2\alpha_\ell^2 \sum_{1\le k<\ell\le 2n} (u_k u_\ell - u_\ell v_k)^2$$

$$= \min_{k<\ell} \alpha_k^2\alpha_\ell^2,$$

So

$$\langle H_g u,u\rangle\langle H_g v,v\rangle - \langle H_g u,v\rangle^2 \ge \frac{4}{a_1^2}$$

using the Lagrange identity twice and observing that u,v orthonormal. Equality holds at $z_0 = (0,\ldots,0,\sqrt{a_n})$ for the tangent vectors $u = (1,0,\ldots,0)$, $v = iu$ of the ellipsoid. Hence,

$$K(u,v) \geq \frac{a_n}{a_1^2} = K\min.$$

Remark In the case $a_{n-1} = a_n$, the estimate for the second term in (32) can be substantially improved as follows:

$$(32'')\quad \mathrm{Ric}(u) \geq -3\frac{\|H\|^2}{\|\nabla f\|^2} + \frac{a_n}{a_1}\left(\frac{1}{a_1} + 2\sum_{k=2}^{n-2}\frac{1}{a_k} + \frac{1}{a_n}\right),$$

where the sum $\sum_{k=2}^{n-2}\frac{1}{a_k}$ is understood to be zero for $n = 3$.

Proof: Choosing an orthonormal basis $e_1,\ldots,e_{2n}$ for $R^{2n} \cong C^n$ such that $e_1,\ldots,e_{2n-3}$ span the tangent space of V' at z, we have in (33),

$$\frac{1}{2}\,\mathrm{tr}\,H_g^T = \frac{1}{2}\,\mathrm{tr}\,H_g - \frac{1}{2}\sum_{k=2n-2}^{2n}\langle H_g e_i, e_i\rangle \geq$$

$$\geq \frac{1}{2}\,\mathrm{tr}\,H_g - \frac{3}{a_n} = 2\left(\frac{1}{a_1} + \ldots + \frac{1}{a_n}\right) - \frac{3}{a_n} =$$

$$= 2\left(\frac{1}{a_1} + \ldots + \frac{1}{a_{n-1}}\right) - \frac{1}{a_n} = \rho > 0.$$

Since $\|H_g^T u\|^2 \leq \|H_g u\|^2$

$$\langle H_g u,u\rangle\,\mathrm{tr}\,H_g^T - \|H_g^T u\|^2 \geq$$

$$\langle(2\rho H_g - H_g^2)u,u\rangle \geq 4\,\min_k \frac{1}{a_k}\left(\rho - \frac{1}{a_k}\right) = 4\frac{1}{a_1}\left(\rho - \frac{1}{a_1}\right),$$

using $a_{n-1} = a_n$. Combining this estimate with (35) yields (32").

We now turn to a result on the Ricci curvature of Brieskorn varieties defined by homogeneous polynomials.

Theorem III.3.

For any $p \geq 2$, there exist an integer $N(p)$ such that in any dimension $n \geq N(p)$, $V'(a_1,\ldots,a_n)$, $a_1 = a_2 = a_n = p$, has strictly positive Ricci curvature.

Proof: Fixing $p \geq 2$ it suffices to show

$$(36) \qquad \frac{2n-4}{p} > 2 \sup_{z \in V'} \frac{\|H\|^2}{\|\nabla f\|^2}$$

for n sufficiently large, according to (32'). To prove this we shall construct a suitable upper bound for $\frac{\|H\|^2}{\|\nabla f\|^2}$.

From (27) and (31) we obtain

$$\frac{\|H\|^2}{\|\nabla f\|^2} = \frac{(\max|z_k|^{p-2})^2}{\frac{1}{(p-1)^2}\sum_{k=1}^{n}|z_k|^{2(p-1)}} = (p-1)^2 \frac{(\max_k |z_k|^2)^{p-2}}{\sum_{k=1}^{n} |z_k|^{2(p-1)}} .$$

Setting $\rho_k = \frac{|z_k|^2}{p}$ we have $\sum_{k=1}^{n} \rho_k = 1$ by (23). We may assume $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_n \geq 0$. Then

$$(37) \qquad \frac{\|H\|^2}{\|\nabla f\|^2} = \frac{(p-1)^2}{p} \frac{\rho_1^{p-2}}{\sum_{k=1}^{n} \rho_k^{p-1}} = \frac{(p-1)^2}{p} A .$$

To maximize (37) we have to minimize the reciprocal

$$\frac{1}{A} = \rho_1\left(1 + \left(\frac{\rho_2}{\rho_1}\right)^{p-1} + \ldots + \left(\frac{\rho_n}{\rho_1}\right)^{p-1}\right)$$

under the above constraints. Put $0 \le \delta_k = \frac{\rho_k}{\rho_1} \le 1$ for $2 \le k \le n$. So we have to consider

$$\frac{1}{A} = \rho_1(1 + \delta_2^{p-1} + \ldots + \delta_n^{p-1})$$

with the constraint

$$\rho_1(1 + \delta_2 + \ldots + \delta_n) = 1, \ \frac{1}{n} \le \rho_1 \le 1.$$

One easily verifies that $\frac{1}{A}$ assumes its minimum necessarily when $\delta_2 = \ldots = \delta_n$. Hence,

$$\frac{1}{A} \ge \min \alpha(x), \ \alpha(x) = \frac{1+(n-1)x^{p-1}}{1+(n-1)x}, \ 0 \le x = \delta_2 \le 1.$$

Since $\alpha(0) = \alpha(1) = 1$ and $x^{p-1} \le x$, α assumes its minimum at some interior point $0 < x_0 < 1$. Observe $\alpha(x) \equiv 1$ for $p = 2$, so let $p \ge 3$. Differentiation yields that x_0 is the unique positive root of the equation

$$(p-2)(n-1)x^{p-1} + (p-1)x^{p-2} - 1 = 0.$$

Therefore,

$$\frac{1}{A} \ge \alpha(x_0) = (p-1)x_0^{p-2} . \tag{38}$$

As we cannot deal explicitly with x_0 for $p \geq 5$, we shall find a lower bound for $\alpha(x_0)$, which suffices to prove the theorem. Clearly,

$$A \leq \frac{1}{p-1} y_0^{p-2}, \text{ where } y_0 = \frac{1}{x_0}$$

satisfies

$$Q(y) = y^{p-1} - (p-1)y - (p-2)(n-1) = 0.$$

It follows that $\frac{1}{p-1} y_0^{p-2} = 1 + \frac{p-2}{p-1} (n-1)\frac{1}{y_0}$, so

$$(39) \qquad A \leq 1 + \frac{p-2}{p-1} (n-1) \frac{1}{y_0}$$

we have to find a lower bound for y_0. Since Q has only one positive root and $Q(0) < 0$, we have $y < y_0$ whenever $Q(y) < 0$. Choose $y_1 = [(p-2)(n-1)]^{\frac{1}{p-1}}$, clearly $Q(y_1) < 0$. Hence, (39) implies

$$(40) \qquad A \leq 1 + \frac{p-2}{p-1} (n-1) \frac{1}{y_1} = 1 + \frac{1}{p-1} [(p-2)(n-1)]^{\frac{p-2}{p-1}}.$$

Combining (37) and (40) yields (36), which completes the argument.

We will now derive a general result for a large class of Brieskorn varieties.

Theorem III.4.

Given arbitrary integers $a_1 \geq a_2 \geq \ldots \geq a_m \geq 2$, then there exist an integer $N(a_1,\ldots,a_m)$ such that the Brieskorn variety $V'(a_1,\ldots,a_m, a_{m+1},\ldots,a_n)$, $a_{m+1} = \ldots = a_{m+n} = 2$, has strictly positive Ricci curvature whenever $n \geq N(a_1,\ldots,a_m)$.

Proof: As in the proof of the last theorem, it suffices to show

$$(39) \qquad \frac{4}{a_1^2}(n+m-2) > 3 \sup_{z\in V'} \frac{\|H\|^2}{\|\nabla f\|^2},$$

for n sufficiently large, according to (32). We shall show that $\frac{\|H\|^2}{\|\nabla f\|^2}$ has an upper bound which only depends on $a_1,\ldots,a_m$ (and not on n). Then from (39) follows. Clearly, $\|H\|^2 = (\max|z_k|^{a_k-2})^2 = \max_k |z_k|^{2(a_k-2)} < a_1^{a_1-2}$

Since $|z_k|^2 < a_k$ for all k by (23). On the other hand,

$$\|\nabla f\|^2 = \sum_{k=1}^{m} \frac{|z_k|^{2(a_k-1)}}{(a_k-1)^2} + \sum_{k=m+1}^{m+n} |z_k|^2 .$$

Using again (23), we conclude that if $\sum_{k=m+1}^{m+n} |z_k|^2 \leq 1$, then $\sum_{k=1}^{m} \frac{|z_k|^2}{a_k} \geq \frac{1}{2}$, so $\sum_{k=1}^{m} \frac{|z_k|^{2(a_k-1)}}{(a_k-1)^2}$ has a positive minimum $B(a_1,\ldots,a_m)$ on the compact set in C^m defined by $\frac{1}{2} \le \sum_{k=1}^{m} \frac{|z_k|^2}{a_k} \le 1$. Hence

$$\|\nabla f\|^2 \ge \min(B(a_1,\ldots,a_m),1).$$

Remark.

The original proof of this theorem is geometrical using the fact that the orthogonal group $O(n)$ acts transitively on the Grassmannian $G_{n,2}$ of 2-planes in R^n.

IV. Further Remarks

We shall briefly discuss the scope of the results proved in the last section and sketch some applications. We mention first that there are Brieskorn varieties which do not admit <u>any</u> metric with positive Ricci curvature (in fact, not even with non-negative Ricci curvature). This follows from Orlik's result in [7] saying that the fundamental group π_1 of $V(a_1, a_2, a_3)$ is infinite for $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \leq 1$ and from Myer's classical theorem according to which π_1 must be finite if there exist a metric of positive Ricci curvature. In higher dimensions, similar counterexamples are not known. However, it is easy to see that $V'(a_1, \ldots, a_n)$ with $a_1 = \ldots = a_n = p$ has negative Ricci curvature at $z = \sqrt{\frac{p}{n}}\,(1, \ldots, 1, -1, \ldots, -1)$ for n even, p odd, and $\dim V' = 2n-3 < 2(p-1)^2$. Thus, $Ric(v) < 0$ for $0 \neq v \perp i\nabla g$ and $Ric(i\nabla g) > 0$. The scalar curvature of V' at z is strictly negative.

On the other hand, working more explicitly with our estimate that appears in the proof of Theorem III.3, one obtains that the Ricci curvature of $V(p, \ldots, p)$ is positive for

$$n > [2(p-1)^2]^{p-1} + 2$$

Detailed computations give better bounds, for example

$N(3) \leq 11$. The fact that the Ricci tensor of $V'(p,\ldots,p)$ is only necessarily positive if the degree p of f is small compared to the dimension n, looks somewhat related to the behavior of the first Chern class of the associated projective variety (over which V' is a circle bundle).

In this context we mention that the "affine quadric" $V'(2,\ldots,2)$ in the Stiefel manifold of real 2-frames in R^n with some homogeneous metric. The computation of the Ricci curvature is a very special and simple case in our general frame work. For a unit tangent vector u at $z \in V'$, $n \geq 3$, we obtain from (21),

$$\mathrm{Ric}(u) = (n-3) + \langle v, \frac{i}{\sqrt{2}} z\rangle^2 .$$

Hence, $n - 3 \leq \mathrm{Ric}(u) \leq n - 2$,

where the bounds are assumed at any point.

In the general case of Theorem III.4, a very rough explicit lower bound for the number of squares that have to be added to yield positive Ricci curvature, is given by

$$N(a_1,\ldots,a_m) \leq (2ma_1)^{a_1}$$

If one is interested in substantially better bounds, then it seems to be more efficient to deal with each example in question separately, rather than to work in the general case which is difficult from a numerical point of view.

For example, if $m = 1$, then $N(3) \leq 5$. The general estimates given in Section III cannot be improved by too much, and even fairly optimal estimates for special examples indicate that $N(a_1,\ldots,a_m)$ will be very large compared to a_1. Aside from this, it does not seem to be possible to generalize Theorem III.4. in any obvious way, say to add cubes instead of squares. Upper bounds for $\frac{\|H\|^2}{\|\nabla f\|^2}$ will depend on the dimension of V' such that (32) cannot be satisfied. We also remark that using

$$g(z) = \sum_{k=1}^{n} \frac{z_k \cdot \overline{z_k}}{a_k} - r^2$$ with some other radius $r > 0$

in (23) does not make much difference in the basic estimates. In fact, $r = 1$ appears to be fairly optimal.

Of course, the essential contents of theorem III.4 is the existence of a very large infinite class of compact manifolds among Brieskorn varieties which carry a metric with strictly positive Ricci curvature in a fairly natural way. Even though we get only finitley many examples in a given dimension n, their number increases rapidly with n. Also, in a certain sense, we can interpret the result to the extent that almost all Brieskorn varieties admit positive Ricci curvature. This is because we may start out with any $V(a_1,\ldots,a_m)$, and "adding squares" makes $V(a_1,\ldots,a_m,2,\ldots,2)$ very similar to $V(a_1,\ldots,a_m)$, except that the dimension rises and topological invariants are shifted as for suspensions.

As an important application, we consider exotic spheres. It is known that every odd dimensional homotopy sphere that bounds a parallelizable manifold is diffeomorphic to some Brieskorn variety $V(a_1,\ldots,a_m)$. For example, the varieties $V(6k-1,3,2,2,2)$, $1 \leq k \leq 28$, represent all the 28 differentiable structures on the 7-sphere. There is a simple condition on $a_1,\ldots,a_m$ such that $V(a_1,\ldots,a_m)$ is a topological sphere, and differentiable structures can be distinguished by invariants computable from $a_1,\ldots,a_m$; compare [1] for details. It follows from that easily that if $V(a_1,\ldots,a_m)$ is an exotic sphere, m odd, then so is $V(a_1,\ldots,a_m,2,2)$. Hence, adding an even number of squares, if m odd, produces distinct exotic spheres from distinct exotic spheres to start with. Moreover, in fact adding an even number of squares in $V(6k-1,3)$, $k \geq 1$, gives already all the exotic spheres that bound parallelizable manifolds in the corresponding dimensions (congruent 1 modulo 4). In the remaining dimensions (congruent -1 modulo 4) there is at most one exotic sphere among Brieskorn varieties, the Kervaire sphere $V(3,2,\ldots,2)$. Applying our results, we obtain:

Theorem IV.1

Among the exotic spheres of odd dimension that bound a parallizable manifold there are infinitely many that admit a metric with strictly positive Ricci curvature. These examples include in particular all the Kervaire spheres in dimension $4k-1$. Moreover, in dimension $4k+1$ there are at least $\nu_k \geq 0$ such exotic spheres, where ν_k is weakly increasing and $\nu_k \to \infty$ for $k \to \infty$.

In contrast to that, Hitchin [6] pointed out that there are examples of exotic spheres in infinitely many dimensions which do not even admit any metric of positive scalar curvature. The very interesting problem whether or not there are exotic spheres with positive sectional curvature K is still unsolved. However, recently Gromoll and Meyer [4] constructed a metric on the Milnor sphere $\Sigma^7 \cong V(5,3,2,2,2)$ with $K \geq 0$, where the set of points with $K > 0$ for all planes is open and dense. They make use of a completely different description of Σ^7 and intrinsic methods. Their example has strictly positive Ricci curvature by the way, which we cannot prove since our bounds are not strong enough in this case.

There are not many topological consequences one can derive for a compact manifold with positive Ricci curvature. Our application is (using Morse theory, [8]) that the loop spaces of $V(a_1,\ldots,a_m,2,\ldots,2)$ and $V(p,p,\ldots,p)$ have the homotopy type of a finite CW-complex with finitely many cells attached in each dimension (n large).

We conclude by mentioning that we have also studied the scalar curvature s of the Brieskorn varieties V'. It is not true that always $s \geq 0$, compare examples given above. It seems one can only quantitatively improve the results of Section III in the case of scalar curvature, which we will discuss elsewhere.

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