PERIOD RELATIONS ON COMPACT RIEMANN SURFACES

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ABSTRACT

Let S be a compact Riemann surface of genus $g \geqq 1$, $\gamma_1, \gamma_2, \ldots, \gamma_g; \delta_1, \delta_2, \ldots, \delta_g$ (abbreviated by (γ, δ)) a canonical homology basis on S, Π the period matrix of S and (γ, δ), u a mapping from S to its Jacobi variety J(S) for a fixed base point P_o on S and $n \geqq 2$ a positive integer.

As a generalization of Abel's theorem, a divisor ζ is a divisor of a multiplicative function f on S with characteristic

$$\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}_n = \begin{bmatrix} \frac{\epsilon_1}{n} & \cdots & \frac{\epsilon_g}{n} \\ \frac{\epsilon'_1}{n} & \cdots & \frac{\epsilon'_g}{n} \end{bmatrix}$$

if and only if its degree $d[\zeta] = 0$ and $u(\zeta) \equiv \binom{-\epsilon}{\epsilon'}$, where $\binom{\nu}{\nu'}_n$ denotes an nth period in J(S).

For any two rational characteristics $\left[\begin{smallmatrix} \epsilon \\ \epsilon' \end{smallmatrix}\right]_n$ and $\left[\begin{smallmatrix} \mu \\ \mu' \end{smallmatrix}\right]_n$, the quotient of two Riemann theta functions $\theta\left[\begin{smallmatrix} \epsilon \\ \epsilon' \end{smallmatrix}\right]_n (u(p), \Pi)$ and $\theta\left[\begin{smallmatrix} \mu \\ \mu' \end{smallmatrix}\right]_n (u(p), \Pi)$ is a multiplicative function on S with characteristic $\left[\begin{smallmatrix} \epsilon-\mu \\ -\epsilon'+\mu' \end{smallmatrix}\right]_n$. If $n = 2$, then its characteristic is $\left[\begin{smallmatrix} \epsilon+\mu \\ \epsilon'+\mu' \end{smallmatrix}\right]$.

Constructing an n-sheeted smooth covering $\hat{S}$ of genus $\hat{g} = n(g-1) + 1$ over S by the method invented by Farkas and Rauch, a canonical homology basis $(\hat{\gamma}, \hat{\delta})$ on $\hat{S}$ can be chosen in a natural way by the lifts of (γ, δ) on S onto $\hat{S}$.

Then there is a basis dv_{jk}, $j = 1, 2, \ldots, n-1$, $k = 1, 2, \ldots, g-1$, for the vector space of multiplicative differentials on S with

characteristic $\left[\begin{smallmatrix}00\ldots0\\ j0\ldots0\end{smallmatrix}\right]_n$ normalized by

$\int_{\gamma_{m+1}} dv_{jk} = \delta_{km}$, $\int_{\delta_{m+1}} dv_{jk} = \tau_{jkm}$, $m = 1,2,\ldots,g-1$. $\tau_j = (\tau_{jkm})$ has positive definite imaginary part for each j, $j = 1,2,\ldots,n-1$. The period matrix $\hat{\Pi}$ of $\hat{S}$ and $(\hat{\gamma},\hat{\delta})$ is

$$\hat{\Pi} = \begin{array}{c} \\ \end{array}\overset{\hspace{1em}\hat{\delta}_1 \hspace{8em} \hat{\delta}_{\ell m}}{\left[\begin{array}{c|c} n\,\Pi_{11} & \Pi_1 \\ \hline \Pi_1 & \frac{1}{n}[\Pi_0 + \sum_{h=1}^{n-1} \omega^{h(\ell-j)}\tau_h] \end{array}\right]}$$

where $\ell = 0,1,2,\ldots,n-1$,
$m = 1,2,3,\ldots,g-1$,
$\omega = \exp\left[\frac{2\Pi i}{n}\right]$,

$$\Pi = \left[\begin{array}{c|c} \Pi_{11} & \Pi_1 \\ \hline \Pi_1 & \Pi_0 \end{array}\right]$$

Now, using the symmetry of $\hat{\Pi}$, ${}^t\tau_j = \tau_{n-j}$. If $n = 2$, ${}^t\tau_1 = \tau_1$ and $\tau_1 \in \mathfrak{S}_{g-1}$ (the Siegel upper-half plane of degree $g-1$). If $g = 2$, $\tau_j = {}^t\tau_j = \tau_{n-j}$, and hence there are $[\frac{n}{2}]$ distinct τ_j's such that $\tau_j \in \mathfrak{S}_1$ for all j, where $[x] = \max\{y = \text{integer} \mid y \leqq x\}$ for a real number x.

For a particular case $n = 4$, in addition to $2^{2(g-1)-1}(2^{2(g-1)}-1)$ even theta constants $\hat{\hat{\theta}}\left[\begin{smallmatrix}0\alpha\alpha\\ 1\beta\beta\end{smallmatrix}\right] = 0$ on $\hat{\hat{S}}$ for any odd $2(g-1)$-characteristic $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$, proved by Farkas and Rauch, $2^{g-2}(2^{g-1}-1)$ even theta constants $\hat{\hat{\theta}}\left[\begin{smallmatrix}0\delta & \delta & \delta & \delta\\ 1\delta' & \delta' & \delta' & \delta'\end{smallmatrix}\right] = 0$ on $\hat{\hat{S}}$ for any odd $(g-1)$-characteristic $\left[\begin{smallmatrix}\delta\\ \delta'\end{smallmatrix}\right]$. Noting that $\hat{\hat{S}}$ is a two-sheeted smooth covering over $\hat{S}$ which is also a two-sheeted smooth covering over S,

$$\frac{\hat{\eta}^2\left[\begin{smallmatrix}\delta & \delta\\ \delta' & \delta'\end{smallmatrix}\right](\hat{\tau})}{\prod_{j=0}^{3} \theta\left[\begin{smallmatrix}0\delta\\ \frac{j}{2}\delta'\end{smallmatrix}\right](\Pi)} = \text{const}$$

for any even (g-1)-characteristic $\left[{\delta \atop \delta'}\right]$, where $\hat{\eta}$ is Schottky theta constant associated with $\hat{S}$ and $(\hat{\gamma},\hat{\delta})$ and

$$\hat{\tau} = \begin{bmatrix} \frac{\tau_1+\tau_3}{2} & \frac{i(\tau_1-\tau_3)}{2} \\ \frac{i(\tau_3-\tau_1)}{2} & \frac{\tau_1+\tau_3}{2} \end{bmatrix} \in \mathfrak{S}_{2(g-1)}.$$

If $g = 2$, then $\hat{\tau} = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_1 \end{pmatrix} = \begin{pmatrix} \tau_3 & 0 \\ 0 & \tau_1 \end{pmatrix} \in \mathfrak{S}_2$.

For a hyperelliptic Riemann surface S of genus $g \geqq 4$, a period relation of Schottky type, $\sqrt{r_1} \pm \sqrt{r_2} \pm \sqrt{r_3} = 0$, holds such that for $g = 4$ and 5 it becomes a period relation of Schottky type exhibited by Farkas and Rauch. For a compact Riemann surface S of genus $g = 6$, a period relation $\sum_{k=1}^{6} \pm\sqrt{r_k} = 0$ of Schottky type holds. For $g = 7$, $\sum_{k=1}^{10} \pm\sqrt{r_k} = 0$ holds. In each case, r_k's are the products of eight theta constants.

For a hyperelliptic Riemann surface S of $w^2 = \prod_{k=1}^{2g+2} (z-\lambda_k)$ of genus $g \geqq 1$, $\theta^4[u(\lambda_1)] = \sum_{k=1}^{g+1} \theta^4[u(\lambda_{2k})]$.

For a compact Riemann surface S of $w^5 = \prod_{\ell=1}^{5} (z-a_\ell)$ (or $w^5 = \prod_{\ell=1}^{5} (z-a_\ell)^4$) of genus $g = 6$, $\theta[K]$ is a vanishing even theta constant with the vanishing order 2, where K is the vector of Riemann constants with respect to any branch point A_ℓ on S over $z = a_\ell$, $\ell = 1,2,3,4,5$.

For a compact Riemann surface S of $w^3 = \prod_{\ell=1}^{6} (z-a_\ell)$

(or $w^3 = \prod_{\ell=1}^{6} (z-a_\ell)^2$) of genus $g = 4$, $\theta[K]$ is the only one vanishing even theta constants with the vanishing order 2, where K is also the vector of Riemann constants with respect to any branch point A_ℓ on S over $z = a_\ell$. Choosing a particular canonical homology basis $\gamma_1, \gamma_2, \gamma_3, \gamma_4; \delta_1, \delta_2, \delta_3, \delta_4$ on S, $K \equiv \binom{1111}{1111}$.

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CONTENTS

INTRODUCTION

For a compact Riemann surface S of genus $g \geqq 1$ endowed with a canonical homology basis $\gamma_1, \gamma_2, \ldots, \gamma_g; \delta_1, \delta_2, \ldots, \delta_g$ (simply abbreviated by (γ, δ)), where γ_i and δ_i are one-cycles on S satisfying $KI(\gamma_i, \gamma_j) = KI(\delta_i, \delta_j) = 0$, $KI(\gamma_i, \delta_j) = \delta_{ij}$, $i, j = 1, 2, \ldots, g$, KI denoting the (skew-symmetric, bilinear, integral-valued) intersection number, the normalized basis $du_1, du_2, \ldots, du_g$ for the vector space $A_1(S)$ of abelian differentials of the first kind on S over the complex number field C is uniquely determined with respect to (γ, δ) by $\int_{\gamma_j} du_i = \delta_{ij}$, $\Pi_{ij}[S, (\gamma, \delta)] = \int_{\delta_j} du_i$, $i, j = 1, 2, \ldots, g$. The $g \times g$ matrix $\Pi[S, (\gamma, \delta)] = (\Pi_{ij}[S, (\gamma, \delta)])$ is called the period matrix of S and (γ, δ).

It was recognized in Riemann's work more than a century ago and was recently proved by Rauch [18, Theorem 3] explicitly that $\Pi_{ij}[S, (\gamma, \delta)]$ are holomorphic functions of $3g-3$ complex parameters, "the" moduli, for $g \geqq 2$, of 1 complex parameter for $g = 1$, for a non-hyperelliptic Riemann surface S, and that they are of $2g-1$ complex parameters for a hyperelliptic Riemann surface S of genus $g \geqq 2$. Consequently, there are $\frac{(g-2)(g-3)}{2}$ holomorphic relations for a non-hyperelliptic Riemann surface S of genus $g \geqq 4$ and $\frac{(g-1)(g-2)}{2}$ holomorphic relations for a hyperelliptic Riemann surface S of genus $g \geqq 3$, among $\Pi_{ij}[S, (\gamma, \delta)]$.

One of the classical problems in the theory of compact Riemann surfaces is to formulate such relations, i.e, period relations. In 1886, Schottky [27] first succeeded in deriving a relation for $g = 4$, and then Schottky and Jung [29] conjectured a similar relation for higher genera. Their original idea was to establish such relations through (Riemann) theta functions, more precisely theta constants, associated with S and (γ,δ). However, those relations were very recently proved and formulated by Rauch and Farkas in the forms of the vanishings of the explicit homogeneous polynomials in theta constants, after establishing the proportionality of the squares of one set of theta constants, the Schottky constants, to certain two-term products of another set, the theta constants, both sets associated with S and a definite canonical homology basis (γ,δ) [5,9,10,11,20,24, and 25].

They also suggested the method how to obtain a period relation for $g > 5$ as a structural generalization for $g = 4$ and 5. But, they noted that the method does not necessarily give all the $\frac{(g-2)(g-3)}{2}$ relations among $\frac{g(g+1)}{2}$ periods. This was one of the remarkable works which have been done for this classical problem so far.

On the other hand, Andreotti and Mayer [4] proved that the existence of Polynomials in theta constants and their derivatives whose vanishings imply all the period relations in quite a different way from Rauch and Farkas.

At any rate, theta functions and, particularily, theta constants associated with S and (γ,δ) are interesting subjects to study for its own.

A modern account of the properties of these theta functions and theta constants is to be found in Lewittes' paper [17], where, in addition to new results, we can find the first correct function-theoretic proofs of some delicate but hitherto obscure assertions of Riemann.

In chapter I, we will primarily study theta functions and theta constants with rational characteristics. Prior to this, some general observations on multiplicative functions and differentials on a compact Riemann surface will be made to have a close look at the behaviors of theta functions and theta constants.

In Chapter II, we will discuss more about the smooth coverings $\hat{S}$ of given S, constructed in a similar way by Farkas and Rauch, and about the propartionality of Schottky constants to theta constants on S with a definite canonical homology basis (γ,δ), along with the Propartionality proved by Accola [1] for a particular case.

In Chapter III, we will derive a period relation on a hyperelliptic Riemann surface S of genus $g \geqq 4$, which has an interesting and noteworthy structure. In additon, period relations of Schottky type for $g = 6$ and 7 will be exhibited by using the method suggested by Farkas and Rauch earlier, and it will be discussed how the latter reduce to the former for $g = 6$ and 7, including for $g = 5$ exhibited by Farkas and Rauch [9,10,11]. A theta identity in 4th power will also be derived for a hyperelliptic Riemann surface S of genus $g \geqq 1$.

In chapter IV, the observations on the three-sheeted branched coverings over the sphere will be made to find vanishing even theta constants on it. Since theta constants are functions of the periods, the vanishings of such even theta constants impose the conditions on the periods, and consequently on "the" moduli, on the surface, while all the odd theta constants are always vanishing. In particular, we will prove that the only one even theta constant does vanish on the Riemann surface of an algebraic function w satisfying $w^3 = \prod_{\ell=1}^{6} (z-a_\ell)$, which is conformally equivalent to the Riemann surface of an algebraic function w satisfying $w^3 = \prod_{\ell=1}^{6} (z-a_\ell)^2$, of genus $g = 4$, and that an even theta constant does vanish on the Riemann surface of w satisfying $w^5 = \prod_{\ell=1}^{5} (z-a_\ell)$, which is conformally equivalent to the Riemann surface of w satisfying $w^5 = \prod_{\ell=1}^{5} (z-a_\ell)^4$, of genus $g = 6$. Finally, choosing a particular canonical homology basis (γ, δ) on the Riemann surface of $w^3 = \prod_{\ell=1}^{6} (z-a_\ell)$ (or $w^3 = \prod_{\ell=1}^{6} (z-a_\ell)^2$) of genus $g = 4$, we will find out the only one vanishing even theta constant associated with the surface and (γ,δ).

As the general references for the theory of Riemann surfaces, we may offer [30] and [34], in particular, to be familiar with functions, differentials, divisors and Abel's theorem, etc, on a compact Riemann surface.

CHAPTER I THETA FUNCTIONS WITH RATIONAL CHARACTERISTICS

1-1 Multiplicative functions and differentials.

We consider a compact Riemann surface S of genus $g \geqq 1$ endowed with a canonical homology basis $\gamma_1, \gamma_2, \ldots, \gamma_g; \delta_1, \delta_2, \ldots, \delta_g$ on S. Then there are normalized abelian differentials $du_1, du_2, \ldots, du_g$ of the first kind on S, uniquely determined by the given homology basis, such that

(1) $\int_{\gamma_j} du_i = \delta_{ij}$, $i, j = 1, 2, \ldots, g$.

It is known that $g \times g$ matrix $\Pi = (\Pi_{ij})$, where

(2) $\Pi_{ij} = \int_{\delta_j} du_i$, $i, j = 1, 2, \ldots, g$,

which we call it the period matrix (by contrast, we call $g \times 2g$ matrix (I_g, Π), where I_g is a $g \times g$ identity matrix, the full period matrix) of S and the given homology basis is complex symmetric with positive definite imaginary part. The set of all such matrices is generally called the Siegel (or generalized) upper half-plane $\mathfrak{S}_g$ of degree (or genus) g. 2g colums of the full period matrix of S are linearly independent over the reals and generates a discrete abelian subgroup L of the space C^g of g complex variables, where C is the complex number field. The quotient group C^g/L is called the Jacobi variety J(S) of S and J(S) is a compact abelian group.

<u>Definition 1.</u>

An integral linear combination of 2g columns $e^{(j)}$, $\Pi^{(j)}$, $j = 1, 2, \ldots, g$, of the full period matrix (I_g, Π), i.e.,

(3) $\epsilon'_1 e^{(1)} + \dots + \epsilon'_g e^{(g)} + \epsilon_1 \Pi^{(1)} + \dots + \epsilon_g \Pi^{(g)}$,

where ϵ_j, ϵ'_j, $j = 1,2,\dots,g$, are integers, is called a period in J(S). We will denote it by $\{{\epsilon \atop \epsilon'}\} = \{{\epsilon_1 \dots \epsilon_g \atop \epsilon'_1 \dots \epsilon'_g}\}$.

For a positive integer $n \geqq 2$, the one nth of a period $\{{\epsilon \atop \epsilon'}\}$ is called an nth period, denoted by $({\epsilon \atop \epsilon'})_n$, i.e.,

(4) $({\epsilon \atop \epsilon'})_n = \frac{1}{n} \{{\epsilon \atop \epsilon'}\}$.

In particular, $({\epsilon \atop \epsilon'})_2$ is called a half period and we will omit the subscript 2, denoting it simply by $({\epsilon \atop \epsilon'})$.

Any two g-vectors u_1 and u_2 in C^g which differ by a period, i.e., there is a period $\{{\epsilon \atop \epsilon'}\}$ such that $u_1 - u_2 = \{{\epsilon \atop \epsilon'}\}$, are said to be congruent (mod. periods), denoted by $u_1 \equiv u_2$. With this congruence, we have a mapping u from S into J(S) defined by

(5) $u(p) = (\int_{p_0}^{p} du_1, \int_{p_0}^{p} du_2, \dots, \int_{p_0}^{p} du_g)$

for each point p on S, where p_0 is a fixed point on S as the base point and a path of integration from P_0 to P differs by a period and hence u has a well defined image in J(S) for each point P on S. Furthermore, this map u can be extended to an arbitrary divisor $\zeta = P_1^{n_1} \dots P_m^{n_m}$ on S by

(6) $u(\zeta) = (u_i(\zeta)) = (\sum_{j=1}^{m} n_j \int_{p_0}^{p} du_i)$, $i = 1,2,\dots,g$, where

n_j, $j = 1,2,\dots,m$, are integers.

<u>Definition 2.</u>

For a positive integer $n \geqq 2$, 2 x g matrix

(7) $\left[{\epsilon \atop \epsilon'}\right]_n = \begin{bmatrix} \frac{\epsilon_1}{n} & \cdots & \frac{\epsilon_g}{n} \\ \frac{\epsilon'_1}{n} & \cdots & \frac{\epsilon'_g}{n} \end{bmatrix}$,

where ϵ_j, ϵ'_j, $j = 1,2,\ldots,g$, are integers, is called a (nth integer g-) characteristic. If

(8) $0 \leqq \epsilon_j, \epsilon'_j \leqq n-1$, $j = 1,2,\ldots,g$,

then $\left[{\epsilon \atop \epsilon'}\right]_n$ is called a (reduced nth integer g-) characteristic. In particular, for $n = 2$, $\left[{\epsilon \atop \epsilon'}\right]_2$ is called a half integer g-characteristic and we will again omit the subscript 2, denoting it by $\left[{\epsilon \atop \epsilon'}\right]$.

Thus an nth integer g-characteristic $\left[{\epsilon \atop \epsilon'}\right]_n$ can be considered as a symbol of an nth period $\left({\epsilon \atop \epsilon'}\right)_n$.

Definition 3.

The character $\left|{\epsilon \atop \epsilon'}\right|_n$ of $\left[{\epsilon \atop \epsilon'}\right]_n$ is, according to [14, 15], defined by

(9) $\left|{\epsilon \atop \epsilon'}\right|_n = \exp\left[\frac{2\pi i}{n} \sum_{j=1}^{g} \epsilon_j \epsilon'_j\right] = \exp\left[\frac{2\pi i}{n} (\epsilon \cdot \epsilon')\right]$.

Any half integer g-characteristic $\left[{\epsilon \atop \epsilon'}\right]$ is called even or odd depending on whether $\epsilon \cdot \epsilon' = \sum_{j=1}^{g} \epsilon_j \epsilon'_j \equiv 0$ or $1 \pmod{2}$, equivalently $\left|{\epsilon \atop \epsilon'}\right|_2 = \left|{\epsilon \atop \epsilon'}\right| = 1$ or -1.

For any given nth integer g-characteristic $\left[{\epsilon \atop \epsilon'}\right]_n$, we can replace it by its reduced nth integer g-characteristic by replacing ϵ_j, ϵ'_j, $j = 1,2,\ldots,g$, by its least non-negative residues modulo n. It is obvious that there are n^g reduced nth integer g-characteristics, and there are $2^{g-1}(2^g+1)$ even and $2^{g-1}(2^g-1)$ odd reduced half integer g-characteristics. Moreover, any half integer g-characteristic is even or odd together with its reduced characteristic.

Definition 4.

A meromorphic multi-valued function f on S is said to be multiplicative with (nth) characteristic $\left[{\epsilon \atop \epsilon'}\right]_n = \left[{\epsilon_1 \ldots \epsilon_g \atop \epsilon'_1 \ldots \epsilon'_g}\right]_n$, where n is a positive integer $\geqq 2$ and ϵ_j, ϵ'_j, $j = 1,2,\ldots,g$, are integers, if a continuation of f along γ_j (δ_j) carries f to $\exp[\frac{2\pi i}{n}\epsilon_j]\cdot f$ ($\exp[\frac{2\pi i}{n}\epsilon'_j]\cdot f$). Similarily, a meromorphic multi-valued differential dv on S is said to be multiplicative with (nth) characteristic $\left[{\epsilon \atop \epsilon'}\right]_n = \left[{\epsilon_1 \ldots \epsilon_g \atop \epsilon'_1 \ldots \epsilon'_g}\right]_n$, where n is a positive integer $\geqq 2$ and ϵ_j, ϵ'_j, $j = 1,2,\ldots,g$, integers, if a continuation of dv along γ_j (δ_j) carries to $\exp[\frac{2\pi i}{n}\epsilon_j].dv$ ($\exp[\frac{2\pi i}{n}\epsilon'_j].dv$).

Since for any integer m, there are integers k and r such that $m = nk + r$, $0 \leqq r \leqq n-1$, and $\exp[\frac{2\pi i}{n}m] = \exp[2\pi ki].\exp[\frac{2\pi i}{n}r] = \exp[\frac{2\pi i}{n}r]$, the characteristic $\left[{\epsilon \atop \epsilon'}\right]_n$ of a multiplicative function and differential is to be understood as the one with $0 \leqq \epsilon_j, \epsilon'_j \leqq n-1$, $j = 1,2,\ldots,g$. Such a characteristic will be called a (reduced nth) charactertic.

Any meromorphic function and abelian differential can be considered as a multiplicative function and differential with characteristic $\left[{\epsilon \atop \epsilon'}\right]_n = \left[{0 \atop 0}\right]_n$.

Theorem 1.

Let S be a compact Riemann surface of genus $g \geqq 1$ and $n \geqq 2$ a postive integer.

A necessary and sufficient condition for a divisor ζ to be a divisor (f) of a multiplicative function f on S with characteristic $\left[{\epsilon \atop \epsilon'}\right]_n$ is that its degree $d[\zeta] = 0$ and $u(\zeta) \equiv \left({-\epsilon \atop \epsilon'}\right)_n$.

The proof of above theorem is completely a generalization of the proof of Abel's theorem and will be omitted here. In the assertion above, if $n = 2$, then $u(\zeta) \equiv \left({-\epsilon \atop \epsilon'}\right) \equiv \left({\epsilon \atop \epsilon'}\right)$. This is a result quoted by Farkas in [9]. Furthermore, if $\left[{\epsilon \atop \epsilon'}\right]_n = \left[{0 \atop 0}\right]_n$, then it is, in fact, Abel's theorem and we may consider Theorem 1 as a generalization of Abel's theorem.

Lemma 1.

For given any integral divisor $\zeta = p_1 p_2 \ldots p_g$ of degree $d[\zeta] = g \geqq 1$, there is a multiplicative function f on S with characteristic $\left[{\epsilon \atop \epsilon'}\right]_n \neq \left[{0 \atop 0}\right]_n$, whose divisor (f) is a multiple of $\frac{1}{\zeta}$.

Proof.

$u(\zeta) = \sum_{i=1}^{g} u(p_i)$ is in C^g (more precisely, in J(S)), and by the Jacobi inversion problem there is a divisor $\omega = Q_1 Q_2 \ldots Q_g$ of degree g such that $u(\omega) \equiv u(\zeta) + \left({-\epsilon \atop \epsilon'}\right)_n$ and $\omega \neq \zeta$. By Theorem 1, there is a multiplicative function f on S with characteristic $\left[{\epsilon \atop \epsilon'}\right]_n$ and its divisor $(f) = \frac{\omega}{\zeta}$, and hence (f) is a multiple of $\frac{1}{\zeta}$.

Lemma 2.

Let $L_{[\epsilon \atop \epsilon']_n}(\frac{1}{\zeta})$ be the vector space of multiplicative functions on S with characteristic $[\epsilon \atop \epsilon']_n$ whose divisors are multiple of $\frac{1}{\zeta}$ for an integral divisor ζ of degree $d[\zeta] = r$ and $n \geqq 2$, and $r_{[\epsilon \atop \epsilon']_n}[\frac{1}{\zeta}]$ the dimension of $L_{[\epsilon \atop \epsilon']_n}(\frac{1}{\zeta})$.

Then either $r_{[\epsilon \atop \epsilon']_n}[\frac{1}{\zeta}] = 0$ or $r_{[\epsilon \atop \epsilon']_n}[\frac{1}{\zeta}] = d[\zeta] - g + 1 + i[\zeta(f)]$, where f is an arbitrary function in $L_{[\epsilon \atop \epsilon']_n}(\frac{1}{\zeta})$.

Proof.

Suppose $r_{[\epsilon \atop \epsilon']_n}[\frac{1}{\zeta}] \neq 0$. Let $f_1, f_2, \ldots, f_k$ be a basis for $L_{[\epsilon \atop \epsilon']_n}(\frac{1}{\zeta})$. For each j, $j = 1,2,\ldots,k$, $\zeta(f_j)$ is an integral divisor, and hence $(\frac{f_j}{f_1})\zeta(f_1)$ is an integral divisor. Now, k linearly independent functions $1, \frac{f_2}{f_1}, \ldots, \frac{f_k}{f_1}$ are elements of $L(\frac{1}{\zeta(f_1)})$, and by Riemann-Roch theorem $r[\frac{1}{\zeta(f_1)}] = d[\zeta(f_1)] - g + 1 + i[\zeta(f_1)] = d[\zeta] - g + 1 + i[\zeta(f_1)]$. Hence $k \leqq d[\zeta] - g + 1 + i[\zeta(f_1)]$. On the other hand, $f_1 f$ is an element of $L_{[\epsilon \atop \epsilon']_n}(\frac{1}{\zeta})$ for any element in $L(\frac{1}{\zeta(f_1)})$, and hence $k \geqq d[\zeta] - g + 1 + i[\zeta(f_1)]$. If f is an arbitrary function in $L_{[\epsilon \atop \epsilon']_n}(\frac{1}{\zeta})$, then $\frac{f}{f_1}$ is a meromorphic function on S and (f) is equivalent to (f_1). Therefore $\zeta(f)$ is equivalent to $\zeta(f_1)$ and $i[\zeta(f)] = i[\zeta(f_1)]$. This completes the proof.

In Lemma 2, if $d[\zeta] = r \geqq g$ and $[^{\epsilon}_{\epsilon'}]_n \neq [^0_0]_n$, $n \geqq 2$, then by Lemma 1 we see that $r_{[^{\epsilon}_{\epsilon'}]_n}[\frac{1}{\zeta}] \neq 0$.

Definition 5.

A multiplicative differential on S with characteristic $[^{\epsilon}_{\epsilon'}]_n$, $n \geqq 2$, which is everywhere finite is called a multiplicative differential of the first kind on S with characteristic $[^{\epsilon}_{\epsilon'}]_n$. We will denote the vector space of all such differentials by $\Omega_{[^{\epsilon}_{\epsilon'}]_n}(1)$, where 1 stands for a unit divisor, and its dimension by $i_{[^{\epsilon}_{\epsilon'}]_n}[1]$.

Lemma 3.

Let S be a compact Riemann surface of genus $g \geqq 2$, $n \geqq 2$ a positive integer and $[^{\epsilon}_{\epsilon'}]_n \neq [^0_0]_n$ an arbitrary characteristic.

Then the degree of a divisor of a multiplicative differential with characteristic $[^{\epsilon}_{\epsilon'}]_n$ is 2g-2.

Proof.

Let du be an abelian differential of the first kind on S. Then $d[(du)] = 2g-2 \geqq g$ and by Lemma 2 $r_{[^{\epsilon}_{\epsilon'}]_n}[\frac{1}{(du)}] = g-1 \geqq 1$. Suppose that dv is a multiplicative differential on S with characteristic $[^{\epsilon}_{\epsilon'}]_n$ and that f is an element in $L_{[^{\epsilon}_{\epsilon'}]_n}(\frac{1}{(du)})$. Then $\frac{dv}{f}$ is an abelian differential on S, and hence $d[(\frac{dv}{f})] = d[(dv)] - d[(f)] = d[(dv)] = 2g-2$.

Lemma 4.

Let S be a compact Riemann surface of genus $g \geqq 2$, $n \geqq 2$ a positive integer and $[{\epsilon \atop \epsilon'}]_n \neq [{0 \atop 0}]_n$ an arbitrary characteristic.

Then $i_{[{\epsilon \atop \epsilon'}]_n}[1] = g-1$.

Proof.

For an abelian differential du of the first kind on S, $d[(du)] = 2g-2 \geqq 2$ and $r_{[{\epsilon \atop \epsilon'}]_n}[\frac{1}{(du)}] = g-1$ by Lemma 2. If $f_1, f_2, \ldots, f_{g-1}$ is basis for $L_{[{\epsilon \atop \epsilon'}]_n}(\frac{1}{(du)})$, then $f_1 du$, $f_2 du, \ldots, f_{g-1} du$ are linearly independent elements in $\Omega_{[{\epsilon \atop \epsilon'}]_n}(1)$, and hence $i_{[{\epsilon \atop \epsilon'}]_n}[1] \geqq g-1$. On the other hand, if dv_1, $dv_2, \ldots, dv_k$ is a basis for $\Omega_{[{\epsilon \atop \epsilon'}]_n}(1)$, then $1, \frac{dv_2}{dv_1}, \ldots, \frac{dv_k}{dv_1}$ are linearly independent elements in $L(\frac{1}{(dv_1)})$, and hence by Riemann-Roch theorem $i_{[{\epsilon \atop \epsilon'}]_n}[1] = k \leqq r[\frac{1}{(dv_1)}] = g-1$.

Definition 6. A multiplicative function on S with characteristic $[{\epsilon \atop \epsilon'}]_n$, $n \geqq 2$, is called special if its divisor of poles has degree $\leqq g-1$.

Lemma 5.

Let f be a special multiplicative function with charactertic $[{\epsilon \atop \epsilon'}]_n \neq [{0 \atop 0}]_n$ on a compact Riemann surface S of genus $g \geqq 2$.

Then f can be expressed as the quotient of a multiplicative differential of the first kind with characteristic $[{\epsilon \atop \epsilon'}]_n$ and an abelian differential of the first kind on S.

Proof.

Let a divisor of f be $\frac{\omega}{\zeta}$. $d[\zeta] \leqq g-1$ by the definition of f, and hence $i[\zeta] \geqq 1$. Then there exists an abelian differential du of the first kind on S such that $\frac{(du)}{\zeta}$ is integral. fdu is a multiplicative differential of the first kind with characteristic $[{\varepsilon \atop \varepsilon'}]_n$, say dv, and $f = \frac{dv}{du}$.

We note that all the results in this section for n = 2 can be found in [9] by Farkas.

1-2 Theta functions with rational characteristics.

Definition 7.

Let $u = (u_1, u_2, \ldots, u_g) \in C^g$, $T = (t_{ij}) \in \mathfrak{S}_g$, $G = (G_1, G_2, \ldots, G_g)$ and $H = (H_1, H_2, \ldots, H_g) \in C^g$.
The function defined by

$$(10) \quad \theta[{G \atop H}](u,T) = \sum_{N \in Z^g} \exp 2\pi i[\tfrac{1}{2}(N+G)T(N+G)+(N+G)(u+H)]$$

is called (the first order) theta function with characteristic $[{G \atop H}]$ and theta matrix T, where Z is the ring of integers.

The theta constant with characteristic $[{G \atop H}]$ at T is

$$(11) \quad \theta[{G \atop H}](0,T) = \theta[{G \atop H}].$$

$\theta[{G \atop H}](u,T)$ converges absolutely and uniformly on compact subsets of $C^g \times \mathfrak{S}_g$, and hence it is an analytic function on $C^g \times \mathfrak{S}_g$.

In particular, if $[{G \atop H}] = [{\varepsilon \atop \varepsilon'}]_2 = [{\varepsilon \atop \varepsilon'}]$ (see Definitions 2 and 3), $\theta[{\varepsilon \atop \varepsilon'}](u,T)$ is even or odd function of u depending on whether $[{\varepsilon \atop \varepsilon'}]$ is even or odd. Consequently, all the odd theta constants vanish at any $T \in \mathfrak{S}_g$, while the even theta constants are not in general. Therefore, all mentions of theta constants will mean even theta constants. There are 2^g first order theta functions with half integer characteristics of which $2^{g-1}(2^g+1)$ are even and $2^{g-1}(2^g-1)$ are odd.

Definition 8.

The period matrix of the theta functions with characteristic $[{G \atop H}]$ and matrix T is g x 2g matrix (I_g, T). With this period matrix, we define the periods $\{{\epsilon \atop \epsilon'}\}$ and the nth periods $({\epsilon \atop \epsilon'})_n$, $n \geqq 2$, by exactly the same ways in Definition 1.

For any $n \geqq 2$, the following properties of theta functions with characteristic $[{\epsilon \atop \epsilon'}]_n = \begin{bmatrix} \frac{\epsilon_1}{n} & \cdots & \frac{\epsilon_g}{n} \\ \frac{\epsilon'_1}{n} & \cdots & \frac{\epsilon'_g}{n} \end{bmatrix}$ were given in [14, 15].

(12) Reduction formula; $\theta[{\epsilon+n\mu \atop \epsilon'+n\mu'}]_n(u,T) = \exp[\frac{2\pi i}{n}(\epsilon\cdot\mu')]\theta[{\epsilon \atop \epsilon'}]_n(u,T)$.

(13) Functional equation; $\theta[{\epsilon \atop \epsilon'}]_n(u+\{{\mu \atop \mu'}\},T)$

$$= \exp\pi i[\frac{2}{n}(\epsilon\cdot\mu'-\epsilon'\cdot\mu)-2\mu\cdot u-\mu T\mu]\theta[{\epsilon \atop \epsilon'}]_n(u,T).$$

(14) $\theta[{\epsilon \atop \epsilon'}]_n(-u,T) = \theta[{-\epsilon \atop -\epsilon'}]_n(u,T)$.

Lemma 6.

For any $n,\ m \geqq 2$,

(15) $\theta[{\epsilon \atop \epsilon'}]_n(u+({u \atop \mu'})_m,T) = \exp\pi i[-\frac{1}{m^2}\mu T\mu-\frac{2}{nm}\mu\cdot(\epsilon'+\mu')-\frac{2}{m}\mu\cdot u)]\cdot$
$\theta[{m\epsilon+n\mu \atop m\epsilon'+n\mu'}]_{nm}(u,T)$ for $u \in C^g$ and $T \in \mathfrak{S}_g$.

Proof.

By the definition, since

$({\mu \atop \mu'})_m = \frac{1}{m}(I_g,T)({u' \atop \mu}) = \frac{u'}{m}+T\frac{u}{m}$,

$$\theta[{}^{\epsilon}_{\epsilon'}]_n(u+({}^{u}_{\mu'})_m, T) = \sum_{N\in Z^g} \exp\Pi i[(N+\frac{\epsilon}{n})T(N+\frac{\epsilon}{n})+2(N+\frac{\epsilon}{n})(u+\frac{u'}{m}+T\frac{u}{m}+\frac{\epsilon'}{n})]$$

$$= \sum_{N\in Z^g}\exp\Pi i[(N+\frac{m\epsilon+n\mu}{nm})T(N+\frac{m\epsilon+n\mu}{nm})+2(N+\frac{m\epsilon+n\mu}{nm})(n+\frac{m\epsilon'+n\mu'}{nm})$$

$$-\frac{1}{m^2}\mu T\mu-\frac{2}{nm}\mu\cdot(\epsilon'+\mu')-\frac{2}{m}\mu\cdot u)]$$

$$= \exp\Pi i[-\frac{1}{m^2}\mu T\mu-\frac{2}{nm}\mu\cdot(\epsilon'+\mu')-\frac{2}{m}\mu\cdot u]\theta[{}^{m\epsilon+n\mu}_{m\epsilon'+n\mu'}]_{nm}(u,T).$$

We note that (15) is true for any complex g-vectors ϵ, ϵ', μ and μ', not necessarily for integer g-vectors.

<u>Corollary 1.</u> (Substitution formula)

For any $n \geqq 2$,

(16) $$\theta[{}^{\epsilon}_{\epsilon'}]_n(u+({}^{u}_{\mu'})_n, T) = \exp\Pi i[-\frac{1}{n^2}\mu T\mu-\frac{2}{n^2}\mu\cdot(\epsilon'+\mu')-\frac{2}{n}\mu\cdot u]\cdot$$

$$\theta[{}^{\epsilon+\mu}_{\epsilon'+\mu'}]_n(u,T) \text{ for } u \in C^g \text{ and } T \in \mathfrak{S}_g.$$

<u>Proof.</u>

Letting $n = m$ in (15), (16) is immediate consequence.

<u>Corollary 2.</u>

For $n \geqq 2$,

(17) $$\frac{\theta[{}^{\epsilon}_{\epsilon'}]_n(u+({}^{\mu}_{\mu'})_n, T)}{\theta[{}^{\delta}_{\delta'}]_n(u+({}^{\mu}_{\mu'})_n, T)} = \exp\frac{2\Pi i}{n^2}[\mu\cdot(\delta'-\epsilon')]\ \frac{\theta[{}^{\epsilon+\mu}_{\epsilon'+\mu'}]_n(u,T)}{\theta[{}^{\delta+\mu}_{\delta'+\mu'}]_n(u,T)}.$$

<u>Proof.</u>

By (16), (17) is trivial.

Lemma 7.

For any n, m $\geqq$ 2, if $\binom{\epsilon}{\epsilon'}_n = \binom{\mu}{\mu'}_n + \{{\nu \atop \nu'}\}$,

$$(18)\quad \theta[\binom{\epsilon}{\epsilon'}_n+\binom{\delta}{\delta'}_m](u,T) = \exp 2\Pi i[\tfrac{1}{n}(\mu\cdot\nu')+\tfrac{1}{m}(\delta\cdot\nu')]\theta[\binom{\mu}{\mu'}_n+\binom{\delta}{\delta'}_m](u,T).$$

Proof.

Let ℓ be the least common multiple of n and m, and $\ell = nn' = mm'$. Then

$$\binom{\epsilon}{\epsilon'}_n+\binom{\delta}{\delta'}_m = \binom{\mu}{\mu'}_n+\{{\nu \atop \nu'}\}+\binom{\delta}{\delta'}_m = \binom{n'u}{n'\mu'}_{nn'}+\binom{m'\delta}{m'\delta}_{mm'}+\binom{\ell\nu}{\ell\nu'}_\ell$$

$$= \binom{n'\mu+m'\delta+\ell\nu}{n'\mu'+m'\delta+\ell\nu'}_\ell, \text{ and}$$

$$\theta[\binom{\epsilon}{\epsilon'}_n+\binom{\delta}{\delta'}_m](u,T) = \theta[{n'\mu+m'\delta+\ell\nu \atop n'\mu'm'\delta'+\ell\nu'}]_\ell(u,T)$$

$$= \exp\frac{2\Pi i}{\ell}[(n'\mu+m'\delta)\cdot\nu']\theta[{n'\mu+m'\delta \atop n'\mu'+m'\delta'}]_\ell(u,T)$$

(by (12))
$$= \exp 2\Pi i[\tfrac{1}{n}(\mu\cdot\nu')+\tfrac{1}{m}(\delta\cdot\nu')]\theta[\binom{\mu}{\mu'}_n+\binom{\delta}{\delta'}_m](u,T).$$

(18) is a modified one of the Reduction formula (12).

Definition 9.

Let S be a compact Riemann surface of genus g $\geqq$ 2 with a canonical homology basis $\gamma_1,\gamma_2,\ldots,\gamma_g$; $\delta_1,\delta_2,\ldots,\delta_g$ on it.

The (Riemann) theta function with characteristic $[{G \atop H}]$, G, H $\in C^g$, associated with S and (γ,δ) is the first order theta function replaced $u \in C^g$ by u(p), where p $\in$ S and u : S $\to$ J(S), and T by Π.

We review here some of the important facts about the Riemann theta functions on S for future discussion. For any characteristic $[{G \atop H}]$, $\theta[{G \atop H}](u(p),\Pi)$ is either identically zero on S or it has a divisor $\zeta = p_1 p_2 \cdots p_g$ of zeros on S such that $u(\zeta)+K \equiv -H-\Pi G$, where K is the vector of Riemann constants depending on the base point $P_0 \in S$ and (γ,δ). Any integral divisor Δ of degree $2g-2$ is a divisor of an abelian differential of the first kind on S if and only if $u(\Delta) \equiv -2K$. All the zeros of $\theta[{0 \atop 0}](u(p),\Pi) = \theta(u(p),\Pi)$ are of the form $W^{g-1}+K$, where W^{g-1} is the image in $J(S)$ under u of all integral divisors of degree $g-1$. In particular, $\theta(K) = 0$.

We need also to recall the Riemann vanishing theorem; Let $e \in C^g$. If $\theta(e) \not\equiv 0$, then there exists a unique integral divisor ζ of degree g such that $e \equiv u(\zeta) + K$ and $i(\zeta) = 0$. If $\theta(e) = 0$, then there is an integral divisor ζ of degree $g-1$ such that $e \equiv u(\zeta)+K$ and $i(\zeta) = s$, where s, $1 \leqq s \leqq g-1$, is the least integer with $\theta(W^s-W^s-e) \not\equiv 0$. Furthermore, all partial derivatives of θ of order less than s vanish at e, while at least one partial derivative of order s does not vanish at e. The integer s is the same for both e and $-e$, and it is called the vanishing order of θ at e.

The proofs of these facts are found in [17], and [14, 15, 24, 26] are recommended for further references about various results of the theta functions.

Lemma 8.

For any characteristics $[{\epsilon \atop \epsilon'}]_n$ and $[{\mu \atop \mu'}]_n$, $n \geqq 2$,

$\dfrac{\theta[{\epsilon \atop \epsilon'}]_n(u(p),\Pi)}{\theta[{\mu \atop \mu'}]_n(u(p),\Pi)}$ is multiplicative function on S with characteristic $[{\epsilon-\mu \atop -\epsilon'+\mu'}]_n$.

Consequently, $\dfrac{\theta^n[{\epsilon \atop \epsilon'}]_n(u(p),\Pi)}{\theta^n[{\mu \atop \mu'}]_n(u(p),\Pi)}$ is a meromorphic function on S.

Proof.

Analytically continuing along γ_k, $k = 1,2,\ldots,g$, by (13)

$$\frac{\theta[{\epsilon \atop \epsilon'}]_n(u(p)+\{{0\ldots\ldots\ldots 0 \atop 0\ldots 010\ldots 0}\},\Pi)}{\theta[{\mu \atop \mu'}]_n(u(p)+\{{0\ldots\ldots\ldots 0 \atop 0\ldots 010\ldots 0}\},\Pi)} = \frac{\exp[\frac{2\Pi i}{n}\epsilon_k]\theta[{\epsilon \atop \epsilon'}]_n(u(p),\Pi)}{\exp[\frac{2\Pi i}{n}\mu_k]\theta[{\mu \atop \mu'}]_n(u(p),\Pi)}$$

$$= \exp[\frac{2\Pi i}{n}(\epsilon_k-\mu_k)]\frac{\theta[{\epsilon \atop \epsilon'}]_n(u(p),\Pi)}{\theta[{\mu \atop \mu'}]_n(u(p),\Pi)}$$

Similarily along δ_k, $k = 1,2,\ldots,g$,

$$\frac{\theta[{\epsilon \atop \epsilon'}]_n(u(p)+\{{0\ldots 010\ldots 0 \atop 0\ldots\ldots\ldots 0}\},\Pi)}{\theta[{\mu \atop \mu'}]_n(u(p)+\{{0\ldots 010\ldots 0 \atop 0\ldots\ldots\ldots 0}\},\Pi)} = \frac{\exp[\frac{2\Pi i}{n}(-\epsilon'_k)]\theta[{\epsilon \atop \epsilon'}]_n(u(p),\Pi)}{\exp[\frac{2\Pi i}{n}(-\mu'_k)]\theta[{\mu \atop \mu'}]_n(u(p),\Pi)}$$

$$= \exp[\frac{2\Pi i}{n}(-\epsilon'_k+\mu'_k)]\frac{\theta[{\epsilon \atop \epsilon'}]_n(u(p),\Pi)}{\theta[{\mu \atop \mu'}]_n(u(p),\Pi)}.$$

Therefore, $\dfrac{\theta[{\epsilon \atop \epsilon'}]_n(u(p),\Pi)}{\theta[{\mu \atop \mu'}]_n(u(p),\Pi)}$ has the characteristic $[{\epsilon-\mu \atop -\epsilon+\mu'}]_n$ and the single-valuedness of $\dfrac{\theta^n[{\epsilon \atop \epsilon'}]_n(u(p),\Pi)}{\theta^n[{\mu \atop \mu'}]_n(u(p),\Pi)}$ is obvious.

Noting that either $\theta[{\epsilon \atop \epsilon'}]_n(u(p),\Pi) \equiv 0$ on S or it has a divisor ζ of zeros of degree g on S such that $u(\zeta)+K \equiv -({\epsilon \atop \epsilon'})_n$,

$$u\left(\left(\frac{\theta[{}^{\varepsilon}_{\varepsilon'}]_n(u(p),\Pi)}{\theta[{}^{\mu}_{\mu'}]_n(u(p),\Pi)}\right)\right) = \left({}^{-\varepsilon+\mu}_{-\varepsilon'+\mu'}\right)_n.$$

For the particular case $n = 2$, its characteristic and the image under u of its divisor are same, i.e., $[{}^{\varepsilon+\mu}_{\varepsilon'+\mu'}]$ and $({}^{\varepsilon+\mu}_{\varepsilon'+\mu'})$.

CHAPTER II SMOOTH COVERINGS OF COMPACT RIEMANN SURFACES

2-1 Period matrices of smooth coverings. We consider a compact Riemann surface S of genus $g \geqq 2$ with a canonical homology basis $\gamma_1, \gamma_2, \ldots, \gamma_g; \delta_1, \delta_2, \ldots, \delta_g$. For our convenience, we will denote $\gamma_2 = \gamma_{01}, \ldots, \gamma_g = \gamma_{0(g-1)}; \delta_2 = \delta_{01}, \ldots, \delta_g = \delta_{0(g-1)}$ and the uniquely determined normalized basis for the vector space $A_1(S)$ of Abelian differentials of the first kind on S by $du_1, du_{01}, \ldots, du_{0(g-1)}$ with respect to the given homology basis (γ, δ).

Let $n \geqq 2$ be a positive integer and $\omega = \exp[\frac{2\pi i}{n}]$. For each j, $j = 1, 2, \ldots, n-1$, by Lemma 4 there is a basis $dv_{j1}, dv_{j2}, \ldots, dv_{j(g-1)}$ for $\Omega_{\left[\begin{smallmatrix}00\ldots.0\\ j0\ldots.0\end{smallmatrix}\right]_n}(1)$, which is the vector space of multiplicative differentials of the first kind on S with characteristic $\left[\begin{smallmatrix}00..0\\ j0..0\end{smallmatrix}\right]_n$, such that

$$\int_{\gamma_{0m}} dv_{jk} = A_{jkm},$$

$$\int_{\delta_{0m}} dv_{jk} = B_{jkm}, \qquad j = 1, 2, \ldots, n-1,$$
$$k, m = 1, 2, \ldots, g-1.$$

At this point, we construct a compact Riemann surface $\hat{S}$ which will be an n-sheeted smooth covering of S. For $n = 2$, this kind of surface $\hat{S}$ was constructed by Farkas and Rauch [9,10,11,24,25]. We follow a similar construction for $\hat{S}$.

More precisely, we take first n copies of S (as a surface in a space), then cut all the copies along γ_1, and then join them along the boundaries, identifying the points on the boundaries, in the obvious manner.

In this way, we have a compact Riemann surface $\hat{S}$ of genus $\hat{g}$ which is an n-sheeted smooth covering of S. By Riemann-Hurwitz formula, $2(\hat{g}-1) = 2n(g-1)$, and hence $\hat{g} = n(g-1)+1$.

Furthermore, it is possible for us to choose a canonical homology basis $(\hat{\gamma},\hat{\delta})$on $\hat{S}$ from the canonical homology basis (γ,δ) on S in the natural way. $\hat{S}$ admits a fixed point free automorphism T of order n, i.e. a covering transformation, which is just the interchange of the "sheets" cyclically in $\hat{S}$.

If we denote a covering map from $\hat{S}$ onto S by f, we can lift the differentials du_1, du_{0k} ; dv_{jk}, $k = 1,2,\ldots,g-1$ and $j = 1,2,\ldots,n-1$, on S through f to the differentials du_1^*, du_{0k}^* ; dv_{jk}^* on $\hat{S}$ defined by

$$du_1^*(\hat{p}) = du_1(f(\hat{p})),$$

$$du_{0k}^*(\hat{p}) = du_{0k}(f(\hat{p})),$$

$$dv_{jk}^*(\hat{p}) = \omega^{j\ell} dv_{jk}(f(\hat{p})), \text{ if } \hat{p} \text{ is on the } \ell\text{th copy of S},$$

where $\ell = 0,1,2,\ldots,n-1$.

These important properties are assembled in the following lemmata.

<u>Lemma 9</u>.

$\hat{S}$ admits a fixed point free automorphism T of order n such that a canonical homologybasis $(\hat{\gamma},\hat{\delta})$ on $\hat{S}$ satisfies the relations

$$(20)\quad T(\hat{\gamma}_1) \sim \hat{\gamma}_1, \qquad T(\hat{\delta}_1) \sim \hat{\delta}_1,$$
$$T(\hat{\gamma}_{\ell m}) \sim \hat{\gamma}_{(\ell+1)m}, \qquad T(\hat{\delta}_{\ell m}) \sim \hat{\delta}_{(\ell+1)m},$$
$$T(\hat{\gamma}_{(n-1)m} \sim \hat{\gamma}_{Om}, \qquad T(\hat{\delta}_{(n-1)m}) \sim \hat{\delta}_{Om},$$

where $\ell = 0,1,2,\ldots,n-1$; $m = 1,2,\ldots,g-1$, and "$\sim$" means homologous to.

du_1^*, du_{OK}^*; dv_{jk}^*, $j = 1,2,\ldots,n-1$, $k = 1,2,\ldots,g-1$, are linearly independent abelian differentials of the first kind on $\hat{S}$ with

$$(21)\quad T(du_1^*) = du_1^*,$$
$$T(du_{OK}^*) = du_{OK}^*$$
$$T(dv_{jk}^*) = \omega^{n-j} dv_{jk}^*,$$

where T is an induced linear transformation acting on $A_1(\hat{S})$ by a covering transformation T on $\hat{S}$, defined by $T(dv) = dv(T^{-1})$ for any $dv \in A_1(\hat{S})$.

<u>Proof</u>.

Except (21), it is clear by the construction of $\hat{S}$ from S. Lewittes proved in [16] that any automorphism T on $\hat{S}$ induces a linear transformation T (we use the same notation with the automorphism T) on $A_1(\hat{S})$ defined by $T(dv) = dv(T^{-1})$ for any $dv \in A_1(\hat{S})$. For each j, k and $\hat{p} \in \hat{S}$,

$$(T(dv_{jk}^*)(\hat{p}) = dv_{jk}^*(T^{-1}(\hat{p})) = \omega^{n-j} dv_{jk}^*(\hat{p}),$$

since $T^{-1}(\hat{p})$ and $\hat{p}$ can be joined by a curve (homologous to δ_1 on S) on $\hat{S}$.

Theorem 2.

Let S be a compact Riemann surface of genus $g \geqq 2$ and $n \geqq 2$ a positive integer.

Then for each j, $j = 1,2,\ldots,n-1$, there exists a basis dv_{jk}, $k = 1,2,\ldots,g-1$ for $\Omega[{}^{00..0}_{j0..0}]_n$ (1) such that

(22) $\int_{\gamma_{m+1}} dv_{jk} = \int_{\gamma_{0m}} dv_{jk} = \delta_{km}$, $m = 1,2,\ldots,g-1$.

Proof.

For each j, let dv_{jk}, $k = 1,2,\ldots,g-1$, be a basis for $\Omega[{}^{00..0}_{j0..0}]_n$ (1) by Lemma 4 and let $A_{jkm} = \int_{\gamma_{0m}} dv_{jk} = \int_{\gamma_{m+1}} dv_{jk}$ and $A_j = (A_{jkm})$.

Then dv_{jk}^* are abelian differentials of the first kind on $\hat{S}$. Furthermore,

(23) $\int_{\hat{\gamma}_1} dv_{jk}^* = \int_{T(\hat{\gamma}_1)} T(dv_{jk}^*) = \omega^{n-j} \int_{\hat{\gamma}_1} dv_{jk}^*$,

and hence

(24) $\int_{\hat{\gamma}_1} dv_{jk}^* = 0$,

since $\omega^{n-j} = \exp[-\frac{2\pi i}{n} j] \neq 0$.

Similarily, for each ℓ, $\ell = 0,1,2,\ldots,n-1$,

(25) $\int_{\hat{\gamma}_{\ell m}} dv_{jk}^* = \int_{T^{n-\ell}(\hat{\gamma}_{\ell m})} T^{n-\ell}(dv_{jk}^*) = \omega^{n-j(n-\ell)} \int_{\hat{\gamma}_{0m}} dv_{jk}^*$

$$= \omega^{j\ell} \int_{\gamma_{0m}} dv_{jk} = \omega^{j\ell} A_{jkm}.$$

In the computations, we used a well known fact that $\int_C du = \int_{T(C)} T(du)$ for any closed curve C, abelian differential du of the first kind and automorphism T on a compact Riemann surface S (Lewittes [16]).

Now, we have a following table of periods of dv_{jk}^* on $\hat{\gamma}_{\ell m}$ on $\hat{S}$;

$$
\begin{array}{lcc} & \hat{\gamma}_1 & \hat{\gamma}_{\ell m} \\ (26) \quad dv_{jk}^* & [0 & \omega^{j\ell} A_{jkm}]. \end{array}
$$

We want to show that $A_j = (A_{jkm})$ is non-singular, i.e., $\det A_j \neq 0$. If $\det A_j = 0$, then there is a non-trivial solution $(C_{j1}, C_{j2}, \ldots, C_{j(g-1)})$ for a simultaneous homogeneous linear equations $\sum_{k=1}^{g-1} C_{jk} A_{jkm} = 0$, $m = 1, 2, \ldots, g-1$.

Let $dv_j = \sum_{k=1}^{g-1} C_{jk} dv_{jk}$. Then dv_j is a non-zero multiplicative differential of the first kind on S with characteristic $\left[\begin{smallmatrix} 00..0 \\ j0..0 \end{smallmatrix}\right]_n$ and the lift dv_j^* on $\hat{S}$ of dv_j on S through a covering map f is an abelian differential of the first kind on $\hat{S}$ by Lemma 9. Now,

$$\int_{\hat{\gamma}_1} dv_j^* = \int_{\hat{\gamma}_1} \sum_{k=1}^{g-1} C_{jk} dv_{jk}^* = \sum_{k=1}^{g-1} C_{jk} \int_{\hat{\gamma}_1} dv_{jk}^* = 0,$$

and

$$\int_{\hat{\gamma}_{\ell m}} dv_j^* = \int_{\hat{\gamma}_{\ell m}} \sum_{k=1}^{g-1} C_{jk} dv_{jk}^* = \sum_{k=1}^{g-1} C_{jk} \int_{\hat{\gamma}_{\ell m}} dv_{jk}^* = \sum_{k=1}^{g-1} C_{jk} \omega^{j\ell} A_{jkm}$$

$$= \omega^{j\ell} \left(\sum_{k=1}^{g-1} C_{jk} A_{jkm} \right) = 0.$$

Since all the $\hat{\gamma}$-periods of dv_j^* on $\hat{S}$ are zero, dv_j^* is identically zero on $\hat{S}$. This implies that dv_j is identically zero on S, in particular, which is a contradiction.

Consequently, $A_j = (A_{jkm})$ is non-singular and there exists $A_j^{-1} = (A'_{jkm})$ for each j.

Let $dv'_{jh} = \sum_{k=1}^{g-1} A'_{jhk} dv_{jk}$, $h = 1,2,\ldots,g-1$.

Then dv'_{jh} are linearly independent in $\Omega\left[\begin{smallmatrix}00\ldots0\\ j0\ldots0\end{smallmatrix}\right]_n (1)$

and

$$\int_{\gamma_{m+1}} dv'_{jh} = \int_{\gamma_{0m}} dv'_{jh} = \int_{\gamma_{0m}} \sum_{k=1}^{g-1} A'_{jhk} dv_{jk} = \sum_{k=1}^{g-1} A'_{jhk} A_{jkm}$$

$$= \delta_{hm}.$$

This completes a proof of the theorem.

<u>Definition 10.</u>

The basis in Theorem 2 is call a normalized basis for $\Omega\left[\begin{smallmatrix}00\ldots0\\ j0\ldots0\end{smallmatrix}\right]_n (1)$ of S and the given homology basis (γ,δ) on S. For this normalized basis, we call $(g-1) \times (g-1)$ matrix $\tau_j = (\tau_{jkm})$, where $\tau_{jkm} = \int_{\delta_{m+1}} dv_{jk} = \int_{\delta_{0m}} dv_{jk}$, $j = 1,2,\ldots,n-1$, $k,m = 1,2,\ldots,g-1$, the period matrix of a normalized basis dv_{jk} for $\Omega\left[\begin{smallmatrix}00\ldots0\\ j0\ldots0\end{smallmatrix}\right]_n (1)$.

We remark that in a discussion from the beginning of this chapter to Lemma 9 a basis dv_{jk} for $\Omega\left[\begin{smallmatrix}00..0\\ j0..0\end{smallmatrix}\right]_n$ can now be assumed normal without any loss of generality.

<u>Lemma 10.</u>

The periods of du_1^*, du_{0k}^*, dv_{jk}^*, $j = 1,2,\ldots,n-1$,

$k = 1,2,\ldots,g-1$, over the canonical homology basis $\hat{\gamma}_1, \hat{\gamma}_{\ell m}; \hat{\delta}_1, \hat{\delta}_{\ell m}, \ell = 0,1,2,\ldots,n-1, m = 1,2,\ldots,g-1$, are

$$(27)\quad \begin{array}{c} \\ du_1^* \\ du_{0k}^* \\ dv_{jk}^* \end{array} \begin{array}{c} \begin{array}{cccc} \hat{\gamma}_1 & \hat{\gamma}_{\ell m} & \hat{\delta}_1 & \hat{\delta}_{\ell m} \end{array} \\ \begin{bmatrix} 1 & 0 & n\Pi_{11} & \Pi_1 \\ 0 & I_{g-1} & n\Pi_1 & \Pi_0 \\ 0 & \omega^{j\ell} I_{g-1} & 0 & \omega^{j\ell}\tau_j \end{bmatrix} \end{array},$$

where $\Pi = \left(\begin{array}{c|c} \Pi_{11} & \Pi_1 \\ \hline \Pi_1 & \Pi_0 \end{array}\right)$ is the period matrix of du_1, du_{0k} of S and (γ,δ) and $\tau_j = (\tau_{jkm})$ is the period matrix of dv_{jk} on S with $\tau_{jkm} = \int_{\delta_{0m}} dv_{jk}$ (Definition 10).

Proof.

Using (20), (21) and (22),

$$\int_{\hat{\gamma}_1} du_1^* = \int_{\gamma_1} du_1 = 1,$$

$$\int_{\hat{\gamma}_{\ell m}} du_1^* = \int_{T^{n-\ell}(\hat{\gamma}_{\ell m})} T^{n-\ell}(du_1^*) = \int_{\hat{\gamma}_{0m}} du_1^* = \int_{\gamma_{0m}} du_1 = 0,$$

$$\int_{\hat{\delta}_1} du_1^* = n\int_{\delta_1} du_1 = n\Pi_{11},$$

$$\int_{\hat{\delta}_{\ell m}} du_1^* = \int_{T^{n-\ell}(\hat{\delta}_{\ell m})} T^{n-\ell}(du_1^*) = \int_{\hat{\delta}_{0m}} du_1^* = \int_{\delta_{0m}} du_1 = \Pi_{1(m+1)}.$$

$$\int_{\hat{\gamma}_1} du_{0k}^* = \int_{\gamma_1} du_{0k} = 0,$$

$$\int_{\hat{\gamma}_{\ell m}} du_{0k}^* = \int_{T^{n-\ell}(\hat{\gamma}_{\ell m})} T^{n-\ell}(du_{0k}^*) = \int_{\hat{\gamma}_{0m}} du_{0k}^* = \int_{\gamma_{0m}} du_{0k} = \delta_{km},$$

$$\int_{\hat{\delta}_1} du_{0k}^* = n\int_{\delta_1} du_{0k} = n\Pi_{(k+1)1} = n\Pi_{1(k+1)},$$

$$\int_{\hat{\delta}_{\ell m}} du_{0k}^* = \int_{T^{n-\ell}(\hat{\delta}_{\ell m})} T^{n-\ell}(du_{0k}^*) = \int_{\hat{\delta}_{0m}} du_{0k}^* = \int_{\delta_{0m}} du_{0k}$$
$$= \Pi_{(k+1)(m+1)},$$

$$\int_{\hat{\gamma}_1} dv_{jk}^{\hat{*}} = \int_{T(\hat{\gamma}_1)} T(dv_{jk}^{\hat{*}}) = \omega^{n-j}\int_{\hat{\gamma}_1} dv_{jk}^{\hat{*}},$$

and hence $\int_{\hat{\gamma}_1} dv_{jk}^{\hat{*}} = 0$, since $\omega^{n-j} \neq 0$.

$$\int_{\hat{\gamma}_{\ell m}} dv_{jk}^{\hat{*}} = \int_{T^{n-\ell}(\hat{\gamma}_{\ell m})} T^{n-\ell}(dv_{jk}^{\hat{*}}) = \omega^{n-j(n-\ell)}\int_{\hat{\gamma}_{0m}} dv_{jk}^{\hat{*}}$$
$$= \omega^{j\ell}\int_{\gamma_{0m}} dv_{jk} = \omega^{j\ell}\delta_{km},$$

$$\int_{\hat{\delta}_1} dv_{jk}^{\hat{*}} = \int_{T(\hat{\delta}_1)} T(dv_{jk}^{\hat{*}}) = \omega^{n-j}\int_{\hat{\delta}_1} dv_{jk}^{\hat{*}},$$

and hence $\int_{\hat{\delta}_1} dv_{jk}^{\hat{*}} = 0$.

$$\int_{\hat{\delta}_{\ell m}} dv_{jk}^{\hat{*}} = \int_{T^{n-\ell}(\hat{\delta}_{\ell m})} T^{n-\ell}(dv_{jk}^{\hat{*}}) = \omega^{n-j(n-\ell)}\int_{\hat{\delta}_{0m}} dv_{jk}^{\hat{*}}$$
$$= \omega^{j\ell}\int_{\delta_{0m}} dv_{jk} = \omega^{j\ell}\tau_{jkm}.$$

Theorem 3.

$$(28)\quad \begin{cases} d\hat{u}_1 = du_1^{\hat{*}}, \\ d\hat{u}_{jk} = \frac{1}{n}[du_{0k}^{\hat{*}} + \sum_{h=1}^{n-1} \omega^{n-hj} dv_{jk}^{\hat{*}}], \quad j = 0,1,2,\ldots,n-1, \\ \qquad\qquad\qquad\qquad\qquad\qquad k = 1,2,\ldots,g-1, \end{cases}$$

is the normal basis for $A_1(\hat{S})$ with a canonical homology basis $(\hat{\gamma},\hat{\delta})$ (Lemma 9), and the period matrix $\hat{\Pi}$ of $\hat{S}$ and $(\hat{\gamma},\hat{\delta})$ is

$$(29)\quad \begin{matrix} d\hat{u}_1 \\ d\hat{u}_{jk} \end{matrix} \overset{\begin{matrix}\hat{\delta}_1 & \qquad\qquad \hat{\delta}_{\ell m}\end{matrix}}{\begin{bmatrix} n\Pi_{11} & \Pi_1 \\ \Pi_1 & \frac{1}{n}[\Pi_0 + \sum_{h=1}^{n-1}\omega^{h(\ell-j)}\tau_h] \end{bmatrix}}, \quad \begin{matrix}\ell = 0,1,2,\ldots,n-1, \\ m = 1,2,\ldots,g-1,\end{matrix}$$

where Π_{11}, Π_1, Π_0 and τ_h are the same as in Lemma 10.

Proof.

This is an immediate consequence of Lemma 10, and we omit here all the computations.

Lemma 11.

${}^t\tau_h = \tau_{n-h}$, $h = 1,2,\ldots,n-1$. In particular, if $g = 2$ then there are $[\frac{n}{2}]$ mutually distinct τ_h's such that $\tau_h = \tau_{n-h}$, and if $n = 2$ then ${}^t\tau_1 = \tau_1$, where ${}^tA$ means a transpose of a matrix A and $[x] = \max\{m \in Z \mid m \leqq x\}$.

Proof.

By Theorem 3, since $\hat{\Pi} \in \mathfrak{S}_{n(g-1)+1}$, i.e., it is complex symmetric, $((j,k),(\ell,m))$ and $((\ell,m),(j,k))$ entries in $\hat{\Pi}$ are equal. Consequently,

$$ {}^t\left(\frac{1}{n}[\Pi_0 + \sum_{h=1}^{n-1} \omega^{h(\ell-j)}\tau_h]\right) = \frac{1}{n}[\Pi_0 + \sum_{h=1}^{n-1} \omega^{h(j-\ell)}\tau_h]. $$

Now,

$$ {}^t\left(\frac{1}{n}[\Pi_0 + \sum_{h=1}^{n-1} \omega^{h(\ell-j)}\tau_h]\right) = \frac{1}{n}[{}^t\Pi_0 + \sum_{h=1}^{n-1} \omega^{h(\ell-j)}\,{}^t\tau_h], $$

$$ = \frac{1}{n}[\Pi_0 + \sum_{h=1}^{n-1} \omega^{h(\ell-j)}\,{}^t\tau_h], $$

$$ \frac{1}{n}[\Pi_0 + \sum_{h=1}^{n-1} \omega^{h(j-\ell)}\tau_h] = \frac{1}{n}[\Pi_0 + \sum_{h=1}^{n-1} \omega^{-h(\ell-j)}\tau_h] $$

$$ = \frac{1}{n}[\Pi_0 + \sum_{h=1}^{n-1} \omega^{(n-h)(\ell-j)}\tau_h] $$

$$ = \frac{1}{n}[\Pi_0 + \sum_{h=1}^{n-1} \omega^{h(\ell-j)}\tau_{n-h}]. $$

This implies that $\sum_{h=1}^{n-1} \omega^{h(\ell-j)}({}^t\tau_h - \tau_{n-h}) = 0.$

In particular, for $\ell = 0$, $\sum_{h=1}^{n-1} \omega^{-hj}({}^t\tau_h - \tau_{n-h}) = 0$.

Noting that $\det(\omega^{-hj}) \neq 0$, $\omega = \exp[\frac{2\pi i}{n}]$, it follows that ${}^t\tau_h - \tau_{n-h} = 0$.

If $g = 2$, then τ_h's are simply 1 x 1 matrices and $\tau_h = {}^t\tau_h = \tau_{n-h}$. Therefore, there are $[\frac{n}{2}]$ τ_h's in this case.

If $n = 2$, then there is the only one τ_1 and ${}^t\tau_1 = \tau_{2-1} = \tau_1$.

We remark that for $n \geqq 2$ ${}^t(\tau_h + \tau_{n-h}) = {}^t\tau_h + {}^t\tau_{n-h} = \tau_{n-h} + \tau_h$, i.e., complex symmetric, and it will be used later. For $n = 2$, ${}^t\tau_1 = \tau_1$ was also proved by Farkas and Rauch [9, 10].

Lemma 12.

For each h, $h = 1,2,\ldots,n-1$, τ_h has a positive definite imaginary part, i.e., $\operatorname{Im} \tau_h \gg 0$.

Proof.

To prove this, we appeal to the Riemann's bilinear relation

$$(30) \quad \int_{\hat{\Gamma}} \lambda\, d\mu = \left(\int_{\hat{\gamma}_1} d\lambda \int_{\hat{\delta}_1} d\mu - \int_{\hat{\delta}_1} d\lambda \int_{\hat{\gamma}_1} d\mu\right) + \sum_{m=1}^{g} \sum_{\ell=1}^{n-1} \left(\int_{\hat{\gamma}_{\ell m}} d\lambda \int_{\hat{\delta}_{\ell m}} d\mu - \int_{\hat{\delta}_{\ell m}} d\lambda \int_{\hat{\delta}_{\ell m}} d\mu\right),$$

where $\hat{\Gamma}$ is the boundary of the simply connected normal form of $\hat{S}$ obtained by a canonical dissection in a usual fashion using the canonical homology basis $(\hat{\gamma}, \hat{\delta})$ on $\hat{S}$, λ is a function on the normal form and $d\mu$ is a differential which is regular in a neighborhood of each point on the boundary $\hat{\Gamma}$.

We take $d\lambda = \sum_{k=1}^{g-1} x_k dv_{hk}^*$ for each h and x_k are reals but not all zero, and $d\mu = \overline{d\lambda}$.

Then

$$\int_{\Gamma}\hat{\lambda}\, d\mu = \int_{\Gamma}\hat{\lambda}\,\overline{d\lambda}$$

$$= \left(\int_{\hat{\gamma}_1} d\lambda \int_{\hat{\delta}_1} \overline{d\lambda} - \int_{\hat{\delta}_1} d\lambda \int_{\hat{\gamma}_1} \overline{d\lambda}\right) + \sum_{m=1}^{g-1} \sum_{\ell=0}^{n-1} \left(\int_{\hat{\gamma}_{\ell m}} d\lambda \int_{\hat{\delta}_{\ell m}} \overline{d\lambda} - \int_{\hat{\delta}_{\ell m}} d\lambda \int_{\hat{\gamma}_{\ell m}} \overline{d\lambda}\right)$$

$$= 2i\,\mathrm{Im}\left[\left(\int_{\hat{\gamma}_1} d\lambda \int_{\hat{\delta}_1} \overline{d\lambda}\right) + \sum_{m=1}^{g-1} \sum_{\ell=0}^{n-1} \left(\int_{\hat{\gamma}_{\ell m}} d\lambda \int_{\hat{\delta}_{\ell m}} \overline{d\lambda}\right)\right]$$

$$= 2i\,\mathrm{Im}\left[\sum_{m=1}^{g-1} \sum_{\ell=0}^{n-1} \left(\sum_{k=1}^{g-1} x_k \omega^{h\ell} \delta_{km}\right) \overline{\left(\sum_{k=1}^{g-1} x_k \omega^{h\ell} \tau_{hkm}\right)}\right]$$

$$= 2i\,\mathrm{Im}\left[\sum_{\ell=0}^{n-1} \sum_{m=1}^{g-1} \sum_{k=1}^{g-1} x_k x_m \overline{\tau_{hkm}}\right]$$

$$= 2ni \sum_{m=1}^{g-1} \sum_{k=1}^{g-1} x_k x_m (\mathrm{Im}\,\overline{\tau_{hkm}})$$

$$= -2ni \sum_{m=1}^{g-1} \sum_{k=1}^{g-1} x_k x_m (\mathrm{Im}\,\tau_{hkm}).$$

On the other hand, since $i\int_{\Gamma}\hat{\lambda}\,\overline{d\lambda} \geqq 0$ and $d\lambda$ is a non-zero abelian differential of the first kind on $\hat{S}$,

$$\sum_{m=1}^{g-1} \sum_{k=1}^{g-1} x_k x_m (\mathrm{Im}\,\tau_{hkm}) > 0,$$

which proves $\mathrm{Im}\,\tau_h \gg 0$.

<u>Theorem 4</u>.

For any $n \geqq 2$, $\tau_h + \tau_{n-h} \in \mathfrak{S}_{g-1}$ for each $h = 1,2,\ldots,n-1$. In particular, if $n = 2$, then $\tau_1 \in \mathfrak{S}_{g-1}$ (in this case, we simply denote τ_1 by τ). If $g = 2$, then $\tau_h \in \mathfrak{S}_1$ for any h such that $\tau_h = \tau_{n-h}$.

Proof.

By Lemma 12, since Im $\tau_h >> 0$ for each h and Im $(\tau_h + \tau_{n-h})$ = Im τ_h + Im τ_{n-h}, Im $(\tau_h + \tau_{n-h}) >> 0$. We already gave a remark that ${}^t(\tau_h + \tau_{n-h}) = \tau_h + \tau_{n-h}$. Hence $\tau_h + \tau_{n-h} \in \mathfrak{S}_{g-1}$. All the other statements are obivious.

2-2 Two-sheeted smooth coverings. In this section, we primarily concern about the case n = 4 in the previous discussion.

Suppose that S is a compact Riemann surface of genus $g \geqq 2$, $\gamma_1, \gamma_{0m}; \delta_1, \delta_{0m}$; m = 1,2,...,g-1, is a canonical homology basis on S, du_1, du_{0k}, k = 1,2,...,g-1, is the uniquely determined normalized basis for $A_1(S)$ with respect to (γ, δ), Π is the period matrix of S and (γ, δ), dv_{jk}, j = 1,2,3, are normalized basi for $\Omega_{\left[\begin{smallmatrix}00..0\\j0..0\end{smallmatrix}\right]_4}(1)$, and that τ_j are the period matrices of dv_{jk}.

Note that $\omega = \exp[\frac{2\pi i}{4}] = \exp[\frac{\pi i}{2}] = i = \sqrt{-1}$ for n = 4.

From S, we construct a compact Riemann surface $\hat{\hat{S}}$ of genus $\hat{\hat{g}} = 4(g-1) + 1$ which is a four-sheeted smooth covering over S, as already shown in 2-1.

On the other hand, we first construct $\hat{S}$ of genus $\hat{g} = 2(g-1) + 1 = 2g-1$ which is a two sheeted smooth covering over S, and then $\hat{\hat{S}}$ of genus $\hat{\hat{g}} = 2(g-1) + 1 = 4(g-1) + 1$ from

$\hat{\hat{S}}$ which is also a two-sheeted smooth covering over $\hat{S}$ and hence it is a four-sheeted smooth covering over S.

Simply considering the identity mapping between two differently constructed surfaces $\hat{\hat{S}}$'s, we easily see that they are conformally equivalent to each other.

We observe that the lifts $du_1\bar{*}$, $du_{0k}\bar{*}$; $dv_{2k}\bar{*}$ on $\hat{S}$ are linearly independent in $A_1(\hat{S})$ and it makes possible to find the normalized basis $d\hat{u}_1$, $d\hat{u}_{jk}$, $j = 0,1$, $k = 1,2,\ldots,g-1$, for $A_1(\hat{S})$ with respect to the canonical homology basis $\hat{\gamma}_1, \hat{\gamma}_{\ell m}; \hat{\delta}_1, \hat{\delta}_{\ell m}$, $\ell = 0,1$, $m = 1,2,\ldots,g-1$, chosen by the lifts of (γ,δ) on S. We denote the period matrix of $\hat{S}$ and $(\hat{\gamma},\hat{\delta})$ by $\hat{\Pi}$.

We next observe that the lifts $dv_{1k}\bar{*}$, $dv_{3k}\bar{*}$ on $\hat{S}$ is a basis for $\hat{\Omega}_{\left[\begin{smallmatrix}00..0\\10..0\end{smallmatrix}\right]}(1)$ on $\hat{S}$. We normalize it to find the normalized basis $d\hat{v}_h$, $h = 1,2,\ldots,2(g-1)$, for $\hat{\Omega}_{\left[\begin{smallmatrix}00..0\\10..0\end{smallmatrix}\right]}(1)$ with respect to $(\hat{\gamma},\hat{\delta})$ and we denote its period matrix by $\hat{\tau}$ (Definition 10 and Theorem 4).

By Lemma 10, the periods of $du_1\bar{*}$, $du_{0k}\bar{*}$; $dv_{jk}\bar{*}$ over $(\hat{\gamma},\hat{\delta})$ on $\hat{S}$ are given by

$$\begin{array}{c} \\ du_1\bar{*} \\ du_{0k}\bar{*} \\ dv_{jk}\bar{*} \end{array}
\begin{array}{c} \begin{array}{cccccc} \hat{\gamma}_1 & \hat{\gamma}_{0m} & \hat{\gamma}_{1m} & \hat{\delta}_1 & \hat{\delta}_{0m} & \hat{\delta}_{1m} \end{array} \\
\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2\Pi_{11} & \Pi_1 & \Pi_1 \\ 0 & I_{g-1} & I_{g-1} & 2\Pi_1 & \Pi_0 & \Pi_0 \\ 0 & I_{g-1} & i^j I_{g-1} & 0 & \tau_j & i^j\tau_j \end{array}\right] \end{array}.$$

By Theorem 3,

$$(31)\quad \begin{cases} d\hat{u}_1 = d\hat{u}_1^{\bar{*}} \\ d\hat{u}_{jk} = \frac{1}{2}[du_{0k}^{\bar{*}} + (-1)^j dv_{2k}^{\bar{*}}], \quad j = 0,1, \quad k = 1,2,\ldots,g-1. \end{cases}$$

To find $\hat{\tau}$, let

$$\begin{cases} d\hat{v}_h = \frac{1}{2}[dv_{3k}^{\bar{*}} + dv_{1k}^{\bar{*}}], \\ d\hat{v}_{(g-1)+k} = \frac{1}{2}[dv_{3k}^{\bar{*}} - dv_{1k}^{\bar{*}}], \quad k = 1,2,\ldots,g-1. \end{cases}$$

Then the periods of $d\hat{v}_k$, $d\hat{v}_{(g-1)+k}$ over $(\hat{\gamma},\hat{\delta})$ on $\hat{S}$ are

$$\begin{array}{c} \\ d\hat{v}_k \\ d\hat{v}_{(g-1)+k} \end{array} \begin{array}{c} \begin{array}{cccccc} \hat{\gamma}_1 & \hat{\gamma}_{0m} & \hat{\gamma}_{1m} & \hat{\delta}_1 & \hat{\delta}_{0m} & \hat{\delta}_{1m} \end{array} \\ \left[\begin{array}{c|cc|c|cc} & I_{g-1} & 0 & & \frac{\tau_1+\tau_3}{2} & \frac{i(\tau_1-\tau_3)}{2} \\ & 0 & I_{g-1} & & \frac{i(\tau_3-\tau_1)}{2} & \frac{\tau_1+\tau_3}{2} \end{array}\right] \end{array}.$$

In particular, we have

$$\hat{\tau} = \begin{bmatrix} \frac{\tau_1+\tau_3}{2} & \frac{i(\tau_1-\tau_3)}{2} \\ \frac{i(\tau_3-\tau_1)}{2} & \frac{\tau_1+\tau_3}{2} \end{bmatrix}.$$

If $g = 2$, then by Theorem 4 $\tau_1 = \tau_3$ and

$$\hat{\tau} = \begin{bmatrix} \tau_1 & 0 \\ 0 & \tau_1 \end{bmatrix} = \begin{bmatrix} \tau_3 & 0 \\ 0 & \tau_3 \end{bmatrix}.$$

By Theorem 4 again, $\hat{\tau} \in \mathfrak{S}_{2g-2}$.

In particular, for $g = 2$ $\hat{\tau} = \begin{bmatrix} \tau_1 & 0 \\ 0 & \tau_1 \end{bmatrix} \in \mathfrak{S}_2$. This is an interesting fact. We summarize these in a following theorem;

Theorem 5.

Let S be a compact Riemann surface of genus $g \geqq 2$, $\gamma_1, \gamma_{0m}; \delta_1, \delta_{0m}$, $m = 1,2,\dots,g-1$, a canonical homology basis on S and dv_{jk}, $j = 1,3$, $k = 1,2,\dots,g-1$, normalized basi for $\Omega_{\left[{00..0 \atop j0..0}\right]_4}(1)$ with the period matrices τ_j. Further, suppose that $\hat{S}$ is a two-sheeted smooth covering over S of genus $\hat{g} = 2g-1$ with a canonical homology basis $\hat{\gamma}_1, \hat{\gamma}_{\ell m}; \hat{\delta}_1, \hat{\delta}_{\ell m}$, $\ell = 0,1$, $m = 1,2,\dots,g-1$, chosen by the lifts of (γ, δ) on S. Then

$$(32)\quad \begin{cases} d\hat{v}_k = \frac{1}{2}[dv_{3k}{}^* + dv_{1k}{}^*], \\ d\hat{v}_{(g-1)+k} = \frac{1}{2}[dv_{3k}{}^* - dv_{1k}{}^*], \qquad k = 1,2,\dots,g-1, \end{cases}$$

where $dv_{jk}{}^*$ are the lifts of dv_{jk} onto $\hat{S}$, is a normalized basis for $\hat{\Omega}_{\left[{00..0 \atop 10..0}\right]}(1)$ on $\hat{S}$, with the period matrix

$$(33)\quad \hat{\tau} = \begin{bmatrix} \dfrac{\tau_1+\tau_3}{2} & \dfrac{i(\tau_1-\tau_3)}{2} \\ \dfrac{i(\tau_3-\tau_1)}{2} & \dfrac{\tau_1+\tau_3}{2} \end{bmatrix} \in \mathfrak{G}_{2g-2}.$$

If $g = 2$, then

$$(34)\quad \hat{\tau} = \begin{bmatrix} \tau_1 & 0 \\ 0 & \tau_1 \end{bmatrix} = \begin{bmatrix} \tau_3 & 0 \\ 0 & \tau_3 \end{bmatrix} \in \mathfrak{G}_2.$$

Definition 11.

The Schottky theta function η, associated with a compact

Riemann surface S of genus $g \geqq 2$, with (complex $(g-1)$-) characteristic $[{G \atop H}]$ is

(35) $\eta[{G \atop H}](v,\tau) = \sum_{N \in Z^{g-1}} \exp 2\Pi i[\frac{1}{2}(N+G)\tau(N+G)+(N+G)(v+G)]$,

where $\tau \in \mathfrak{S}_{g-1}$ and $v \in C^{g-1}$.

(36) $\eta[{G \atop H}](0,\tau) = \eta[{G \atop H}]$

is called the Schottky theta constant.

Thus we now have two kinds of theta functions θ and η associated with a compact Riemann surface S of genus $g \geqq 2$. We will denote these functions associated with $\hat{S}$ by $\hat{\theta}$ and $\hat{\eta}$. It is very useful to review some of recent important results about these two kinds of theta functions, primarily done by Farkas and Rauch [5,8,9,10,11,24,25]. For a compact Riemann surface S of genus $g \geqq 2$, $2^{g-2}(2^{g-1}-1)$ even theta constants on $\hat{S}$

(37) $\hat{\theta}[{0\epsilon \; \epsilon \atop 1\epsilon' \; \epsilon'}](\hat{\Pi}) = 0$

for all reduced odd $(g-1)$-characteristics $[{\epsilon \atop \epsilon'}]$. Consequently from this fact, on S

(38) $$\frac{\eta[{\delta \atop 0}](2\tau_2)}{\theta[{0\delta \atop 10}](2\Pi)} = k = \text{constant}$$

for all reduced $(g-1)$-characteristics $[{\delta \atop 0}]$, and furthermore, the Schottky - Jung relation on S

(39) $$\frac{\eta^2[{\epsilon \atop \epsilon'}](\tau_2)}{\theta[{0\epsilon \atop 0\epsilon'}](\Pi)\theta[{0\epsilon \atop 1\epsilon'}](\Pi)} = k^2$$

for all reduced even $(g-1)$-characteristics $[{\epsilon \atop \epsilon'}]$. Besides, Accola established the quotient of two theta functions on a smooth abelian covering $\hat{S}$ over S and on S, respectively, which seems to be a generalization of (39) in [1], and Fay also derived a quotient, very similar to one by Accola, of these two theta functions in [12]. But we independently derived such a quotient almost at the same time slightly in a different way. We found recently that this computational method was used once by Farkas ("Relations between quatratic differentials", Advances in the theory of Riemann surfaces, Princetion University Press, 1971, pp. 141-156.)

We need two formulas; for any $g \geqq 2$

$$(40)\quad \hat{\theta}[{g_1\,\delta\,\delta \atop h_1\,\delta'\,\delta'}](\hat{u},\hat{\Pi}) = \sum_{p\in Z_2^{g-1}} (-1)^{(\delta+p)\cdot\delta'}\,\eta[{p \atop 0}](v,2\tau_2)\,\theta[{g_1\,\delta+p \atop h_1\,\;0}](u,2\Pi),$$

where $[{g_1 \atop h_1}]$ is any reduced 1-characteristic, $[{\delta \atop \delta'}]$ is any reduced $(g-1)$-characteristic, $u \in C^g$, $v \in C^{g-1}$, $\hat{u} \in C^{2g-1}$ such that $\hat{u}_1 = u_1$ $\hat{u}_\ell = \frac{1}{2}(u_\ell+v_{\ell-1})$, $\hat{u}_{(g-1)+\ell} = \frac{1}{2}(u_\ell-v_{\ell-1})$, $\ell = 2,3,\ldots,g$, and the summation runs over all the 2^{g-1} $(g-1)$-vector p with entries 0 and 1. The proof of (40) can be found in [9].

$$(41)\quad \theta[{\epsilon \atop \epsilon'}](u,2T)\,\theta[{\hat{\epsilon} \atop \epsilon'}](0,2T)$$

$$= \frac{1}{2^g}\sum_{\delta\in Z_2^g}(-1)^{\delta\cdot\epsilon}\,\theta[{\epsilon+\hat{\epsilon} \atop \epsilon'+\delta}](\tfrac{1}{2}u,T)\,\theta[{\epsilon+\hat{\epsilon} \atop \delta}](\tfrac{1}{2}u,T),$$

where $g \geqq 1$, $T \in \mathfrak{S}_g$, $u \in C^g$, $[{\epsilon \atop \epsilon'}]$ and $[{\hat{\epsilon} \atop \epsilon'}]$ are any reduced g-characteristics and the summation runs over all the g-vectors δ with entries 0 and 1. The proof of (41) is in [13].

Lemma 13.

$$\hat{\theta}\left[\begin{smallmatrix}00\ldots0\\10\ldots0\end{smallmatrix}\right](2u_1,u_\ell,u_\ell;\hat{\Pi})$$

$$= k\theta\left[\begin{smallmatrix}0\ldots0\\0\ldots0\end{smallmatrix}\right](u_1,u_\ell;\Pi)\theta\left[\begin{smallmatrix}00\ldots0\\10\ldots0\end{smallmatrix}\right](u_1,u_\ell;\Pi),$$

where $(u_1,u_\ell) \in C^g$, $\ell = 2,3,\ldots,g$ and k is given by (38).

Proof.

Let $S_0(\delta) = \{p \in Z_2^{g-1} | p\cdot\delta \equiv 0 \pmod 2\}$ and $S_1(\delta) = \{p \in Z_2^{g-1} | p\cdot\delta \equiv 1 \pmod 2\}$ for any $\delta \in Z_2^{g-1}$.

Letting $v = 0 \in C^{g-1}$ in (40) and applying (41),

$$\hat{\theta}\left[\begin{smallmatrix}00\ldots0\\10\ldots0\end{smallmatrix}\right](2u_1,u_\ell,u_\ell;\hat{\Pi})$$

$$= \sum_{p\in Z_2^{g-1}} \eta\left[\begin{smallmatrix}p\\0\end{smallmatrix}\right](2\tau_2)\theta\left[\begin{smallmatrix}0p\\10\end{smallmatrix}\right](2u_1,2u_\ell;2\Pi)$$

$$= k\sum_{p\in Z_2^{g-1}} \theta\left[\begin{smallmatrix}0p\\10\end{smallmatrix}\right](2\Pi)\theta\left[\begin{smallmatrix}0p\\10\end{smallmatrix}\right](2u_1,2u_\ell;2\Pi)$$

$$= k\sum_{p\in Z_2^{g-1}} \frac{1}{2^g}\sum_{\substack{\delta\in Z_2^{g-1}\\ \delta'\in Z_2}} (-1)^{(\delta',\delta)\cdot(0,p)}\theta\left[\begin{smallmatrix}0 & 2p\\1+\delta' & \delta\end{smallmatrix}\right](u_1,u_\ell;\Pi)\,\theta\left[\begin{smallmatrix}0 & 2p\\\delta' & \delta\end{smallmatrix}\right](u_1,u_\ell;\Pi)$$

$$= \frac{k}{2^g}\Big[\sum_{\delta'\in Z_2}\Big\{\sum_{p\in Z_2^{g-1}}\theta\left[\begin{smallmatrix}0 & 0\ldots0\\1+\delta' & 0\ldots0\end{smallmatrix}\right](u_1,u_\ell;\Pi)\theta\left[\begin{smallmatrix}0 & 0\ldots0\\\delta' & 0\ldots0\end{smallmatrix}\right](u_1,u_\ell;\Pi)$$

$$+ \sum_{0\neq\delta\in Z_2^{g-1}}\Big(\sum_{p\in S_0(\delta)}(-1)^{p\cdot\delta} + \sum_{p\in S_1(\delta)}(-1)^{p\cdot\delta}\Big)\theta\left[\begin{smallmatrix}0 & 0\\1+\delta' & \delta\end{smallmatrix}\right](u_1,u_\ell:\Pi)\,X$$

$$\theta\left[\begin{smallmatrix}0 & 0\\\delta' & \delta\end{smallmatrix}\right](u_1,u_\ell:\Pi)\Big\}\Big]$$

$$= \frac{k}{2}\Big[\theta\left[\begin{smallmatrix}00\ldots0\\10\ldots0\end{smallmatrix}\right](u_1,u_\ell;\Pi)\theta\left[\begin{smallmatrix}0\ldots0\\0\ldots0\end{smallmatrix}\right](u_1,u_\ell;\Pi)$$

$$+ \theta\left[\begin{smallmatrix}0\ldots0\\0\ldots0\end{smallmatrix}\right](u_1,u_\ell;\Pi)\theta\left[\begin{smallmatrix}00\ldots0\\10\ldots0\end{smallmatrix}\right](u_1,u_\ell;\Pi)\Big]$$

$$= k\theta\left[\begin{smallmatrix}0\cdots0\\0\cdots0\end{smallmatrix}\right](u_1,u_\ell;\Pi)\theta\left[\begin{smallmatrix}00\cdots0\\10\cdots0\end{smallmatrix}\right](u_1,u_\ell;\Pi).$$

Theorem 6.

$$(42)\quad \hat{\theta}\left[\begin{smallmatrix}\epsilon & \delta & \delta\\ 1+\epsilon' & \delta' & \delta'\end{smallmatrix}\right](2u_1,u_\ell,u_\ell;\hat{\Pi}) = k\theta\left[\begin{smallmatrix}\epsilon & \delta\\ \frac{\epsilon'}{2} & \delta'\end{smallmatrix}\right](u_1,u_\ell;\Pi)\theta\left[\begin{smallmatrix}\epsilon & \delta\\ \frac{1+\epsilon'}{2} & \delta\end{smallmatrix}\right](u_1,u_\ell;\Pi),$$

where $\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right]$ is any reduced 1-characteristic, $\left[\begin{smallmatrix}\delta\\ \delta'\end{smallmatrix}\right]$ is any reduced (g-1)-characteristic, $(u_1,u_\ell) \in \mathbb{C}^g$, $\ell = 2,3,\ldots,g$ and k is the same as in Lemma 13.

Proof.

By Substitution formula (16) in Corollary 1 with n = 2,

$$\hat{\theta}\left[\begin{smallmatrix}\epsilon & \delta & \delta\\ 1+\epsilon' & \delta' & \delta'\end{smallmatrix}\right](2u_1,u_\ell,u_\ell;\hat{\Pi})$$

$$= \exp[A\Pi i]\hat{\theta}\left[\begin{smallmatrix}00\cdots00\cdots0\\10\cdots00\cdots0\end{smallmatrix}\right](w,\hat{\Pi}),$$

where $A = \frac{1}{4}(\epsilon,\delta,\delta)\hat{\Pi}(\epsilon,\delta,\delta) + \frac{1}{2}(\epsilon,\delta,\delta)\cdot(1+\epsilon',\delta',\delta')$

$$+ (\epsilon,\delta,\delta)\cdot(2u_1,u_\ell,u_\ell)$$

$$= \frac{1}{2}(\epsilon,\delta)\Pi(\epsilon,\delta)+\frac{1}{2}\epsilon+\frac{1}{2}\epsilon\epsilon'+\delta\cdot\delta'+2\epsilon u_1+2\delta\cdot u_\ell,$$

and

$$w = \begin{pmatrix}2u_1\\ u_\ell\\ u_\ell\end{pmatrix} + \frac{1}{2}\left[\begin{smallmatrix}\epsilon & \delta & \delta\\ \epsilon' & \delta' & \delta'\end{smallmatrix}\right]$$

$$= \begin{pmatrix}2(u_1+\frac{\epsilon'}{4}+\Pi_{11}(\frac{\epsilon}{2})+\Pi_1\cdot(\frac{\delta}{2}))\\ u_\ell+\frac{\delta'}{2}+\Pi_1\cdot(\frac{\epsilon}{2})+\Pi_0\cdot(\frac{\delta}{2})\\ u_\ell+\frac{\delta'}{2}+\Pi_1\cdot(\frac{\epsilon}{2})+\Pi_0\cdot(\frac{\delta}{2})\end{pmatrix} = \begin{pmatrix}2v_1\\ v_\ell\\ v_\ell\end{pmatrix}.$$

And we obtain

$$\begin{bmatrix} v_1 \\ v_\ell \end{bmatrix} = \begin{bmatrix} u_1+\frac{\epsilon'}{4}+\Pi_{11}(\frac{\epsilon}{2})+\Pi_1\cdot(\frac{\delta}{2}) \\ u_\ell+\frac{\delta'}{2}+\Pi_1\cdot(\frac{\epsilon}{2})+\Pi_0\cdot(\frac{\delta}{2}) \end{bmatrix} = \begin{bmatrix} u_1 \\ u_\ell \end{bmatrix} + \frac{1}{2}\left\{\begin{matrix} \epsilon & \delta \\ \frac{\epsilon'}{2} & \delta' \end{matrix}\right\}.$$

Then, by Lemma 13 and again applying (16),

$$\hat{\theta}\left[\begin{smallmatrix}00..00..0\\10..00..0\end{smallmatrix}\right](w,\hat{\Pi}) = \hat{\theta}\left[\begin{smallmatrix}00..00..0\\10..00..0\end{smallmatrix}\right](2v_1,v_\ell,v_\ell;\hat{\Pi})$$

$$= k\theta\left[\begin{smallmatrix}0..0\\0..0\end{smallmatrix}\right](v_1,v_\ell;\Pi)\theta\left[\begin{smallmatrix}00..0\\10..0\end{smallmatrix}\right](v_1,v_\ell;\Pi)$$

$$= k\exp[B\Pi i]\theta\left[\begin{smallmatrix}\epsilon & \delta\\ \frac{\epsilon'}{2} & \delta'\end{smallmatrix}\right](u_1,u_\ell;\Pi)\theta\left[\begin{smallmatrix}\epsilon & \delta\\ 1+\frac{\epsilon'}{2} & \delta'\end{smallmatrix}\right](u_1,u_\ell;\Pi),$$

where $B = -\frac{1}{4}(\epsilon,\delta)\Pi(\epsilon,\delta)-\frac{1}{2}(\epsilon,\delta)\cdot(\frac{\epsilon'}{2},\delta')-(\epsilon,\delta)\cdot(u_1,u_\ell)$

$$-\frac{1}{4}(\epsilon,\delta)\Pi(\epsilon,\delta)-\frac{1}{2}(\epsilon,\delta)\cdot(1+\frac{\epsilon'}{2},\delta')-(\epsilon,\delta)\cdot(u_1,u_\ell)$$

$$= -\frac{1}{2}(\epsilon,\delta)\Pi(\epsilon,\delta)-\frac{1}{2}\epsilon-\frac{1}{2}\epsilon\epsilon'-\delta\cdot\delta'-2\epsilon u_1-2\delta\cdot u_\ell.$$

Since $A + B = 0$, it follows that

$$\hat{\theta}\left[\begin{smallmatrix}\epsilon & \delta & \delta\\ 1+\epsilon' & \delta' & \delta'\end{smallmatrix}\right](2u_1,u_\ell,u_\ell;\hat{\Pi})$$

$$= k\exp[(A+B)\Pi i]\theta\left[\begin{smallmatrix}\epsilon & \delta\\ \frac{\epsilon'}{2} & \delta'\end{smallmatrix}\right](u_1,u_\ell;\Pi)\theta\left[\begin{smallmatrix}\epsilon & \delta\\ 1+\frac{\epsilon'}{2} & \delta'\end{smallmatrix}\right](u_1,u_\ell;\Pi)$$

$$= k\theta\left[\begin{smallmatrix}\epsilon & \delta\\ \frac{\epsilon'}{2} & \delta'\end{smallmatrix}\right](u_1,u_\ell;\Pi)\theta\left[\begin{smallmatrix}\epsilon & \delta\\ 1+\frac{\epsilon'}{2} & \delta'\end{smallmatrix}\right](u_1,u_\ell;\Pi).$$

Theorem 7.

For a four-sheeted smooth covering $\hat{S}$ of genus $\hat{g} = 4(g-1)+1$ over a compact Riemann surface S of genus $g \geqq 2$, $2^{g-2}(2^{g-1}-1)$ even theta constants

(43) $\hat{\theta}\left[\begin{smallmatrix}0\delta & \delta & \delta & \delta\\ 1\delta' & \delta' & \delta' & \delta'\end{smallmatrix}\right](\hat{\Pi}) = 0$

for any reduced odd $(g-1)$-characteristic $\left[\begin{smallmatrix}\delta\\ \delta'\end{smallmatrix}\right]$.

Proof.

As we mentioned in the very beginning of this section, $\hat{S}$

is a two-sheeted smooth covering over $\hat{S}$ which is also a two-sheeted smooth covering over S. Repeating exactly the same argument as a proof of Theorem 6, we can obtain the same result on $\hat{\hat{S}}$ as on $\hat{S}$. That is, setting $\hat{u}_1 = 0$, $\hat{u}_\ell = 0$ and $\epsilon = \epsilon' = 0$,

$$(44) \quad \hat{\hat{\theta}}\begin{bmatrix} 0\delta & \delta & \delta & \delta \\ 1\delta' & \delta' & \delta' & \delta' \end{bmatrix}(\hat{\hat{\Pi}}) = \hat{k}\hat{\theta}\begin{bmatrix} 0\delta & \delta \\ 0\delta' & \delta' \end{bmatrix}(\hat{\Pi})\hat{\theta}\begin{bmatrix} 0\delta & \delta \\ 1\delta' & \delta' \end{bmatrix}(\hat{\Pi}),$$

where $\hat{k} = \text{constant} = \dfrac{\hat{\eta}\begin{bmatrix} \epsilon \\ 0 \end{bmatrix}(2\hat{\tau})}{\hat{\theta}\begin{bmatrix} 0\epsilon \\ 10 \end{bmatrix}(2\hat{\Pi})}$ for any reduced (2g-2)-characteristic $\begin{bmatrix} \epsilon \\ 0 \end{bmatrix}$.

However, $\hat{\theta}\begin{bmatrix} 0\delta & \delta \\ 1\delta' & \delta' \end{bmatrix}(\hat{\Pi}) = 0$ by (37), and hence $2^{g-2}(2^{g-1}-1)$ even theta constants $\hat{\hat{\theta}}\begin{bmatrix} 0\delta & \delta & \delta & \delta \\ 1\delta' & \delta' & \delta' & \delta' \end{bmatrix}(\hat{\hat{\Pi}}) = 0$ on $\hat{\hat{S}}$.

The significance of Theorem 7 is that $2^{2(g-1)-1}(2^{2(g-1)}-1)$ even theta constants $\hat{\hat{\theta}}\begin{bmatrix} 0\alpha\alpha \\ 1\beta\beta \end{bmatrix}(\hat{\hat{\Pi}}) = 0$ on $\hat{\hat{S}}$ for all odd reduced 2(g-1)-characteristics $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ by (37), and that, in addition to them, there are more $2^{g-2}(2^{g-1}-1)$ vanishing even theta constants $\hat{\hat{\theta}}\begin{bmatrix} 0\delta & \delta & \delta & \delta \\ 1\delta' & \delta' & \delta' & \delta' \end{bmatrix}(\hat{\hat{\Pi}})$ on $\hat{\hat{S}}$ for any odd reduced (g-1)-characteristic $\begin{bmatrix} \delta \\ \delta' \end{bmatrix}$ such that 2(g-1)-characteristic $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \delta & \delta \\ \delta' & \delta' \end{bmatrix}$ is even.

Corollary 3.

$$(45) \quad \frac{\hat{\hat{\theta}}\begin{bmatrix} \epsilon\delta & \delta & \delta & \delta \\ 1\delta' & \delta' & \delta' & \delta' \end{bmatrix}(4u_1, u_\ell, u_\ell, u_\ell, u_\ell; \hat{\hat{\Pi}})}{\theta\begin{bmatrix} \epsilon\delta \\ 0\delta' \end{bmatrix}(u_1, u_\ell; \Pi)\theta\begin{bmatrix} \epsilon\delta \\ \frac{1}{2}\delta' \end{bmatrix}(u_1, u_\ell; \Pi)\theta\begin{bmatrix} \epsilon\delta \\ 1\delta' \end{bmatrix}(u_1, u_\ell; \Pi)\theta\begin{bmatrix} \epsilon\delta \\ \frac{3}{2}\delta' \end{bmatrix}(u_1, u_\ell; \Pi)}$$

$$= (-1)^\epsilon \hat{k} k^2,$$

where $\epsilon = 0$ or 1, $[{\delta \atop \delta'}]$ is any reduced (g-1)-characteristic, $(u_1, u_\ell) \in C^g$, $\ell = 2,3,\ldots,g$,

$$\hat{k} = \frac{\hat{\eta}[{\alpha \atop 0}](2\hat{\tau})}{\hat{\hat{\theta}}[{0\alpha \atop 10}](2\hat{\Pi})} \quad \text{for any reduced } 2(g-1)\text{-characteristic}$$

$[{\alpha \atop 0}]$, $\hat{\tau}$ is given by (33) and k is given by (38).

Proof.

Remembering the relations between $\hat{\hat{S}}$, $\hat{S}$ and S as in the proof of Theorem 7, we apply (42) repeatedly to obtain

$$\hat{\hat{\theta}}[{\epsilon\delta \atop 1\delta'}{\delta \atop \delta'}{\delta \atop \delta'}{\delta \atop \delta'}](4u_1, u_\ell, u_\ell, u_\ell, u_\ell; \hat{\hat{\Pi}})$$

$$= \hat{k}\hat{\theta}[{\epsilon\delta \atop 0\delta'}{\delta \atop \delta'}](2u_1, u_\ell, u_\ell; \hat{\Pi})\hat{\theta}[{\epsilon\delta \atop 1\delta'}{\delta \atop \delta'}](2u_1, u_\ell, u_\ell; \hat{\Pi})$$

$$= (-1)^\epsilon \hat{k}\hat{\theta}[{\epsilon \atop 1+1}{\delta \atop \delta'}{\delta \atop \delta'}](2u_1, u_\ell, u_\ell; \hat{\Pi})\hat{\theta}[{\epsilon\delta \atop 1\delta'}{\delta \atop \delta'}](2u_1, u_\ell, u_\ell; \hat{\Pi})$$

$$= (-1)^\epsilon \hat{k}k^2\theta[{\epsilon\delta \atop \frac{1}{2}\delta'}](u_1, u_\ell; \Pi)\theta[{\epsilon\delta \atop \frac{\epsilon}{2}\delta'}](u_1, u_\ell; \Pi)\theta[{\epsilon\delta \atop 0\delta'}](u_1, u_\ell; \hat{\Pi}) \times$$

$$\theta[{\epsilon\delta \atop 1\delta'}](u_1, u_\ell; \Pi),$$

which gives a quotient we wanted.

Lemma 14.

$$(46) \qquad \frac{\hat{\eta}^2[{\delta \atop \delta'}{\delta \atop \delta'}](\hat{\tau})}{\theta[{0\delta \atop 0\delta'}](\Pi)\theta[{0\delta \atop \frac{1}{2}\delta'}](\Pi)\theta[{0\delta \atop 1\delta'}](\Pi)\theta[{0\delta \atop \frac{3}{2}\delta'}](\Pi)} = \hat{k}^2 k^2$$

for any even reduced (g-1)-characteristic $[{\delta \atop \delta'}]$, where $\hat{k}$ and k are the same as in Corollary 3.

Proof.

Schottky - Jung relation on $\hat{S}$ is

$$(47)\qquad \frac{\hat{\eta}^2\left[{\delta \atop \delta'}{\delta \atop \delta'}\right](\hat{\tau})}{\hat{\theta}\left[{0\delta \atop 0\delta'}{\delta \atop \delta'}\right](\hat{\Pi})\hat{\theta}\left[{0\delta \atop 1\delta'}{\delta \atop \delta'}\right](\hat{\Pi})} = \hat{k}^2.$$

by (42),

$$\hat{\theta}\left[{0\delta \atop 0\delta'}{\delta \atop \delta'}\right](\hat{\Pi})\hat{\theta}\left[{0\delta \atop 1\delta'}{\delta \atop \delta'}\right](\hat{\Pi}) = k^2\theta\left[{0\delta \atop \frac{1}{2}\delta'}\right](\Pi)\theta\left[{0\delta \atop \frac{3}{2}\delta'}\right](\Pi)\theta\left[{0\delta \atop 0\delta'}\right](\Pi)\theta\left[{0\delta \atop 1\delta'}\right](\Pi).$$

Now, a quotient (46) is obvious.

Corollary 4.

If $g = 2$, then

$$(48)\qquad \frac{\hat{\eta}^4\left[{\delta \atop \delta'}\right](\tau_1)}{\theta\left[{0\delta \atop 0\delta'}\right](\Pi)\theta\left[{0\delta \atop \frac{1}{2}\delta'}\right](\Pi)\theta\left[{0\delta \atop 1\delta'}\right](\Pi)\theta\left[{0\delta \atop \frac{3}{2}\delta'}\right](\Pi)} = \hat{k}^2\, k^2$$

for any even reduced 1-characteristic $\left[{\delta \atop \delta'}\right]$.

Proof.

For $g = 2$, $\hat{\tau} = \begin{bmatrix} \tau_1 & 0 \\ 0 & \tau_1 \end{bmatrix}$ by (34) in Theorem 5.

$$\hat{\eta}\left[{\delta \atop \delta'}{\delta \atop \delta'}\right](\hat{\tau}) = \sum_{N,M\in Z^{g-1}} \exp 2\Pi i\left[\tfrac{1}{2}(N+\tfrac{\delta}{2},M+\tfrac{\delta}{2})\hat{\tau}(N+\tfrac{\delta}{2},M+\tfrac{\delta}{2})+(N+\tfrac{\delta}{2},M+\tfrac{\delta}{2})(\tfrac{\delta'}{2},\tfrac{\delta'}{2})\right]$$

$$= \sum_{N\in Z^{g-1}} \exp 2\Pi i\left[\tfrac{1}{2}(N+\tfrac{\delta}{2})\tau_1(N+\tfrac{\delta}{2})+(N+\tfrac{\delta}{2})(\tfrac{\delta'}{2})\right]$$

$$\cdot \sum_{M\in Z^{g-1}} \exp 2\Pi i\left[\tfrac{1}{2}(M+\tfrac{\delta}{2})\tau_1(M+\tfrac{\delta}{2})+(M+\tfrac{\delta}{2})(\tfrac{\delta'}{2})\right]$$

$$= \hat{\eta}^2\left[{\delta \atop \delta'}\right]$$

and (46) imply (48).

The quotient (48) gives us a theta identity for $g = 2$ with rational characteristics, i.e., from $\eta^4\left[{0 \atop 0}\right]-\eta^4\left[{1 \atop 0}\right]-\eta^4\left[{0 \atop 1}\right] = 0$, we get

$$(49)\qquad \theta\left[{00 \atop 00}\right]_4\theta\left[{00 \atop 10}\right]_4\theta\left[{00 \atop 20}\right]_4\theta\left[{00 \atop 30}\right]_4 - \theta\left[{02 \atop 00}\right]_4\theta\left[{02 \atop 10}\right]_4\theta\left[{02 \atop 20}\right]_4\theta\left[{02 \atop 30}\right]_4$$

$$- \theta\begin{bmatrix}00\\02\end{bmatrix}_4 \theta\begin{bmatrix}00\\12\end{bmatrix}_4 \theta\begin{bmatrix}00\\22\end{bmatrix}_4 \theta\begin{bmatrix}00\\32\end{bmatrix}_4 = 0.$$

CHAPTER III PERIOD RELATIONS ON HYPERELLIPTIC SURFACES

3-1 Period relation of Schottky type.

If a compact Riemann surface S of genus $g \geqq 1$ does admit a meromorphic function Z which assumes each complex value twice, then S is a two-sheeted branched covering over the sphere S_0, and is usually called a hyperelliptic (Riemann) surface. For a moment, we pay an attention on this kind of surface, especially trying to establish a period relation.

On a hyperelliptic surface S of genus $g \geqq 1$, a meromorphic function w on S can be found, satisfying an irreducible algebraic equation

(50) $w^2 = Z(Z-1)(Z-\lambda_1)(Z-\lambda_2)\dots(Z-\lambda_{2g-1})$,

where λ_j, $j = 1,2,\dots,2g-1$, are mutually distinct finite different from 0 and 1, and thus S can be realized as the Riemann surface of an algebraic function w. By Riemann - Hurwitz formula, there are 2g+2 branch points 0, 1, ∞ and λ_j, $j = 1,2,\dots,2g-1$, of order 1. Furthermore, these points are precisely 2g+2 weierstrass points on S with the gap sequence $(1,3,\dots,2g-1)$. g differentials, defined by

(51) $dv_k = \frac{Z^{k-1}dZ}{w}$, $k = 1,2,\dots,g$,

is a base for $A_1(S)$.

Suppose that $g \geqq 4$. We choose a canonical homology basis (γ,δ) on S as in Figure 1.

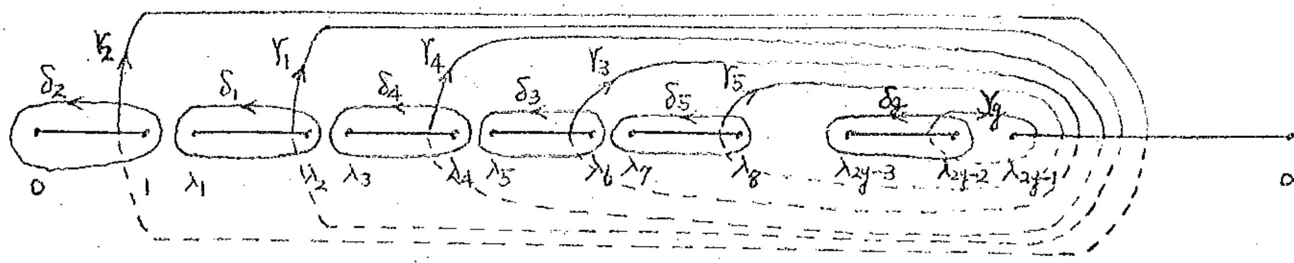

Figure 1.

Now, taking a point P_0 (with $Z(p_0) = 0$) on S as a base point and finding the uniquely determined normalized basis du_k, $k = 1,2,\ldots,g$, for $A_1(S)$ with respect to the chosen (γ,δ) on S, we have a mapping u from S to its Jacobi variety J(S). In particular, we can find all the images of 2g+2 branch points under u in J(S);

(52)

$$u(0) \equiv \begin{pmatrix} 00000\ldots0 \\ 00000\ldots0 \end{pmatrix}$$

$$u(1) \equiv \begin{pmatrix} 01000\ldots0 \\ 00000\ldots0 \end{pmatrix},$$

$$u(\lambda_1) \equiv \begin{pmatrix} 01000\ldots0 \\ 11000\ldots0 \end{pmatrix},$$

$$u(\lambda_2) \equiv \begin{pmatrix} 11000\ldots0 \\ 11000\ldots0 \end{pmatrix},$$

$$u(\lambda_3) \equiv \begin{pmatrix} 11000\ldots0 \\ 01010\ldots0 \end{pmatrix},$$

$$u(\lambda_4) \equiv \begin{pmatrix} 11010\ldots0 \\ 01010\ldots0 \end{pmatrix},$$

$$u(\lambda_5) \equiv \begin{pmatrix} 11010\ldots0 \\ 01100\ldots0 \end{pmatrix},$$

$$u(\lambda_6) \equiv \begin{pmatrix} 11110\ldots0 \\ 01100\ldots0 \end{pmatrix},$$

$$u(\lambda_7) \equiv \begin{pmatrix} 111100\ldots0 \\ 010010\ldots0 \end{pmatrix},$$

$$u(\lambda_8) \equiv \begin{pmatrix} 111110\ldots0 \\ 010010\ldots0 \end{pmatrix},$$

- - - - - - - - - - -

$$u(\lambda_{2g-3}) \equiv \begin{pmatrix} 11111\ldots10 \\ 01000\ldots01 \end{pmatrix},$$

$$u(\lambda_{2g-2}) \equiv \begin{pmatrix} 11111\ldots11 \\ 01000\ldots01 \end{pmatrix},$$

$$u(\lambda_{2g-1}) \equiv \begin{pmatrix} 11111\ldots11 \\ 01000\ldots00 \end{pmatrix},$$

$$u(\infty) \equiv \begin{pmatrix} 00000\ldots0 \\ 01000\ldots0 \end{pmatrix},$$

in which they are all half periods. A vector K of Riemann constants with respect to a base point P_0 is given by

$$(53)\quad K \equiv \sum_{h=1}^{g} u(\lambda_{2h-1}) \equiv \begin{pmatrix} g-1 & g & g-3 & g-2 & g-4 & \cdots & 1 \\ 1 & g & 1 & 1 & 1 & \cdots & 1 \end{pmatrix},$$

which is the sum of all g odd half periods. The proofs of (52) and (53) are found in [14]. (Chapter 10, §1 and §2)

Now, at $Z = 0$ the local parameter t is given by $t = \sqrt{Z}$, and

$$Z = t^2, \quad dZ = 2tdt,$$

$$w = [t^2(t^2-1)(t^2-\lambda_1)\cdots(t^2-\lambda_{2g-1})]^{\frac{1}{2}} = t[(t^2-1)(t^2-\lambda_1)\cdots(t^2-\lambda_{2g-1})]^{\frac{1}{2}}.$$

At $Z = 1$, the local parameter t is $t = \sqrt{Z-1}$, and

$$Z = 1+t^2, \quad dZ = 2tdt,$$

$$w = [(t^2+1)t^2(t^2+1-\lambda_1)\cdots(t^2+1-\lambda_{2g-1})]^{\frac{1}{2}}$$

$$= t[(t^2+1)(t^2+1-\lambda_1)\cdots(t^2+1-\lambda_{2g-1})]^{\frac{1}{2}}.$$

At $Z = \lambda_j$, $j = 1,2,\ldots,2g-1$, the local parameter t is $t = \sqrt{Z-\lambda_j}$, and

$$Z = \lambda_j+t^2, \quad dZ = 2tdt,$$

$$w = [(t^2+\lambda_j)(t^2+\lambda_j-1)t^2 \prod_{\substack{j'=1 \\ j'\neq j}}^{2g-1} (t^2+\lambda_j-\lambda_{j'})]^{\frac{1}{2}}$$

$$= t[(t^2+\lambda_j)(t^2+\lambda_j-1) \prod_{\substack{j'=1 \\ j'\neq j}}^{2g-1} (t^2+\lambda_j-\lambda_{j'})]^{\frac{1}{2}}.$$

At $Z = \infty$, the local parameter t is $t = \frac{1}{\sqrt{Z}}$, and

$$Z = \frac{1}{t^2}, \quad dZ = -\frac{1}{t^3}\,dt,$$

$$w = \left[\frac{1}{t^2}\left(\frac{1}{t^2}-1\right)\prod_{j=1}^{2g-1}\left(\frac{1}{t^2}-\lambda_j\right)\right]^{\frac{1}{2}} = \frac{1}{t^{2g+1}}\left[(1-t^2)\prod_{j=1}^{2g-1}(1-\lambda_j t^2)\right]^{\frac{1}{2}}.$$

From these, we have a following table of orders of zeros (or poles) of various functions and differentials at each branch points;

	0	1	λ_j	∞	
dZ	1	1	1	-3	
Z^μ	2μ	0	0	-2μ	$j = 1,2,\ldots,2g-1,$
w	1	1	1	$-(2g+1)$	$\mu = 0,1,2,\ldots,g-1.$

Table 1.

Let $\zeta = P_1P_2\ldots P_g$ is an integral divisor of g distinct branch points. We want to show that $i(\zeta) = 0$.

Suppose that there is an element du of $A_1(S)$ such that its divisor (du) is a multiple of ζ. Then du vanishes at $P_1,\ldots,P_g$. Since $dv_k = \frac{Z^{k-1}dZ}{w}$, $k = 1,2,\ldots,g$, is a basis for $A_1(S)$, there exist constants C_k such that

$$du = \sum_{k=1}^{g} C_k dv_k$$

and
$$\sum_{k=1}^{g}\left(C_k\left(\frac{Z^{k-1}dZ}{w}\right)\right)(P_h) = \left(\sum_{k=1}^{g} C_k dv_k\right)(P_h) = du(P_h) = 0,$$

$h = 1,2,\ldots,g$. If $P_h \neq \infty$, $h = 1,2,\ldots,g$, then

$$\det(dv_k(P_h)) = \det\left(\left(\frac{Z^{k-1}dz}{w}\right)(P_h)\right) = \prod_{h=1}^{g}\frac{dZ}{w}(P_h)\cdot\det(Z^{k-1}(P_h))$$

$$= \prod_{h=1}^{g}\frac{dZ}{w}(P_h)\cdot\prod_{h>h'}(Z(P_h)-Z(P_{h'})) \neq 0,$$

since $\frac{dZ}{w}(P_h) \neq 0$ for $P_h \neq \infty$ and P_h are mutually distinct. If any one of P_h, $h = 1,2,\ldots,g$, is the branch point ∞, say $P_1 = \infty$, then

$$\det(dv_k(P_h)) = \det\left(\left(\frac{Z^{k-1}dZ}{w}\right)(P_h)\right)$$
$$= (-1)^{g+1}\frac{Z^{g-1}dZ}{w}(P_1)\cdot\prod_{h=2}^{g}\frac{dZ}{w}(P_h)\cdot\det\begin{bmatrix}1 & Z(P_2)\ldots\ldots Z^{g-2}(P_2)\\ - & - - - - - - -\\ 1 & Z(P_g)\ldots\ldots Z^{g-2}(P_g)\end{bmatrix}$$
$$\neq 0,$$

since $dv_1(P_1) = dv_2(P_1) = \ldots = dv_{g-1}(P_1) = 0$ and $dv_g(P_1) \neq 0$. In any case, $\det(dv_k(P_h)) \neq 0$ and hence $C_k = 0$, $k = 1,2,\ldots,g$. This implies that $du = \sum_{k=1}^{g} C_k dv_k = 0$, and that $i(\zeta) = 0$.

If we let $e \equiv u(\zeta) + K$, then $\theta[e](u(p),\Pi)$ is a non-zero theta function associated with S and (γ,δ). Furthermore, ζ is precisely the divisor of zeros of $\theta[e](u(p),\Pi)$.

Applying this argument, we can find many non-zero theta functions associated with S and (γ,δ). In particular,

	Theta functions	zeros
	$\theta\left[\begin{smallmatrix}01000\ldots0\\11000\ldots0\end{smallmatrix}\right](u(p),\Pi)$	$0,\gamma_2,\gamma_4,\gamma_6,\gamma_8,\ldots,\gamma_{2g-2}$,
	$\theta\left[\begin{smallmatrix}01000\ldots0\\00000\ldots0\end{smallmatrix}\right](u(p),\Pi)$	$\infty,\lambda_2,\lambda_4,\lambda_6,\lambda_8,\ldots,\lambda_{2g-2}$,
(54)		
	$\theta\left[\begin{smallmatrix}01000\ldots0\\01010\ldots0\end{smallmatrix}\right](u(p),\Pi)$	$0,\lambda_2,\lambda_5,\lambda_7,\ldots,\lambda_{2g-1}$,

$\theta[{}^{01000\ldots0}_{00010\ldots0}](u(p),\Pi)$ $\qquad \infty,\lambda_2,\lambda_5,\lambda_7,\ldots,\lambda_{2g-1},$

$\theta[{}^{01000\ldots0}_{01100\ldots0}](u(p),\Pi)$ $\qquad 0,\lambda_2,\lambda_4,\lambda_7,\ldots,\lambda_{2g-1},$

$\theta[{}^{01000\ldots0}_{00100\ldots0}](u(p),\Pi)$ $\qquad \infty,\lambda_2,\lambda_4,\lambda_7,\ldots,\lambda_{2g-1},$

$\theta[{}^{01000\ldots0}_{11110\ldots0}](u(p),\Pi)$ $\qquad 0,\lambda_3,\lambda_4,\lambda_7,\ldots,\lambda_{2g-1},$

$\theta[{}^{01000\ldots0}_{10110\ldots0}](u(p),\Pi)$ $\qquad \infty,\lambda_3,\lambda_4,\lambda_7,\ldots,\lambda_{2g-1},$

$\theta[{}^{00000\ldots0}_{01000\ldots0}](u(p),\Pi)$ $\qquad 1,\lambda_2,\lambda_4,\lambda_6,\ldots,\lambda_{2g-2},$

$\theta[{}^{01000\ldots0}_{00000\ldots0}](u(p),\Pi)$ $\qquad \infty,\lambda_2,\lambda_4,\lambda_6,\ldots,\lambda_{2g-2},$

$\theta[{}^{00000\ldots0}_{01010\ldots0}](u(p),\Pi)$ $\qquad 1,\lambda_2,\lambda_5,\lambda_7,\ldots,\lambda_{2g-1},$

$\theta[{}^{01000\ldots0}_{00010\ldots0}](u(p),\Pi)$ $\qquad \infty,\lambda_2,\lambda_5,\lambda_7,\ldots,\lambda_{2g-1},$

$\theta[{}^{00000\ldots0}_{01100\ldots0}](u(p),\Pi)$ $\qquad 1,\lambda_2,\lambda_4,\lambda_7,\ldots,\lambda_{2g-1},$

$\theta[{}^{01000\ldots0}_{00100\ldots0}](u(p),\Pi)$ $\qquad \infty,\lambda_2,\lambda_4,\lambda_7,\ldots,\lambda_{2g-1},$

$\theta[{}^{00000\ldots0}_{11110\ldots0}](u(p),\Pi)$ $\qquad 1,\lambda_3,\lambda_4,\lambda_7,\ldots,\lambda_{2g-1},$

$\theta[{}^{01000\ldots0}_{10110\ldots0}](u(p),\Pi)$ $\qquad \infty,\lambda_3,\lambda_4,\lambda_7,\ldots,\lambda_{2g-1}.$

$\dfrac{\theta[{}^{01000\ldots0}_{01000\ldots0}](u(p),\Pi)}{\theta[{}^{01000\ldots0}_{00000\ldots0}](u(p),\Pi)}$ is a non-zero multiplicative function on S with characteristic $[{}^{00000\ldots0}_{01000\ldots0}]$ by Lemma 8, with a zero 0 and a pole ∞. On the other hand, we easily see that $\sqrt{Z}$ is a multiplicative function on S with characteristic $[{}^{00000\ldots0}_{01000\ldots0}]$ and with a zero 0 and a pole ∞. Consequently, for a constant C,

$$C\frac{\theta[{}^{01000\ldots0}_{01000\ldots0}](u(p),\Pi)}{\theta[{}^{01000\ldots0}_{00000\ldots0}](u(p),\Pi)} = \sqrt{Z}.$$

Putting P with Z(p) = 1,

$$C = \frac{\theta\left[\begin{smallmatrix}01000\ldots0\\00000\ldots0\end{smallmatrix}\right]\left(\left(\begin{smallmatrix}01000\ldots0\\00000\ldots0\end{smallmatrix}\right),\Pi\right)}{\theta\left[\begin{smallmatrix}01000\ldots0\\01000\ldots0\end{smallmatrix}\right]\left(\left(\begin{smallmatrix}01000\ldots0\\00000\ldots0\end{smallmatrix}\right),\Pi\right)}$$

$$= \frac{\exp\pi i\left[-\frac{1}{4}\Pi_{22}\right]\theta\left[\begin{smallmatrix}02000\ldots0\\00000\ldots0\end{smallmatrix}\right]}{\exp\pi i\left[-\frac{1}{4}\Pi_{22}-\frac{1}{2}\right]\theta\left[\begin{smallmatrix}02000\ldots0\\01000\ldots0\end{smallmatrix}\right]}$$

$$= i\frac{\theta\left[\begin{smallmatrix}00000\ldots0\\00000\ldots0\end{smallmatrix}\right]}{\theta\left[\begin{smallmatrix}00000\ldots0\\01000\ldots0\end{smallmatrix}\right]}\;.$$

Hence, we have

$$\sqrt{\lambda_1} = i\frac{\theta\left[\begin{smallmatrix}00000\ldots0\\00000\ldots0\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}01000\ldots0\\01000\ldots0\end{smallmatrix}\right]\left(\left(\begin{smallmatrix}01000\ldots0\\11000\ldots0\end{smallmatrix}\right),\Pi\right)}{\theta\left[\begin{smallmatrix}00000\ldots0\\01000\ldots0\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}01000\ldots0\\00000\ldots0\end{smallmatrix}\right]\left(\left(\begin{smallmatrix}01000\ldots0\\11000\ldots0\end{smallmatrix}\right),\Pi\right)}$$

$$= i\frac{\theta\left[\begin{smallmatrix}00000\ldots0\\00000\ldots0\end{smallmatrix}\right]\exp\pi i\left[-\frac{1}{4}\Pi_{22}-1\right]\theta\left[\begin{smallmatrix}02000\ldots0\\12000\ldots0\end{smallmatrix}\right]}{\theta\left[\begin{smallmatrix}00000\ldots0\\01000\ldots0\end{smallmatrix}\right]\exp\pi i\left[-\frac{1}{4}\Pi_{22}-\frac{1}{2}\right]\theta\left[\begin{smallmatrix}02000\ldots0\\11000\ldots0\end{smallmatrix}\right]}$$

$$= \frac{\theta\left[\begin{smallmatrix}00000\ldots0\\00000\ldots0\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}00000\ldots0\\10000\ldots0\end{smallmatrix}\right]}{\theta\left[\begin{smallmatrix}00000\ldots0\\01000\ldots0\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}00000\ldots0\\11000\ldots0\end{smallmatrix}\right]}\;.$$

We do the similar computations to obtain the following expressions of $\sqrt{\lambda_1}$ and $\sqrt{\lambda_1-1}$;

$$(55)\quad \sqrt{\lambda_1} = \frac{\theta\left[\begin{smallmatrix}00000\ldots0\\00000\ldots0\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}00000\ldots0\\10000\ldots0\end{smallmatrix}\right]}{\theta\left[\begin{smallmatrix}00000\ldots0\\01000\ldots0\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}00000\ldots0\\11000\ldots0\end{smallmatrix}\right]}$$

$$= \frac{\theta\left[\begin{smallmatrix}00000\ldots0\\00010\ldots0\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}00000\ldots0\\10010\ldots0\end{smallmatrix}\right]}{\theta\left[\begin{smallmatrix}00000\ldots0\\01010\ldots0\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}00000\ldots0\\11010\ldots0\end{smallmatrix}\right]}$$

$$= \frac{\theta\begin{bmatrix}00000\ldots0\\00100\ldots0\end{bmatrix}\theta\begin{bmatrix}00000\ldots0\\10100\ldots0\end{bmatrix}}{\theta\begin{bmatrix}00000\ldots0\\01100\ldots0\end{bmatrix}\theta\begin{bmatrix}00000\ldots0\\11100\ldots0\end{bmatrix}}$$

$$= \frac{\theta\begin{bmatrix}00000\ldots0\\00110\ldots0\end{bmatrix}\theta\begin{bmatrix}00000\ldots0\\10110\ldots0\end{bmatrix}}{\theta\begin{bmatrix}00000\ldots0\\01110\ldots0\end{bmatrix}\theta\begin{bmatrix}00000\ldots0\\11110\ldots0\end{bmatrix}},$$

$$(56)\ \sqrt{\lambda_1 - 1} = \frac{\theta\begin{bmatrix}01000\ldots0\\00000\ldots0\end{bmatrix}\theta\begin{bmatrix}01000\ldots0\\10000\ldots0\end{bmatrix}}{\theta\begin{bmatrix}00000\ldots0\\01000\ldots0\end{bmatrix}\theta\begin{bmatrix}00000\ldots0\\11000\ldots0\end{bmatrix}}$$

$$= \frac{\theta\begin{bmatrix}01000\ldots0\\00010\ldots0\end{bmatrix}\theta\begin{bmatrix}01000\ldots0\\10010\ldots0\end{bmatrix}}{\theta\begin{bmatrix}00000\ldots0\\01010\ldots0\end{bmatrix}\theta\begin{bmatrix}00000\ldots0\\11010\ldots0\end{bmatrix}}$$

$$= \frac{\theta\begin{bmatrix}01000\ldots0\\00100\ldots0\end{bmatrix}\theta\begin{bmatrix}01000\ldots0\\10100\ldots0\end{bmatrix}}{\theta\begin{bmatrix}00000\ldots0\\01100\ldots0\end{bmatrix}\theta\begin{bmatrix}00000\ldots0\\11100\ldots0\end{bmatrix}}$$

$$= \frac{\theta\begin{bmatrix}01000\ldots0\\00110\ldots0\end{bmatrix}\theta\begin{bmatrix}01000\ldots0\\10110\ldots0\end{bmatrix}}{\theta\begin{bmatrix}00000\ldots0\\01110\ldots0\end{bmatrix}\theta\begin{bmatrix}00000\ldots0\\11110\ldots0\end{bmatrix}}.$$

For (56), we only need to consider a multiplicative function $\sqrt{Z-1}$ on S with characteristic $\begin{bmatrix}01000\ldots0\\01000\ldots0\end{bmatrix}$ and with a zero 1 and a pole ∞.

Since for any complex number λ $\lambda-(\lambda-1) = 1$, and hence

$$(57)\ \sqrt{\sqrt{\lambda_1}\sqrt{\lambda_1}\sqrt{\lambda_1}\sqrt{\lambda_1}} \pm \sqrt{\sqrt{\lambda_1-1}\sqrt{\lambda_1-1}\sqrt{\lambda_1-1}\sqrt{\lambda_1-1}} = \pm 1.$$

Putting (55) and (56) in (57), we get

$$(58)\ \sqrt{r_1} \pm \sqrt{r_2} \pm \sqrt{r_3} = 0,$$

where $r_1 = \theta\begin{bmatrix}00000\ldots0\\00000\ldots0\end{bmatrix}\theta\begin{bmatrix}00000\ldots0\\10000\ldots0\end{bmatrix}\theta\begin{bmatrix}00000\ldots0\\00010\ldots0\end{bmatrix}\theta\begin{bmatrix}00000\ldots0\\10010\ldots0\end{bmatrix}$
$\times\ \theta\begin{bmatrix}00000\ldots0\\00100\ldots0\end{bmatrix}\theta\begin{bmatrix}00000\ldots0\\10100\ldots0\end{bmatrix}\theta\begin{bmatrix}00000\ldots0\\00110\ldots0\end{bmatrix}\theta\begin{bmatrix}00000\ldots0\\10110\ldots0\end{bmatrix}$

(59)
$$r_2 = \theta\left[{01000\ldots0 \atop 00000\ldots0}\right]\theta\left[{01000\ldots0 \atop 10000\ldots0}\right]\theta\left[{01000\ldots0 \atop 00010\ldots0}\right]\theta\left[{01000\ldots0 \atop 10010\ldots0}\right]$$
$$\times\ \theta\left[{01000\ldots0 \atop 00100\ldots0}\right]\theta\left[{01000\ldots0 \atop 10100\ldots0}\right]\theta\left[{01000\ldots0 \atop 00110\ldots0}\right]\theta\left[{01000\ldots0 \atop 10110\ldots0}\right],$$
$$r_3 = \theta\left[{00000\ldots0 \atop 01000\ldots0}\right]\theta\left[{00000\ldots0 \atop 11000\ldots0}\right]\theta\left[{00000\ldots0 \atop 01010\ldots0}\right]\theta\left[{00000\ldots0 \atop 11010\ldots0}\right]$$
$$\times\ \theta\left[{00000\ldots0 \atop 01100\ldots0}\right]\theta\left[{00000\ldots0 \atop 11100\ldots0}\right]\theta\left[{00000\ldots0 \atop 01110\ldots0}\right]\theta\left[{00000\ldots0 \atop 11110\ldots0}\right].$$

For $g = 4$, (58) becomes a period relation of Schottky type (of three terms) derived by Farkas and Rauch [9,10,11]. In this point of view, we could call (58) a period relation of Schottky type on a hyperelliptic Riemann surface S of genus $g \geqq 4$.

Theorem 8.

On a hyperelliptic surface S of genus $g \geqq 4$ with a canonical homology basis (γ, δ) shown in Figure 1, a period relation of Schottky type $\sqrt{r_1} \pm \sqrt{r_2} \pm \sqrt{r_3} = 0$ holds, where r_k, $k = 1,2,3$, are given by (59).

In fact, Theorem 8 is a generalization to genus $g > 4$ of a work which was recently done by Farkas. We now know that a relation (58) for $g > 4$ can be obtained from a relation for $g = 4$ simply by adjoinning a $(g-4)$-characteristic $\left[{00..0 \atop 00..0}\right]$ to each 4-characteristic in a relation for $g = 4$ on a hyperelliptic surface.

3-2 Period relations of Schottky type for genera 6 and 7.

Definition 12.

Two (reduced g-theta) characteristics $[\epsilon] = \left[{\epsilon_1 \ldots \epsilon_g \atop \epsilon'_1 \ldots \epsilon'_g}\right]$

and $[\delta] = [\begin{smallmatrix}\delta_1 \cdots \delta_g \\ \delta'_1 \cdots \delta'_g\end{smallmatrix}]$, $g \geqq 1$, are said to be syzygetic (or azygetic) if

(60) $$|\epsilon, \delta| = (-1)^{\sum_{k=1}^{g} (\epsilon_k \delta'_k - \epsilon'_k \delta_k)} = (-1)^{\sum_{k=1}^{g} (\epsilon_k \delta'_k + \epsilon'_k \delta_k)} = 1 \quad (\text{or} -1).$$

Any three characteristics $[\epsilon]$, $[\delta]$ and $[\mu]$ are said to be syzygetic (or azygetic) if

(61) $$|\epsilon, \delta, \mu| = |\epsilon| \cdot |\delta| \cdot |\mu| \cdot |\epsilon+\delta+\mu| = 1 \ (\text{or} -1),$$

where $[\epsilon+\delta+\mu] = [\begin{smallmatrix}(\epsilon+\delta+\mu)_1 \cdots (\epsilon+\delta+\mu)_g \\ (\epsilon+\delta+\mu)'_1 \cdots (\epsilon+\delta+\mu)'_g\end{smallmatrix}]$ is defined by

(62) $$\begin{matrix}(\epsilon+\delta+\mu)_k \equiv \epsilon_k + \delta_k + \mu_k \pmod 2, \\ (\epsilon+\delta+\mu)'_k \equiv \epsilon'_k + \delta'_k + \mu'_k \pmod 2,\end{matrix} \quad k = 1, 2, \ldots, g,$$

and $|\epsilon|$ is the character of $[\epsilon]$ (Definition 3.)

It was proved in [14,15] that

(63) $$|\epsilon, \delta, \mu| = |\epsilon, \delta| \cdot |\delta, \mu| \cdot |\mu, \epsilon|.$$

In particular, any three characteristics are syzygetic (azygetic) if any two of them are syzygetic (azygetic).

Definition 13.

$2g+2$ (reduced g-theta) characteristics $[\epsilon_1], [\epsilon_2], \ldots, [\epsilon_{2g+2}]$, $g \geqq 1$, is called the fundamental system of theta characteristics (abbriviated by F.S. of Th. Ch.) if any mutually distinct three of them are azygetic.

Farkas and Rauch [9,10,11] obtained a period relation of Schottky type for $g = 5$ in the following steps; First, choose a syzgetic group $G = \{(\begin{smallmatrix}0000 \\ 0000\end{smallmatrix}), (\begin{smallmatrix}0000 \\ 0100\end{smallmatrix}), (\begin{smallmatrix}0000 \\ 0010\end{smallmatrix}), (\begin{smallmatrix}0000 \\ 0110\end{smallmatrix})\}$ of half periods of degree 2 and an azygetic set $\{[\begin{smallmatrix}0000 \\ 0000\end{smallmatrix}], [\begin{smallmatrix}1000 \\ 0000\end{smallmatrix}], [\begin{smallmatrix}0000 \\ 1000\end{smallmatrix}], [\begin{smallmatrix}1001 \\ 1001\end{smallmatrix}]\}$ of four 4-characteristics. Then, combining each characteristic in the later with G and forming the product of the four Schottky theta constants whose characteristics are thus obtained, a general theta identity for $g = 4$ is found, i.e.,

(64)

$$\eta\left[\begin{smallmatrix}0000\\0000\end{smallmatrix}\right]\eta\left[\begin{smallmatrix}0000\\0100\end{smallmatrix}\right]\eta\left[\begin{smallmatrix}0000\\0010\end{smallmatrix}\right]\eta\left[\begin{smallmatrix}0000\\0110\end{smallmatrix}\right] \pm \eta\left[\begin{smallmatrix}1000\\0000\end{smallmatrix}\right]\eta\left[\begin{smallmatrix}1000\\0100\end{smallmatrix}\right]\eta\left[\begin{smallmatrix}1000\\0010\end{smallmatrix}\right]\eta\left[\begin{smallmatrix}1000\\0110\end{smallmatrix}\right]$$

$$\pm \eta\left[\begin{smallmatrix}0000\\1000\end{smallmatrix}\right]\eta\left[\begin{smallmatrix}0000\\1100\end{smallmatrix}\right]\eta\left[\begin{smallmatrix}0000\\1010\end{smallmatrix}\right]\eta\left[\begin{smallmatrix}0000\\1110\end{smallmatrix}\right] \pm \eta\left[\begin{smallmatrix}1001\\1001\end{smallmatrix}\right]\eta\left[\begin{smallmatrix}1001\\1101\end{smallmatrix}\right]\eta\left[\begin{smallmatrix}1000\\1010\end{smallmatrix}\right]\eta\left[\begin{smallmatrix}1001\\1111\end{smallmatrix}\right]$$

$$= 0.$$

Finally, by Schottky - Jung relation (39), (64) implies

(65) $\sqrt{r_1} \pm \sqrt{r_2} \pm \sqrt{r_3} \pm \sqrt{r_4} = 0,$

where $r_1 = \theta\left[\begin{smallmatrix}00000\\00000\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}00000\\10000\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}00000\\00100\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}00000\\10100\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}00000\\00010\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}00000\\10010\end{smallmatrix}\right] \times \theta\left[\begin{smallmatrix}00000\\00110\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}00000\\10110\end{smallmatrix}\right],$

$r_2 = \theta\left[\begin{smallmatrix}01000\\00000\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}01000\\10000\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}01000\\00100\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}01000\\10100\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}01000\\00010\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}01000\\10010\end{smallmatrix}\right] \times \theta\left[\begin{smallmatrix}01000\\00110\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}01000\\10110\end{smallmatrix}\right],$

$r_3 = \theta\left[\begin{smallmatrix}00000\\01000\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}00000\\11000\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}00000\\01100\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}00000\\11110\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}00000\\01010\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}00000\\11010\end{smallmatrix}\right] \times \theta\left[\begin{smallmatrix}00000\\01110\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}00000\\11110\end{smallmatrix}\right],$

$r_4 = \theta\left[\begin{smallmatrix}01001\\01001\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}01001\\11001\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}01001\\01101\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}01001\\11101\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}01001\\01011\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}01001\\11011\end{smallmatrix}\right] \times \theta\left[\begin{smallmatrix}01001\\01111\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}01001\\11111\end{smallmatrix}\right].$

On a hyperelliptic surface S of genus $g \geqq 1$, all the even half periods of the form

(66) $e(n,j) \equiv u(A_1^{2n}A_{j_1}A_{j_2}\ldots A_{j_{g-1-2n}}) + K,$

where $A_1, A_2, \ldots, A_{2g+2}$ are branch points on S, n is odd integer with $0 \leqq 2n \leqq g-1$, $j_k \neq j_m$ if $k \neq m$, A_1 is a base point and K is the vector of Riemann constants, vanish, i.e., $\theta[e(n,j)] = 0$ (Lewittes [17]). In particular, for a hyperelliptic surface S of genus $g = 5$ with a canonical homology basis as in Figure 1, $K \equiv \binom{01011}{11111}$ and $u(\lambda_3\lambda_5) + K \equiv \binom{01001}{11001}$, $u(\lambda_2\lambda_4) + K \equiv \binom{01001}{01101}$, $u(\lambda_3\lambda_4) + K \equiv \binom{01001}{11111}$, $u(\lambda_2\lambda_5) + K \equiv \binom{01001}{01011}$ are vanishing even half periods. This leads $r_4 = 0$ in (65), and consequently (65) becomes (58) for $g = 5$.

Lemma 15.

For $g = 3$

(67) $$\theta^4\begin{bmatrix}000\\000\end{bmatrix}-\theta^4\begin{bmatrix}100\\000\end{bmatrix}-\theta^4\begin{bmatrix}000\\100\end{bmatrix}-\theta^4\begin{bmatrix}110\\110\end{bmatrix}-\theta^4\begin{bmatrix}111\\101\end{bmatrix}-\theta^4\begin{bmatrix}101\\111\end{bmatrix} = 0$$

for an azygetic set $\{\begin{bmatrix}000\\000\end{bmatrix},\begin{bmatrix}100\\000\end{bmatrix},\begin{bmatrix}000\\100\end{bmatrix},\begin{bmatrix}110\\110\end{bmatrix},\begin{bmatrix}111\\101\end{bmatrix},\begin{bmatrix}101\\111\end{bmatrix}\}$.

For $g = 4$

(68) $$\theta^2\begin{bmatrix}0000\\0000\end{bmatrix}\theta^2\begin{bmatrix}0000\\1000\end{bmatrix}-\theta^2\begin{bmatrix}1000\\0000\end{bmatrix}\theta^2\begin{bmatrix}1000\\0100\end{bmatrix}-\theta^2\begin{bmatrix}0000\\1000\end{bmatrix}\theta^2\begin{bmatrix}0000\\1100\end{bmatrix}$$
$$-\theta^2\begin{bmatrix}1010\\1010\end{bmatrix}\theta^2\begin{bmatrix}1010\\1110\end{bmatrix}-\theta^2\begin{bmatrix}1011\\1001\end{bmatrix}\theta^2\begin{bmatrix}1011\\1101\end{bmatrix}-\theta^2\begin{bmatrix}1001\\1011\end{bmatrix}\theta^2\begin{bmatrix}1001\\1111\end{bmatrix} = 0,$$

and for $g = 5$

(69) $$\theta\begin{bmatrix}00000\\00000\end{bmatrix}\theta\begin{bmatrix}00000\\01000\end{bmatrix}\theta\begin{bmatrix}00000\\00100\end{bmatrix}\theta\begin{bmatrix}00000\\01100\end{bmatrix}$$
$$-\theta\begin{bmatrix}10000\\00000\end{bmatrix}\theta\begin{bmatrix}10000\\01000\end{bmatrix}\theta\begin{bmatrix}10000\\00100\end{bmatrix}\theta\begin{bmatrix}10000\\01100\end{bmatrix}$$
$$-\theta\begin{bmatrix}00000\\10000\end{bmatrix}\theta\begin{bmatrix}00000\\11000\end{bmatrix}\theta\begin{bmatrix}00000\\10100\end{bmatrix}\theta\begin{bmatrix}00000\\11100\end{bmatrix}$$
$$-\theta\begin{bmatrix}10010\\10010\end{bmatrix}\theta\begin{bmatrix}10010\\11010\end{bmatrix}\theta\begin{bmatrix}10010\\10110\end{bmatrix}\theta\begin{bmatrix}10010\\11110\end{bmatrix}$$
$$-\theta\begin{bmatrix}10011\\10001\end{bmatrix}\theta\begin{bmatrix}10011\\11001\end{bmatrix}\theta\begin{bmatrix}10011\\10101\end{bmatrix}\theta\begin{bmatrix}10011\\11101\end{bmatrix}$$
$$-\theta\begin{bmatrix}10001\\10011\end{bmatrix}\theta\begin{bmatrix}10001\\11011\end{bmatrix}\theta\begin{bmatrix}10001\\10111\end{bmatrix}\theta\begin{bmatrix}10001\\11111\end{bmatrix} = 0.$$

Proof.

In Riemann-theta formula [13,14,15]

$$(70)\quad 2^g y_{[\eta]} = \sum_{[\epsilon]} |\epsilon,\eta|\; x_{[\epsilon]},$$

where

$$y_{[\eta]} = \theta[\eta](u^{(1)},\Pi)\theta[\eta+\rho](u^{(2)},\Pi)\theta[\eta+\sigma](u^{(3)},\Pi)\theta[\eta-\rho-\sigma](u^{(4)},\Pi),$$

$$x_{[\epsilon]} = \theta[\epsilon][v^{(1)},\Pi)\theta[\epsilon+\rho](v^{(2)},\Pi)\theta[\epsilon+\sigma](v^{(3)},\Pi)\theta[\epsilon-\rho-\sigma](u^{(4)},\Pi),$$

$$\begin{bmatrix} v^{(1)} \\ v^{(2)} \\ v^{(3)} \\ v^{(4)} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} u^{(1)} + u^{(2)} + u^{(3)} + u^{(4)} \\ u^{(1)} + u^{(2)} - u^{(3)} - u^{(4)} \\ u^{(1)} - u^{(2)} + u^{(3)} - u^{(4)} \\ u^{(1)} - u^{(2)} - u^{(3)} - u^{(4)} \end{bmatrix},$$

and ρ,σ are arbitrary characteristics. $u^{(1)} = u^{(2)} = u^{(3)} = u^{(4)} = 0$ implies $v^{(1)} = v^{(2)} = v^{(3)} = v^{(4)} = 0$. If we denote the left sides of (67), (68) and (69), respectively, by A, then (70) leads us to the form $2^g A = -2^{g-1}A$, $g = 3,4,5$, and hence $3A = 0$, i.e., $A = 0$.

Corollary 5.

For a compact Riemann surface S of genus $g = 6$,

$$(71)\quad \sqrt{r_1} \pm \sqrt{r_2} \pm \sqrt{r_3} \pm \sqrt{r_4} \pm \sqrt{r_5} \pm \sqrt{r_6} = 0,$$

where $r_1 = \theta\begin{bmatrix}000000\\000000\end{bmatrix}\theta\begin{bmatrix}000000\\100000\end{bmatrix}\theta\begin{bmatrix}000000\\001000\end{bmatrix}\theta\begin{bmatrix}000000\\101000\end{bmatrix} \times$
$\theta\begin{bmatrix}000000\\000100\end{bmatrix}\theta\begin{bmatrix}000000\\100100\end{bmatrix}\theta\begin{bmatrix}000000\\001100\end{bmatrix}\theta\begin{bmatrix}000000\\101100\end{bmatrix},$

$r_2 = \theta\begin{bmatrix}010000\\000000\end{bmatrix}\theta\begin{bmatrix}010000\\100000\end{bmatrix}\theta\begin{bmatrix}010000\\001000\end{bmatrix}\theta\begin{bmatrix}010000\\101000\end{bmatrix} \times$
$\theta\begin{bmatrix}010000\\000100\end{bmatrix}\theta\begin{bmatrix}010000\\100100\end{bmatrix}\theta\begin{bmatrix}010000\\001100\end{bmatrix}\theta\begin{bmatrix}010000\\101100\end{bmatrix},$

$$r_3 = \theta\begin{bmatrix}000000\\010000\end{bmatrix}\theta\begin{bmatrix}000000\\110000\end{bmatrix}\theta\begin{bmatrix}000000\\011000\end{bmatrix}\theta\begin{bmatrix}000000\\111000\end{bmatrix} \times \theta\begin{bmatrix}000000\\010100\end{bmatrix}\theta\begin{bmatrix}000000\\110100\end{bmatrix}\theta\begin{bmatrix}000000\\011100\end{bmatrix}\theta\begin{bmatrix}000000\\111100\end{bmatrix},$$

$$r_4 = \theta\begin{bmatrix}010010\\010010\end{bmatrix}\theta\begin{bmatrix}010010\\110010\end{bmatrix}\theta\begin{bmatrix}010010\\011010\end{bmatrix}\theta\begin{bmatrix}010010\\111010\end{bmatrix} \times \theta\begin{bmatrix}010010\\010110\end{bmatrix}\theta\begin{bmatrix}010010\\110110\end{bmatrix}\theta\begin{bmatrix}010010\\011110\end{bmatrix}\theta\begin{bmatrix}010010\\111110\end{bmatrix},$$

$$r_5 = \theta\begin{bmatrix}010011\\010001\end{bmatrix}\theta\begin{bmatrix}010011\\110001\end{bmatrix}\theta\begin{bmatrix}010011\\011001\end{bmatrix}\theta\begin{bmatrix}010011\\111001\end{bmatrix} \times \theta\begin{bmatrix}010011\\010101\end{bmatrix}\theta\begin{bmatrix}010011\\110101\end{bmatrix}\theta\begin{bmatrix}010011\\011101\end{bmatrix}\theta\begin{bmatrix}010011\\111101\end{bmatrix},$$

$$r_6 = \theta\begin{bmatrix}010001\\010011\end{bmatrix}\theta\begin{bmatrix}010001\\110011\end{bmatrix}\theta\begin{bmatrix}010001\\011011\end{bmatrix}\theta\begin{bmatrix}010001\\111011\end{bmatrix} \times \theta\begin{bmatrix}010001\\010111\end{bmatrix}\theta\begin{bmatrix}010001\\110111\end{bmatrix}\theta\begin{bmatrix}010001\\011111\end{bmatrix}\theta\begin{bmatrix}010001\\111111\end{bmatrix}.$$

Proof.

By Schottky - Jung relation (39), (71) immediately follows from (69).

If S is hyperelliptic, then by (52) and (66) $K \equiv \binom{101001}{101111}$, and $u(\lambda_2\lambda_4\lambda_6) + K \equiv \binom{010001}{010011}(r_6 = 0)$, $u(\lambda_2\lambda_4\lambda_8)+K\equiv\binom{010011}{011001}(r_5=0)$, $u(\lambda_2\lambda_4\lambda_{10})+K\equiv\binom{010010}{011010}(r_4=0)$ are vanishing even half periods (out of 364 such half periods). Again (71) becomes (58) for $g = 6$. We can do a similar work for $g = 7$.

Lemma 16.

For $g = 4$

$$(72)\quad \theta^4\begin{bmatrix}0000\\0000\end{bmatrix}-\theta^4\begin{bmatrix}1000\\0000\end{bmatrix}-\theta^4\begin{bmatrix}0000\\1000\end{bmatrix}-\theta^4\begin{bmatrix}1100\\1100\end{bmatrix}-\theta^4\begin{bmatrix}1110\\1010\end{bmatrix} -\theta^4\begin{bmatrix}1111\\1001\end{bmatrix}-\theta^4\begin{bmatrix}1001\\1101\end{bmatrix}-\theta^4\begin{bmatrix}1011\\1110\end{bmatrix}-\theta^4\begin{bmatrix}1010\\1111\end{bmatrix}-\theta^4\begin{bmatrix}1101\\1011\end{bmatrix} = 0$$

for an azygetic set of ten 4-characteristics.

For $g = 5$

$$(73)\quad \theta^2\begin{bmatrix}00000\\00000\end{bmatrix}\theta^2\begin{bmatrix}00000\\01000\end{bmatrix}-\theta^2\begin{bmatrix}10000\\00000\end{bmatrix}\theta^2\begin{bmatrix}10000\\01000\end{bmatrix}$$
$$-\theta^2\begin{bmatrix}00000\\10000\end{bmatrix}\theta^2\begin{bmatrix}00000\\11000\end{bmatrix}-\theta^2\begin{bmatrix}10100\\10100\end{bmatrix}\theta^2\begin{bmatrix}10100\\11100\end{bmatrix}$$
$$-\theta^2\begin{bmatrix}10110\\10010\end{bmatrix}\theta^2\begin{bmatrix}10110\\11010\end{bmatrix}-\theta^2\begin{bmatrix}10111\\10001\end{bmatrix}\theta^2\begin{bmatrix}10111\\11001\end{bmatrix}$$
$$-\theta^2\begin{bmatrix}10001\\10101\end{bmatrix}\theta^2\begin{bmatrix}10001\\11101\end{bmatrix}-\theta^2\begin{bmatrix}10011\\10110\end{bmatrix}\theta^2\begin{bmatrix}10011\\11110\end{bmatrix}$$
$$-\theta^2\begin{bmatrix}10010\\10111\end{bmatrix}\theta^2\begin{bmatrix}10010\\11111\end{bmatrix}-\theta^2\begin{bmatrix}10101\\10011\end{bmatrix}\theta^2\begin{bmatrix}10101\\11011\end{bmatrix} = 0$$

and for $g = 6$

$$(74)\quad \theta\begin{bmatrix}000000\\000000\end{bmatrix}\theta\begin{bmatrix}000000\\010000\end{bmatrix}\theta\begin{bmatrix}000000\\001000\end{bmatrix}\theta\begin{bmatrix}000000\\011000\end{bmatrix}$$
$$-\theta\begin{bmatrix}100000\\000000\end{bmatrix}\theta\begin{bmatrix}100000\\010000\end{bmatrix}\theta\begin{bmatrix}100000\\001000\end{bmatrix}\theta\begin{bmatrix}100000\\011000\end{bmatrix}$$
$$-\theta\begin{bmatrix}000000\\100000\end{bmatrix}\theta\begin{bmatrix}000000\\110000\end{bmatrix}\theta\begin{bmatrix}000000\\101000\end{bmatrix}\theta\begin{bmatrix}000000\\111000\end{bmatrix}$$
$$-\theta\begin{bmatrix}100100\\100100\end{bmatrix}\theta\begin{bmatrix}100100\\110100\end{bmatrix}\theta\begin{bmatrix}100100\\101100\end{bmatrix}\theta\begin{bmatrix}100100\\111100\end{bmatrix}$$
$$-\theta\begin{bmatrix}100110\\100010\end{bmatrix}\theta\begin{bmatrix}100110\\110010\end{bmatrix}\theta\begin{bmatrix}100110\\101010\end{bmatrix}\theta\begin{bmatrix}100110\\111010\end{bmatrix}$$
$$-\theta\begin{bmatrix}100111\\100001\end{bmatrix}\theta\begin{bmatrix}100111\\110001\end{bmatrix}\theta\begin{bmatrix}100111\\101001\end{bmatrix}\theta\begin{bmatrix}100111\\111001\end{bmatrix}$$
$$-\theta\begin{bmatrix}100001\\100101\end{bmatrix}\theta\begin{bmatrix}100001\\110101\end{bmatrix}\theta\begin{bmatrix}100001\\101101\end{bmatrix}\theta\begin{bmatrix}100001\\111101\end{bmatrix}$$
$$-\theta\begin{bmatrix}100011\\100110\end{bmatrix}\theta\begin{bmatrix}100011\\110110\end{bmatrix}\theta\begin{bmatrix}100011\\101110\end{bmatrix}\theta\begin{bmatrix}100011\\111110\end{bmatrix}$$
$$-\theta\begin{bmatrix}100010\\100111\end{bmatrix}\theta\begin{bmatrix}100010\\110111\end{bmatrix}\theta\begin{bmatrix}100010\\101111\end{bmatrix}\theta\begin{bmatrix}100010\\111111\end{bmatrix}$$
$$-\theta\begin{bmatrix}100101\\100011\end{bmatrix}\theta\begin{bmatrix}100101\\110011\end{bmatrix}\theta\begin{bmatrix}100101\\101011\end{bmatrix}\theta\begin{bmatrix}100101\\111011\end{bmatrix} = 0.$$

Proof.

As in Lemma 15, if we denote the left sides of (72), (73) and (74), respectively, by A, then by Riemann theta formula (70) we have again $2^g A = -2^{g-1} A$, $g=4,5,6$, and hence $2A=-A$, $3A=0$, i.e., $A=0$.

Corollary 6.

For a compact Riemann surface S of genus $g = 7$

(75) $\sum_{k=1}^{10} \pm \sqrt{r_k} = 0,$

where r_k is the product of eight theta constants with characteristics obtained by adjoining $[{0 \atop 0}]$ and $[{0 \atop 1}]$ as the first column to each characteristic of kth term in (74), and k is counted in order in (74).

Proof.

By Schottky - Jung relation (39), (75) follows immediately from (74).

Again, if S is hyperelliptic, then by (52) and (66)

$k \equiv \binom{0101101}{1111111}$ and

$u(\lambda_3\lambda_5\lambda_6\lambda_7)+K \equiv \binom{0100101}{1110011}, \ (r_{10}=0),$

$u(\lambda_3\lambda_5\lambda_6\lambda_{13})+K \equiv \binom{0100010}{1110111}, \ (r_9=0),$

$u(\lambda_3\lambda_5\lambda_6\lambda_{11})+K \equiv \binom{0100010}{1110111}, \ (r_8=0),$

$u(\lambda_3\lambda_5\lambda_6\lambda_9)+K \equiv \binom{0100001}{1110101}, \ (r_7=0),$

$u(\lambda_3\lambda_5\lambda_8\lambda_{10})+K \equiv \binom{0100111}{1100001}, \ (r_6=0),$

$u(\lambda_3\lambda_5\lambda_8\lambda_{12})+K \equiv \binom{0100110}{1100010}, \ (r_5=0),$

$u(\lambda_3\lambda_5\lambda_{10}\lambda_{12})+K \equiv \binom{0100100}{1100100}, \ (r_4=0),$

are vanishing even half periods (out of 1366 such half periods). Thus (75) reduces to (58) for g = 7.

According to our observations, we conjecture that a period relation of Schottky type for $g \geqq 5$, $\sum_{k=1}^{m} \pm \sqrt{r_k} = 0,$

might have

$$m = 3+2^0+2^1+2^2+\dots+2^{g-5} = 2(2^{g-5}+1)$$

terms by the induction on genus g.

3-3 Theta identity on hyperelliptic surfaces.

Let S be a hyperelliptic surface of genus $g \geqq 1$,

(76) $$w^2 = \prod_{j=1}^{2g+2} (Z-\lambda_j),$$

where λ_j are mutually distinct finite, its concrete algebraic representation and $\gamma_1,\gamma_2,\dots,\gamma_g;\ \delta_1,\delta_2,\dots,\delta_g$ a canonical homology basis on S chosen as in Figure 2.

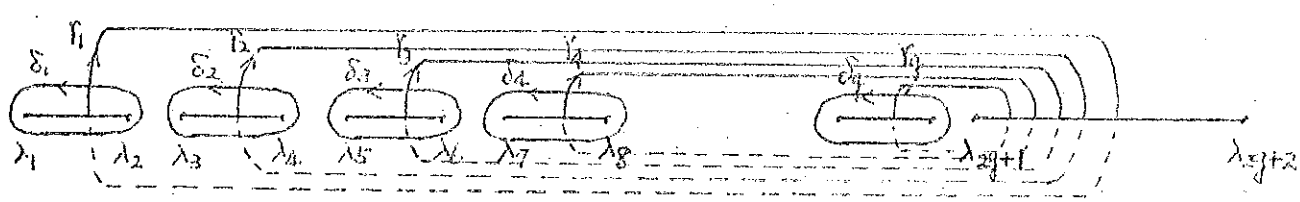

Figure 2

(77) $$dv_k = \frac{Z^{k-1}dZ}{w},\ k = 1,2,\dots,g,$$

is a basis for $A_1(S)$, and we denote

(78) $$A_{k\ell} = \int_{\delta_\ell} dv_k,\ \ell = 1,2,\dots,g,$$

(79) $$A = \det(A_{k\ell}).$$

Taking the point P_0 on S with $Z(p_0) = \lambda_1$ as a base point, we can define a mapping u from S to its Jacobi variety J(S) and, as in 3-1, all the images $u(\lambda_j)$ in J(S) of all the branch points λ_j under u can be computed. They are, in fact, half periods;

$$u(\lambda_1) \equiv (\binom{0}{0}^g),$$

$$u(\lambda_2) \equiv (\binom{1}{0}\binom{0}{0}^{g-1}),$$

$$u(\lambda_3) \equiv (\binom{1}{1}\binom{0}{1}\binom{0}{0}^{g-2}),$$

$$u(\lambda_4) \equiv (\binom{1}{1}\binom{1}{1}\binom{0}{0}^{g-2}),$$

$$u(\lambda_5) \equiv (\binom{1}{1}\binom{1}{0}\binom{0}{1}\binom{0}{0}^{g-3}),$$

$$u(\lambda_6) \equiv (\binom{1}{1}\binom{1}{0}\binom{1}{1}\binom{0}{0}^{g-3}),$$

(80) $$u(\lambda_7) \equiv (\binom{1}{1}\binom{1}{0}^2\binom{0}{1}\binom{0}{0}^{g-4}),$$

$$u(\lambda_8) \equiv (\binom{1}{1}\binom{1}{0}^2\binom{1}{1}\binom{0}{0}^{g-4}),$$

- - - - - - - -

$$u(\lambda_{2g-1}) \equiv (\binom{1}{1}\binom{1}{0}^{g-2}\binom{0}{1}),$$

$$u(\lambda_{2g}) \equiv (\binom{1}{1}\binom{1}{0}^{g-2}\binom{1}{1}),$$

$$u(\lambda_{2g+1}) \equiv (\binom{1}{1}\binom{1}{0}^{g-1}),$$

$$u(\lambda_{2g+2}) \equiv (\binom{0}{1}\binom{0}{0}^{g-1}).$$

According to Krazer's prescription (chapter 10, §1, Theorem 2 and §2), the vector K of Riemann constants with respect to a chosen base point P_0 on S with $Z(P_0) = \lambda_1$ is given by

$$(81) \quad K \equiv (\text{the sum of all odd half periods among } u(\lambda_j)) \equiv (\sum_{\substack{j=3 \\ j=\text{odd}}}^{2g+1} u(\lambda_j)) \equiv (\sum_{\substack{j=1 \\ j=\text{odd}}}^{2g+1} u(\lambda_j)) \equiv \left(\begin{smallmatrix} g & g-1 & g-2 & \cdots & 1 \\ g & 1 & 1 & \cdots & 1 \end{smallmatrix}\right).$$

As before, we also denote the period matrix of S and (γ, δ) by Π.

Suppose that for a g-characteristic $[\epsilon]$ $\theta[\epsilon] = \theta[\epsilon](\Pi) \neq 0$, and that there are two complementary sets $\{\lambda_{j_1}, \lambda_{j_2}, \ldots, \lambda_{j_{g+1}}\}$ and $\{\lambda_{j'_1}, \lambda_{j'_2}, \ldots, \lambda_{j'_{g+1}}\}$ of 2g+2 branch points λ_j, $j = 1, 2, \ldots, 2g+2$, with

$$(82) \quad [\epsilon] \equiv [K + \sum_{h=1}^{g+1} u(\lambda_{j_h})] \equiv [K + \sum_{h=1}^{g+1} u(\lambda_{j'_h})].$$

Then, Thomae's theorem [15, 31] tells us that

$$(83) \quad \theta[\epsilon] = \sqrt{\frac{A}{(2\pi i)^g}} \sqrt[4]{\Delta(\lambda_j) \cdot \Delta(\lambda_{j'})},$$

where $\Delta(\lambda_j) = \det \begin{bmatrix} 1 & \lambda_{j_1} & \lambda_{j_1}^2 & \cdots & \lambda_{j_1}^g \\ 1 & \lambda_{j_2} & \lambda_{j_2}^2 & \cdots & \lambda_{j_2}^g \\ - & - & - & - & - \\ 1 & \lambda_{j_{g+1}} & \lambda_{j_{g+1}}^2 & \cdots & \lambda_{j_{g+1}}^g \end{bmatrix} = \prod_{m'>m} (\lambda_{j_{m'}} - \lambda_{j_m})$

and $\Delta(\lambda_{j'})$ is defined as $\Delta(\lambda_j)$.

By (80), we can easily see that 2g+2 characteristics $[u(\lambda_j)]$ form a F.S. of Th. Ch., and that there are g+2 even half periods $u(\lambda_1)$, $u(\lambda_2)$, $u(\lambda_4),\dots,u(\lambda_{2g+2})$ and g odd half periods $u(\lambda_3)$, $u(\lambda_5),\dots,u(\lambda_{2g+1})$ with $[\sum_{j=1}^{2g+2} u(\lambda_j)] \equiv \left[\begin{smallmatrix}00\dots0\\00\dots0\end{smallmatrix}\right]$.

On S, any theta constant with characteristic $[\varepsilon] \equiv [K+\sum_{h=1}^{g+1} u(\lambda_{j_h})]$ for g+1 branch points $\lambda_{j_1},\lambda_{j_2},\dots,\lambda_{j_{g+1}}$ does not vanish by (66). In particular, by (80) and (81)

(84)

$$[u(\lambda_1)] \equiv [K+u(\lambda_1)+\sum_{\substack{j=3\\ j=\text{odd}}}^{2g+1} u(\lambda_j)] \equiv [K+\sum_{\substack{j=2\\ j=\text{even}}}^{2g+2} u(\lambda_j)],$$

$$[u(\lambda_{2k})] \equiv [K+u(\lambda_{2k})+\sum_{\substack{j=3\\ j=\text{odd}}}^{2g+1} u(\lambda_j)] \equiv [K+u(\lambda_1)+\sum_{\substack{j=2\\ j=\text{even}\neq 2k}}^{2g+2} u(\lambda_j)].$$

<u>Lemma 17.</u>

(85) $$\sum_{k=1}^{g+1} \Delta(\lambda_3,\lambda_5,\dots,\lambda_{2k-1},\lambda_{2k},\lambda_{2k+1},\dots,\lambda_{2g+1})\cdot$$

$$\Delta(\lambda_1,\lambda_2,\lambda_4,\dots,\lambda_{2k-2},\lambda_{2k+2},\dots,\lambda_{2g+2})$$

$$= \Delta(\lambda_1,\lambda_3,\lambda_5,\dots,\lambda_{2g+1})\cdot\Delta(\lambda_2,\lambda_4,\lambda_6,\dots,\lambda_{2g+2}).$$

<u>Proof.</u>

By the definition of Δ, $\Delta(\lambda_1,\lambda_3,\lambda_5,\dots,\lambda_{2g+1})\cdot$ $\Delta(\lambda_2,\lambda_4,\lambda_6,\dots,\lambda_{2g+2}) = \prod_{\substack{k,k'=1\\ k'>k}}^{g+1} (\lambda_{2k'-1}-\lambda_{2k'-1})(\lambda_{2k'}-\lambda_{2k})$. We use a simple notation for a proof, i.e., $(1,x,x^2,\dots,x^g) = (x)$ for any variable x.

If $\lambda_{2k'} = \lambda_{2k}$, $1 \leqq k < k' \leqq g+1$., then

$$\sum_{k=1}^{g+1} \Delta(\lambda_3, \lambda_5, \ldots, \lambda_{2k-1}, \lambda_{2k}, \lambda_{2k+1}, \ldots, \lambda_{2g+1}) \cdot \Delta(\lambda_1, \lambda_2, \lambda_4, \ldots, \lambda_{2k-2}, \lambda_{2k+2}, \ldots, \lambda_{2g+2})$$

$$= \sum_{k=1}^{g+1} \begin{vmatrix} \lambda_3 \\ \lambda_5 \\ \vdots \\ \lambda_{2k-1} \\ \lambda_{2k} \\ \lambda_{2k+1} \\ \vdots \\ \lambda_{2g+1} \end{vmatrix} \cdot \begin{vmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{2k-2} \\ \lambda_{2k+2} \\ \vdots \\ \lambda_{2g+2} \end{vmatrix} = \begin{vmatrix} \lambda_3 \\ \lambda_5 \\ \vdots \\ \lambda_{2k-1} \\ \lambda_{2k} \\ \lambda_{2k+1} \\ \vdots \\ \lambda_{2g+1} \end{vmatrix} \cdot \begin{vmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{2k-2} \\ \lambda_{2k+2} \\ \vdots \\ \lambda_{2k'} \\ \vdots \\ \lambda_{2g+2} \end{vmatrix} + \begin{vmatrix} \lambda_3 \\ \lambda_5 \\ \vdots \\ \lambda_{2k'-1} \\ \lambda_{2k'} \\ \lambda_{2k'+1} \\ \vdots \\ \lambda_{2g+1} \end{vmatrix} \begin{vmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{2k} \\ \vdots \\ \lambda_{2k'-2} \\ \lambda_{2k'+2} \\ \vdots \\ \lambda_{2g+2} \end{vmatrix}$$

$$= \begin{vmatrix} \begin{pmatrix} \lambda_3 \\ \lambda_5 \\ \vdots \\ \lambda_{2k-1} \\ \lambda_{2k} \\ \lambda_{2k+1} \\ \vdots \\ \lambda_{2g+1} \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{2k-2} \\ \lambda_{2k+2} \\ \vdots \\ \lambda_{2k'} \\ \vdots \\ \lambda_{2g+2} \end{pmatrix} \end{vmatrix} + \begin{vmatrix} \begin{pmatrix} \lambda_3 \\ \lambda_5 \\ \vdots \\ \lambda_{2k'-1} \\ \lambda_{2k'} \\ \lambda_{2k'+1} \\ \vdots \\ \lambda_{2g+1} \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{2k} \\ \vdots \\ \lambda_{2k'-2} \\ \lambda_{2k'+2} \\ \vdots \\ \lambda_{2g+2} \end{pmatrix} \end{vmatrix}$$

$$= (-1)^{(g+1)+(g-k+1)+(k-1)} \begin{vmatrix} 0 & (\lambda_1) \\ \begin{pmatrix} \lambda_3 \\ \lambda_5 \\ \vdots \\ \lambda_{2g+1} \end{pmatrix} & \\ 0 & \begin{pmatrix} \lambda_2 \\ \lambda_4 \\ \vdots \\ \lambda_{2k-2} \end{pmatrix} \\ (\lambda_{2k}) & 0 \\ 0 & \begin{pmatrix} \lambda_{2k+2} \\ \vdots \\ \lambda_{2k'} \\ \vdots \\ \lambda_{2g+2} \end{pmatrix} \end{vmatrix}$$

$$+(-1)^{(g+1)+(g-k'+1)+(k'-1)} \begin{vmatrix} 0 & (\lambda_1) \\ \begin{bmatrix} \lambda_3 \\ \lambda_5 \\ \vdots \\ \lambda_{2g+1} \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} \lambda_2 \\ \lambda_4 \\ \vdots \\ \lambda_{2k} \\ \vdots \\ \lambda_{2k'-2} \end{bmatrix} \\ (\lambda_{2k'}) & 0 \\ & \begin{bmatrix} \lambda_{2k'+2} \\ \vdots \\ \lambda_{2g+2} \end{bmatrix} \end{vmatrix}$$

$$
= -\begin{vmatrix}
0 & \begin{bmatrix}\lambda_1\end{bmatrix} \\
\begin{bmatrix}\lambda_3\\ \lambda_5\\ \vdots\\ \lambda_{2g+1}\end{bmatrix} & \begin{bmatrix}\lambda_3\\ \lambda_5\\ \vdots\\ \lambda_{2g+1}\end{bmatrix} \\
0 & \begin{bmatrix}\lambda_2\end{bmatrix} \\
 & \begin{bmatrix}\lambda_4\\ \vdots\\ \lambda_{2k-2}\end{bmatrix} \\
(\lambda_{2k}) & \begin{bmatrix}\lambda_{2k}\end{bmatrix} \\
 & \begin{bmatrix}\lambda_{2k+2}\\ \vdots\end{bmatrix} \\
0 & \begin{bmatrix}\lambda_{2k'}\\ \vdots\\ \lambda_{2g+2}\end{bmatrix}
\end{vmatrix}
-\begin{vmatrix}
0 & \begin{bmatrix}\lambda_1\end{bmatrix} \\
\begin{bmatrix}\lambda_3\\ \lambda_5\\ \vdots\\ \lambda_{2g+1}\end{bmatrix} & \begin{bmatrix}\lambda_3\\ \lambda_5\\ \vdots\\ \lambda_{2g+1}\end{bmatrix} \\
 & \begin{bmatrix}\lambda_2\end{bmatrix} \\
0 & \begin{bmatrix}\lambda_4\\ \vdots\end{bmatrix} \\
 & \begin{bmatrix}\lambda_{2k}\\ \vdots\end{bmatrix} \\
(\lambda_{2k'}) & \begin{bmatrix}\lambda_{2k'}\\ \vdots\end{bmatrix} \\
0 & \begin{bmatrix}\lambda_{2g+2}\end{bmatrix}
\end{vmatrix}
$$

$$= \{(-1)-(-1)^{(k'-k)+(k'-k-1)}\} \left| \begin{array}{cc} \begin{matrix} 0 \\ \begin{bmatrix} \lambda_3 \\ \lambda_5 \\ \vdots \\ \lambda_{2g+1} \end{bmatrix} \end{matrix} & \begin{bmatrix} \lambda_1 \\ \lambda_3 \\ \lambda_5 \\ \vdots \\ \lambda_{2g+1} \end{bmatrix} \\ \begin{matrix} 0 \\ (\lambda_{2k}) \\ 0 \end{matrix} & \begin{bmatrix} \lambda_2 \\ \lambda_4 \\ \vdots \\ \lambda_{2k} \\ \vdots \\ \lambda_{2k'} \\ \vdots \\ \lambda_{2g+2} \end{bmatrix} \end{array} \right|$$

$= 0.$

Here, we used a fact that $\det(A)\cdot\det(B) = \det\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ for any square matrices A and B. Thus we showed that the left side of (85) is divided by the factors $\lambda_{2k'}-\lambda_{2k}$, $1 \leqq k < k' \leqq g + 1$. Similarily, if $\lambda_{2h'-1} = \lambda_{2h-1}$, $2 \leqq h < h' \leqq g + 1$, then the left side of (85) is zero and it is divided by the factors $\lambda_{2h'-1} - \lambda_{2h-1}$. In fact, for each k, $k = 1,2,\ldots,g+1$,

$\Delta(\lambda_3,\lambda_5,\ldots,\lambda_{2h-1},\ldots,\lambda_{2k-1},\lambda_{2k+1},\ldots,\lambda_{2h'-1},\ldots,\lambda_{2g+1}) = 0$

for any h and h' with $2 \leqq h < h' \leqq g + 1$, which are the first factors of each term in the summation of the left side of (85).

Finally, if $\lambda_{2h-1} = \lambda_1$, $h = 2,3,\ldots,g+1$, then we have to use the induction on g to show that the left side of (85) is again zero. For example, for $g = 1$, the left side of (85) is

$$\Delta(\lambda_2,\lambda_3)\cdot\Delta(\lambda_1,\lambda_4)+\Delta(\lambda_3,\lambda_4)\cdot\Delta(\lambda_1,\lambda_2)=\Delta(\lambda_2,\lambda_1)\cdot\Delta(\lambda_1,\lambda_4)$$
$$+\Delta(\lambda_1,\lambda_4)\cdot\Delta(\lambda_1,\lambda_2)=-\Delta(\lambda_1,\lambda_2)\Delta(\lambda_1,\lambda_4)$$
$$+\Delta(\lambda_1,\lambda_2)\Delta(\lambda_1,\lambda_4)=0$$

if $\lambda_1 = \lambda_3$. In this case, the sum will be zero, but no each term in the sum will be zero.

Therefore, both sides of (85) have the factors $\prod_{\substack{k,k'=1 \\ k'>k}}^{g+1} (\lambda_{2k'-1}-\lambda_{2k-1})\cdot(\lambda_{2k'}-\lambda_{2k})$. In the right side of (85), the sign of the term $\lambda_{2g+1}^{g}\lambda_{2g-1}^{g-1}\cdots\lambda_3\lambda_{2g+2}^{g}\lambda_{2g}^{g-1}\cdots\lambda_4$ is (+), taking all the first terms of the factors. Likewise, the sign of the same term in the left side of (85) is also (+), since the first term of the sum is $\Delta(\lambda_2,\lambda_3,\lambda_5,\ldots,\lambda_{2g+1})\cdot\Delta(\lambda_1,\lambda_4,\ldots,\lambda_{2g+2})$ and $\lambda_{2g+1}^{g}\lambda_{2g-1}^{g-1}\cdots\lambda_3\lambda_{2g+2}^{g}\lambda_{2g}^{g-1}\cdots\lambda_4$ has the sign (+) and it does not appear in any other terms in the sum. This completes a proof of Lemma.

Theorem 9.

For a hyperelliptic surface S of $w^2 = \prod_{j=1}^{2g+2} (Z-\lambda_j)$ of genus $g \geqq 1$ with a canonical homology basis as shown in Figure 2,

(86) $\theta^4[u(\lambda_1)] - \sum_{k=1}^{g+1} \theta^4[u(\lambda_{2k})] = 0,$

where u is a mapping from S to its Jacobi variety J(S) with respect to a base point P_0 on S with $Z(P_0) = \lambda_1$.

Proof.

By (83), (84) and (85)

$$\theta^4[u(\lambda_1)] = (\frac{A}{(2\pi i)^g})^2 \Delta(\lambda_1,\lambda_3,\dots,\lambda_{2g+1})\cdot\Delta(\lambda_2,\lambda_4,\dots,\lambda_{2g+2})$$

$$= (\frac{A}{(2\pi i)^g})^2 \sum_{k=1}^{g+1} \Delta(\lambda_3,\lambda_5,\dots,\lambda_{2k-1},\lambda_{2k},\lambda_{2k+1},\dots,\lambda_{2g+1})\cdot \Delta(\lambda_1,\lambda_2,\lambda_4,\dots,\lambda_{2k-2},\lambda_{2k+2},\dots,\lambda_{2g+2})$$

$$= \sum_{k=1}^{g+1} \theta^4[u(\lambda_{2k})],$$

which is (86).

CHAPTER IV BRANCHED COVERINGS OVER THE SPHERE

4-1 n-sheeted branched coverings over the sphere.

If a compact Riemann surface S of genus $g \geqq 1$ permits a meromorphic function Z on it which assumes every complex value n times, $n \geqq 2$, then there is a meromorphic function w on S satisfying an irreducible algebraic equation

$$(87) \quad F(Z,w) = w^n + r_1(Z)w^{n-1} + r_2(Z)w^{n-2} + \dots + r_{n-1}(Z)w + r_n(Z) = 0,$$

where $r_h(Z)$, $h = 1,2,\dots,n$, are rational functions of Z, such that any other meromorphic function v on S can be represented as a rational function of Z and w,

$$v = R_1(Z)w^{n-1} + R_2(Z)w^{n-2} + \dots + r_{n-1}(Z)w + R_n(Z),$$

where $R_j(Z)$, $j = 1,2,\dots,n$ are also rational functions of Z. The set of all such meromorphic functions v on S is the field $\mathfrak{M}(S)$ of algebraic functions on S which is generated by Z and w satisfying $F(Z,w) = 0$. Furthermore, the mapping $p \leftrightarrow (Z,w)$, where $P \in S$, gives an one-to-one conformal mapping from S onto the Riemann surface of an algebraic function w(Z), and hence S can now be realized as the Riemann surface of an algebraic function w(Z). Conservely, all the Riemann surfaces of algebraic functions are known as the compact Riemann surfaces.

If $r_1(Z) = r_2(Z) = \dots = r_{n-1}(Z) = 0$ in (87), then $F(Z,w) = w^n + r_n(Z) = 0$ can be reduced to the form

$$(88) \quad w^n = \frac{A \prod_k (Z-a_k)^{m_k}}{\prod_h (Z-b_h)^{n_h}},$$

where A is a complex constant and $1 \leqq m_k, n_h$ for all k and h.

Writing $m_k = n\ell_k + e_k$, $n_h = nq_h + f_h$, where ℓ_k and q_h are integers and $1 \leqq e_k, f_h \leqq n-1$ for each k and h,

$$(89)\begin{cases} Z' = Z \\ w' = \dfrac{\prod_h (Z-b_h)^{q_h}}{A^{\frac{1}{n}} \prod_k (Z-a_k)^{\ell_k}} w \end{cases}$$

defines a birational transformation from $\mathfrak{M}(S)$ onto $\mathfrak{M}'(S)$ generated by Z' and w' satisfying an irreducible algebraic equation

$$(90) \quad F'(Z',w') = w'^n - \frac{\prod_k (Z'-a_k)^{e_k}}{\prod_h (Z'-b_h)^{f_h}} = 0.$$

Again,

$$(91) \quad \begin{cases} Z'' = Z' \\ w'' = \prod_h (Z'-b_h) w' \end{cases}$$

defines a birational transformation from $\mathfrak{M}'(S)$ onto $\mathfrak{M}''(S)$ generated by Z" and w" satisfying an irreducible algebraic function

$$(92) \quad F''(Z'',w'') = w''^n - \prod_k (Z''-a_k)^{e_k} \cdot \prod_h (Z''-b_h)^{n-f_h} = 0.$$

Note that $1 \leqq n-f_h \leqq n-1$ for all h, since $1 \leqq f_h \leqq n-1$. Since a mapping defined by $P \leftrightarrow (Z,\omega) \leftrightarrow (Z',w') \leftrightarrow (Z'',w'')$ is conformal, S is comformally equivalent to the Riemann surface of an algebraic function $w''(Z'')$ satisfying (92). Consequently, we may assume that a concrete algebraic representation of a given compact Riemann surface S is for a particular case

$$(93)\quad w^n = \prod_{\ell=1}^{b} (Z-a_\ell)^{m_\ell}$$

where $n \geqq 2$, $1 \leqq m_\ell \leqq n-1$ and m_ℓ are mutually distinct. S can now be realized as an n-sheeted branched covering over the sphere. We, further, assume that

$$(94)\quad \sum_{\ell=1}^{b} m_\ell \equiv 0 \pmod{n}$$

to make the points on S over $Z = \infty$ as regular points.

For each ℓ, $\ell = 1,2,\ldots,b$, let

$$(95)\quad \frac{m_\ell}{n} = \frac{h_\ell}{\nu_\ell},\ (h_\ell,\nu_\ell) = 1 \text{ and } \mu_\ell = \frac{n}{\nu_\ell}.$$

Then $\mu_\ell \cdot \nu_\ell = n$ and, since $\mu_\ell = \frac{n}{\nu_\ell} = \frac{m_\ell}{h_\ell}$, $\mu_\ell \cdot h_\ell = m_\ell$ for each ℓ.

Now, there are μ_ℓ points $A_{\ell 1}, A_{\ell 2}, \ldots\ldots\ldots\ldots, A_{\ell\mu_\ell}$ on S over a point $Z = a_\ell$ such that each point $A_{\ell k}$, $k = 1,2,\ldots,\mu_\ell$, is a branch point on S of order $\nu_\ell - 1$ for each ℓ. By Riemann - Hurwitz formula, the genus g of S is given by

$$\begin{aligned} 2g-2 &= n(-2)+ \sum_{\ell=1}^{b}(\nu_\ell-1)\mu_\ell \\ &= -2n+ \sum_{\ell=1}^{b}(\nu_\ell-1)\frac{n}{\nu_\ell} \\ &= -2n+n \sum_{\ell=1}^{b}\left(1-\frac{1}{\nu_\ell}\right), \end{aligned}$$

and hence

$$(96)\quad g = 1-n+\frac{n}{2} \sum_{\ell=1}^{b}\left(1-\frac{1}{\nu_\ell}\right).$$

Note that for $g \geqq 1$ and $n \geqq 2$

(97) $$\sum_{\ell=1}^{b}\left(1-\frac{1}{\nu_\ell}\right) \geqq 2.$$

For each ℓ, the local parameter t at $A_{\ell k}$, $k = 1,2,\ldots,\mu_\ell$, is given by

(98) $$Z-a_\ell = t^{\nu_\ell},$$

and hence

(99) $$w = \prod_{\ell=1}^{b}(Z-a_\ell)^{\frac{m_\ell}{n}} = t^{\frac{\nu_\ell m_\ell}{n}} \prod_{\substack{\ell'=1 \\ \ell'\neq\ell}}^{b}(t^{\nu_\ell}+a_\ell-a_{\ell'})^{\frac{m_{\ell'}}{n}}$$

$$= t^{h_\ell} \prod_{\substack{\ell'=1 \\ \ell'\neq\ell}}^{b}(t^{\nu_\ell}+a_\ell-a_{\ell'})^{\frac{m_{\ell'}}{n}},$$

(100) $$dZ = \nu_\ell t^{\nu_\ell-1} dt.$$

At each point $\infty_1, \infty_2, \ldots, \infty_n$ on S over the point $Z = \infty$, the local parameter t is given by

(101) $$Z = \frac{1}{t},$$

and hence

(102) $$Z-a_\ell = \frac{1}{t} - a_\ell, \quad \ell = 1,2,\ldots,b,$$

(103) $$w = \prod_{\ell=1}^{b}(Z-a_\ell)^{\frac{m_\ell}{n}} = \prod_{\ell=1}^{b}\left(\frac{1}{t}-a_\ell\right)^{\frac{m_\ell}{n}}$$

$$= t^{-\frac{1}{n}\sum_{\ell=1}^{b} m_\ell} \prod_{\ell=1}^{b}(1-a_\ell t)^{\frac{m_\ell}{n}}$$

(104) $$dZ = -\frac{1}{t^2}dt.$$

From these, we have a following table of orders of zeros (or poles) of functions and differential;

	$A_{\ell k}$	∞_h	
dZ	$\nu_\ell - 1$	-2	$\mu=1,2,\ldots,n-1,$ $h=1,2,\ldots,n,$
w^μ	μh_ℓ	$-\frac{\mu}{n}\sum_{\ell=1}^{b} m_\ell$	$k=1,2,\ldots,\mu_\ell$ $\ell=1,2,\ldots,b,$
$Z-a_{\ell'}$	$\nu_\ell\,\delta_{\ell\ell'}$	-1	$\delta_{\ell\ell'}=\begin{cases}0, & \text{if } \ell\neq\ell',\\ 1, & \text{if } \ell=\ell'.\end{cases}$

Table 2

A divisor of a meromorphic function w on S is

$$(105)\quad (w) = \frac{\prod_{\ell=1}^{b}\left(\prod_{k=1}^{\mu_\ell} A_{\ell k}\right)^{h_\ell}}{\left(\prod_{h=1}^{n}\infty_h\right)^{\frac{1}{n}\sum_{\ell=1}^{b} m_\ell}}$$

by Table 2.

At this point, we choose a canonical homology basis (γ,δ) on S and an arbitrary point P_0 on S as a base point. Denoting a mapping from S to J(S) with respect to a base point P_0 on S by u as usual, we have

$$(106)\quad \sum_{\ell=1}^{b} h_\ell\left(\sum_{k=1}^{\mu_\ell} u(A_{\ell k})\right) \equiv \frac{1}{n}\sum_{\ell=1}^{b} m_\ell\left(\sum_{h=1}^{n} u(\infty_h)\right)$$

by (105) and Abel's theorem.

For each ℓ, $\ell = 1,2,\ldots,b$, $Z-a_\ell$ is a meromorphic function on S with a divisor

$$(107)\quad (Z-a_\ell) = \frac{\prod_{k=1}^{\mu_\ell} A_{\ell k}^{\nu_\ell}}{\prod_{h=1}^{n}\infty_h},$$

and hence

(108) $\nu_\ell \sum_{k=1}^{\mu_\ell} u(A_{\ell k}) \equiv \sum_{h=1}^{n} u(\infty_h)$, $\ell = 1,2,\ldots,b$.

Note that, since $\frac{h_\ell}{\nu_\ell} = \frac{m_\ell}{n} < 1$, $1 \leqq h_\ell < \nu_\ell$, h_ℓ and ν_ℓ are integers, and hence

(109) $1 \leqq h_\ell \leqq \nu_\ell - 1$, $\ell = 1,2,\ldots,b$.

We now assume that

(110) $\frac{1}{n} \sum_{\ell=1}^{b} m_\ell \geqq 2$.

With this assumption, we can find an abelian differential dv of the first kind on S. First, we compute by (96)

(111) $$2g-2 = -2n+\sum_{\ell=1}^{b} (\nu_\ell - 1)\mu_\ell = -2n+\sum_{\ell=1}^{b} \mu_\ell \nu_\ell - \sum_{\ell=1}^{b} \mu_\ell$$
$$= (b-2)n - \sum_{\ell=1}^{b} \mu_\ell.$$

Let

(112) $dv = \frac{dZ}{W}$.

Then its divisor is by Table 2

(113) $$(dv) = \prod_{\ell=1}^{b} \left(\prod_{k=1}^{\mu_\ell} A_{\ell k}\right)^{\nu_\ell - 1 - h_\ell} \cdot \left(\prod_{h=1}^{n} \infty_h \right)^{\left(\frac{1}{n} \sum_{\ell=1}^{b} m_\ell - 2\right)}.$$

Its degree is

(114) $$d[(dv)] = \sum_{\ell=1}^{b} \mu_\ell(\nu_\ell - 1 - h_\ell) + \sum_{\ell=1}^{b} m_\ell - 2n$$
$$= bn - \sum_{\ell=1}^{b} \mu_\ell - \sum_{\ell=1}^{b} \mu_\ell h_\ell + \sum_{\ell=1}^{b} m_\ell - 2n$$
$$= (b-2)n - \sum_{\ell=1}^{b} \mu_\ell = 2g-2.$$

Therefore, dv defined by (112) is an abelian differential of the first kind on S and $-2K(p_0) \equiv u((dv))$, where $K(p_0)$ is the vector of Riemann constants with respect to a base point P_0 on S. We collect this result in the following theorem;

Theorem 10.

On a compact Riemann surface S of an algebraic function $w^n = \prod_{\ell=1}^{b}(Z-a_\ell)^{m_\ell}$, where $n \geqq 2$, $1 \leqq m_\ell \leqq n-1$, $\ell = 1,2,\ldots,b$, and $\sum_{\ell=1}^{b} m_\ell \equiv 0 \pmod n$ of genus $g \geqq 1$, let $\frac{m_\ell}{n} = \frac{h_\ell}{\nu_\ell}$, $(h_\ell,\nu_\ell)=1$, $\mu_\ell = \frac{n}{\nu_\ell}$ for each ℓ, and $A_{\ell k}$, $k = 1,2,\ldots,\mu_\ell$, be the branch points on S over $Z = a_\ell$ and $\infty_1, \infty_2,\ldots,\infty_n$ the points on S over $Z = \infty$. Then the vector $K(P_0)$ of Riemann constants with respect to a base point P_0 on S satisfies

$$(115)\quad -2K(p_0) \equiv \sum_{\ell=1}^{b}((\nu_\ell-1-h_\ell)\sum_{k=1}^{\mu_\ell} u(A_{\ell k}))+(\frac{1}{n}\sum_{\ell=1}^{b} m_\ell-2)\sum_{h=1}^{n} u(\infty_h),$$

if $\frac{1}{n}\sum_{\ell=1}^{b} m_\ell \geqq 2$.

Consider two Riemann surfaces of $w^n = \prod_{\ell=1}^{b}(Z-a_\ell)$ and $w^n = \prod_{\ell=1}^{b}(Z-a_\ell)^{n-1}$. They are conformally equivalent to each other. For, if we denote the Riemann surface of $w_1^n = \prod_{\ell=1}^{b}(Z_1-a_\ell)$ by S_1 and the Riemann surface of $w_2^n = \prod_{\ell=1}^{b}(Z_2-a_\ell)^{n-1}$ by S_2, then a mapping

$$\begin{cases} Z_1 = Z_2 \\ w_1 = \dfrac{\prod_{\ell=1}^{b}(Z_2-a_\ell)}{w_2} \end{cases}$$

defines a birational transformation between two algebraic function fields $\mathfrak{M}(S_1)$, generated by Z_1 and w_1, and $\mathfrak{M}(S_2)$, generated by Z_2 and w_2. Consequently, a mapping $(Z_1,w_1) \rightarrow (Z_2,w_2)$ is a conformal mapping from S_1 onto S_2, and hence they are conformally equivalent to each other. We do not need to make any distinction between those two Riemann surfaces.

For the Riemann surface S of $w^n = \prod_{\ell=1}^{b} (Z-a_\ell)^{n-1}$, $m_\ell = n-1, \nu_\ell=n$, $h_\ell = n-1$ and $\mu_\ell = 1$, i.e., A_ℓ is a branch point on the surface of order n-1 over $Z = a_\ell$, for each ℓ, $\ell = 1,2,\ldots,b$.

By (108), $nu(A_\ell) \equiv \sum_{h=1}^{n} u(\infty_h)$ for all ℓ. If we take a branch point A_ℓ as a base point P_0, $0 \equiv nu(A_\ell) \equiv nu(A_{\ell'}) \equiv \sum_{h=1}^{n} u(\infty_h)$ for any ℓ and ℓ', $\ell,\ell' = 1,2,\ldots,b$. Furthermore, by Riemann-Hurwitz formula, $0 \leqq 2g-2 = -2n+b(n-1) = -2n+\sum_{\ell=1}^{b} m_\ell$ for $g \geqq 1$, and hence $\frac{1}{n}\sum_{\ell=1}^{b} m_\ell = \frac{b(n-1)}{n} \geqq 2$.

Then, by (115) in Theorem 10, $-2K(A_\ell) \equiv 0$, i.e., $K(A_\ell)$ is a half period for each ℓ.

Theta function $\theta[K(A_\ell)](u(p),\Pi)$, where $p \in S$ and Π is the period matrix of S and (γ,δ), associated with S and (γ,δ) has the same order of vanishing at the origin as the order of vanishing of $\theta(u(p),\Pi) = \theta\left[\begin{smallmatrix}0\cdots0\\0\cdots0\end{smallmatrix}\right](u(p),\Pi)$ at $K(A_\ell)$. We recall that an odd function always vanishes at the origin, and that a partial derivative of an even (odd) function is odd (even). Therefore, an even function does have the even order at the origin, while an odd function has the odd order at the origin.

Since $K(A_\ell) \equiv u(A_\ell^{g-1})+K(A_\ell)$ and θ vanishes at $K(A_\ell)$ with the order $i(A_\ell^{g-1})$ by Riemann vanishing theorem, a half period $K(A_\ell)$ is even or odd depending on whether $i(A_\ell^{g-1})$ is even or odd.

If a compact Riemann surface S of genus $g \geqq 1$ has a concrete algebraic representation $w^n = \prod_{\ell=1}^{b}(Z-a_\ell)^{n-1}$, n is the minimal number for S and $g = n+1$, then

$$\begin{aligned} i(A_\ell^{g-1}) &= (\text{the number of gaps at } A_\ell \geqq g-1) \\ &= g-(\text{the number of gaps at } A_\ell < g-1) \\ &= (n+1) - (n-1) = 2. \end{aligned}$$

That is, $K(A_\ell)$ is an even half period and $\theta[K(A_\ell)]$ vanishes with the order 2 at the origin for each ℓ, $\ell = 1,2,\ldots,b$. Furthermore, for all ℓ' and ℓ, $\ell',\ell = 1,2,\ldots,b$,

$$\begin{aligned} K(A_{\ell'}) &\equiv K(A_\ell)+u(A_{\ell'}^{g-1}) \equiv K(A_\ell)+u(A_{\ell'}^{n}) \\ &\equiv K(A_\ell)+nu(A_{\ell'}) \equiv K(A_\ell) \equiv K. \end{aligned}$$

This means that all the vectors of Riemann constants with respect to any brauch point are equivalent to each other.

However, by (96),

$b =$ (the number of branch points on S)

$$= \frac{2g-2+2n}{n-1} = \frac{(4n-4)+4}{n-1} = 4 + \frac{4}{n-1} = \text{integer}.$$

Hence $n-1 = 1, 2$ or 4. Finally, we obtain three possible cases; (1) $n = 2$, $g = 3$, (2) $n = 3$, $g = 4$, (3) $n = 5$, $g = 6$. The first case is a well known result (Lewittes [17]). For the second case, K is the only one vanishing even half period on S of genus $g = 4$. For, otherwise, S is hyperelliptic, i.e., $n = 2$, according to Farkas ([6], Theorem 11).

Similarily, if a compact Riemann surface S of genus $g \geqq 1$ has a concrete algebraic representation $w^n = \prod_{\ell=1}^{b} (Z - a_\ell)$, n is the minimal number for S and $g = n + 1$, then $0 \leqq 2g-2 = -2n+b(n-1)$ and $\frac{1}{n} \sum_{\ell=1}^{b} m_\ell = \frac{b}{n}$. In this case, we can not apply Theorem 10. However, since $2 \leqq \frac{(n-1)b}{n}$, by Table 2

$$dv = \frac{1}{w^{n-1}}\, dz$$

is an abelian differential of the first kind on S with a divisor

$$(dv) = \left(\prod_{h=1}^{n} \infty_h\right)^{\left(\frac{(n-1)b}{n} - 2\right)} .$$

Taking a branch point A_ℓ, $\ell = 1, 2, \ldots, b$, as a base point P_0 on S, we have again $0 \equiv nu(A_\ell) \equiv nu(A_{\ell'}) \equiv \sum_{h=1}^{n} u(\infty_h)$ for any ℓ and ℓ', $\ell, \ell' = 1, 2, \ldots, b$. Hence $-2K(A_\ell) \equiv ((n-1)b-2n) \sum_{h=1}^{n} u(\infty_h) \equiv 0$, i.e.,

$K(A_\ell)$ is a half period. $i(A_\ell^{g-1})$ = g-(the number of gaps at $A_\ell < g-1$) = (n+1)-(n-1) = 2. This shows that $K(A_\ell)$ is an even half period for each ℓ. For any ℓ and ℓ', $K(A_{\ell'}) \equiv K(A_\ell)+u(A_{\ell'}{}^{g-1}) \equiv K(A_\ell)+nu(A_{\ell'}) \equiv K(A_\ell) \equiv K$. Now, the rest of all arguments for $w^n = \prod_{\ell=1}^{b}(Z-a_\ell)^{n-1}$ is true for $w^n = \prod_{\ell=1}^{b}(Z-a_\ell)$.

We summarize these in following theorems;

Theorem 11.

Let S be a compact Riemann surface of $w^3 = \prod_{\ell=1}^{6}(Z-a_\ell)$ (or $w^3 = \prod_{\ell=1}^{6}(Z-a_\ell)^2$) of genus g = 4 and K the vector of Riemann constants with respect to any branch point A_ℓ, $\ell = 1,2,\ldots,6$, on S over $Z = a_\ell$. Then K is an even half period and $\theta[K]$ is the only one vanishing even theta constant associated with S and arbirtarily chosen canonical homology basis (γ,δ) on S. $\theta[K]$ has the vanishing order 2.

Theorem 12.

Let S be a compact Riemann surface of $w^5 = \prod_{\ell=1}^{5}(Z-a_\ell)$(or $w^5 = \prod_{\ell=1}^{5}(Z-a_\ell$ of genus g = 6 and K the vector of Riemann constants with respect to any branch point A_ℓ, $\ell = 1,2,\ldots,5$, on S over $Z = a_\ell$. Then K is an even half period and $\theta[K]$ is a vanishing even theta constant associated with S and arbitrarily chosen canonical homology basis (γ, δ) on S. $\theta[K]$ has the vanishing order 2.

In the next section, we will find K in Theorem 11 for a canonical homology basis.

Lewittes [17] gave a hyperelliptic surface of odd genus as an example of a surface having a property that any two vectors of Riemann constants with respect to any two branch points are congruent to each other. For the Riemann surface of $w^n = \prod_{\ell=1}^{b} (Z-a_\ell)$ (or $w^n = \prod_{\ell=1}^{b} (Z-a_\ell)^{n-1}$) of genus $g = nq+1$, where q is an integer, i.e., $g \equiv 1 \pmod n$, $n \geqq 2$,

$$K(A_{\ell'}) \equiv K(A_\ell) + u(A_{\ell'}{}^{g-1}) \equiv K(A_\ell) + u(A_{\ell'}{}^{nq}) \equiv K(A_\ell) + nqu(A_{\ell'}) \equiv K(A_\ell)$$

for any ℓ and ℓ', $\ell, \ell' = 1,2,\ldots,b$, since $0 \equiv nu(A_\ell)$ $\equiv nu(A_{\ell'}) \equiv \sum_{h=1}^{n} u(\infty_h)$. Thus, we have more examples having such a property. Indeed, our examples include Lewittes' as a special case.

4-2 Three-sheeted branched coverings over the sphere.

In (93) $w^n = \prod_{\ell=1}^{b} (Z-a_\ell)^{m_\ell}$, if $n = 3$ then $m_\ell = 1$ or 2.

Hence we can rewrite a concrete algebraic representation (93) of a compact Riemann surface S of genus $g \geqq 1$ as follows;

(116) $$w^3 = \prod_{\ell=1}^{j} (Z-\alpha_\ell)^2 \prod_{m=1}^{k} (Z-\beta_m),$$

where $b = j + k$, α_ℓ and β_m are mutually distinct finite for all ℓ and m, $\ell = 1,2,\ldots,j$, $m = 1,2,\ldots,k$. Recalling a condition $\sum_{\ell=1}^{b} m_\ell \equiv 0 \pmod{n}$ on (93),

(117) $2j+k \equiv 0 \pmod{3}$.

Further, we may assume that

(118) $j \leqq k$.

For, If $j > k$, then denoting the Riemann surface of $w_1^3 = \prod_{\ell=1}^{j}(Z_1-\alpha_\ell)^2 \prod_{m=1}^{k}(Z_1-\beta_m)$ by S_1 and the Riemann surface of $w_2^3 = \prod_{\ell=1}^{j}(Z_2-\alpha_\ell) \prod_{m=1}^{k}(Z_2-\beta_m)^2$ by S_2, a mapping

$$\begin{cases} Z_1 = Z_2 \\ w_1 = \dfrac{\prod_{\ell=1}^{j}(Z_2-\alpha_\ell)\prod_{m=1}^{k}(Z_2-\beta_m)}{w_2} \end{cases}$$

defines a birational transformation between algebraic function fields $\mathfrak{M}(S_1)$, generated by Z_1 and w_1, and $\mathfrak{M}(S_2)$, generated by Z_2 and w_2. Then a mapping $(Z_1,w_1) \to (Z_2,w_2)$ is a conformal mapping from S_1 onto S_2, and they are conformally equivalent to each other.

By (116) and (95) in 4-1, $\nu_\ell = \nu_m = 3$ and $\mu_\ell = \mu_m = 1$ for all ℓ and m. But $h_\ell = 2$ for all $\ell = 1,2,\ldots,j$ and $h_m = 1$ for all m, $m = 1,2,\ldots,k$. From Table 2, we have a following table;

	A_ℓ	B_m	∞_h
dZ	2	2	-2
w	2	1	$-\frac{1}{3}(2j+k)$
w^2	4	2	$-\frac{2}{3}(2j+k)$
$Z-\alpha_{\ell'}$	$3\delta_{\ell\ell'}$	0	-1
$Z-\beta_{m'}$	0	$3\delta_{mm'}$	-1

$h = 1,2,3,$
$\ell,\ell' = 1,2,\ldots,j,$
$m,m' = 1,2,\ldots,k,$
$\delta_{\ell\ell'} = \begin{cases} 0, & \text{if } \ell \neq \ell' \\ 1, & \text{if } \ell = \ell'. \end{cases}$

Table 3

In the table, A_ℓ and B_m are branch points on S over $Z = \alpha_\ell$ and $Z = \beta_m$, respectively.

By Riemann - Hurwitz formula,

(119) $2g - 2 = 3(-2)+2(j+k),$

and hence

(120) $b = j+k=g+2,$

(121) $2j+k=j+(j+k)=j+g+2 \equiv 0 \pmod 3.$

For $g=1$, $b=j+k=3$ and $j+3 \equiv 0 \pmod 3$. Hence $j=0$ and $k=3$ is the only one possible case and $\frac{2j+k}{3} = 1 < 2$, but $\frac{2(2j+k)}{3} = 2.$

For $g=2$, $b=j+k=4$ and $j+4 \equiv 0 \pmod 3$. Again, $j=2$ and $k=2$ is the only one possible case and $\frac{2j+k}{3} = 2.$

For $g=3$, $b=j+k=5$ and $j+5 \equiv 0 \pmod 3$. $j=1$ and $k=4$ is the only one possible case and $\frac{2j+k}{3} = 2.$

For $g \geqq 4$, $\frac{2j+k}{3} = \frac{j+g+2}{3} = 2+\frac{j}{3} \geqq 2$.

By these observations, we can find many elements in $A_1(S)$.

For $g = 1$,

(122) $$dv = \frac{dZ}{w^2}$$

is a basis for $A_1(S)$ and it has a divisor

(123) $(dv) = 1 =$ unit divisor.

For $g = 3q+1$, $q > 0$, $2j+k = j+g+2 = j+3q+3 \equiv 0 \pmod 3$. let $j = 3r$, $r \geqq 0$ and

(124) $$dv = \frac{\prod_{\ell=1}^{j}(Z-\alpha_\ell)\prod_{m=1}^{2q-r}(Z-\beta_m)}{w^2}\,dZ.$$

Then

(125) $$(dv) = \prod_{\ell=1}^{j} A_\ell \cdot \prod_{m=1}^{2q-r} B_m^3,$$

and $d[(dv)] = j+3(2q-r) = 6q = 2(3q+1)-2 = 2g-2$. Note that $k = g+2-j = 3(q-r+1) \geqq j = 3r$, $q-2r+1 \geqq 0$ and $k-(2q-r) = q-2r+3 \geqq 2$.

Let

(126) $$dv = \frac{dZ}{w}.$$

(127) $$(dv) = \prod_{m=1}^{k} B_m \cdot \left(\prod_{h=1}^{3} \infty_h\right)^{\left(\frac{j+g+2}{3}-2\right)}$$

and $d[(dv)] = k+3\left(\frac{j+g+2}{3}-2\right) = j+k+g+2-6 = 2(g+2)-6 = 2g-2$.

For $g = 3q+2$, $q \geqq 0$, $2j+k = j+g+2 = j+3q+4 \equiv 0 \pmod 3$.

Let $j = 3r+2$, $r \geqq 0$ and

(128) $$dv = \frac{\prod_{\ell=1}^{j}(Z-\alpha_\ell)\prod_{m=1}^{2q-r}(Z-\beta_m)}{w^2}\, dZ.$$

Then

(129) $$(dv) = \prod_{\ell=1}^{j} A_\ell \cdot \prod_{m=1}^{2q-r} B_m^3,$$

and $d[(dv)] = j+3(2q-r) = 6q+2 = 2(3q+2)-2 = 2g-2$. Since $k = g+2-j = 3q-3r+2 \geqq j = 3r+2$, $q-2r \geqq 0$ and $k-(2q-r)=q-2r+2 \geqq 2$. In this case, $dv = \frac{dZ}{w}$ is also in $A_1(S)$ and it has a divisor (127), since $\frac{j+g+2}{3} = \frac{1}{3}(3r+2+3q+2+2) = (r+q)+2 \geqq 2$.

For $g = 3q+3$, $q \geqq 0$, $2j+k = j+g+2 = j+3q+5 \equiv 0 \pmod 3$. Let $j = 3r+1$, $r \geqq 0$ and

(130) $$dv = \frac{\prod_{\ell=1}^{j}(Z-\alpha_\ell)\prod_{m=1}^{2q-r+1}(Z-\beta_m)}{w^2}\, dZ$$

Then

(131) $$(dv) = \prod_{\ell=1}^{j} A_\ell \cdot \prod_{m=1}^{2q-r+1} B_m^3,$$

and $d[(dv)] = j+3(2q-r+1) = 6q+4 = 2(3q+3)-2 = 2g-2$. Since $k = g+2-j = 3(q-r)+4 \geqq j = 3r+1$, $q-2r+1 \geqq 0$ and $k-(2q-r+1) = q-2r+3 \geqq 2$. $dv = \frac{dZ}{w}$ is in $A_1(S)$ and its divisor is given by (127), since $\frac{1}{3}(j+g+2) = \frac{1}{3}(3r+1+3q+3+2) = (r+q)+2 \geqq 2$.

<u>Theorem 13.</u>

Let S be a compact Riemann surface of $w^3 = \prod_{\ell=1}^{j}(Z-\alpha_\ell)^2 \prod_{m=1}^{k}(Z-\beta_m)$, where α_ℓ and β_m are mutually distinct finite, $0 \leqq j \leqq k$ and

$2j+k \equiv 0 \pmod 3$, and let A_ℓ and B_m are branch points on S over $Z = \alpha_\ell$ and $Z = \beta_m$, respectively. Then the vector $K(P_0)$ of Riemann constants with respect to a base point P_0 on S satisfies

$$(132) \quad -2K(P_0) \equiv \sum_{\ell=1}^{j} u(A_\ell) \equiv \sum_{m=1}^{k} u(B_m),$$

where $P_0 = A_1$ if $j \neq 0$ or B_1 if $j = 0$.

Proof.

Under the assumption that $P_0 = A_1$ or B_1, we have $3u(A_\ell) \equiv 3u(B_m) \equiv \sum_{h=1}^{3} u(\infty_h) \equiv 0$ for all ℓ and m. Since $-2K(p_0) \equiv u((dv))$ for $dv \in A_1(S)$, by (123), (125), (129) and (131), $-2K(p_0) \equiv \sum_{\ell=1}^{j} u(A_\ell)$. On the other hand, by (127) $-2K(p_0) \equiv \sum_{m=1}^{k} u(B_m)$.

Corollary 7.

With the same notations as in Theorem 13, if $j = 0$ or 1, then $K(p_0)$ is a half period and $\theta[K(p_0)]$ vanishes with the order $i(P_0^{g-1})$. Hence $\theta[K(p_0)]$ is even or odd depending on whether $i(p_0^{g-1})$ is even or odd.

Proof.

If $j = 0$, $P_0 = B_1$ and by (123), (125), (129) and (131) $-2K(P_0) \equiv -2K(B_1) \equiv 0$. If $j = 1$, then $P_0 = A_1$ and by (132) $-2K(P_0) \equiv -2K(A_1) \equiv u(A_1) \equiv 0$. The rest of the assertion is trivial by Riemann vanishing theorem.

According to Theorem 13 and Corollary 7, if $j = 0$ then $K(B_1)$ is a half period and $K(B_m) = K(B_1)+u(B_m^{g-1})$ are all 6th periods for all m, $m = 2,3,\ldots,k$. If $j = 1$, then $K(A_1)$ is a half period and $K(B_m) = K(A_1)+u(B_m^{g-1})$ are all 6th periods for all $m = 1,2,3,\ldots,k$. If $j \geqq 2$, then $K(A_\ell)$ and $K(B_m)$ are all 6th periods for all ℓ and m, $\ell = 1,2,\ldots,j$, $m = 1,2,\ldots,k$.

We return to the Riemann surface S of

$$(133) \quad w^3 = \prod_{m=1}^{6}(Z-\beta_m) \quad \left(\text{or } w^3 = \prod_{\ell=1}^{6}(Z-\alpha_\ell)^2\right)$$

of genus $g = 4$ ($j = 0$ and $k = 6$). In this case, we already proved that $K = K(B_m)$ with respect to any branch point B_m, $m = 1,2,\ldots,6$, are congruent to each other and it is the only one vanishing half period on S such that $\theta[K]$ has the vanishing order 2 at the origin in Theorem 11. This can also be checked by Corollary 7.

To find this K, we choose a canonical homology basis (γ,δ) on S as in Figure 3.

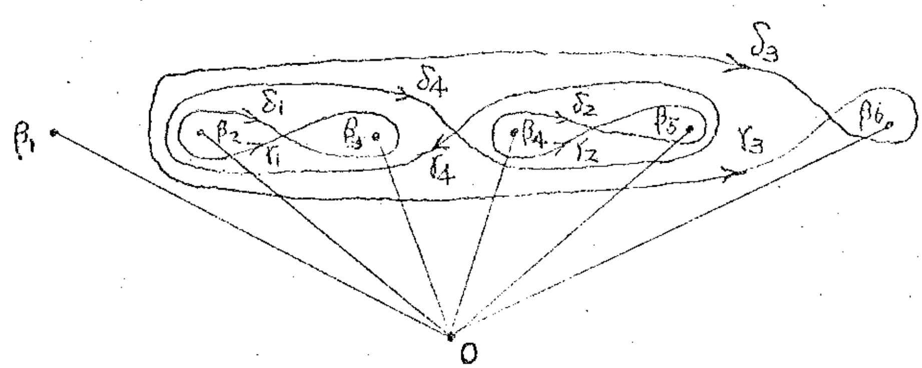

Figure 3

We give more explations about Figure 3.

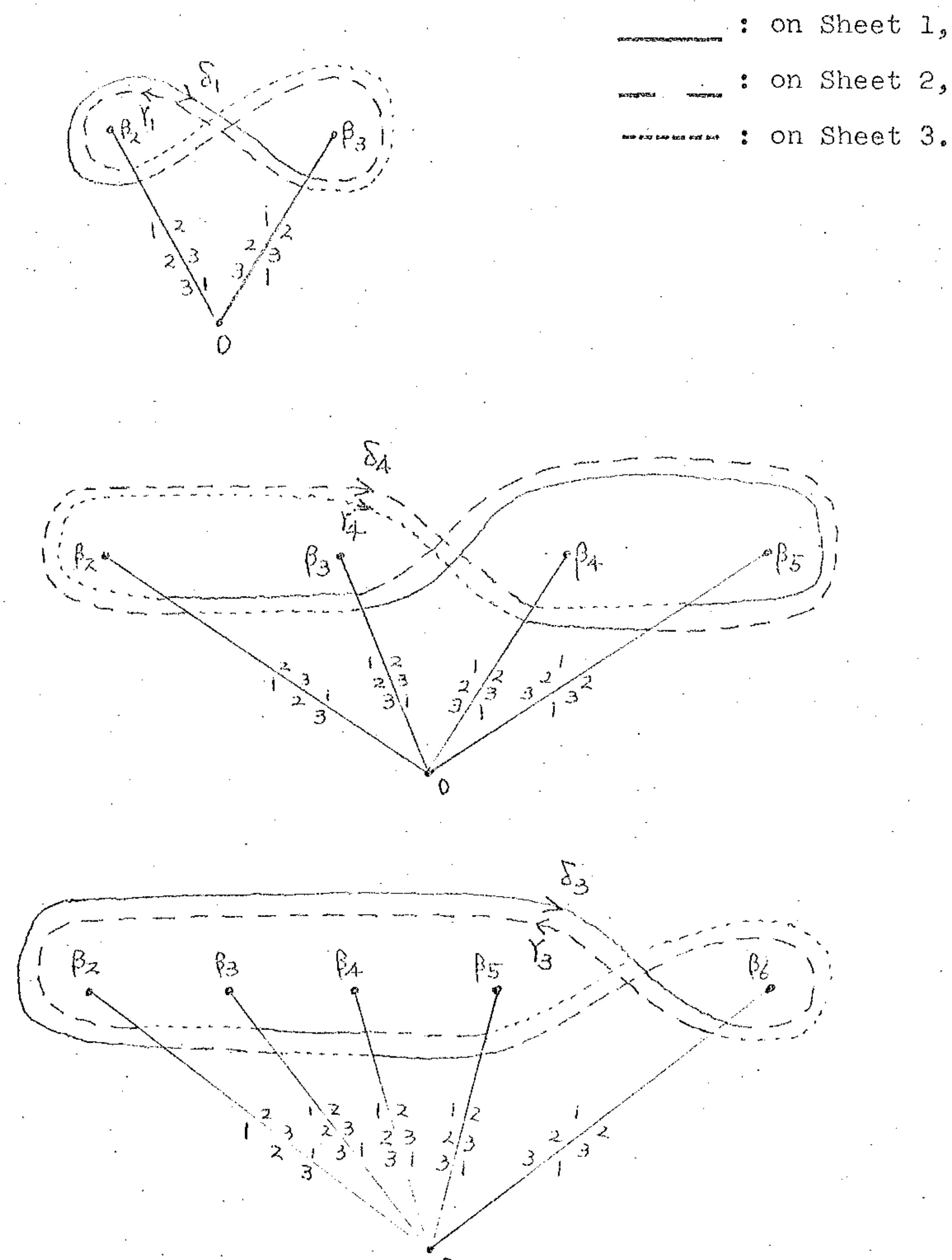

Figure 4

γ_2 and δ_2 are chosen in a similar way that γ_1 and δ_1 are chosen. The point O is a point on S such that no any two branch points are colinear through O. The numbers 1, 2 and 3 are expressing the "sheets" of S, on which the points are expressed by $(Z_0, w(Z_0))$, $(Z_0, \rho w(Z_0))$ and $(Z_0, \rho^2 w(Z_0))$ over $Z = Z_0$, respectively, where $\rho = \exp[\frac{2\Pi i}{3}]$. Each pair of (γ_k, δ_k), $k = 1,2,3,4$, meet on the sheet 2. This canonical homology basis was found by Wellstein [33].

We denote an automorphism on S of order 3, which is the cyclic interchange of three sheets, by T. From Figure 4, we have

(134) $T(\gamma_1) \sim -\gamma_1 + \delta_1$, $\quad T(\delta_1) \sim -\gamma_1$,

$T(\gamma_2) \sim -\gamma_2 + \delta_2$, $\quad T(\delta_2) \sim -\gamma_2$,

$T(\gamma_3) \sim -\gamma_3 + \delta_3$, $\quad T(\delta_3) \sim -\gamma_3$,

$T(\gamma_4) \sim -\gamma_4 - \delta_4$, $\quad T(\delta_4) \sim \gamma_4$.

Hence, we know that

(135) $(\gamma', \delta') = (T(\gamma), T(\delta)) = M \circ (\gamma, \delta) = \binom{AB}{CD} \circ (\gamma, \delta)$

$$= \left(\begin{array}{cccc|cccc} -1 & & & & 1 & & & \\ & -1 & & & & 1 & & \\ & & -1 & & & & 1 & \\ & & & -1 & & & & -1 \\ \hline -1 & & & & & & & \\ & -1 & & & & & & \\ & & -1 & & & & 0 & \\ & & & 1 & & & & \end{array}\right) \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix}$$

is agin a canonical homology basis on S. We further know that M is an element of the Siegel modular group $Sp(4,\mathbb{Z})$. The period matrix Π' of S and (γ',δ') is given by

$\Pi' = M \circ \Pi = (A\Pi+B)(C\Pi+D)^{-1}$.

But, Torelli's theorem tells us that

(136) $\Pi' = M \circ \Pi = \Pi$.

On the other hand, there is a non-zero constant L such that

(137) $\theta[M \circ [{\epsilon \atop \epsilon'}]](M\circ\Pi) = L\, \theta[{\epsilon \atop \epsilon'}](\Pi)$

for any g-characteristic $[{\epsilon \atop \epsilon'}]$, where

(138) $M\circ[{\epsilon \atop \epsilon'}] \equiv \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}\begin{pmatrix} \epsilon \\ \epsilon' \end{pmatrix} + \begin{pmatrix} \mathrm{diag}(C\cdot {}^tD) \\ \mathrm{diag}(A\cdot {}^tB) \end{pmatrix} \pmod 2$, (Igusa [13]).

Now, let $K \equiv \binom{k}{k'} = \begin{pmatrix} k_1 k_2 k_3 k_4 \\ k_1' k_2' k_3' k_4' \end{pmatrix}$, k_ℓ, $k_\ell' = 0$ or 1, and $K' = M\circ K$. Then, by (135) and (138),

$$(139)\quad K' \equiv \left[\begin{array}{cccc|cccc} & & & & 1 & & & \\ & 0 & & & & 1 & & 0 \\ & & & & & & 1 & \\ & & & & & 0 & & -1 \\ \hline -1 & & & & -1 & & & \\ & -1 & 0 & & & -1 & & 0 \\ & & -1 & & & & -1 & \\ 0 & & & 1 & 0 & & & -1 \end{array}\right] \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_1' \\ k_2' \\ k_3' \\ k_4' \end{bmatrix} + \left[\begin{array}{c} \\ \\ 0 \\ \\ \hline -1 \\ -1 \\ -1 \\ 1 \end{array}\right]$$

$$= \begin{bmatrix} k_1' \\ k_2' \\ k_3' \\ -k_4' \\ -k_1-k_1'-1 \\ -k_2-k_2'-1 \\ -k_3-k_3'-1 \\ k_4-k_4'+1 \end{bmatrix}$$

$$\equiv \begin{pmatrix} k_1' & k_2' & k_3' & k_4' \\ k_1+k_1'+1 & k_2+k_2'+1 & k_3+k_3'+1 & k_4+k_4'+1 \end{pmatrix} \text{ (half period).}$$

Furthermore,

$$(140)\quad k_1'(k_1+k_1'+1)+k_2'(k_2+k_2'+1)+k_3'(k_3+k_3'+1)+k_4'(k_4+k_4'+1)$$

$$= (k_1k_1'+k_2k_2'+k_3k_3'+k_4k_4')+(k_1'^2+k_1')+(k_2'^2+k_2')+(k_3'^2+k_3')$$

$$+ (k_4'^2+k_4')$$

$$= (k_1k_1'+k_2k_2'+k_3k_3'+k_4k_4')+2(k_1'+k_2'+k_3'+k_4')$$

$$\equiv 0 \pmod 2,$$

since $K = \binom{k}{k'}$ is even and $d^2 = d$ for any $d = 0$ or 1. This shows us that $K' = MoK$ is also an even half period. By (137),

$$\theta[K'](\Pi') = \theta[K'](\Pi) = L\theta[K](\Pi) = 0$$

and K is the only one vanishing half period. Therefore, $K \equiv K'$. From (139),

$$k_1 \equiv k_1', \; k_2 \equiv k_2', \; k_3 \equiv k_3', \; k_4 \equiv k_4',$$

$$k_1' \equiv k_1+k_1'+1, \; k_2' \equiv k_2+k_2'+1, \; k_3' \equiv k_3+k_3'+1, \; k_4' \equiv k_4+k_4'+1, \quad \text{(mod 2)}$$

and hence $k_\ell = k_\ell' = 1$ for all ℓ, $\ell = 1,2,3,4$. Thus we obtain

$$K \equiv \binom{1111}{1111}.$$

<u>Theorem 14</u>. Let S be a compact Riemann surface of $w^3 = \prod_{m=1}^{6} (Z-\beta_m)$ (or $w^3 = \prod_{\ell=1}^{6} (Z-\alpha_\ell)^2$) of genus $g = 4$ with a canonical homology basis $\gamma_1,\gamma_2,\gamma_3,\gamma_4;\delta_1,\delta_2,\delta_3,\delta_4$ as shown in Figure 3 and 4. Then, the only one vanishing even half period K, which is the vector of Riemann constants with respect to any branch point B_m, $m = 1,2,\ldots,6$, (or A_ℓ, $\ell = 1,2,\ldots,6$) on S, is

(141) $K \equiv \binom{1111}{1111}$.

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