Structure of Entropy: The hidden dimensions

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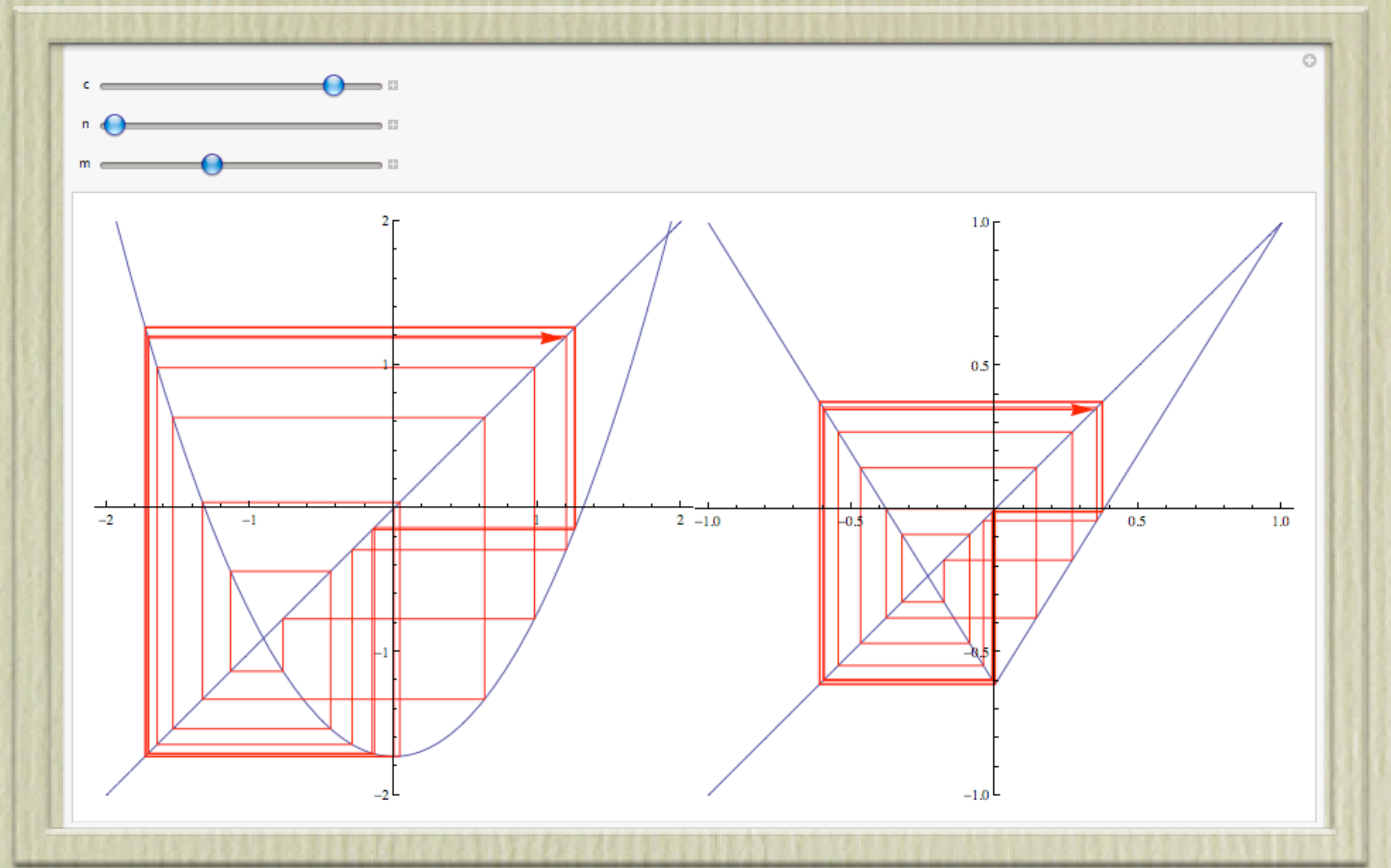
Jackfest 2011



Real polynomials have PL uniformly expanding models: growth factor λ = exp(entropy)

PostCritically Finite Maps

- What λ's can occur for PCF maps?
- λ is an algebraic integer.
  - Identifications in orbit of 0 implay a polynomial equation in λ.
- λ is a positive eigenvalue for a nonnegative eigenvector of a nonnegative matrix:
  - incidence matrix for intervals bounded by postcritical orbit.

Perron-Frobenius

- Perron-Frobenius theorem: an ergodic non-negative matrix has a unique non-negative eigenvector.

- The corresponding "positive" eigenvalue λ is larger than all others.

- $\Rightarrow$ if the matrix is integral, then λ is an algebraic integer larger than all its other Galois conjugates.

- Converse Perron-Frobenius (Lind): if λ is an algebraic integer larger than all other Galois conjugates, there is a non-negative ergodic integer matrix having λ as its positive eigenvalue.

- If the largest conjugate λ' of a positive real algebraic number λ
  - satisfies $|\lambda'| < \lambda$, it is a Perron number
  - satisfies $|\lambda'| < 1$, it is a Pisot or PV number
  - satisfies $|\lambda'| = 1$, it is a Salem number

- The dimension of the smallest matrix for λ is not bounded by the degree of λ, for $d > 2$.

Real Polynomial Entropy

- Theorem: for every Perron number λ, there exist λ-uniformly expanding postcritically finite selfmaps of an interval. (But no guarantee as to the number of laps).

Perron to Semigroup to Positive Matrix

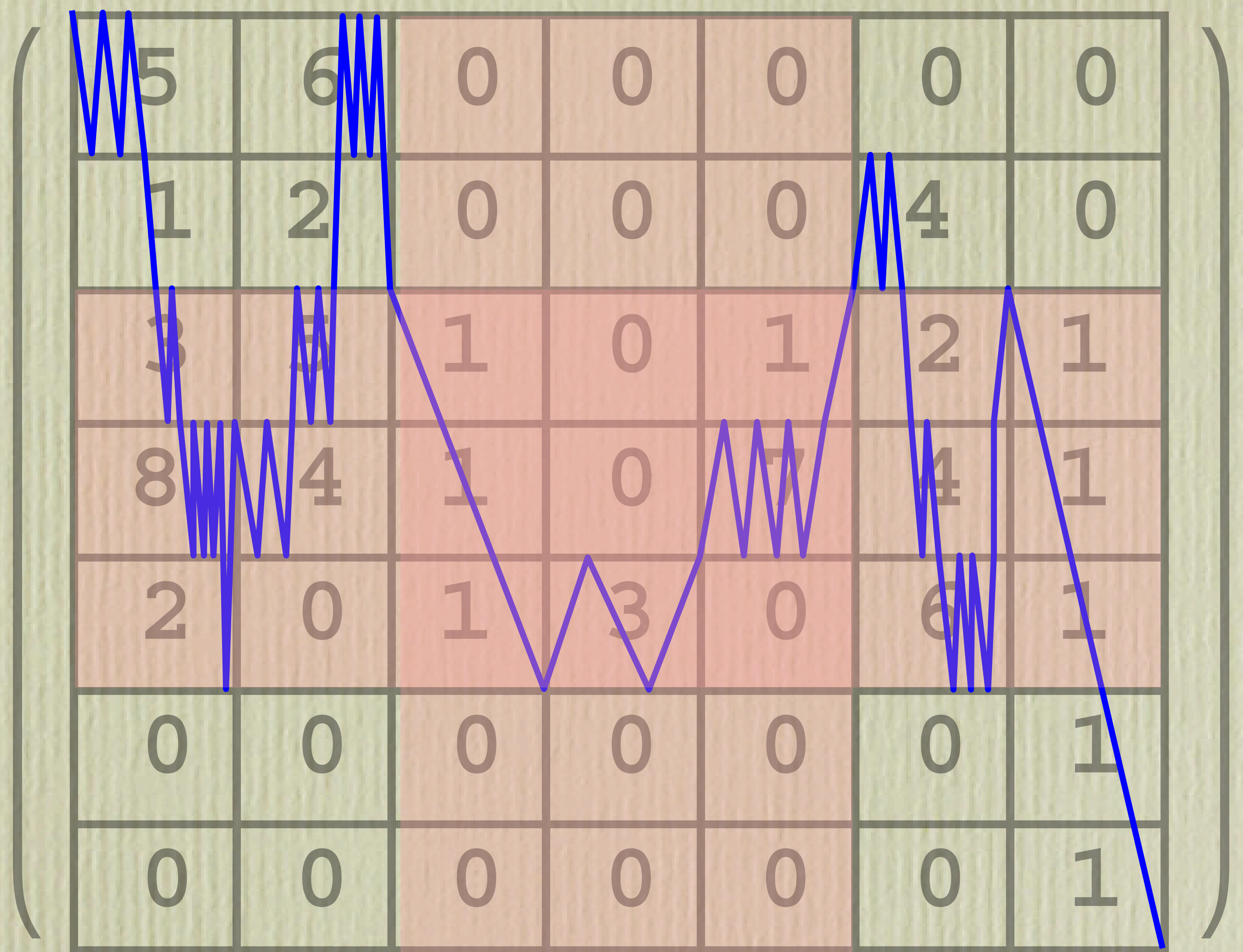
- Start with ring of algebraic integers in Q(λ), where λ action is in GL(d,Z).
- Find a subsemigroup (under addition) excluding 0 and invariant by λ.
- Using a polyhedral approximation, cut it the semigroup down to be finitely generated and S invariant by λ.
- Consider A =Free Abelian(generators) → S and

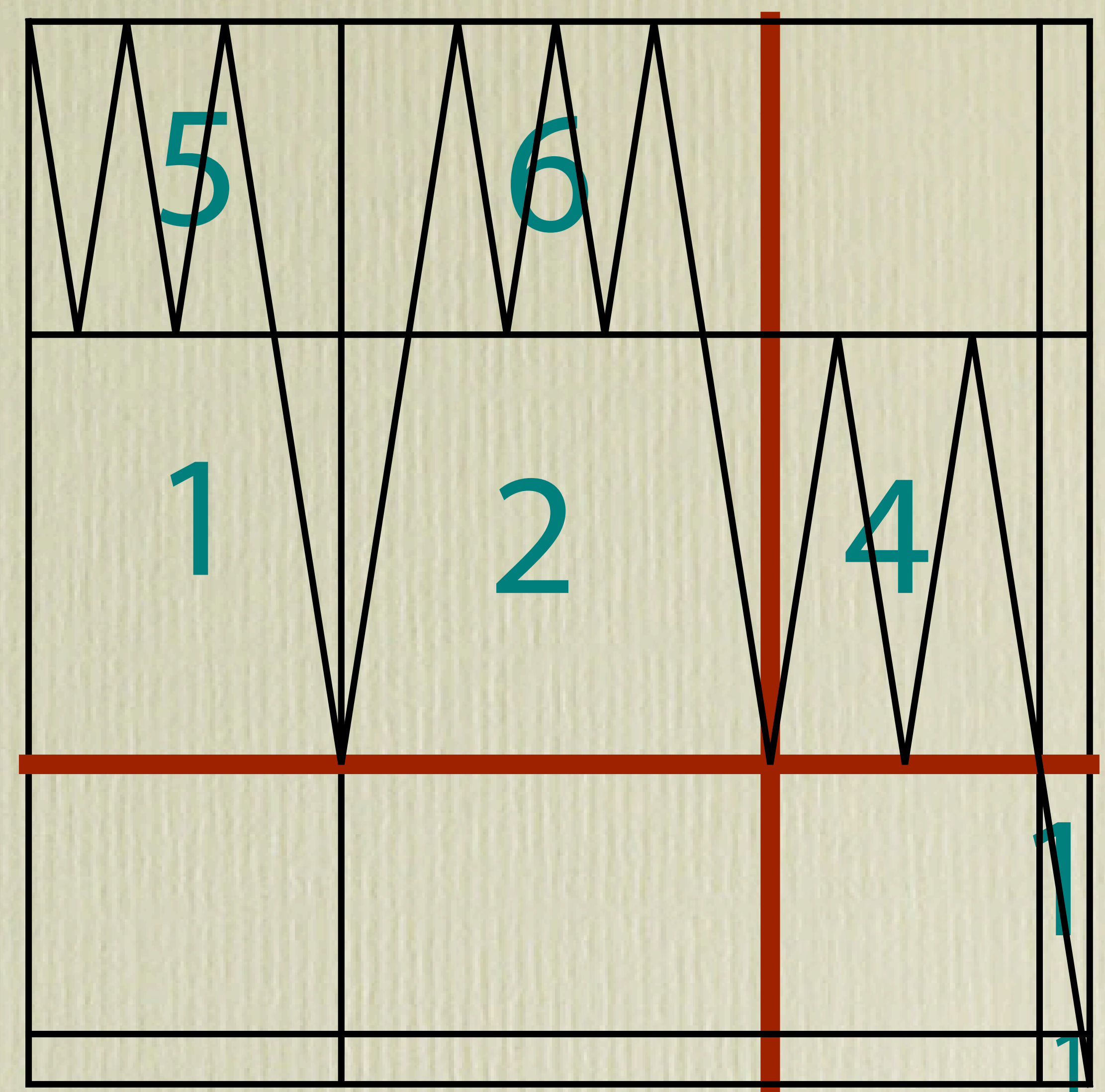
  lift multiplication by λ to A; it becomes nonnegative.

Semigroup to Incidence Matrix

- Use post-critical-**value** partition
- Conditions on nonnegative matrix:
    - in each column, {nonzero entries} is connected, length>0.
    - in each column, {odd entries} is connected, and the odd blocks link together in a chain.

$$\begin{pmatrix} 5 & 6 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 4 & 0 \\ 3 & 5 & 1 & 0 & 1 & 2 & 1 \\ 8 & 4 & 1 & 0 & 7 & 4 & 1 \\ 2 & 0 & 1 & 3 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$


5 6 0 0 0 0 0
1 2 0 0 0 4 0
3 5 1 0 1 2 1
8 4 1 0 7 4 1
2 0 1 3 0 6 1
0 0 0 0 0 0 1
0 0 0 0 0 0 1

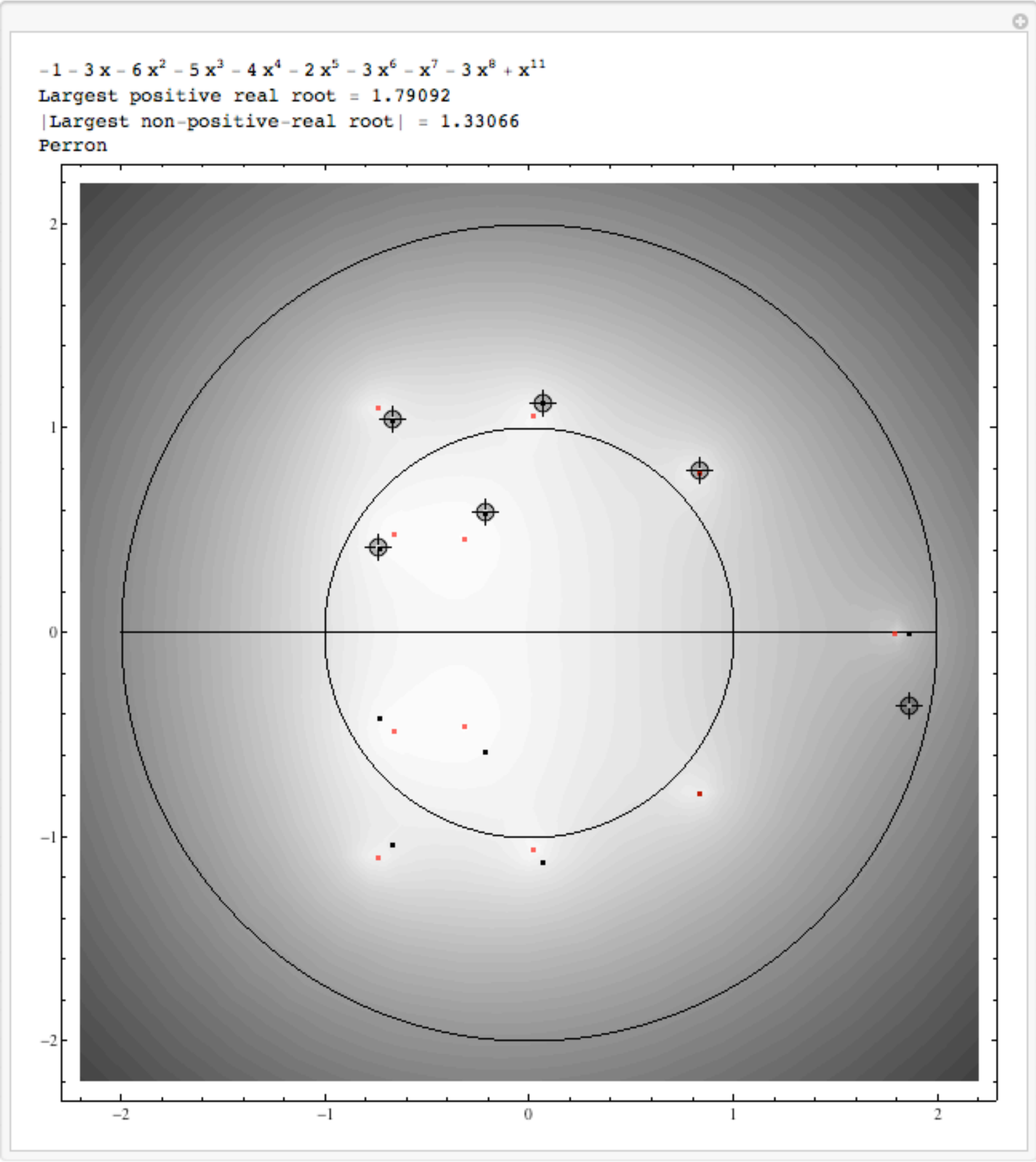

5
6
1
2
4
1
1

Semigroup to incidence matrix to power of λ to λ

- Given a positive semigroup S invariant by λ,
  - Choose generators G for representing every parity class (lattice / 2*lattice)
  - Let T = sum(G).
    - Make T=0 / 2*lattice (extra gens)
  - Choose a power of λ sending S to 3T+S
- Arrange G in order on a line, and make matrix.
- Implant as return map along a periodic orbit of system for lower entropy.

Given a Perron number λ, Does it occur for PCF map?

- Degree 2:
- If the critical point is periodic, there is a polynomial for λ having {1,-1} coefficients
- If the critical point is preperiodic, there is a polynomial for λ having {-2,-1,0,1,2} coefficients
- Minimal polynomial divides.
- Norm is 1 or 2



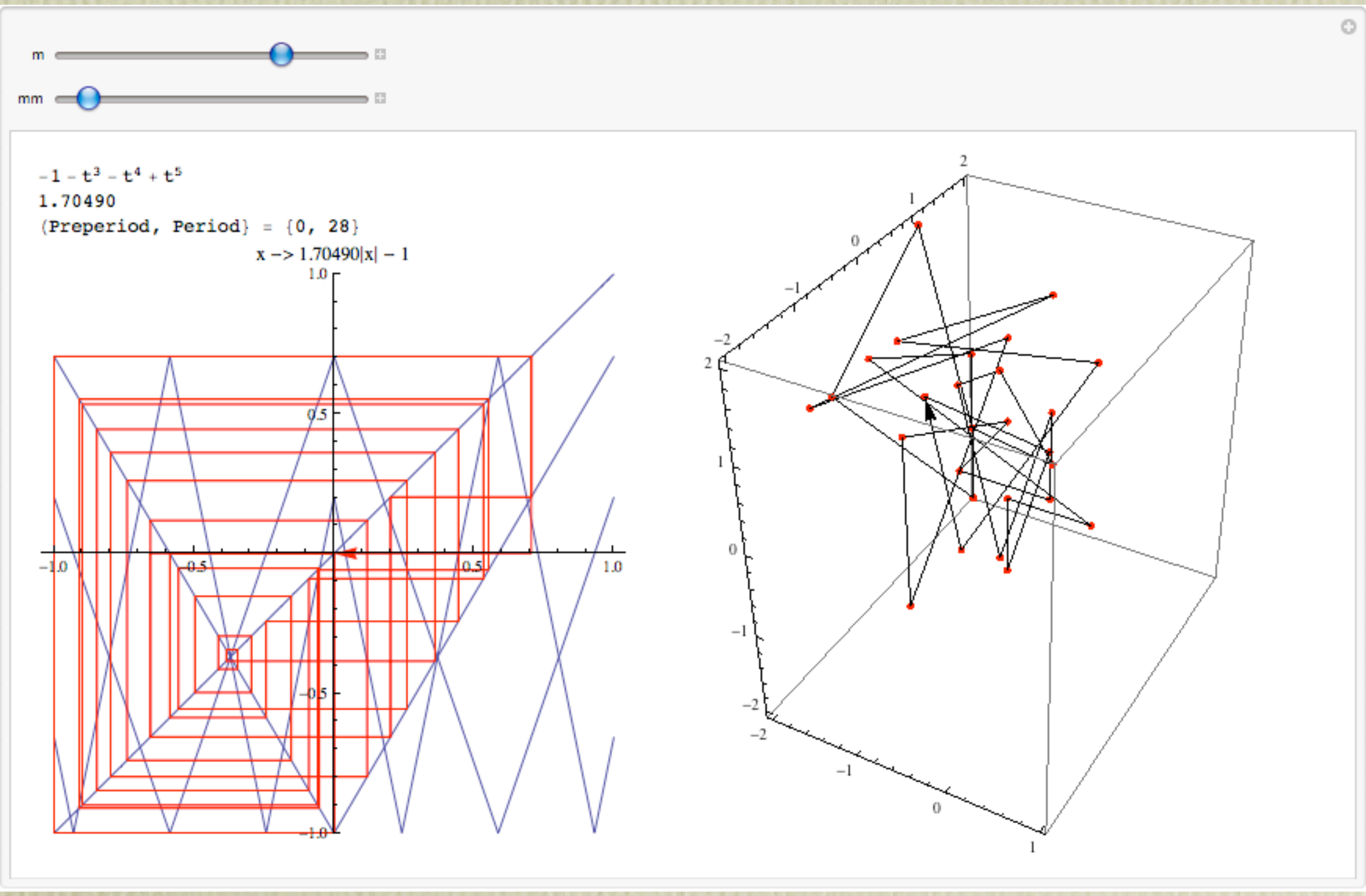
Multi-dimensional picture

- It is very hard to tell numerically whether a PL function is PCF.

- For an algebraic number λ defined by a polynomial P(x), just do polynomial arithmetic mod P(x): $R(x) \to \pm\, x\, R(x) - 1 \mod P(x)$,

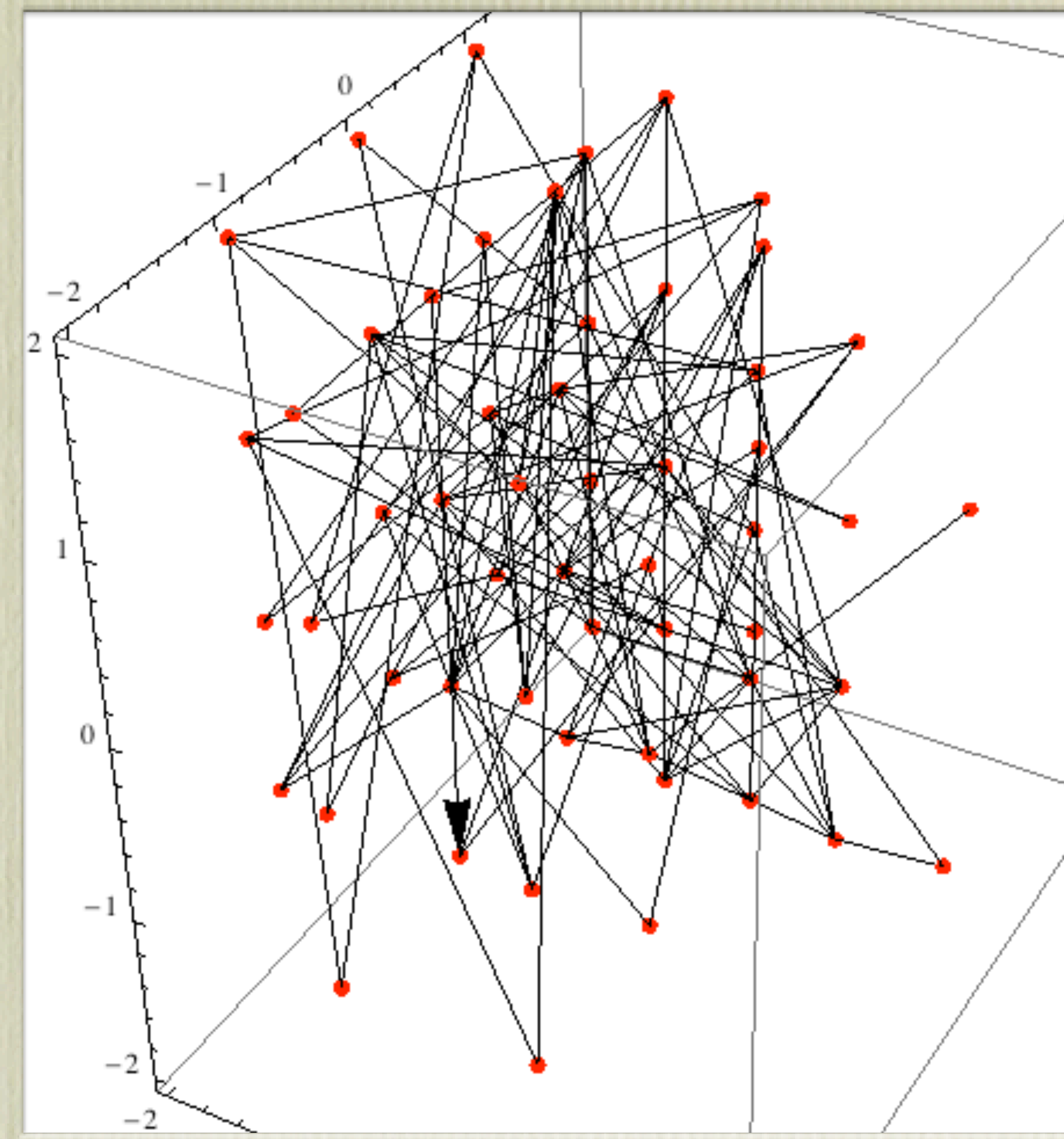
  where the sign is the sign of R(λ). Look at the coefficient space (or the set of values R(λ′)).

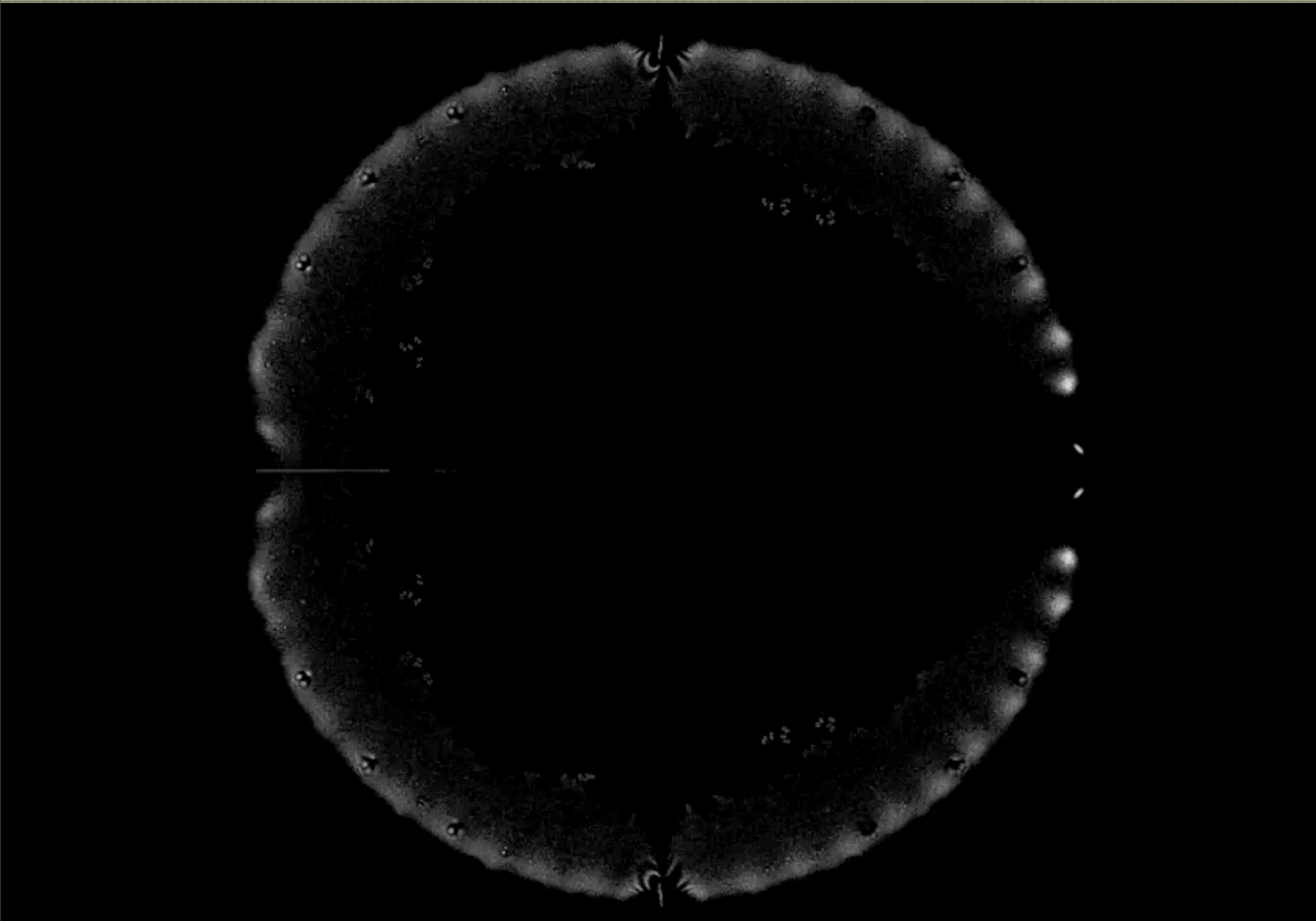
3-dimensional projection

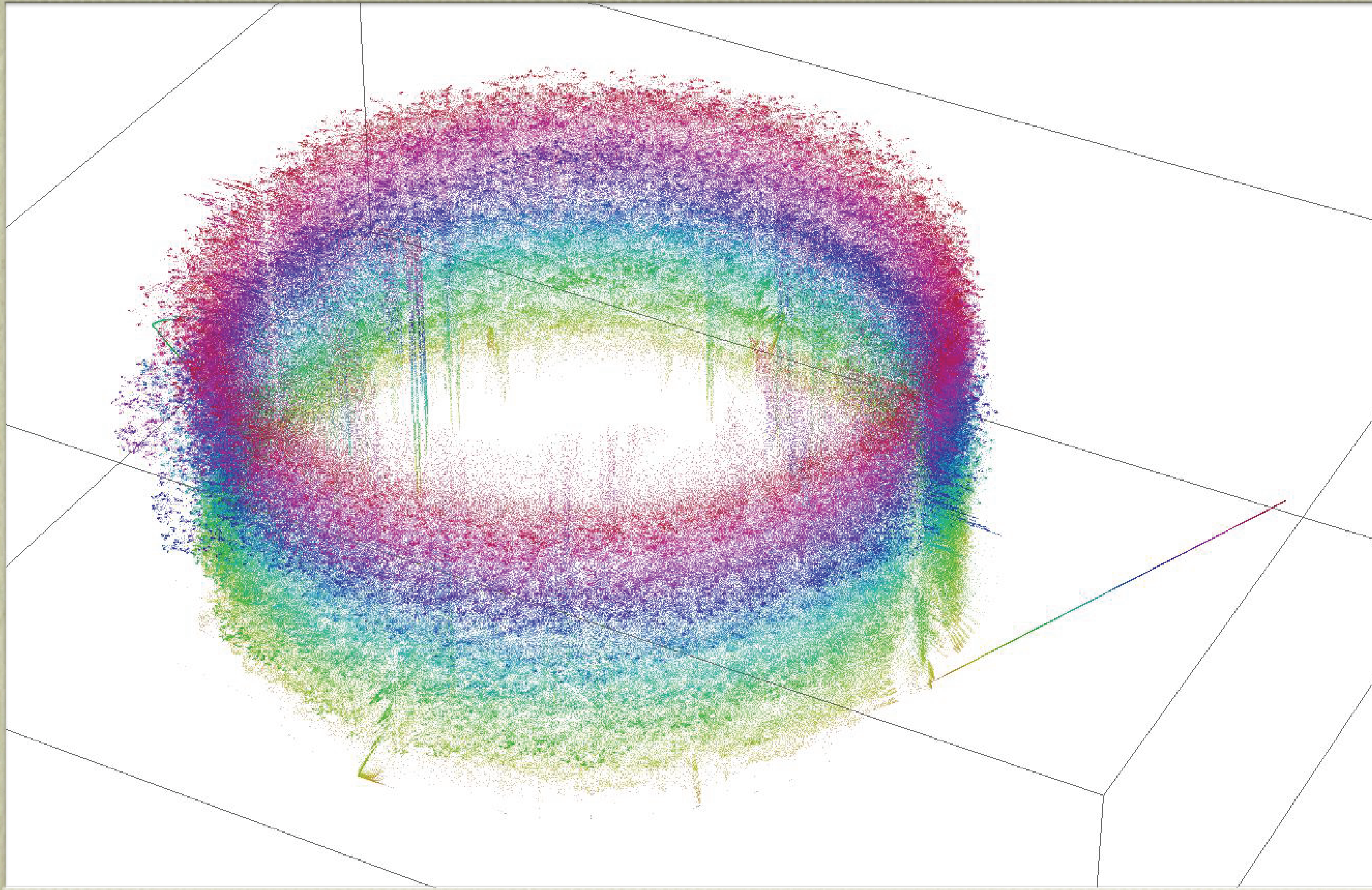


Pisot numbers follow the leader

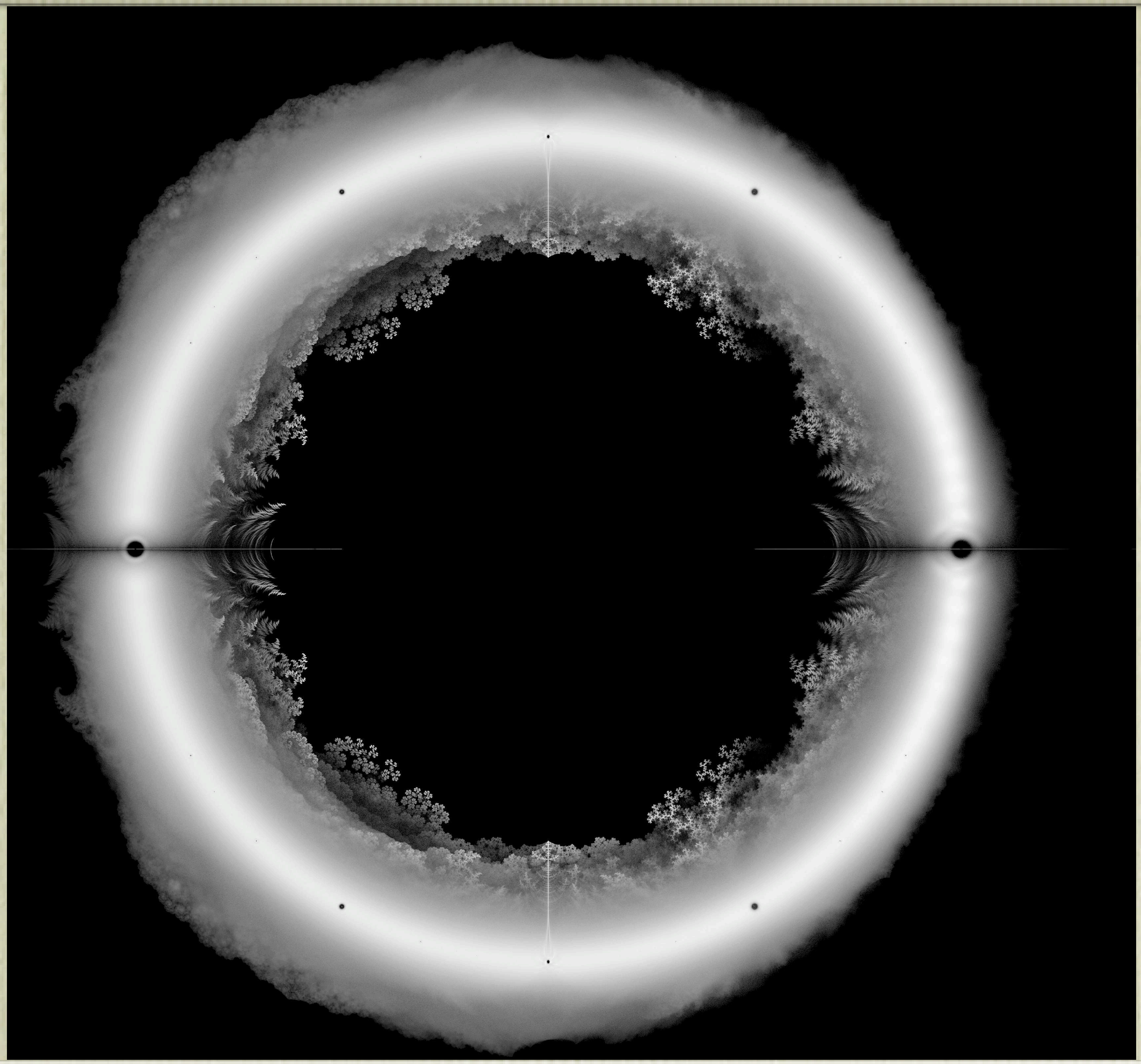
- Thm: λ is a Pisot number and if critical points are in Q(λ), f is PCF.
- In the multidimensional picture, all the λ' axes contract. So if the λ axis is controlled, the iteration stays bounded, and must ultimately repeat.



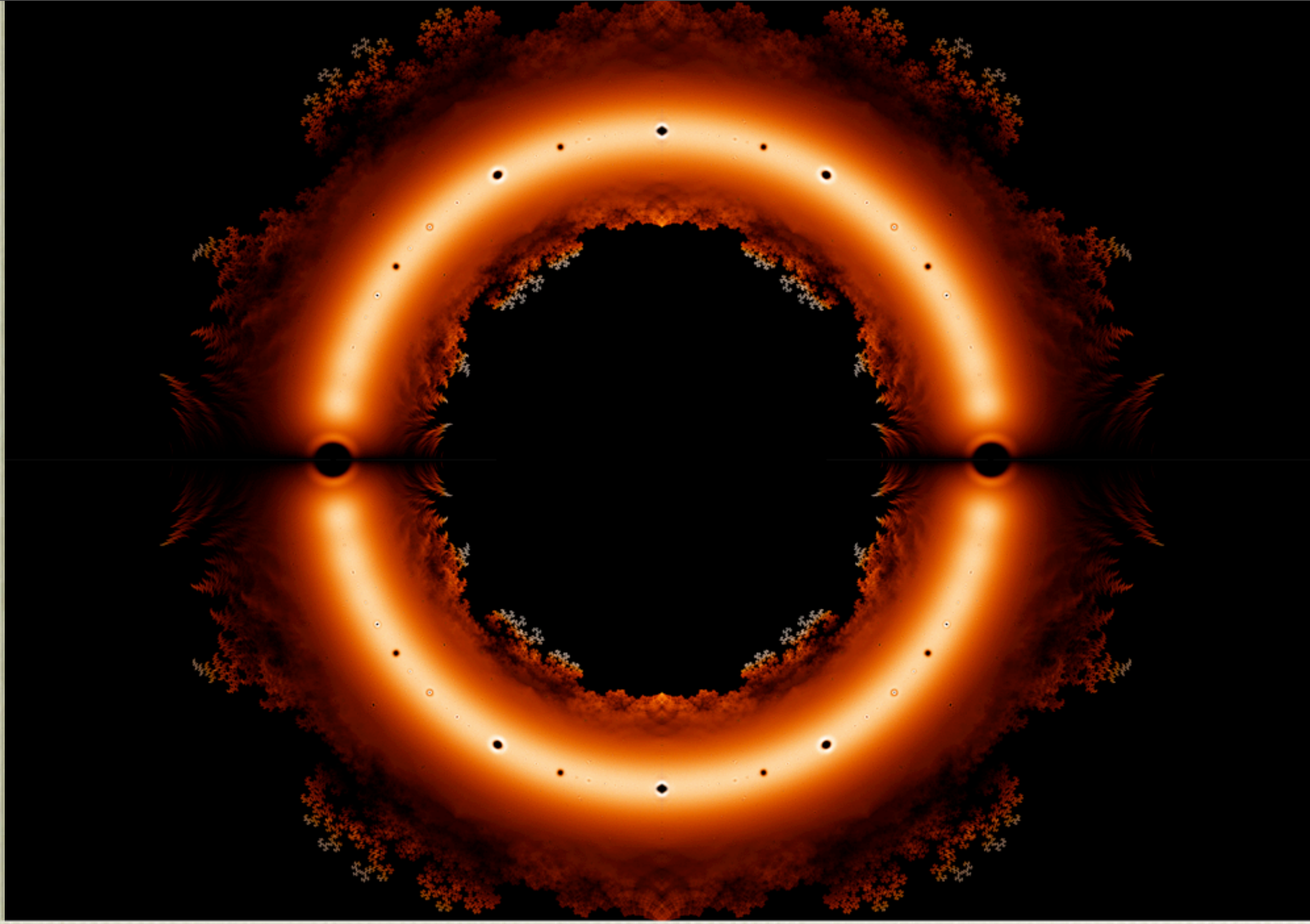




Kneading Roots vs. growth



Galois Conjugates: PCF growth



Roots of polynomials with 1,-1 coefficients
(Sam Derbyshire via John Baez)

Automorphisms of free groups

- A train track self-homotopy-equivalence of a graph, ϕ: $\Gamma \rightarrow \Gamma$ is a train track map if all

  forward images of every edge map as local homeomorphisms. The growth factor is an invariant of the associated free group outer automorphism ϕ. It is the maximal rate of growth of lengths of conjugacy classes.

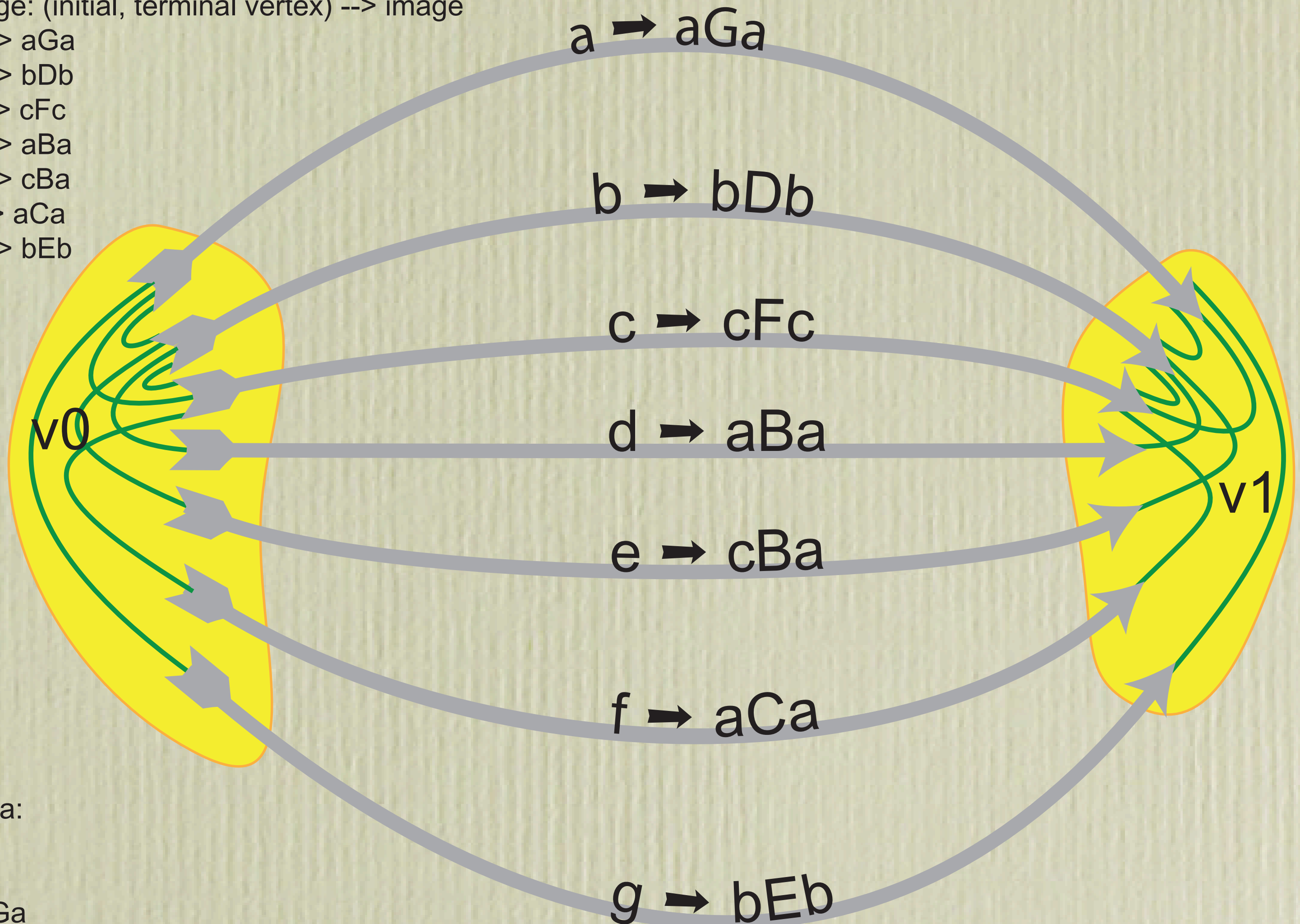
- Theorem (Bestvina-Handel) For any outer automorphism ϕ that has no invariant free factors is represented by a train track map of some Γ

Characterization of growth for free group automorphisms

- Theorem: a number λ is the growth factor for some train track automorphism of some free group if and only if λ is a weak Perron number
- Theorem: Any pair of Perron units (λ, λ’) is the growth factor for some free group outer automorphism φ along with its inverse φ^-1 provided each is also greater than the conjugates of the inverse of the other.

An **auto**morphism with growth 3(!)

7 // number of edges
2 // number of vertices
// format: edge: (initial, terminal vertex) --> image
a: (v0, v1) --> aGa
b: (v0, v1) --> bDb
c: (v0, v1) --> cFc
d: (v0, v1) --> aBa
e: (v0, v1) --> cBa
f: (v0, v1) --> aCa
g: (v0, v1) --> bEb

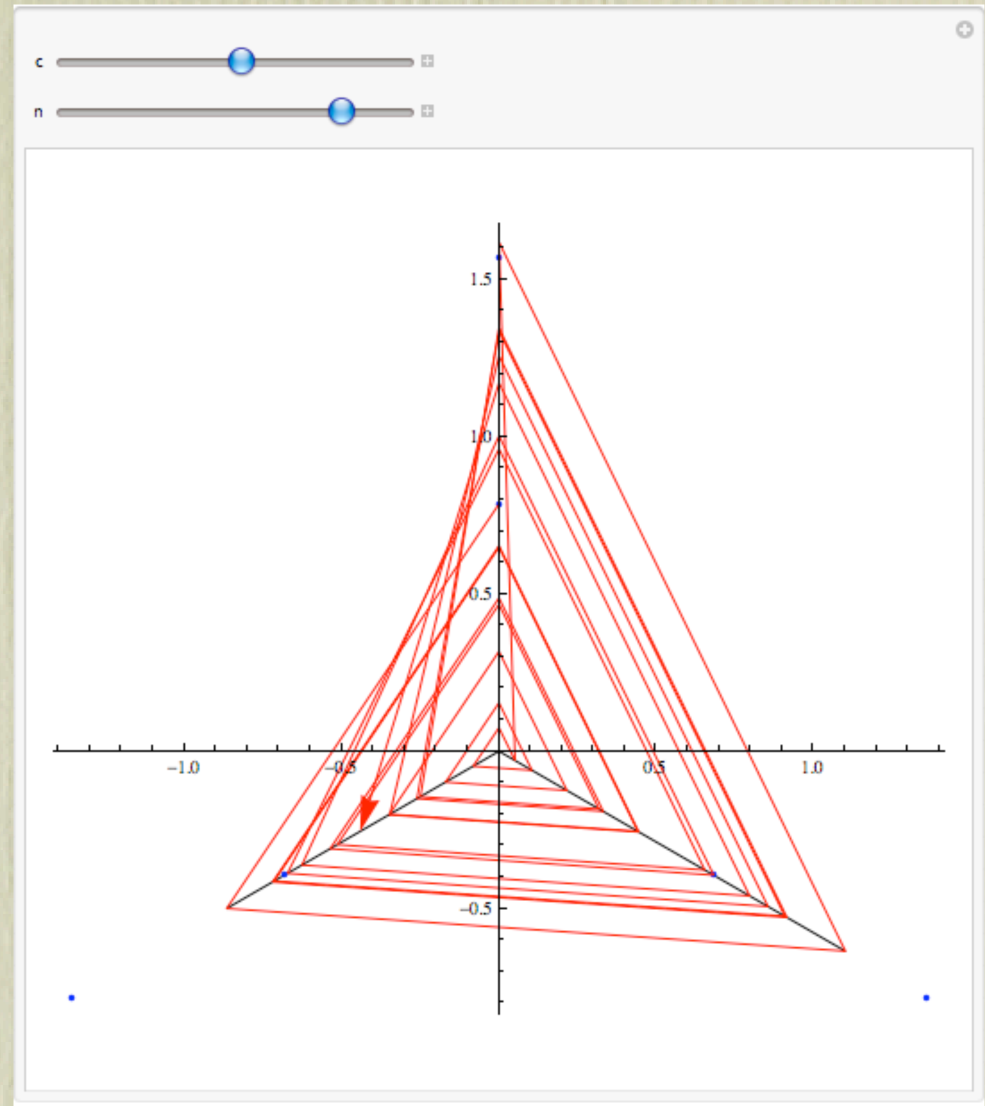


Iterates of a:
a
aGa
aGaBeBaGa
aGaBeBaGaBdBcBaBdBaGaBeBaGa
aGaBeBaGaBdBcBaBdBaGaBeBaGaBdBaBaBdBcFcBdBaGaBdBaBaBdBaGaBeBaGaBdBcBaB\dBaGaBeBaGa

An automorphism of F(6) with $\lambda = 3$.

Splitting Hairs

- Construct a self-map of an asterisk-shaped tree with expansion constant λ.
- Split each hair into seven strands, joined at their endpoints.
- Use *3 traintrack as model for lifting tree-map to free group automorphism.



Limit set for small eigenvalues