Dynamics of Foliations by Riemann Surfaces.

Example

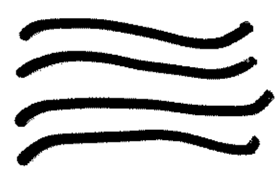
In $\mathbb{C}^k$

$$z_j'(\zeta) = f_j(z(\zeta)) \qquad 1 \le j \le k$$

f_j polynomials.

Singular pts in $\mathbb{C}^k$ $(f = 0)$

Locally away from sing.



The foliation can be extended to $\mathbb{P}^k$.

It is the image of the foliation of $\mathbb{C}^{k+1}$ given by

$$F(z) = \sum_{j=0}^{k} F_j(z) \frac{\partial}{\partial z_j}, \quad \deg F_i = d$$

F_i homogeneous polynomial.

4) Foliations are parametrized

$F = [F_0 : \dots : F_k]$. $\deg F_j = d \geq 2$

Assume $\bigcap F_j^{-1}(0) = 0$.

Singular points

$F[p] = [p]$

$\# \mathrm{Sing}\, \mathcal{F} = \dfrac{d^{k+1} - 1}{d - 1}$.

[redacted]. Can also consider atlas.

$(z,t) \to (z',t') = (h(z,t), t'(t))$.

Fol$_d$ open Zariski set in some $\mathbb{P}^N$

Always sing pt.

$DZ(0) = \{\lambda_1, \dots, \lambda_k\}$.

Sing hyperbolic if $\lambda_i / \lambda_j \notin \mathbb{R}$ $\forall (i,j)$

Chaperon.

3) Global behavior of leaves ?

Ilyashenko. Hudai-Veronov.
<u>when Line at infinity invariant in $\mathbb{P}^2$</u>

generically leaves are dense

JOUANOLOU.
Generic foliations in $\mathbb{P}^k$ have no algebraic leaves.

$H^d_{yp}(\mathbb{P}^2)$. Foliations with only hyperbolic singularities and without algebraic leaves is a real Zariski dense set in $\mathcal{F}ol_d(\mathbb{P}^2)$.

So analysis cannot start at an alg. leave.

4)

Leaves ?

Lins-Neto, Glutsyuk

If the singularities are non degenerate, Then leaves are covered by unit disc.

$$\begin{array}{c} \Delta \\ \downarrow \phi \\ L \end{array}$$

Gomez-Mont, Candel

If singularities are hyperbolic Poincaré metric is continuous.

$\omega_p(z)$

$$\eta(z) = \frac{\|v\|}{\|v\|_p} \longrightarrow 0 \text{ at sing.}$$

5) Discrete dyn. syst
Invariant measure

$\downarrow$

Foliations or Laminations
?

measure invariant
by holonomy.

$$T = \int_B [V_\alpha]\, d\mu(\alpha).$$

μ inv. by holonomy $\Leftrightarrow$ T closed

Sullivan..

In general they don't
exist.

Harmonic measures

$T = \int [V_\alpha] h_\alpha \, d\mu(\alpha)$ $\partial\bar{\partial}$-closed current.

$m_\beta = T \wedge \beta$.

$h_\alpha > 0$ harmonic on V_α.

Lucy Garnett. in Riemannian case (without sing.).

sets of μ measure zero inv. by sliding.

$(X, E, \mathcal{L})$ E closed set. "small"

$X \setminus E$ Lamination by Riemann Surfaces.

Then there is a $(1,1) \geq 0$ current $\partial\bar{\partial}$ closed, directed by Lamination.

Berndtsson - S.

$\int T \wedge T$. $\qquad \int T \wedge_g T$

Main pt

$$\int T \wedge T = 0 \quad \text{if } T \geq 0$$

T direct. by $\mathcal{F}$.

$i\partial\bar\partial T = 0$.

Hodge - Riemann. But

cohomology space Hilbert-space

7)

what happens in arbitrary dim?

Birkhoff Thm. for foliations?

... Garnett. Diffusion with heat equation requires bounded geometry of leaves.

we have sing.

Dinh, VA Nguyen, S.

- Laminations by R.S. transversally $\mathcal{C}^{2+\delta}$ in a complex manifold M.
- Holomorphic foliations with finite number of sing in M.
- Abstract lamination. ex: suspensions.

Poincaré metric ω_P.

measurable.

$T \geq 0$ $\partial\bar\partial T = 0$ directed by Foliation.

<u>Thm</u>. $\mathcal{F}$ foliation with finite number of linearizable sing.

Then:

$$\omega_P \sim \frac{1}{\|z\|^2 (\log \|z\|)^2}$$

near sing $(z=0)$

Then.

$$\int T \wedge \omega_P < \infty$$

If $\partial T = \tau \wedge T$

$\mu_T = i\, \tau \wedge \bar{\tau} \wedge T \qquad \int \mu_T < \infty$

$$\partial T = \int \frac{\partial h_\alpha}{h_\alpha} \wedge h_\alpha [V_\alpha]\, d\mu(\alpha)$$

Th. $(X, \mathcal{L}, E)$ compact Laminat.
Sing. E. in a complex manifold M.

T extremal positive current direct by $\mathcal{L}$. $\partial\bar\partial$-closed.
without mass on Parabolic leaves.

$$\phi_a(0) = a$$

$$m_P = T \wedge \omega_P$$

Define $m_{a,R} = \frac{1}{M_R} (\phi_a)_* \left(\log^+ \frac{r}{|\zeta|} \, \omega_P \right)$

$(M_R \sim R)$ Then

$$m_{a,R} \to m_P \qquad m_P\text{. a.e.}$$

We can use another metric

$\omega_P \rightsquigarrow$ metric. ω

$$\frac{1}{c_r} (\phi_a)_* \left(\log^+ \frac{r}{|\zeta|} \right) \longrightarrow \frac{T}{\|T\|} \quad T\text{.a.e.}$$

Cog

$$\int_{rD} |\phi_a'(z)|^2 \log^+ \frac{r}{|z|} \, i dz \wedge d\bar z \underset{r \to 1.}{\simeq} \log \frac{1}{1-r}$$

$$B_R u(a) = \frac{1}{H_R} \int \log^+ \frac{r}{|s|} (\Phi_a)_* (u\,\omega_p)$$

$$= \langle m_{a,R} , u \rangle$$

<u>Lemma</u>

$u \in L^1(m_p)$ implies

$$\int (B_R u)\, T \wedge \omega_p = \int u\, T \wedge \omega_p .$$

Fubini !

L^p ergodicity.

$$(\Delta_p v)\, \omega_p := i \partial \bar\partial v$$

$u = \Delta_p v$

$$B_R u(a) = B_R (\Delta_p v)(a) = \langle m_{a,R} , (\Delta_p v)\omega_p \rangle$$

$$= \langle i \partial \bar\partial T_{a,R} , v \rangle \longrightarrow 0 .$$

9)

Pointwise ergodic thm.

$$\widetilde{B}u(a) := \overline{\lim_{R\to\infty}} \, |B_R u(a)|.$$

<u>Lemma</u> If $u \in L^1(m_p)$.

$$m_p\{\widetilde{B}u > \varepsilon\} \leq c\,\varepsilon^{-1} \, \|u\|_{L^1(m_p)}.$$

Then comparison.

$$\left| B_R u(a) - \frac{2\pi}{M_R} \int_0^{\frac{M_R}{2\pi}} S_t u(a)\, dt \right| \leq C R^{-1/2} \sqrt{\log R}$$

Heat equation for $T \geq 0$ $(1,1)$ $i\partial\bar{\partial}T = 0$
directed by a Lamination (singular).

[illegible]

$$T = \int h_\alpha [V_\alpha] \, d\mu(\alpha)$$

$$\partial T = \tau \wedge T$$

Choose a $(1,1)$ form $\beta \geq 0$.
(ex ω_P, $i\tau \wedge \bar{\tau}$)

$$\int T \wedge \beta < \infty .$$

Dirichlet Space w. respect to T.

$$\|u\|^2 = \int |u|^2 \, T \wedge \beta + \int i\partial u \wedge \bar{\partial} u \wedge T .$$

on smooth functions and complete.

<u>Def:</u> $(\Delta_\beta u) \, T \wedge \beta := i\partial\bar{\partial} u \wedge T .$

$$\tilde{\Delta}_\beta := \Delta_\beta + \tfrac{1}{2} \nabla_\beta$$

$$(\nabla_\beta u) \, T \wedge \beta = i(\partial u \wedge \tau - \bar{\partial} u \wedge \bar{\tau}) \wedge T .$$

9) $m_\beta := T \wedge \beta$ harmonic measure.

Th: $-\widetilde{\Delta}_\beta$ (m_β $-\Delta_\beta$) are max. monotone on $L^2(m_\beta)$.

Then (Hille Yosida) they generate a semi-groupe of contractions on $L^2(m_\beta)$. $S(t)$

m_β is $S(t)$ invariant.

contract. in $L^p(m_\beta)$.

$i\partial\bar\partial u^2 = 2iu\,\partial\bar\partial u + 2i\,\partial u \wedge \bar\partial u$.

Solutions are only in L^2.

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \Delta_\beta u(x,t) & \text{leavewise} \\ u(x,0) = u_0. \end{cases}$$

Pr. $S(t)u \geq 0$ when $u \geq 0$

(Stampacchia's trick).

<u>Corollary</u>

$$\frac{1}{R}\int_0^R S(t)u_0\,dt \xrightarrow[\text{a.e.}]{} u_0^*$$

and in $L^p(m_\beta)$.

<u>Th</u> If T extemal (mixing) $-\widetilde{\Delta}_\beta$

$S(t)u_0 \to \langle u_0, T\wedge\omega_\beta\rangle$

9) m_β a.e. and in $L^2(m_\beta)$.

Poincaré Metric
Entropy.

- $\mathcal{C}^{2+\delta}$ lamination in complex manifold M.
- Foliation with linearizable Singularities.

Leaves covered by unit disc.

Entropy.
Quantify how leaves get apart.

For general smooth lamination without sing.
Ghys - Langevin - Walczak.
Finiteness of entropy
$\approx$ lipschitz maps.

Entropy.

(X, d) metric space totally bounded. $Y \subset X$

$\mathcal{D} = \{d_t\} \nearrow$ metrics. $\left[d_n(x,y) := \sup_{j \le n} d(f^j(x), f^j(y)) \right]$

$N(Y, t, \varepsilon)$: Min number of balls $B_t(x, \varepsilon)$ necessary to cover Y.

$$h_{\mathcal{D}}(Y) = \overline{\lim}\ \frac{1}{t} \log N(Y, t, \varepsilon).$$

<u>Ex 1</u> $M \xrightarrow{f} M$ meromorphic

M comp. Kähler.

$X := M \setminus f^{-n}(I)$

$X \xrightarrow{f} X$

f has finite entropy (T. C. Dinh – S.)

Calculus on currents.

9)

$(X, \mathcal{L}, E)$ hyperbolic.

$\tau_a : D \to L$

$\tau_a(0) = a$.

$d_R(a,b) := \inf_{\theta \in \mathbb{R}} \sup_{\zeta \in D_R} d(\tau_a(e^{i\theta}\zeta), \tau_b(\zeta))$.

R- hyperbolic time.

(R, ε) (a, b) close

Leave through a, b

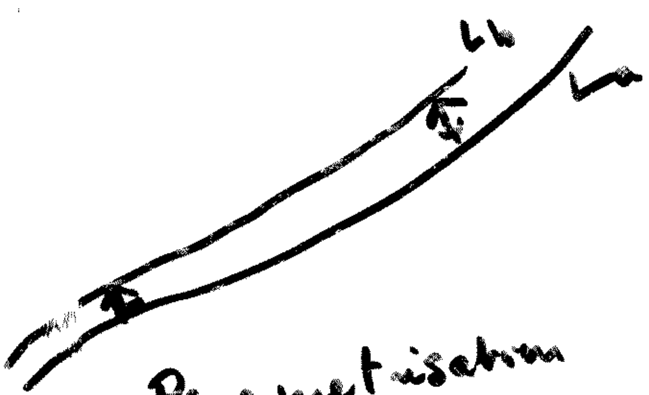
ε- close $\longrightarrow$ time R.

Thm $\mathcal{C}^{2+\delta}$ laminat. (without sing)

Poincaré metric is Hölder cont.

entropy is finite.

$$d_R \leq A^R d_0 + c e^{-R}.$$



Parametrisation

connection Laplace Beltrami

estimates.

9)

M compact. complex.

$\mathcal{F}$ foliation linearizable sing.

Precise modulus of continuity

for $\eta = \frac{\|v\|_e}{\|v\|_P}$ Poinc. Metric.

$$\sim \frac{1}{|\log \operatorname{dist}(x, L)| + 1}$$

"Thm" $\mathcal{F}$ foliat. Lin sing.

Entropy is finite.

Difficulty (R, ε) close near sing.

$r \sim \varepsilon e^{-e^R}$.

but situation inhomogeneous

dim $M = 2$. O.K Fol. transversally

conformal holonomy conformal.

9)

$(X, \mathcal{L}, E)$ Lamination
by R.S in a complex manifold
M. E finite set.

$$\Delta \xrightarrow{\phi} L$$

$$T_r^L := \frac{1}{c_r} \int_0^r \frac{ds}{s} \phi_*(\Delta_s) = \frac{1}{c_r} \phi_*\left(\log^+ \frac{r}{|s|}\right)$$

Fornæss, S:
cluster points are positif
directed $\partial\bar\partial$- closed currents
(1,1) T_r is like a Birkhoff
sum on the "orbit".

<u>Th</u> (Fornæss, S) $\mathcal{F}$ foliation in $\mathbb{P}^2$
without algebraic leaves and
only hyperbolic singularities.
Then there is a <u>unic</u> $T \geq 0$ $\partial\bar\partial$-closed
directed current of mass 1.
In particular
$T_r^L \to T$ independently of
L.

Dim 2. Sing. Intersection theory
?) for $\partial\bar\partial$-closed currents.