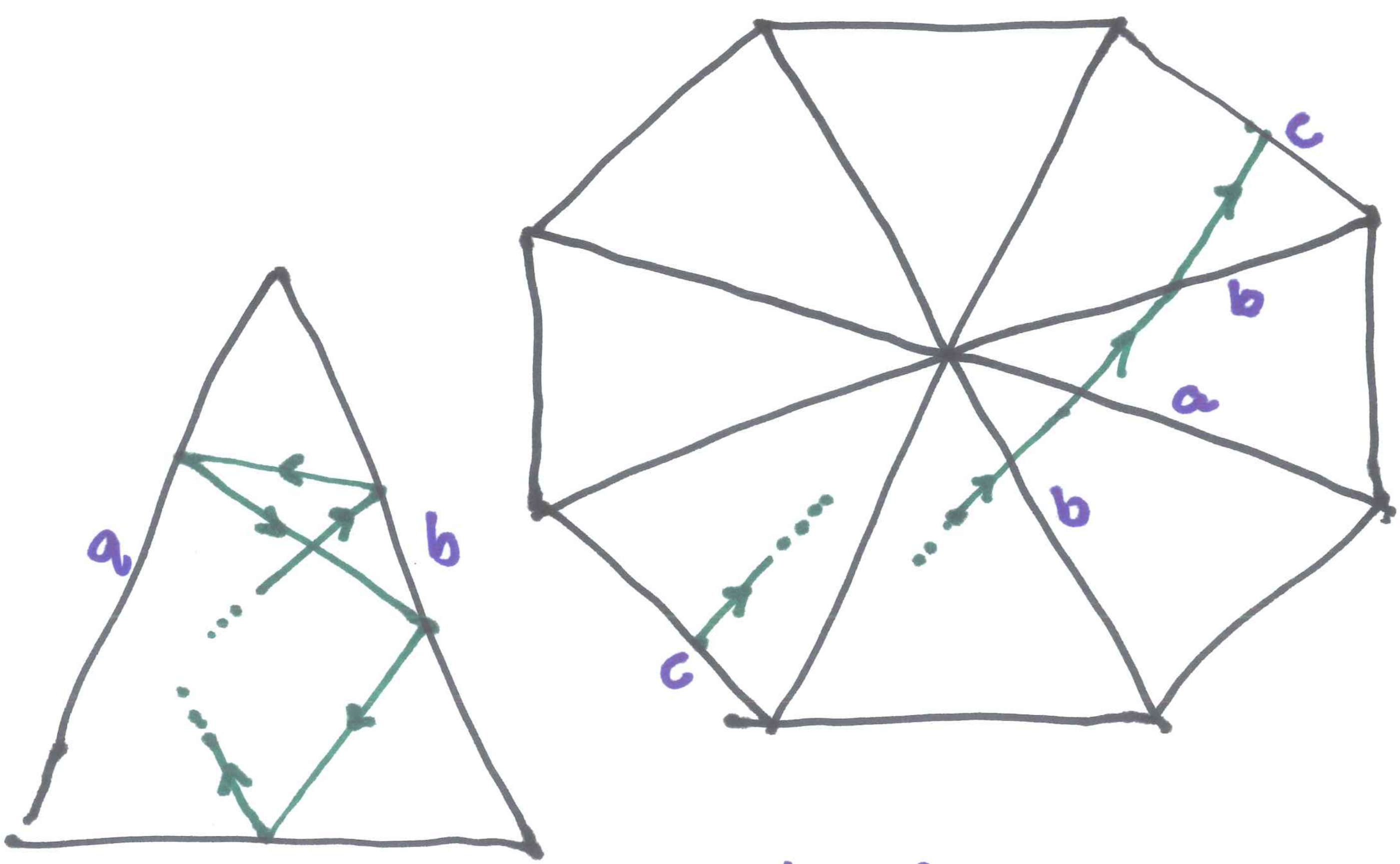
Polygonal Billiards: An Example

John Smillie

Joint work with Corinna Ulcigrai.

Certain triangles are distinguished by the Diophantine properties of their orbits.

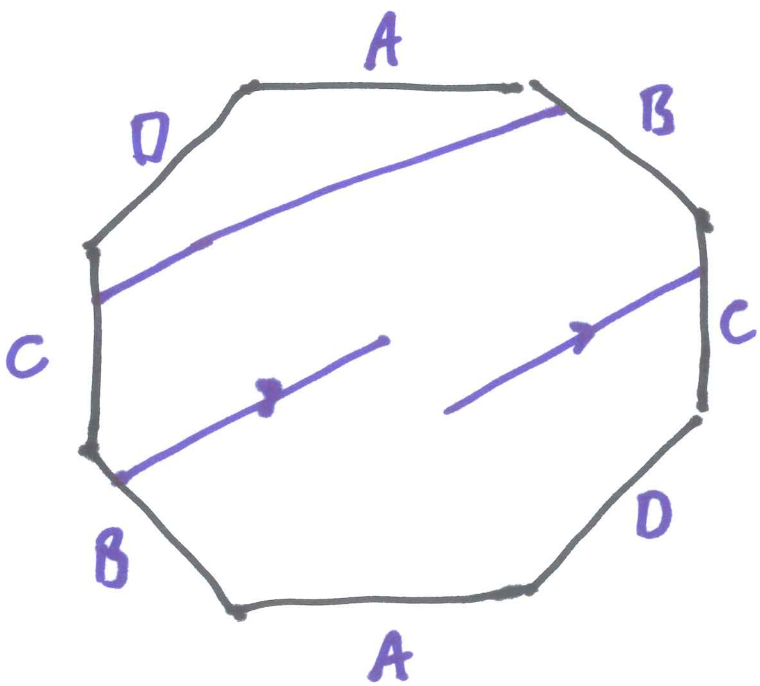


...babc...

What are the hitting sequences?

What can you say about the distribution of letters in these sequences?

Slight modification of problem:

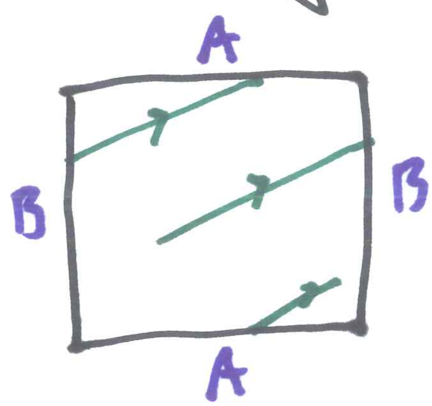


...CB...

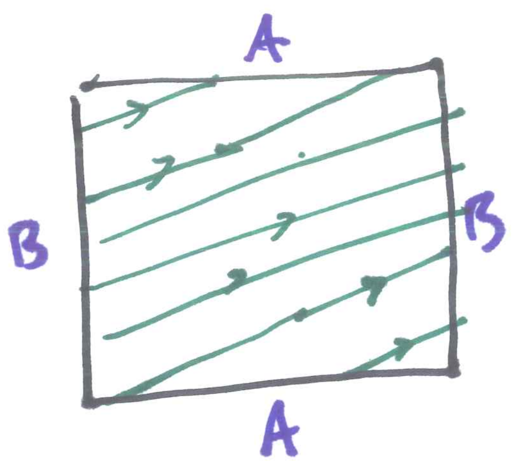
Original problem is resolved in Robyn Miller's thesis.

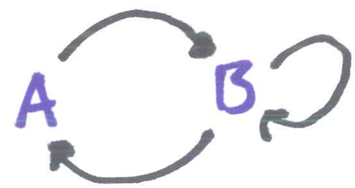
Warm up problem:

Which sequences of A's and B's are cutting sequences in the square?

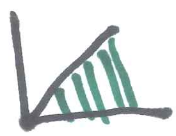


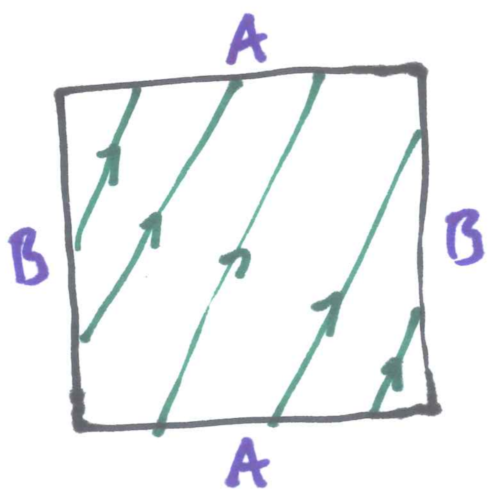
These are called Sturmian sequences.

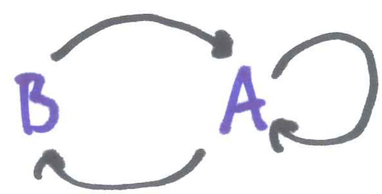




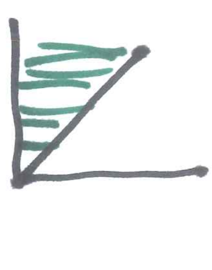
$\mathcal{D}_0$

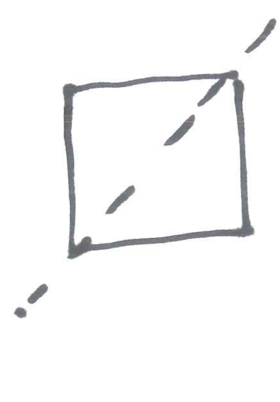






$\mathcal{D}_1$

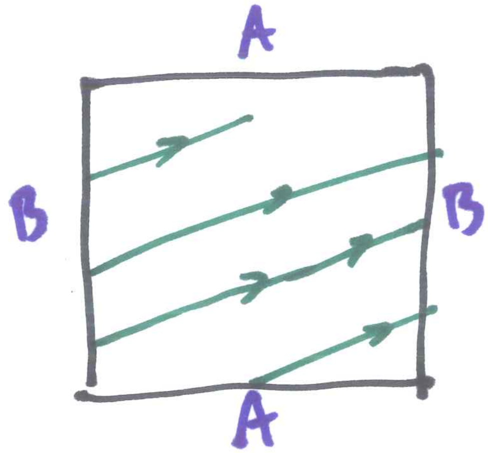




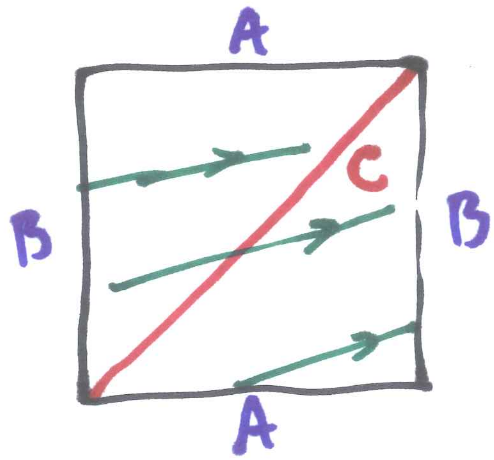
A sequence is <u>admissable</u> if it is compatible with diagram D_0 or D_1.

A cutting sequence is admissable.

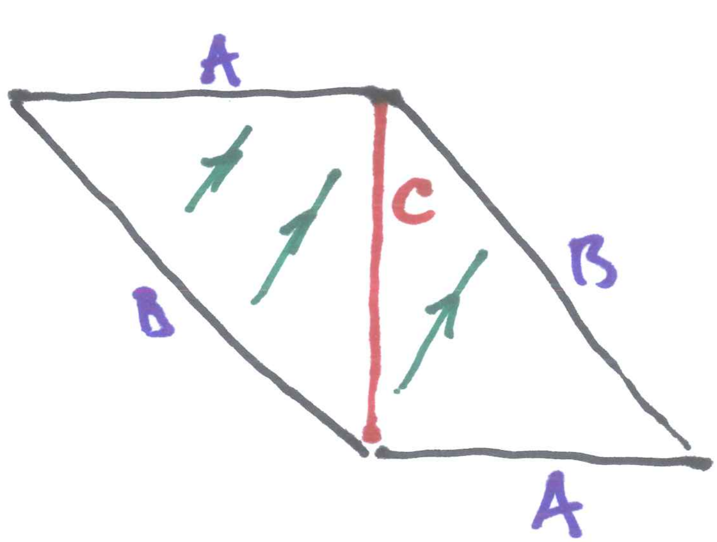
An operation on admissable sequences.

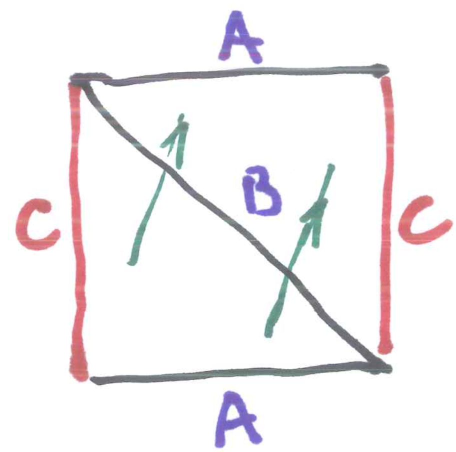


ABBBABBA

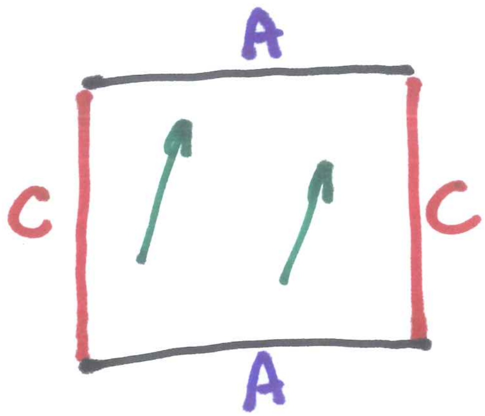


ABCBCBABCBA

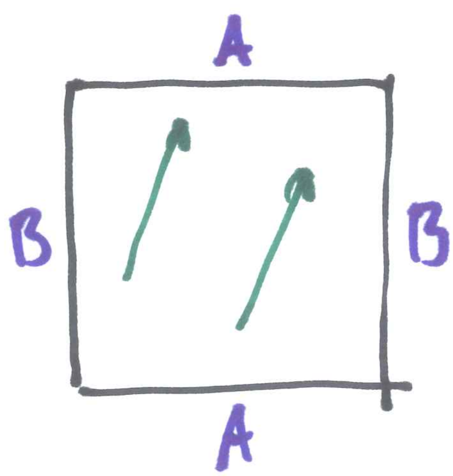




ABCBCBABCBA



ACCACA

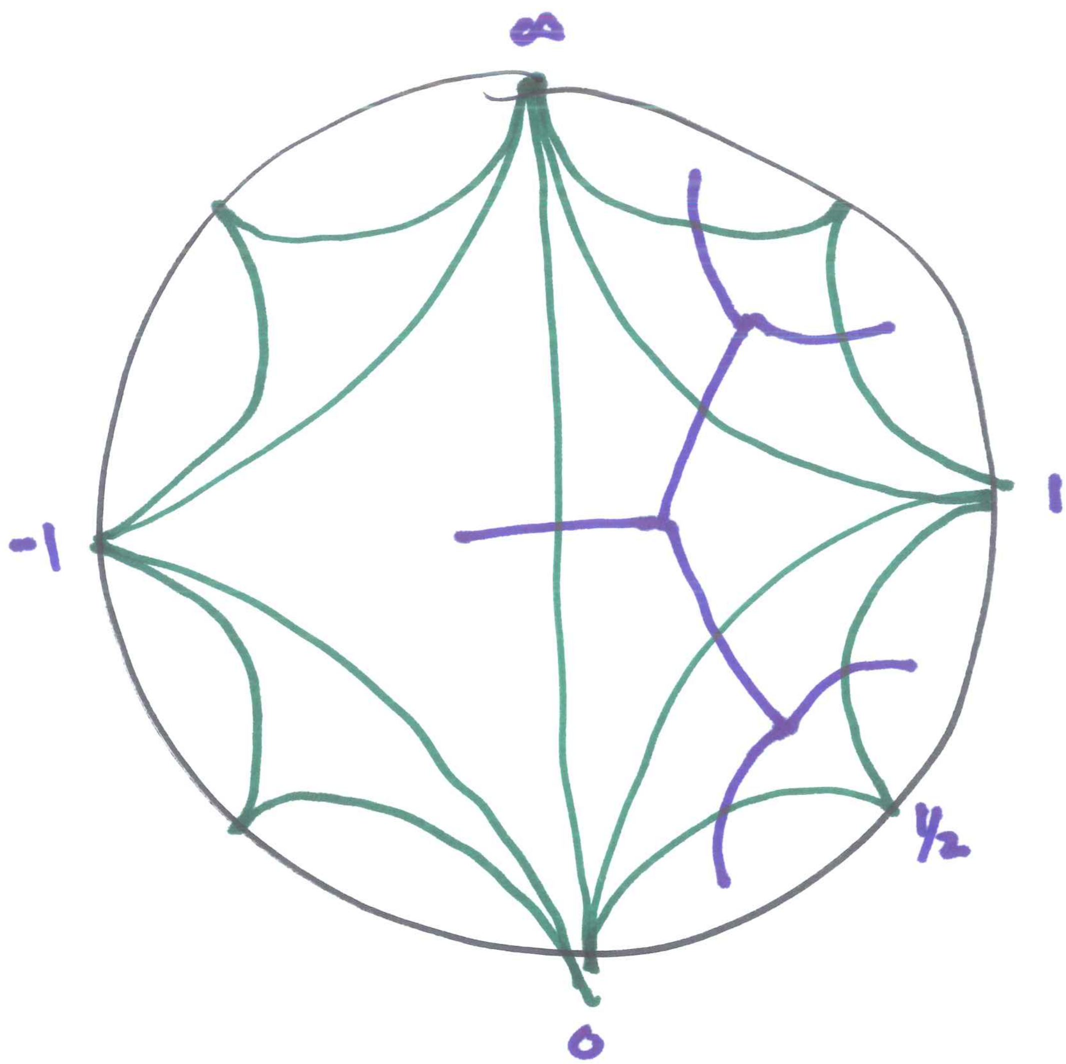


ABBABA

The operation of <u>derivation</u> of an admissable sequence removes one letter from each block.

A sequence is <u>characteristic</u> if it is admissable and the result of an arbitrary number of derivations is admissable.

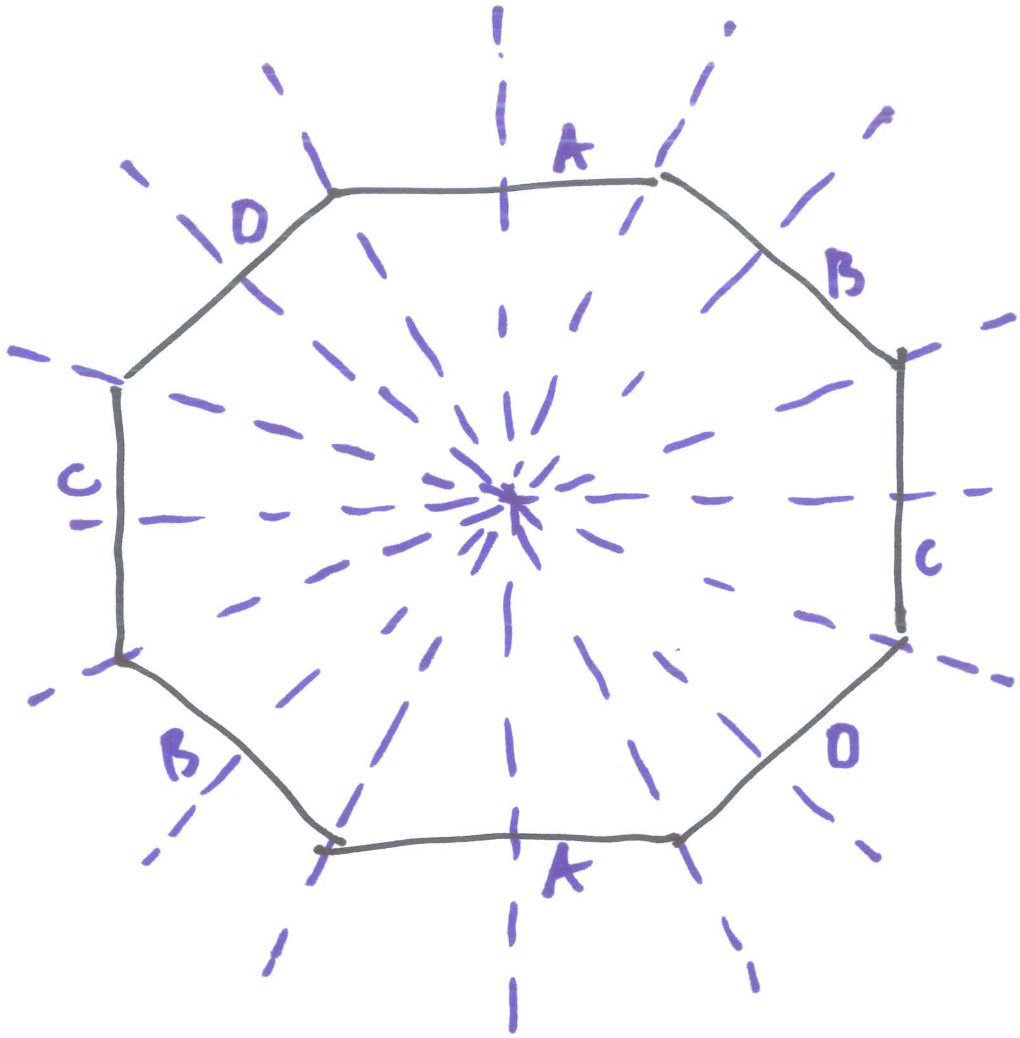
Thm. Every cutting sequence is characteristic. Converse is essentially true.



Knowing the sequence of types of derivations corresponds to knowing the direction of the trajectory.

Affine transformations take orbits to orbits but change the speed with which these orbits are traversed.

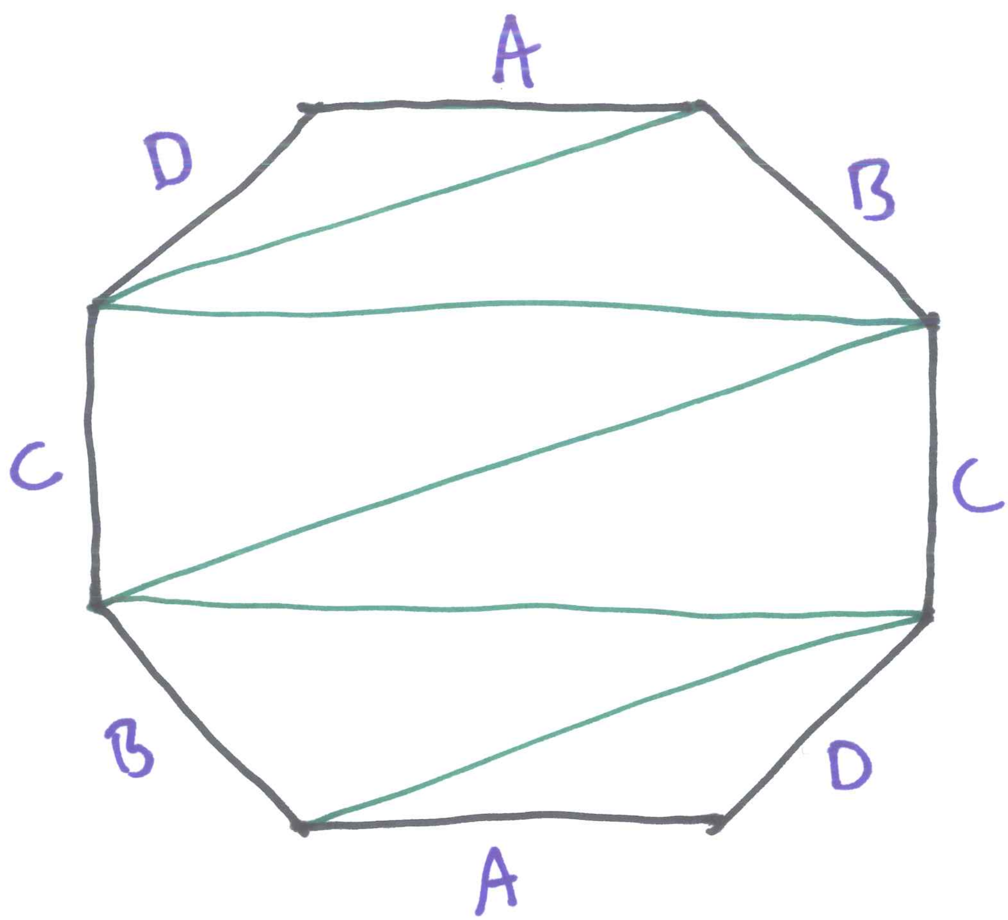
Example of renormalization as described by Mitsu.

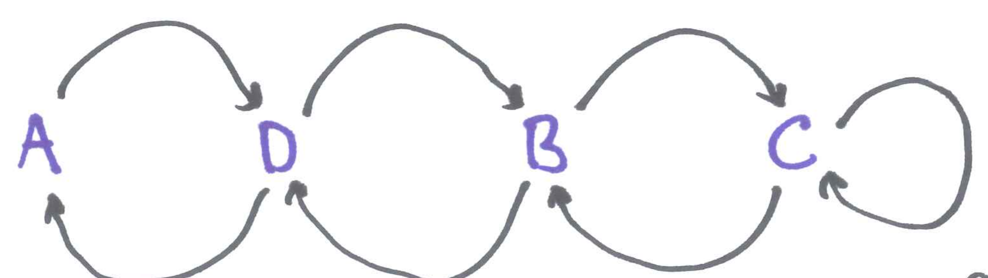


D_8 acts.

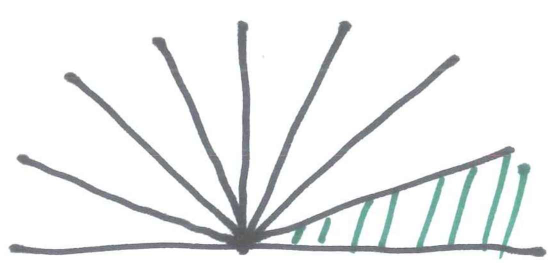
$\pm I$ preserves labels.

May assume direction is in the upper half-plane.

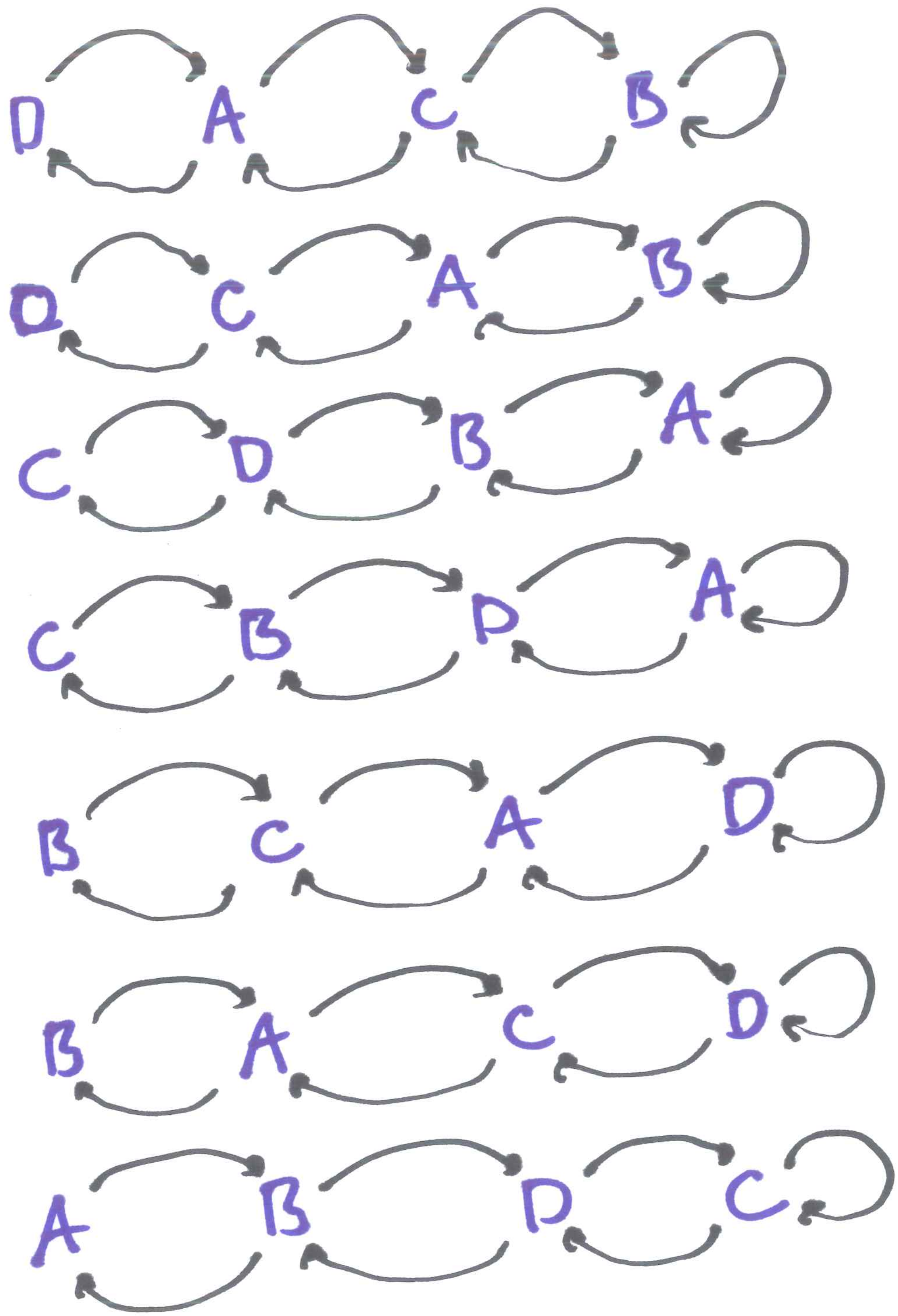




θ_0



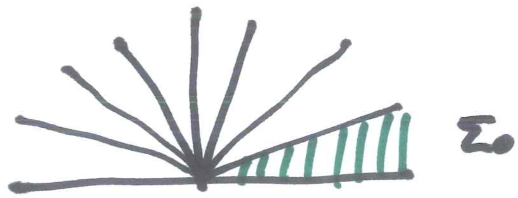
Σ_0



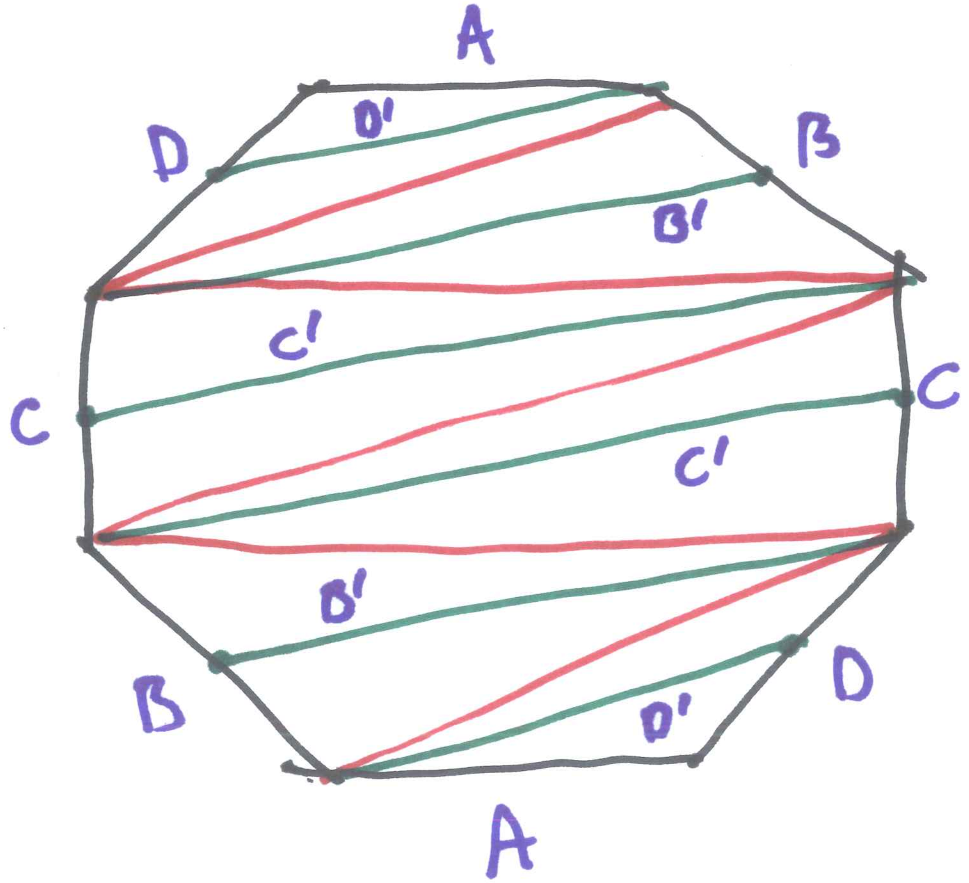
θ_1 θ_2 θ_3 θ_4 θ_5 θ_6 θ_7

A sequence in the letters A, B, C, D is <u>admissable</u> if it is compatible with a diagram $\mathcal{D}_0 \ldots \mathcal{D}_7$.

Consider a trajectory with direction in sector Σ_0.

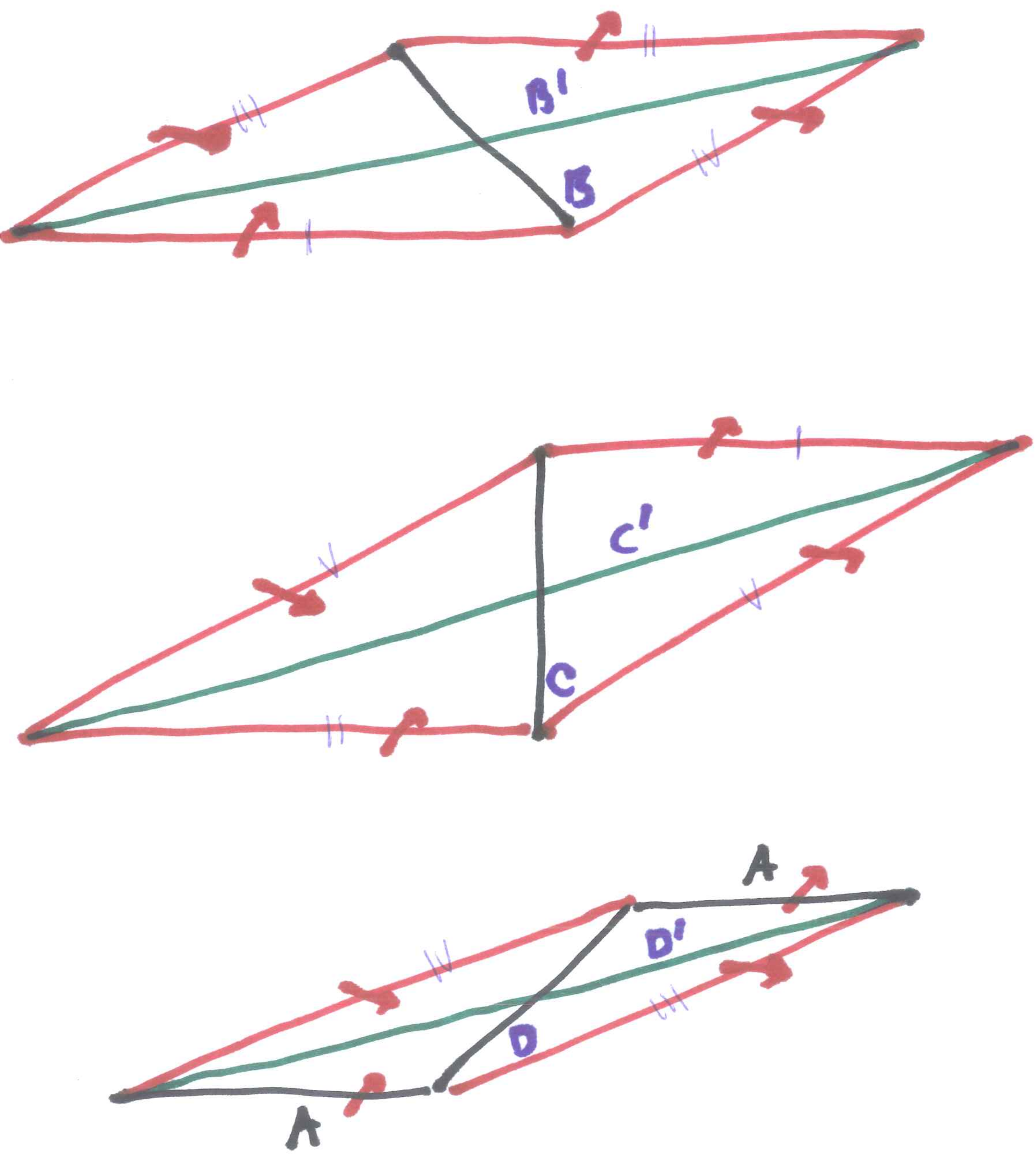


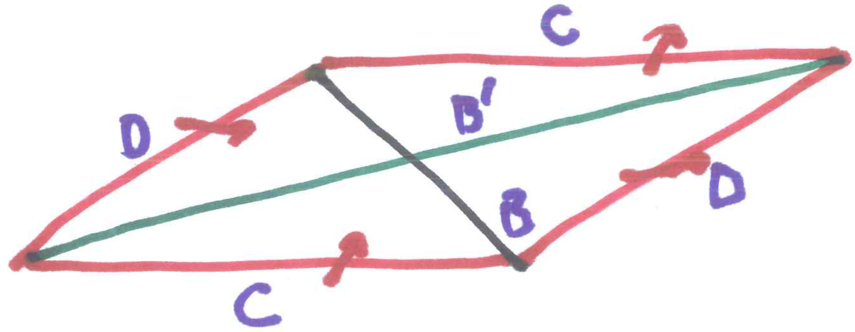
Introduce auxilliary edges.

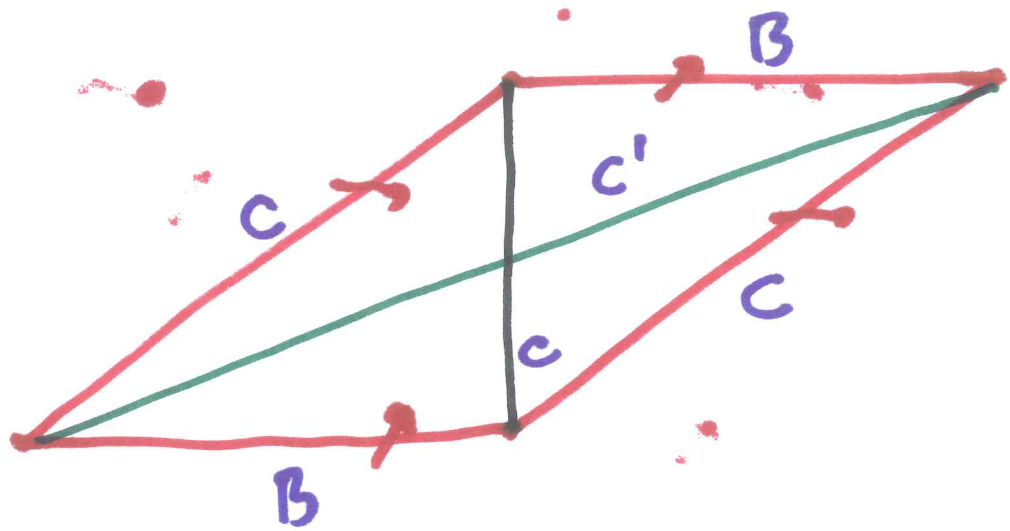


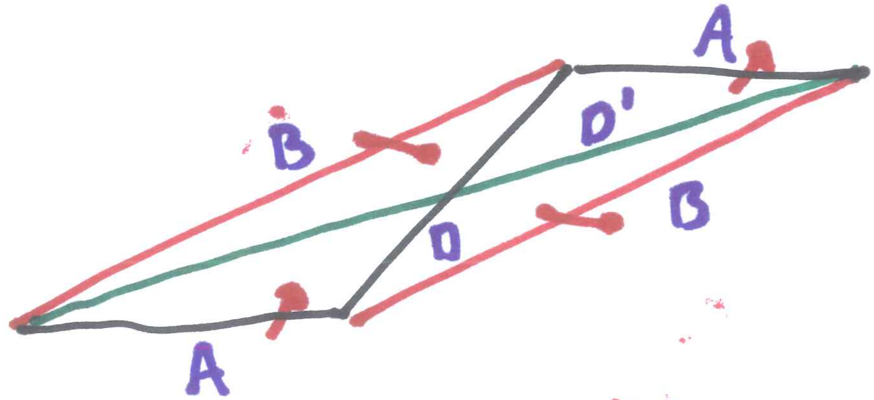
Claim: Knowing the intersections of a trajectory with A, B, C, D determine the intersections with A, B', C', D'.

We can break the octagon into 3 quadrilaterals.




C
D
B'
D
B
C

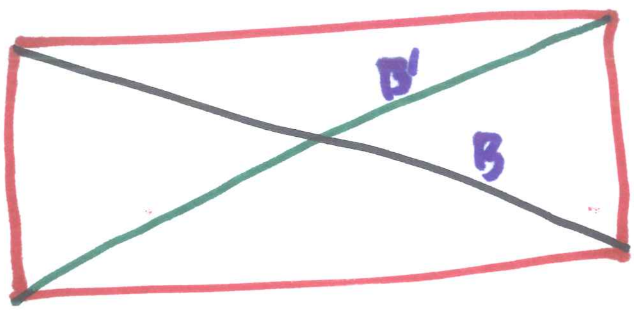

B
C
C'
C
c
B

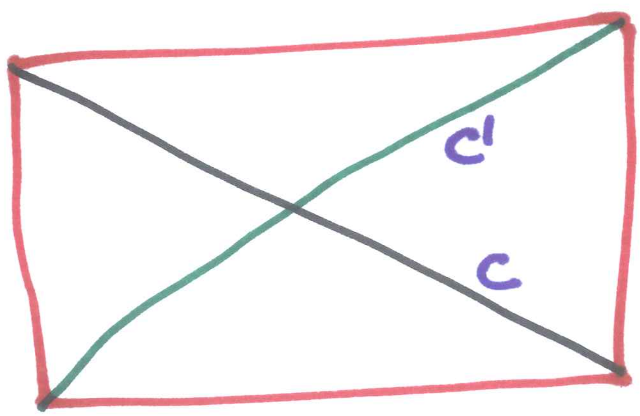

A
B
D'
B
D
A

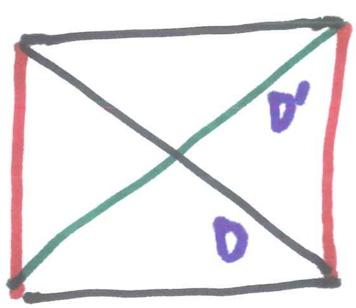
A letter in a sequence is <u>sandwiched</u> if it is preceded and followed by the same letter.

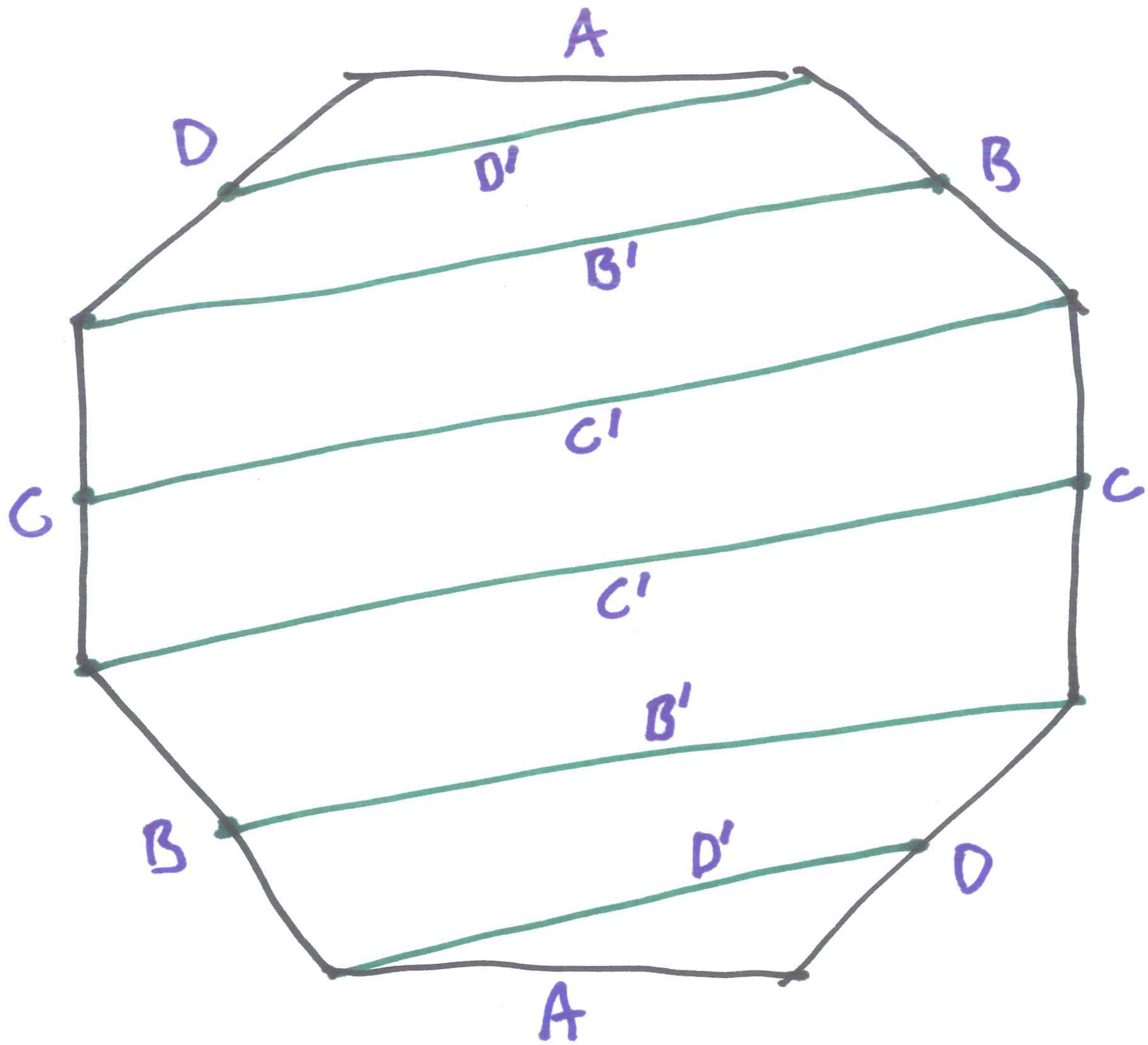
To go from the A, B, C, D coding to the A, B', C', D' coding we keep the sandwiched letters.

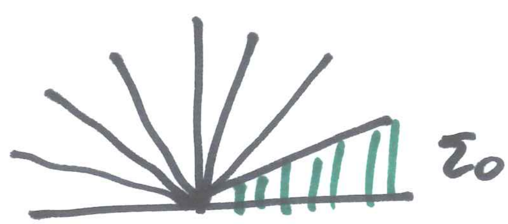
There is an affine involution which takes $A \to A$, $B \to B'$, $C \to C'$ and $D \to D'$!

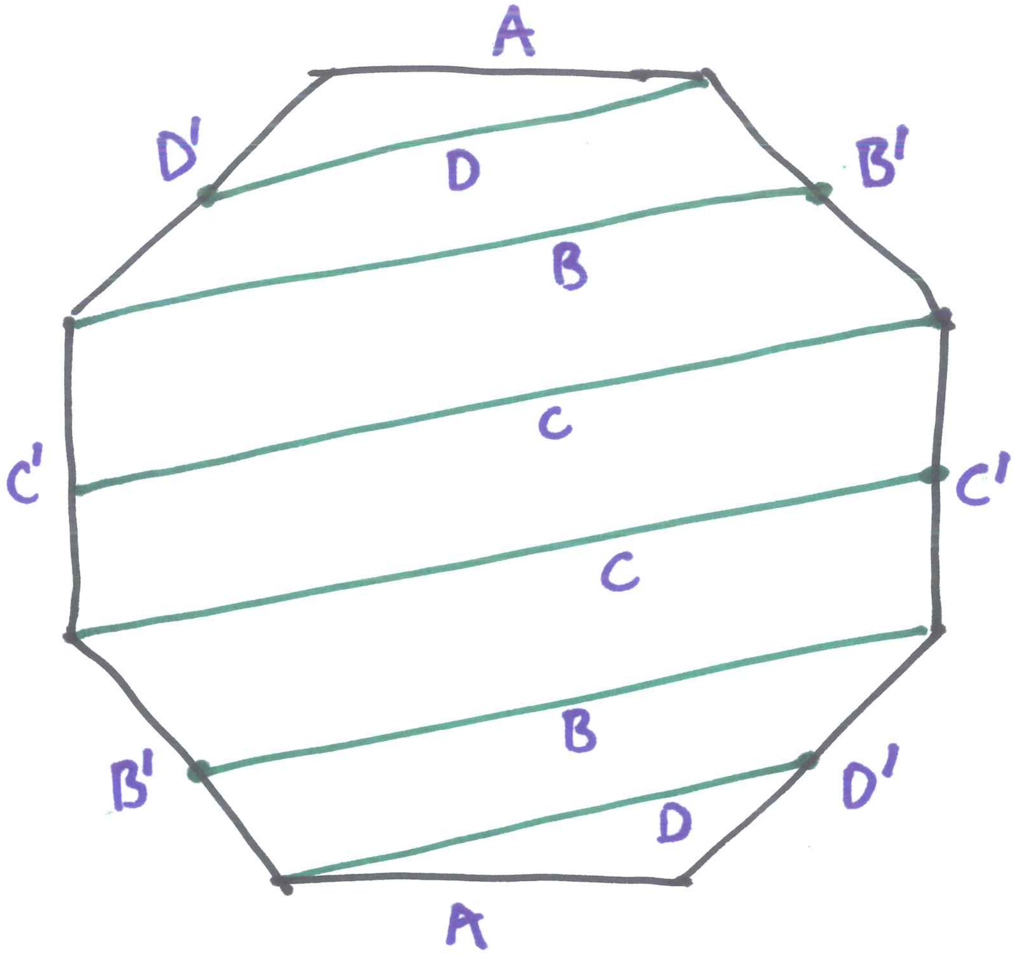


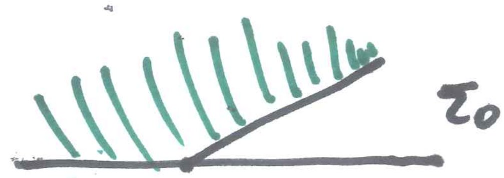





A
D
D'
B
B'
C'
C
C
C'
B'
B
D'
D
A


Σ_0


A
D'
D
B'
B
C
C'
C'
C
B
B'
D'
D
A


τ_0

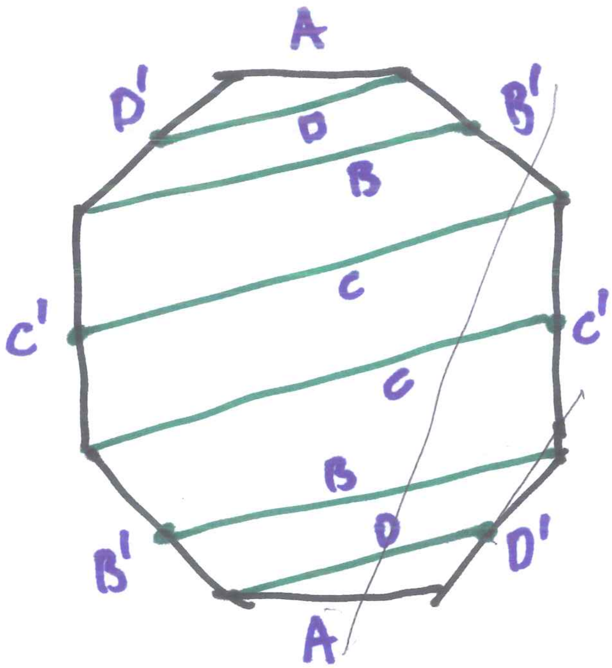
The *derivation* of an admissable sequence is the sequence obtained by dropping all non-sandwiched letters.

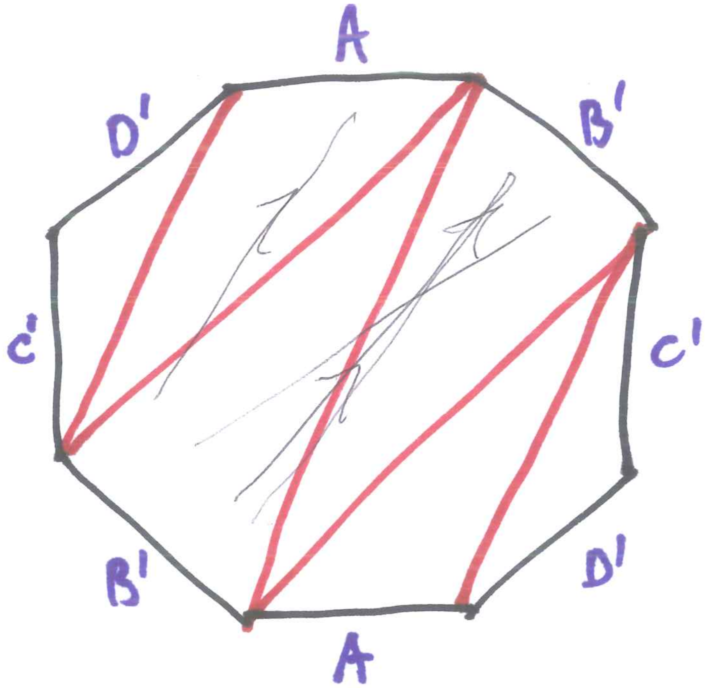
The derivation of a cutting sequence is a cutting sequence.

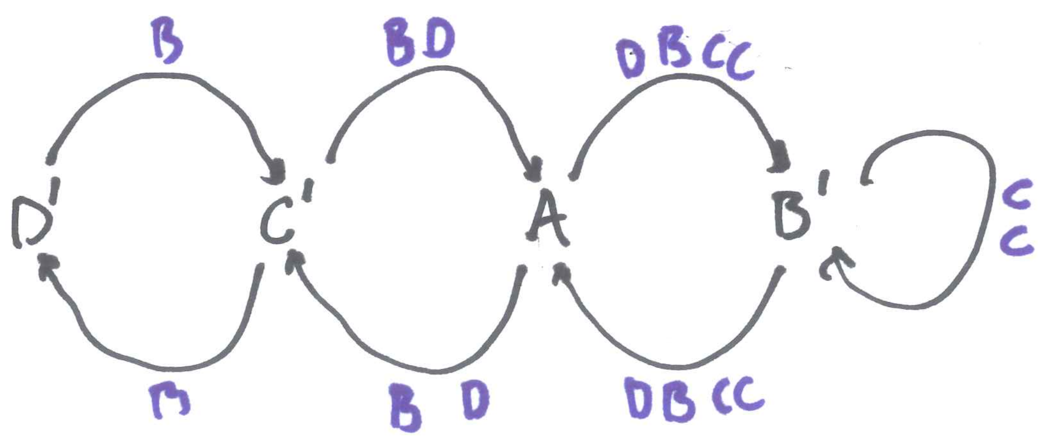
Derivation:

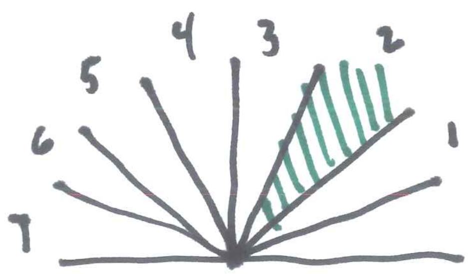
$\sigma =$ B D B C B D A D B C C B C C B D A D B

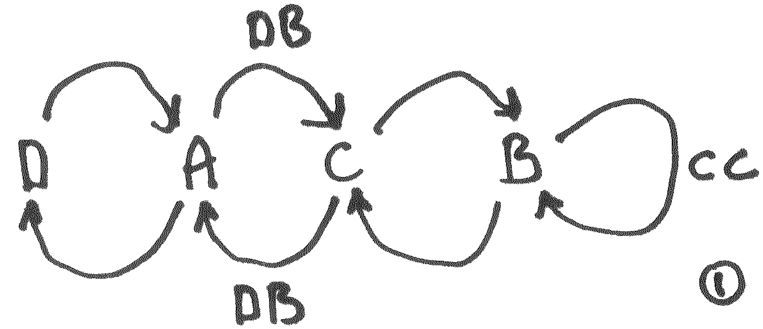
$\sigma' =$ D C A B A


A
D'
B'
D
B
C'
C
C'
C
B
D
B'
D'
A

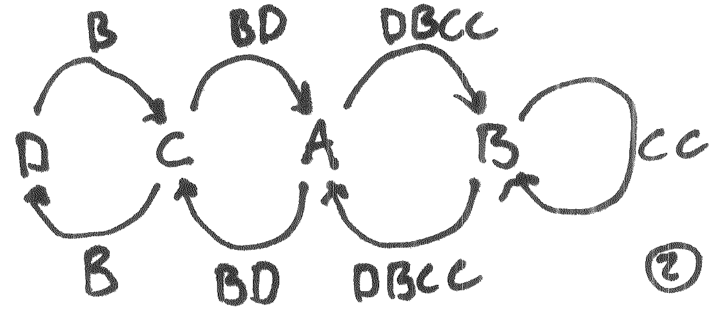

A
D'
B'
C'
C'
B'
D'
A


B
BD
DBCC
D'
C'
A
B'
C
C
B
B D
DBCC

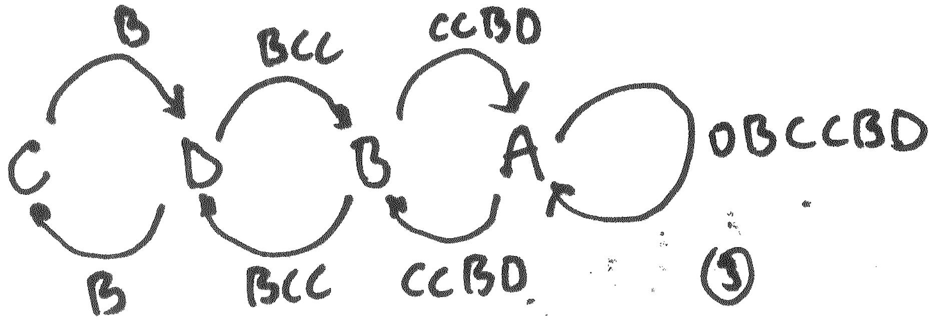

4
3
2
5
6
1
7


DB
D A C B
CC
DB

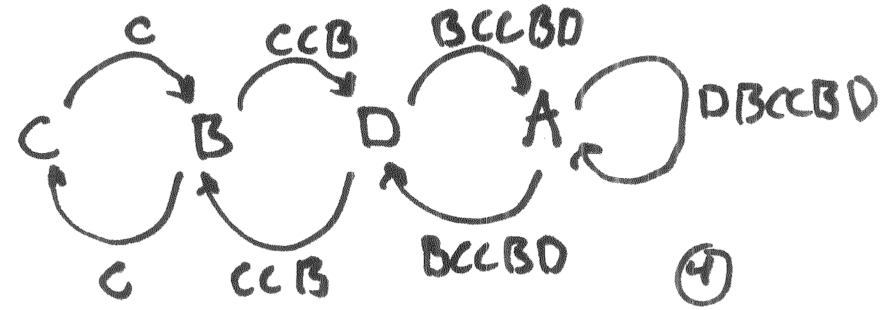
1


B BD DBCC
D C A B
CC
B BD DBCC

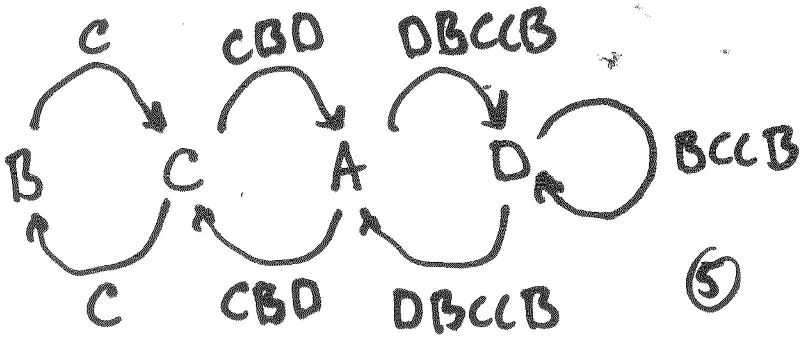
2


B BCC CCBD
C D B A
DBCCBD
B BCC CCBD

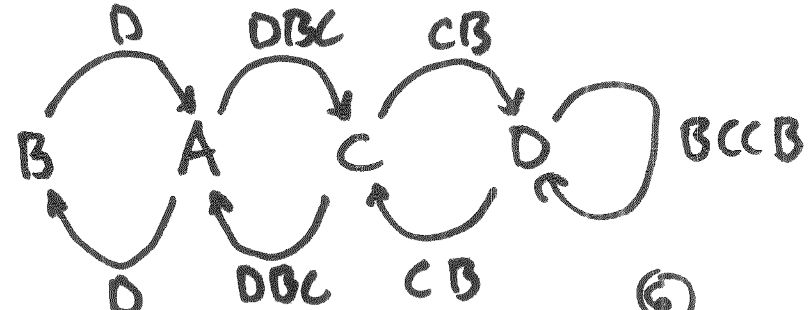
3


C CCB BCCBD
C B D A
DBCCBD
C CCB BCCBD

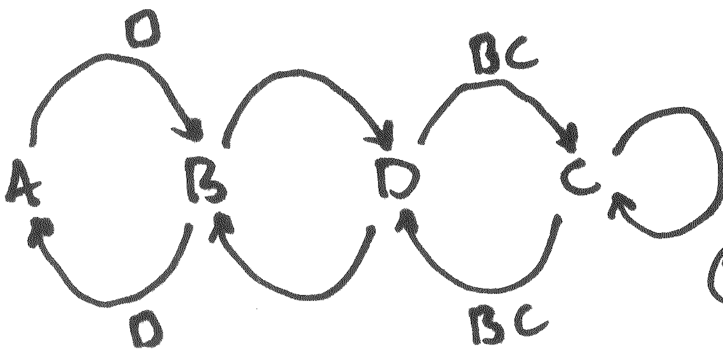
4


C CBD DBCCB
B C A D
BCCB
C CBD DBCCB

5


D DBC CB
B A C D
BCCB
D DBC CB

6

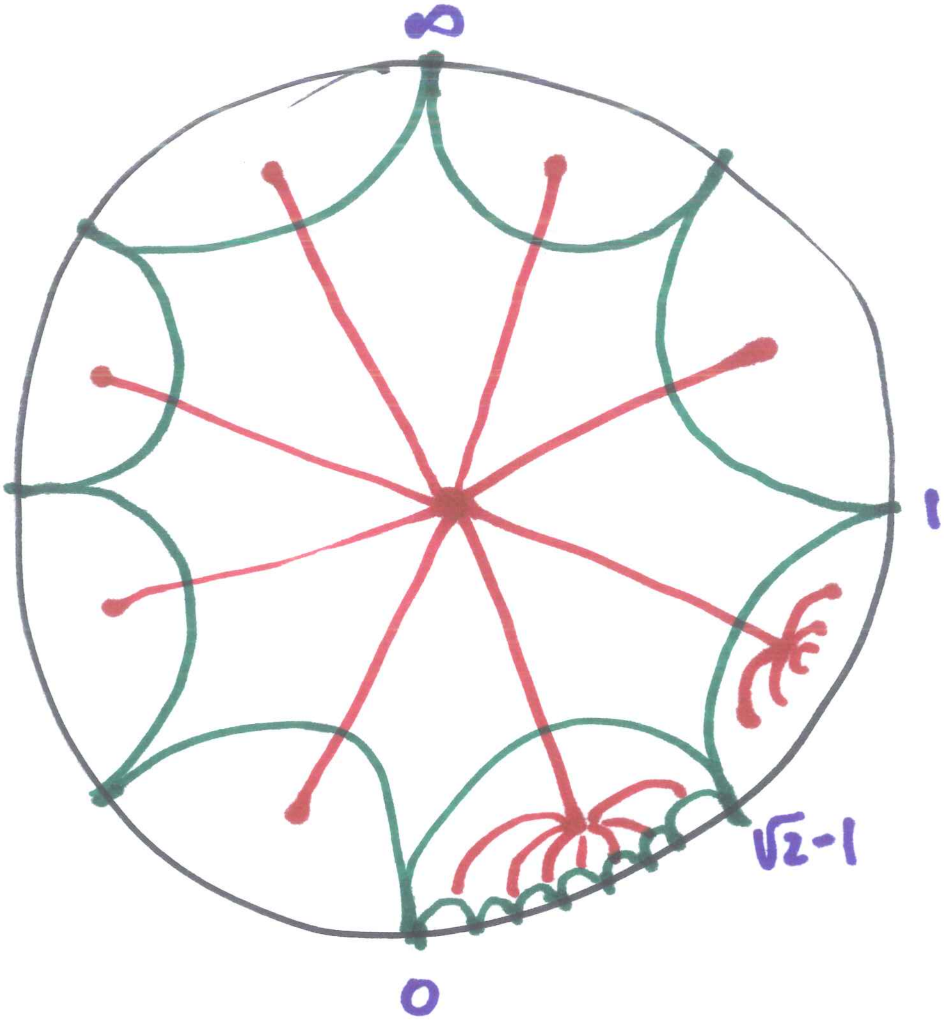

D BC
A B D C
D BC

7

A sequence is *coherent* if the pattern of sandwiched letters is compatible with one of the diagrams on 1...7.

An admissable, coherent sequence is *characteristic* if it can be derived arbitrarily many times and the result is still admissable and coherent.

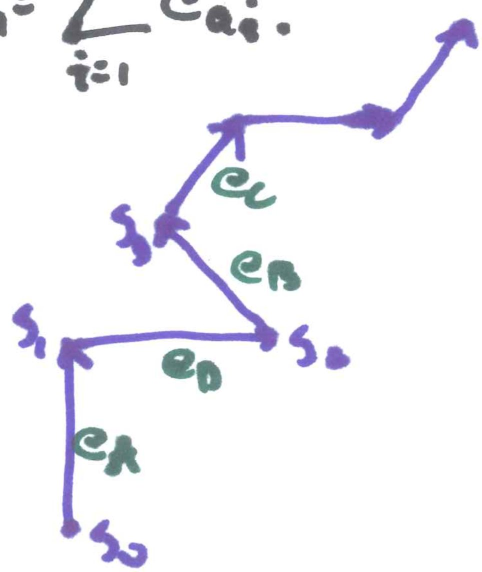
Thm. A cutting sequence is characteristic and the converse is essentially true.



Knowing the sequence of types
derivations corresponds to
nowing the direction of the
rajectory.

Let e_A, e_B, e_C, e_D be a basis for $\mathbb{R}^4$. The cutting sequence of a trajectory (a_i) determines a "snake" in $\mathbb{R}^4$.

Let $s_n = \sum_{i=1}^{n} e_{a_i}$.

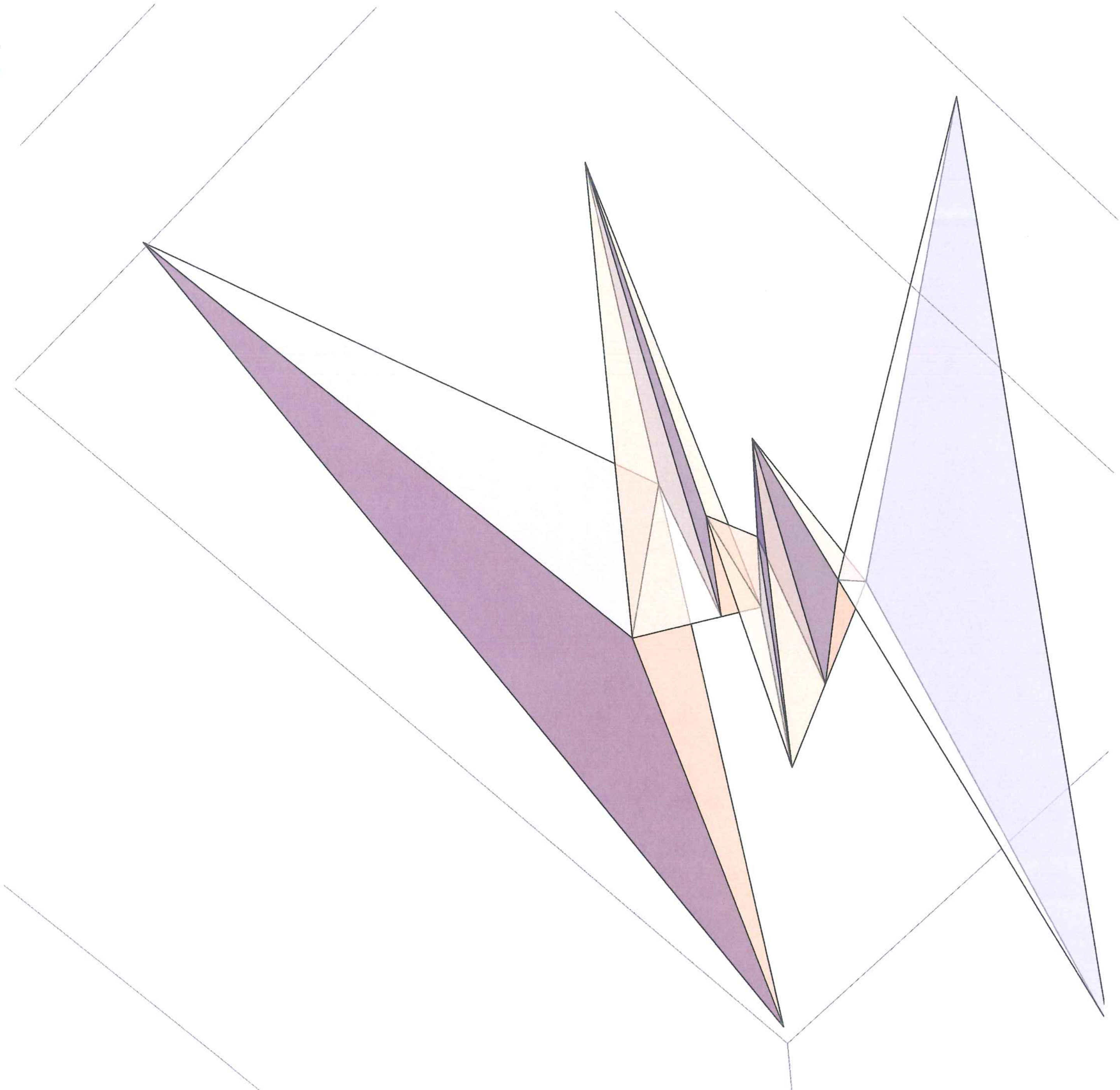


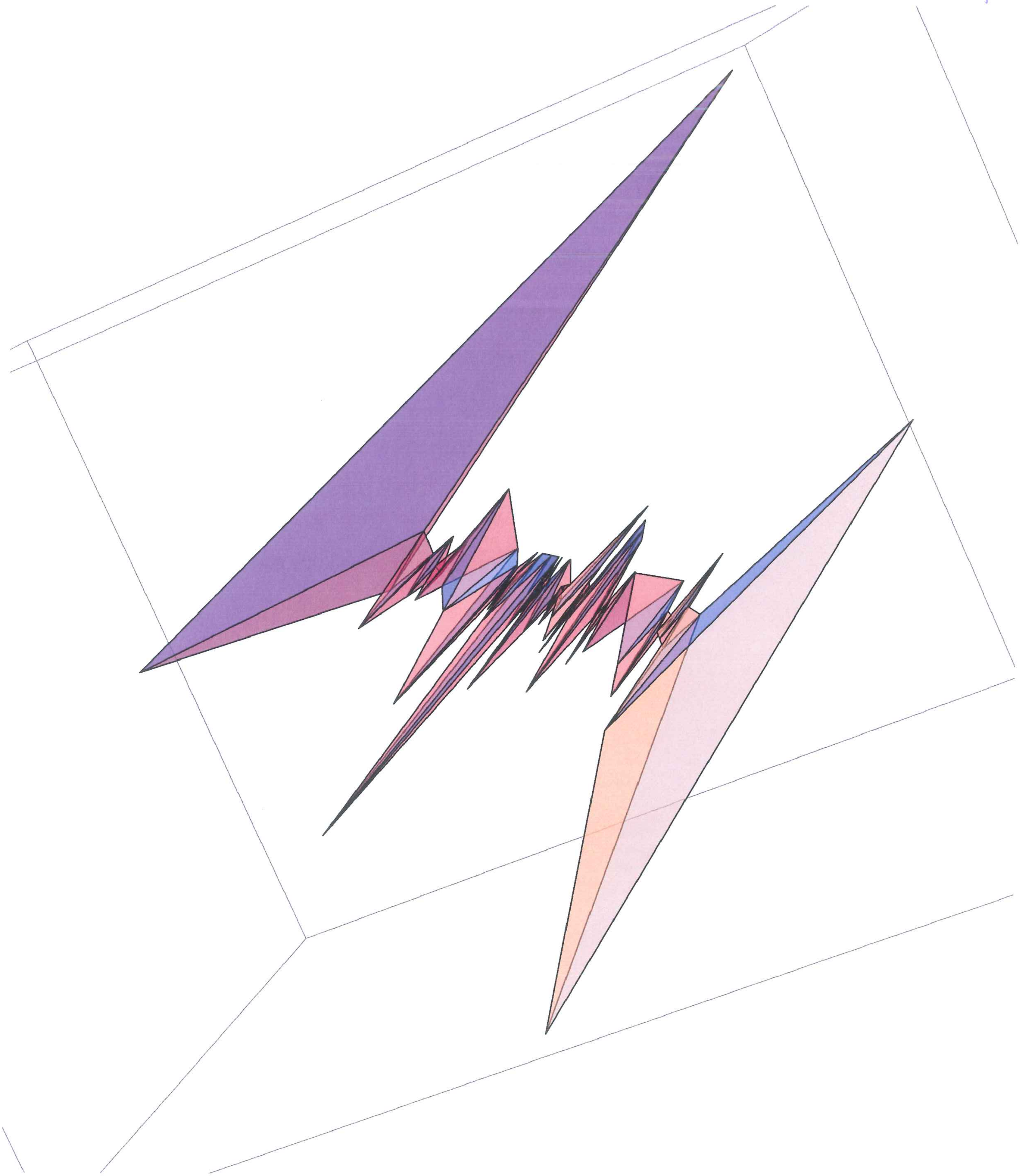
The direction of the snake is determined by the relative frequency of letters.

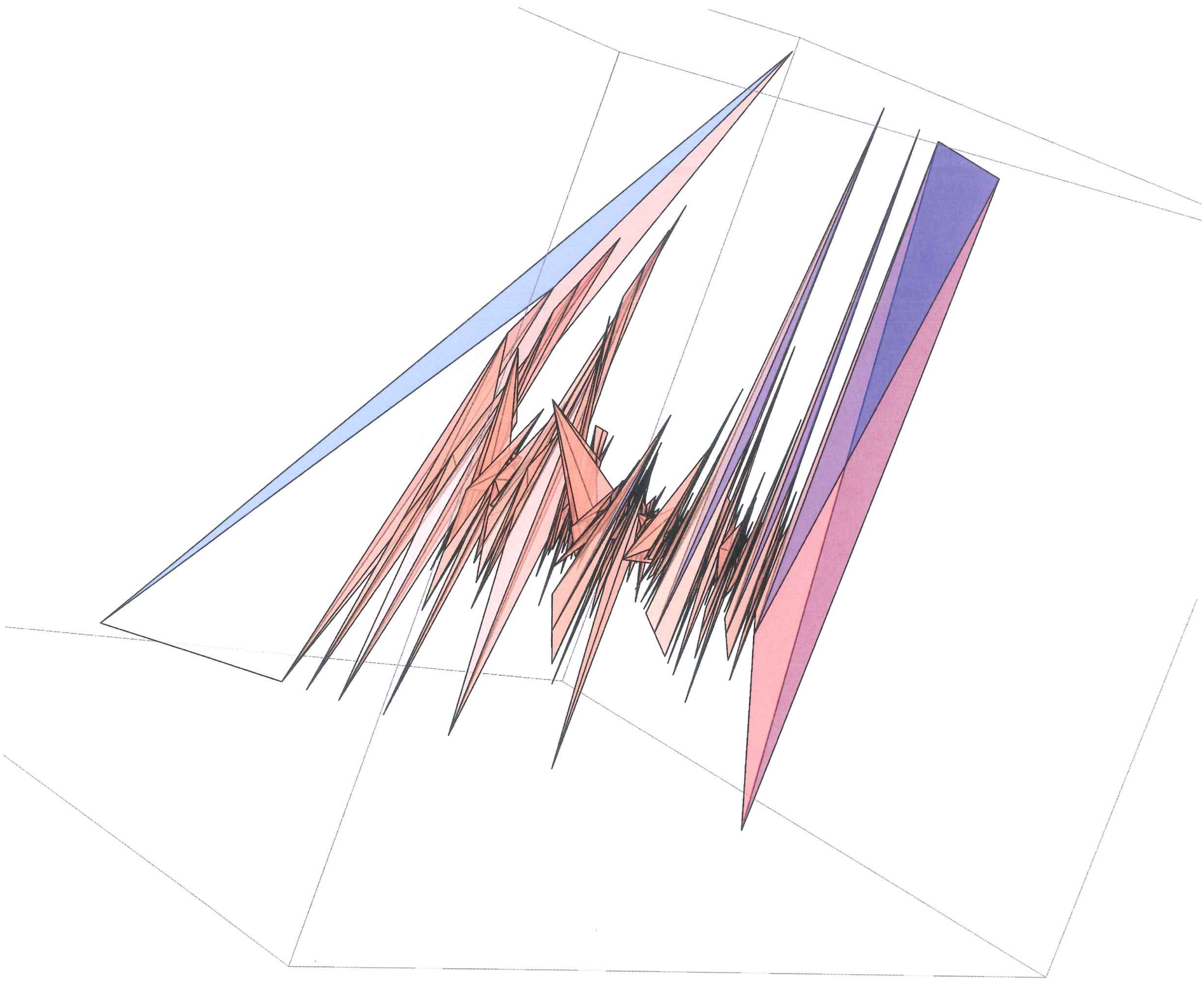
Thm. (Veech) For every non-periodic direction in the octagon

$$\lim_{n\to\infty} \frac{S_n}{\|S_n\|}$$

converges to a well defined asymptotic direction.



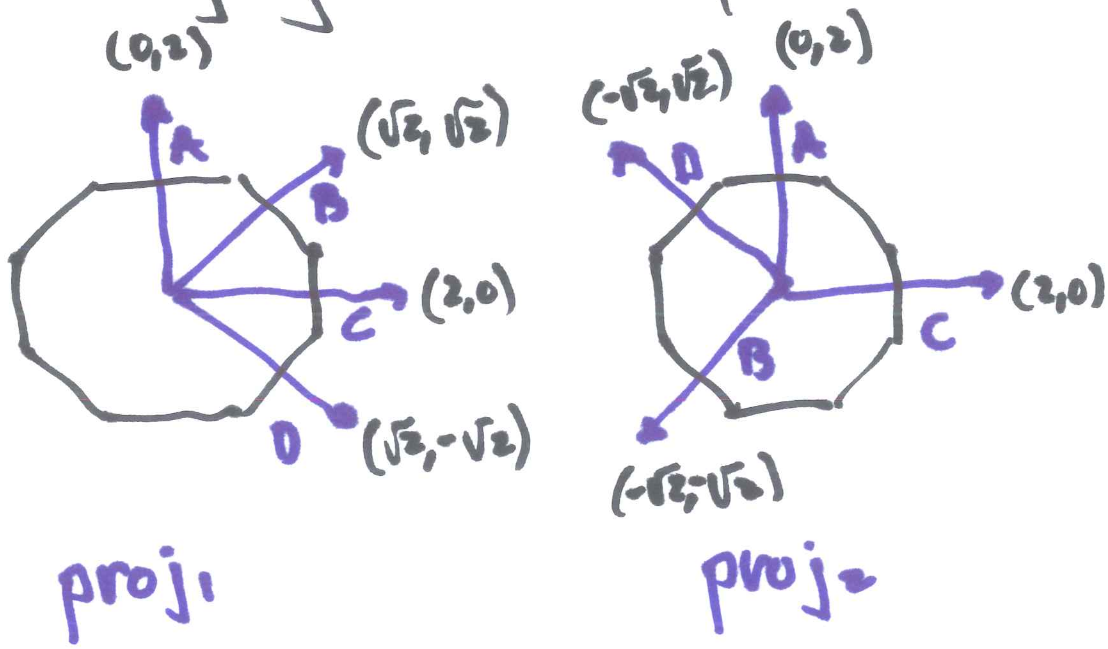


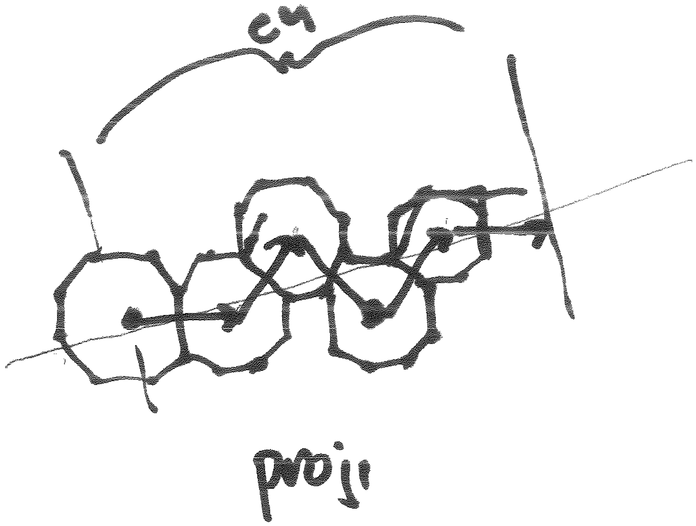


The asymptotic directions lie in a 2-dimensional subspace of $\mathbb{R}^4$. There is a natural complementary subspace obtained by Galois conjugation.
"Hidden Dimensions"

Let proj_1 be the projection onto the asymptotic subspace.

Let proj_2 be the projection onto the conjugate subspace.



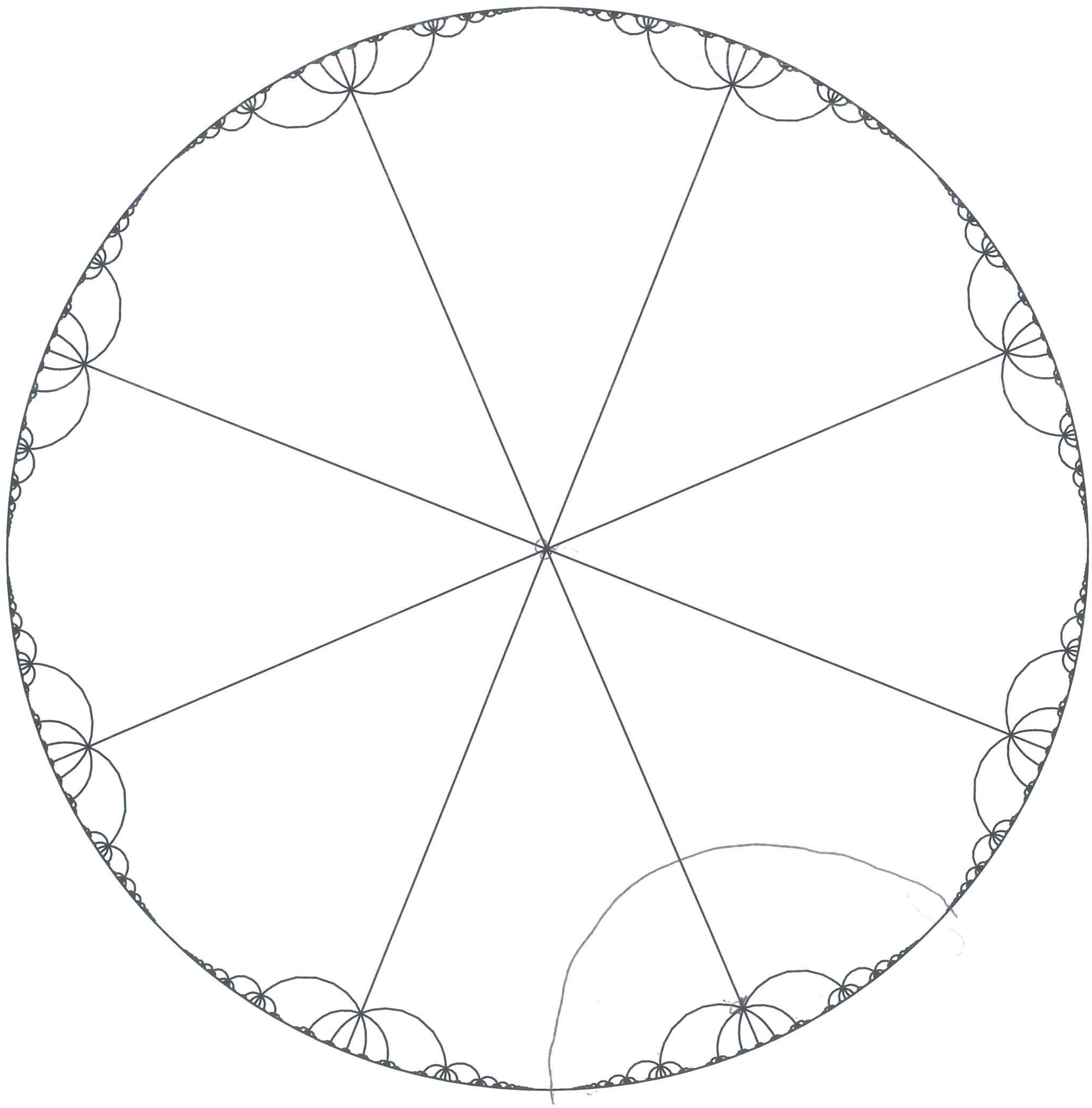


Thm. (Zorich, Möller) For almost every direction in the octagon we have

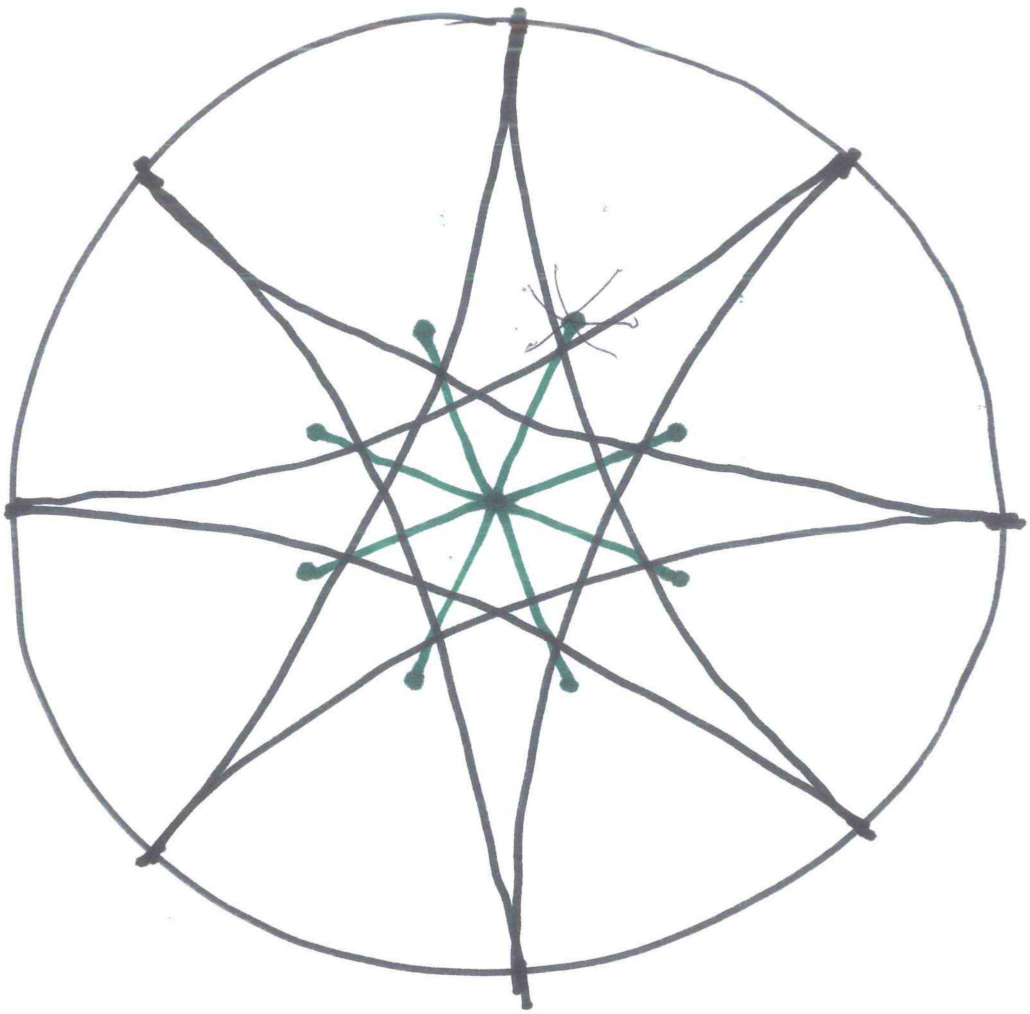
$$\limsup \frac{\log \|\mathrm{proj}_2(s_n)\|}{\log n} = \frac{1}{3}$$

and there is a well defined direction of deviation.

Thm. (S-Ulcigrai) The direction of deviation is a discontinuous function of the direction of the trajectory.



(a) The tree dual to the tesselation.



Galois tesselation is non-discrete.

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dualpic[{1, 2, 3, 2, 4, 2}]
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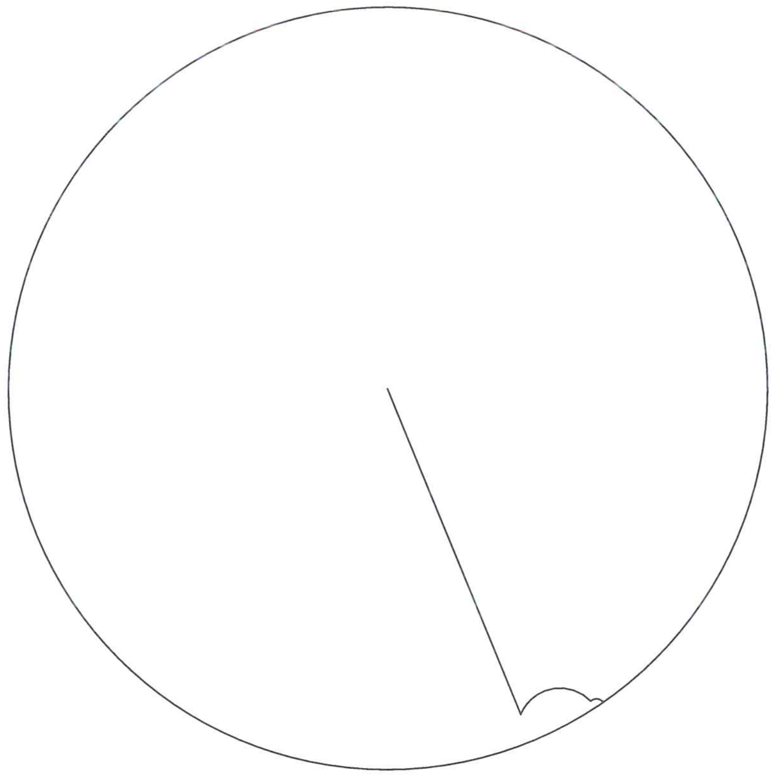
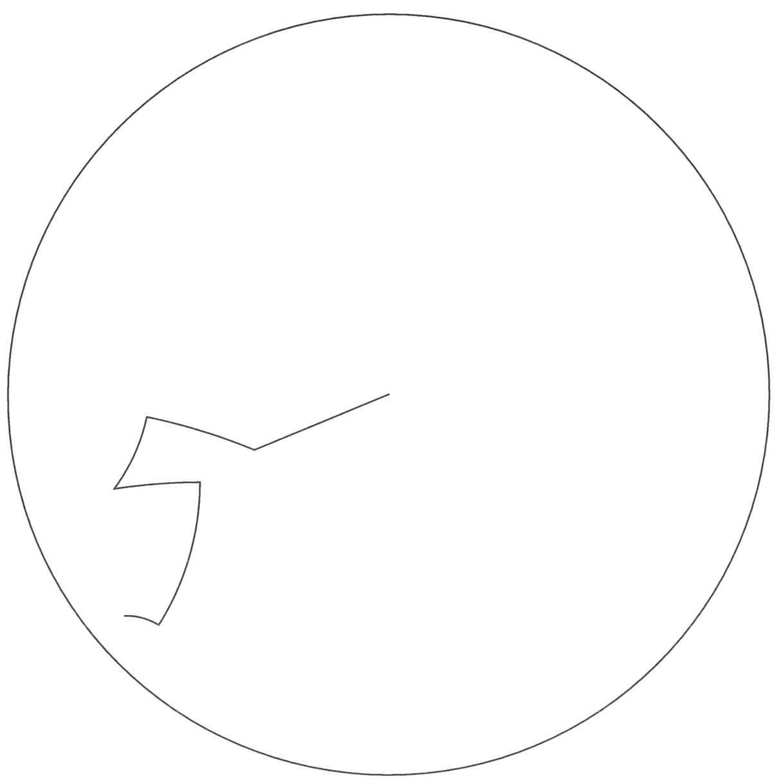
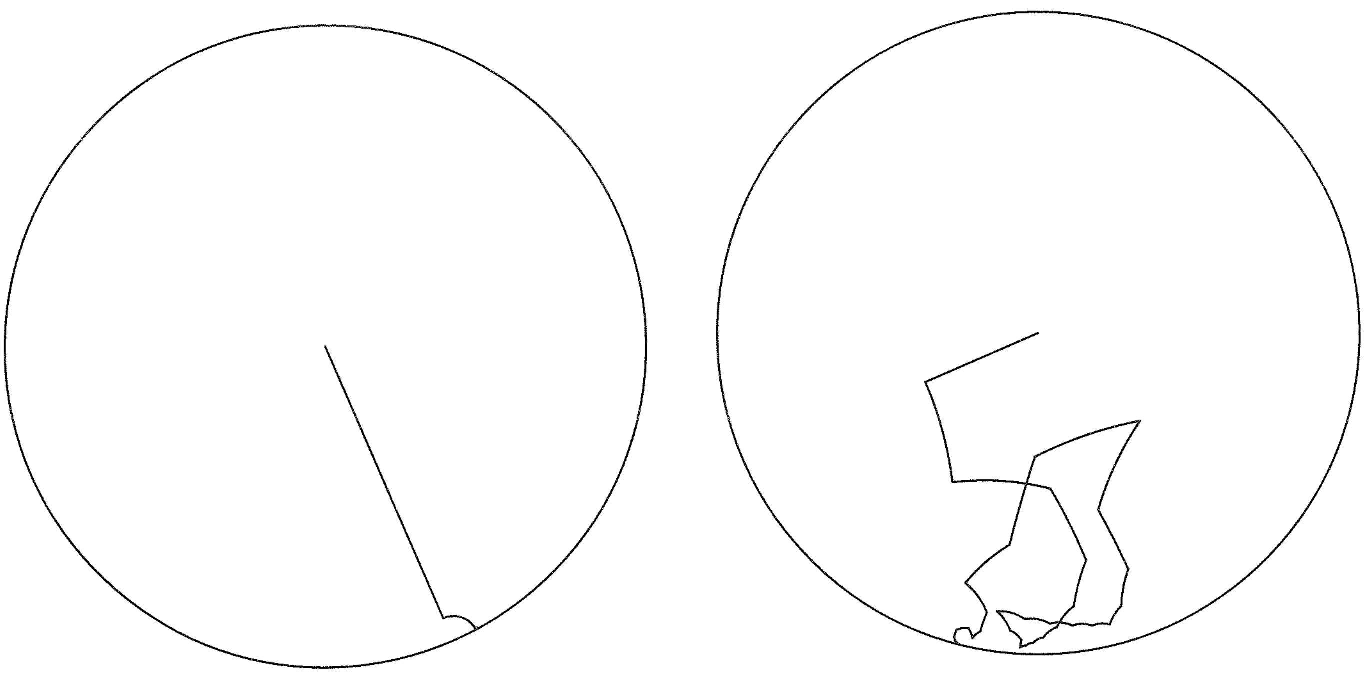
{  ,

 }



68.2977