Boundary Behavior of Thurston's Pullback Map and Pilgrim's conjecture

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Banff, February 2011

Notations

- $f\colon \mathbb{S}^2 \to \mathbb{S}^2$ is an orientation-preserving branched self-cover of $\mathbb{S}^2$ of degree $d_f \geq 2$
- Ω_f is the set of critical points of f
- P_f is the postcritical set: $P_f = \cup_{i\geq 1} f^i(\Omega_f)$.

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Definition

A branched cover f is called *postcritially finite* or a *Thurston map* if P_f is finite. Denote $p_f = \#P_f$.

Thurston equivalence

Definition

Two Thurston maps f and g are Thurston equivalent if and only if there exist two homeomorphisms $h_1, h_2 : \mathbb{S}^2 \to \mathbb{S}^2$ such that the diagram

$$\begin{tikzcd}
(\mathbb{S}^2, P_f) \arrow[r, "h_1"] \arrow[d, "f"'] & (\mathbb{S}^2, P_g) \arrow[d, "g"'] \\
(\mathbb{S}^2, P_f) \arrow[r, "h_2"] & (\mathbb{S}^2, P_g)
\end{tikzcd}$$

commutes, $h_1|_{P_f} = h_2|_{P_f}$, and h_1 and h_2 are homotopic relative to P_f.

Theorem (Thurston's Theorem)

A postcritically finite branched cover $f : \mathbb{S}^2 \to \mathbb{S}^2$ with hyperbolic orbifold is either Thurston-equivalent to a rational map g (which is then necessarily unique up to conjugation by a Möbius transformation), or f has a Thurston obstruction.

Teichmüller space

Some further notations

- $\mathcal{T}_f$ is the Teichmüller space modeled on the marked surface $(\mathbb{S}^2, P_f)$
- $\mathcal{M}_f$ is the corresponding moduli space
- Recall that $\mathcal{T}_f$ can be defined as the quotient of the space of all diffeomorphisms from $(\mathbb{S}^2, P_f)$ to the Riemann sphere. We write $\tau = \langle h \rangle$ if a point τ is represented by a homeomorphism h
- $Q(\mathbb{P}, h(P_f))$ is the cotangent space at a point $\tau = \langle h \rangle$ in the Teichmüller space T_f which is canonically isomorphic to the space of all integrable meromorphic quadratic differentials on the marked Riemann surface corresponding to τ

Thurston's iteration

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Definition of Thurston's iteration

$$\begin{array}{ccc} (\mathbb{S}^2, P_f) & \xrightarrow{h_1} & (\mathbb{P}, h_1(P_f)) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_\tau} \\ (\mathbb{S}^2, P_f) & \xrightarrow{h_\tau} & (\mathbb{P}, h_\tau(P_f)) \end{array}$$

Fixed Points of σ_f

Definition of Thurston's iteration

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\end{tikzcd}
$$

Lemma

A Thurston map f is equivalent to a rational function if and only if σ_f has a fixed point.

Definition

The push-forward operator is locally defined by the formula

$$g_* q|_U = \sum_i g_i^* q,$$

where g_i are all inverse branches of g.

Co-derivative of σ_f

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Lemma

σ_f is a holomorphic self-map of $\mathcal{T}_f$ and the co-derivative of σ_f satisfies $(d\sigma_f(\tau))^* = (f_\tau)_*$ where $(f_\tau)_*$ is the push-forward operator on quadratic differentials.

Metric definitions

For a meromorphic integrable quadratic differential on $\mathbb{P}$ we define

- its Teichmüller norm

$$\|q\|_T = 2\int_{\mathbb{P}} |q|$$

  and
- its Weil-Petersson norm

$$\|q\|_{WP} = \left(\int_{\mathbb{P}} \rho^{-2}|q|^2\right)^{1/2}$$

Estimates on the norm of $d\sigma_f^*$

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Proof.

$$\int_U |g_* q| = \int_U |\sum_i g_i^* q| \leq \sum_i \int_U |g_i^* q| = \sum_i \int_{U_i} |q|$$

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Corollary (almost)

There exists at most one fixed point of σ_f, hence the uniqueness in Thurston's theorem follows.

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$$= d \sum_i \int_{U_i} \frac{|q|^2}{g^* \rho^2} \leq d \int_{g^{-1}(U)} \frac{|q|^2}{\rho_1^2},$$

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Corollary

σ_f is Lipschitz with respect to the WP-metric.

Invariant multicurves

Yet some more definitions:

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- A multicurve Γ is *completely invariant* if $f^{-1}(\Gamma) = \Gamma$

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- Each point in the stratum $\mathcal{S}_\Gamma$ is a collection of complex structures on the components of the corresponding topological noded surface with marked points.
- $\mathcal{S}_\Gamma$ is the product of Teichmüller spaces of these components.
- Within each stratum one can define its own natural Teichmüller and Weil-Petersson metrics.
- The quotient $\overline{\mathcal{M}}_f$ of $\overline{\mathcal{T}}_f$ by the action of the mapping class group is compact.

Theorem (Masur)

The augmented Teichmüller space $\overline{\mathcal{T}}_f$ is homeomorphic to the WP-completion of the Teichmüller space.

Extension to the boundary of Teichmüller space

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Corollary

σ_f *extends continuously to* $\overline{\mathcal{T}}_f$.

Definition of σ_f on the boundary

We represent points in $\overline{\mathcal{T}}_f$ not only by homeomorphisms but also by continuous maps from $(\mathbb{S}^2, P_f)$ to a noded Riemann surface that are allowed to send a whole simple closed curve to a node. Consider such an h representing some point in $\overline{\mathcal{T}}_f$.

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We complete this diagram as before

$$\begin{tikzcd}
(\mathbb{S}^2, P_f) \arrow[r, "h_1"] \arrow[d, "f"'] & (R_1, P_f) \arrow[d, "\{f^{C_i}\}"'] \\
(\mathbb{S}^2, P_f) \arrow[r, "h"] & (R, P_f)
\end{tikzcd}$$

Action of σ_f on $\overline{\mathcal{T}}_f$

Theorem

The map σ_f as defined above is continuous on $\overline{\mathcal{T}}_f$.

Remark.

Note that by definition σ_f maps any stratum $\mathcal{S}_\Gamma$ into the stratum $\mathcal{S}_{f^{-1}(\Gamma)}$, therefore invariant boundary strata are in one-to-one correspondence with completely invariant multicurves.

Thurston matrix and obstructions

Definition

Every invariant multicurve Γ has its associated *Thurston matrix* $M_\Gamma = (m_{i,j})$ with

$$m_{i,j} = \sum_{\gamma_{i,j,k}} (\deg f|_{\gamma_{i,j,k}} : \gamma_{i,j,k} \to \gamma_j)^{-1}$$

where $\gamma_{i,j,k}$ ranges through all preimages of γ_j that are homotopic to γ_i.

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Definition

Since all entries of M_Γ are non-negative real, the leading eigenvalue λ_Γ of M_Γ is real and non-negative. The multicurve Γ is a *Thurston obstruction* if $\lambda_\Gamma \geq 1$.

Lemma

If $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ *is a completely invariant simple obstruction, then* $\mathcal{S}_\Gamma$ *is weakly attracting.*

Classification of invariant boundary strata

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Lemma

If $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ *is a completely invariant multicurve and* $\lambda_\Gamma < 1$*, then* $\mathcal{S}_\Gamma$ *is weakly repelling.*

Pick any starting point $\tau \in \mathcal{T}_f$ and consider $\tau_n = \sigma_f^n(\tau)$. Take an accumulation point in $\overline{\mathcal{M}}_f$ of the projection of τ_n to the moduli space on the stratum of smallest possible dimension.
For simplicity we assume that τ_n accumulates on some $\tau_0 \in \mathcal{S}_\Gamma$.

Sketch of the proof of Thurston's theorem

Pick any starting point $\tau \in \mathcal{T}_f$ and consider $\tau_n = \sigma_f^n(\tau)$. Take an accumulation point in $\overline{\mathcal{M}}_f$ of the projection of τ_n to the moduli space on the stratum of smallest possible dimension.
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- If $\Gamma = \emptyset$ then τ_0 is a fixed point of σ_f.

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For simplicity we assume that τ_n accumulates on some $\tau_0 \in \mathcal{S}_\Gamma$.

- If $\Gamma = \emptyset$ then τ_0 is a fixed point of σ_f.
- If $\Gamma \neq \emptyset$ then Γ must be a Thurston obstruction. Otherwise, $\mathcal{S}_\Gamma$ is weakly repelling and therefore τ_n can not have an accumulation point there.

Definition

The *canonical* obstruction Γ_f is the set of all homotopy classes of curves γ that satisfy $l(\gamma, \sigma_f^n(\tau)) \to 0$ for all $\tau \in \mathcal{T}_f$

Pilgrim's theorems

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Theorem (Canonical Obstruction Theorem)

If for a Thurston map with hyperbolic orbifold its canonical obstruction is empty then it is Thurston equivalent to a rational function. If the canonical obstruction is not empty then it is a Thurston obstruction.

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For any point $\tau \in \mathcal{T}_f$ there exists a bound $L = L(\tau, f) > 0$ such that for any essential simple closed curve $\gamma \notin \Gamma_f$ the inequality $l(\gamma, \sigma_f^n(\tau)) \geq L$ holds for all n.

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Pilgrim's conjecture

Recall that the action on any invariant stratum is given by pullbacks of complex structures by a collection of maps σ_{fC} for all components C of any surface in the stratum. Combinatorics of the process is very simple: we have a map from a finite set into itself, every component is pre-periodic. The whole action, therefore, can be characterized by studying cycles of components. For each cycle Y there are three cases, the composition f^Y of all coverings in the cycle is either of the following:

- a homeomorphism,
- a Thurston map with a parabolic orbifold,
- a Thurston map with a hyperbolic orbifold.

Pilgrim's conjecture

Theorem

If a cycle Y of components a topological surface corresponding to the stratum $\mathcal{S}_{\Gamma_f}$ has hyperbolic orbifold then f^Y is not obstructed and, hence, equivalent to a rational map.

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Idea of the proof

We may assume that our component is mapped to itself. Take an accumulation point $\tau_0 \in \mathcal{S}_{\Gamma_f}$ of $\sigma_f^n(\tau)$. Let τ' be the coordinate corresponding to Y. Let Γ_Y be the canonical obstruction for f^Y.

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Then the accumulation set of $\sigma_{f^Y}^n(\tau')$ must be a subset of the closure of $\mathcal{S}_{\Gamma_f \cup \Gamma_Y}$. On the other hand, it is clearly a subset of the accumulation set of $\sigma_f^n(\tau) \subset \mathcal{S}_{\Gamma_f}$. This means Γ_Y must be empty.