

The Surface Subgroup Theorem

and

The Ehrenpreis Conjecture

by Jeremy Kahn

Joint with Vladimir Markovic

Theorem (KM) (Ehrenpreis conjecture)

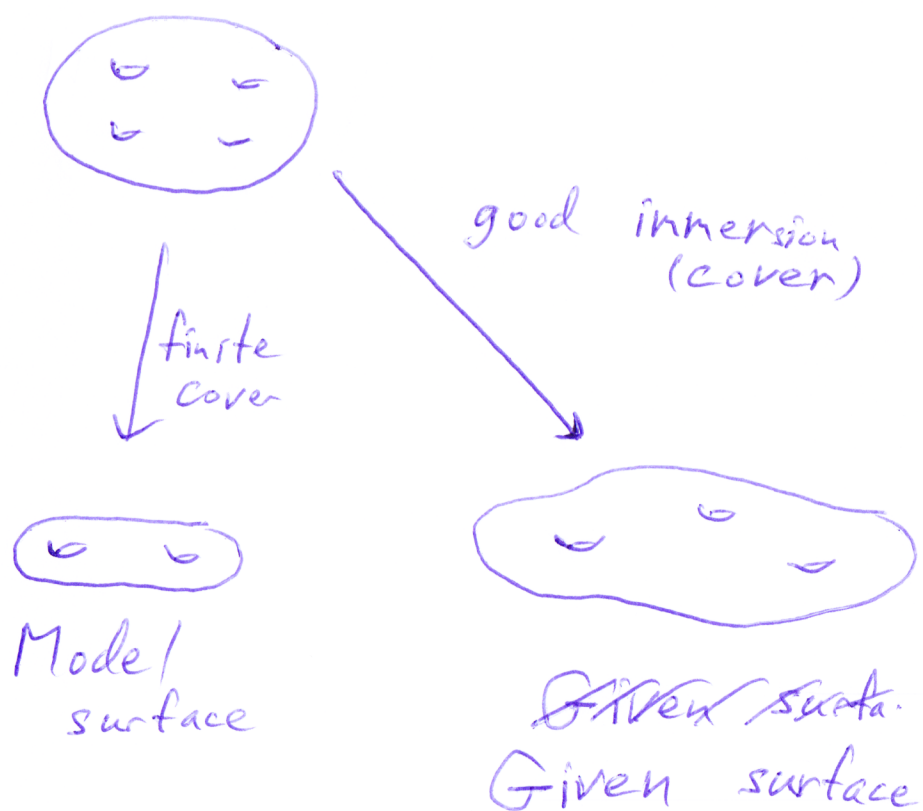
Let S and T be closed
hyperbolic Riemann surfaces. Then

S and T have nearby finite
covers (in the Teichmüller metric).

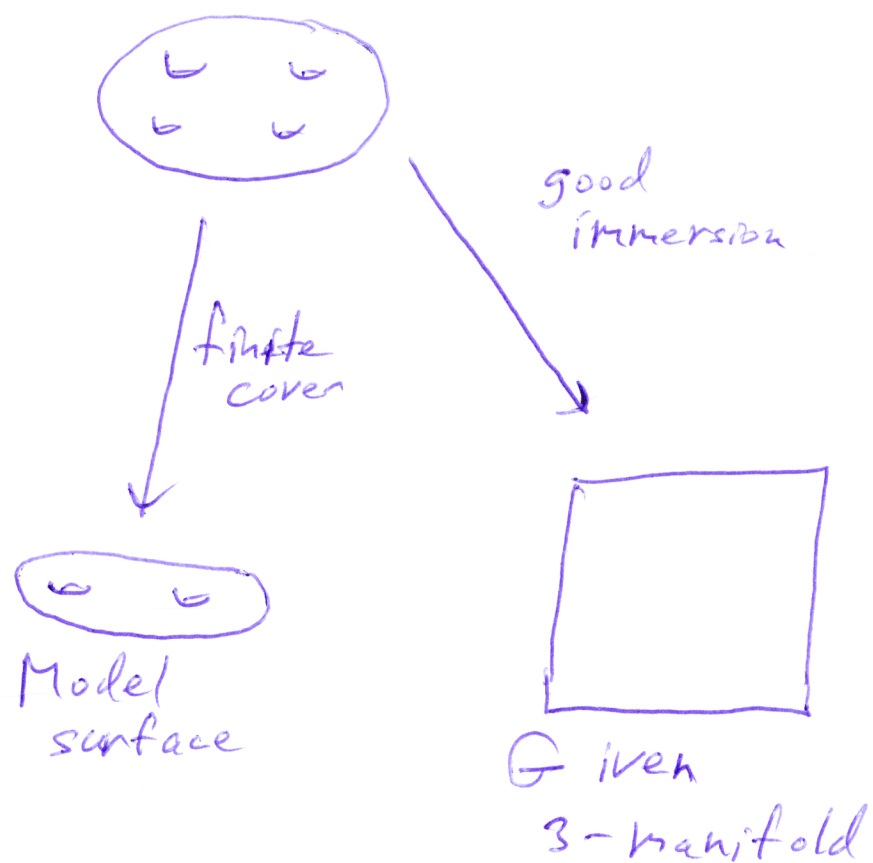
Theorem (KM) (Surface subgroup conjecture)

Let M be a closed hyperbolic 3-manifold. Then $\pi_1(M)$ has a surface subgroup (the isomorphic image of π_1 of a hyperbolic surface).

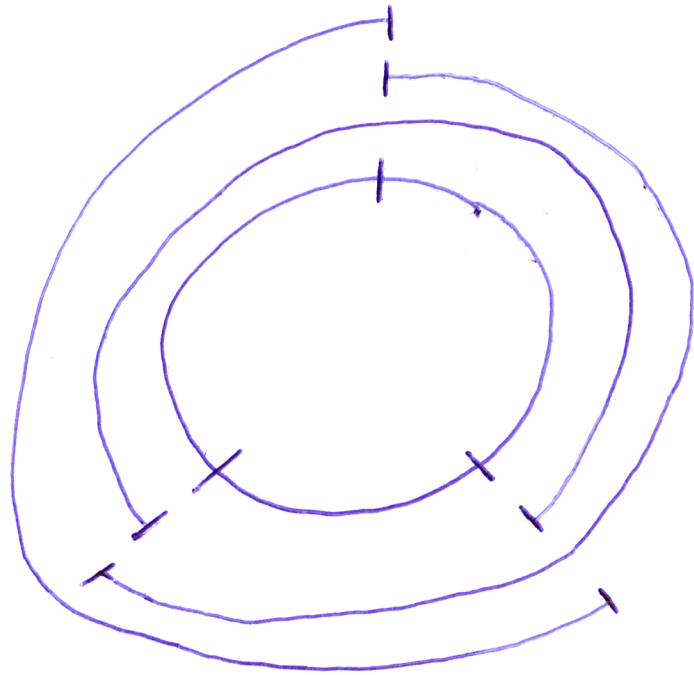
2D



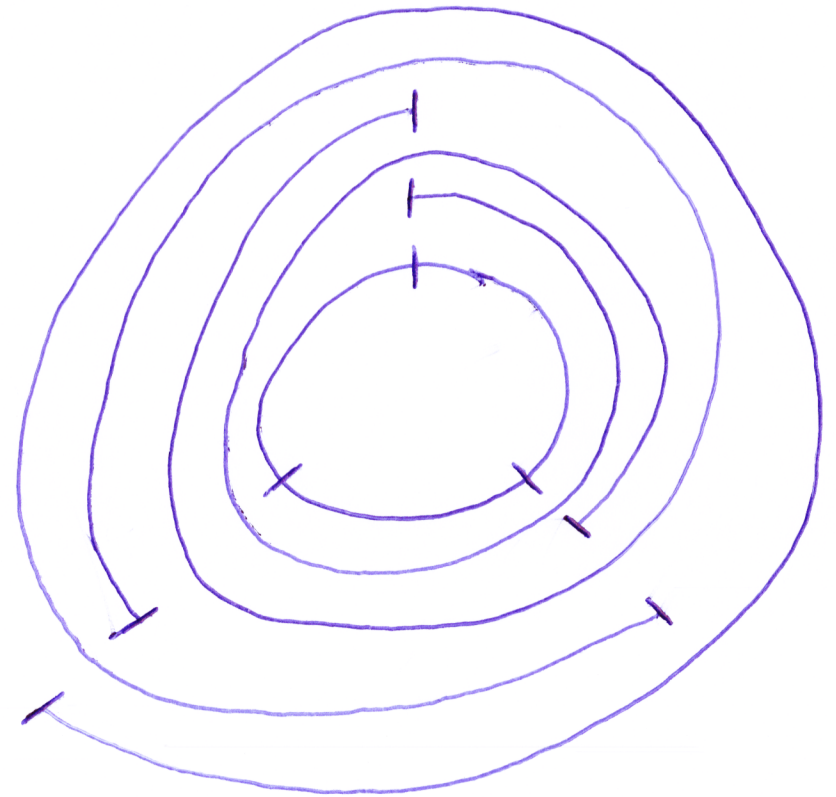
3D



Building Covers of the Circle



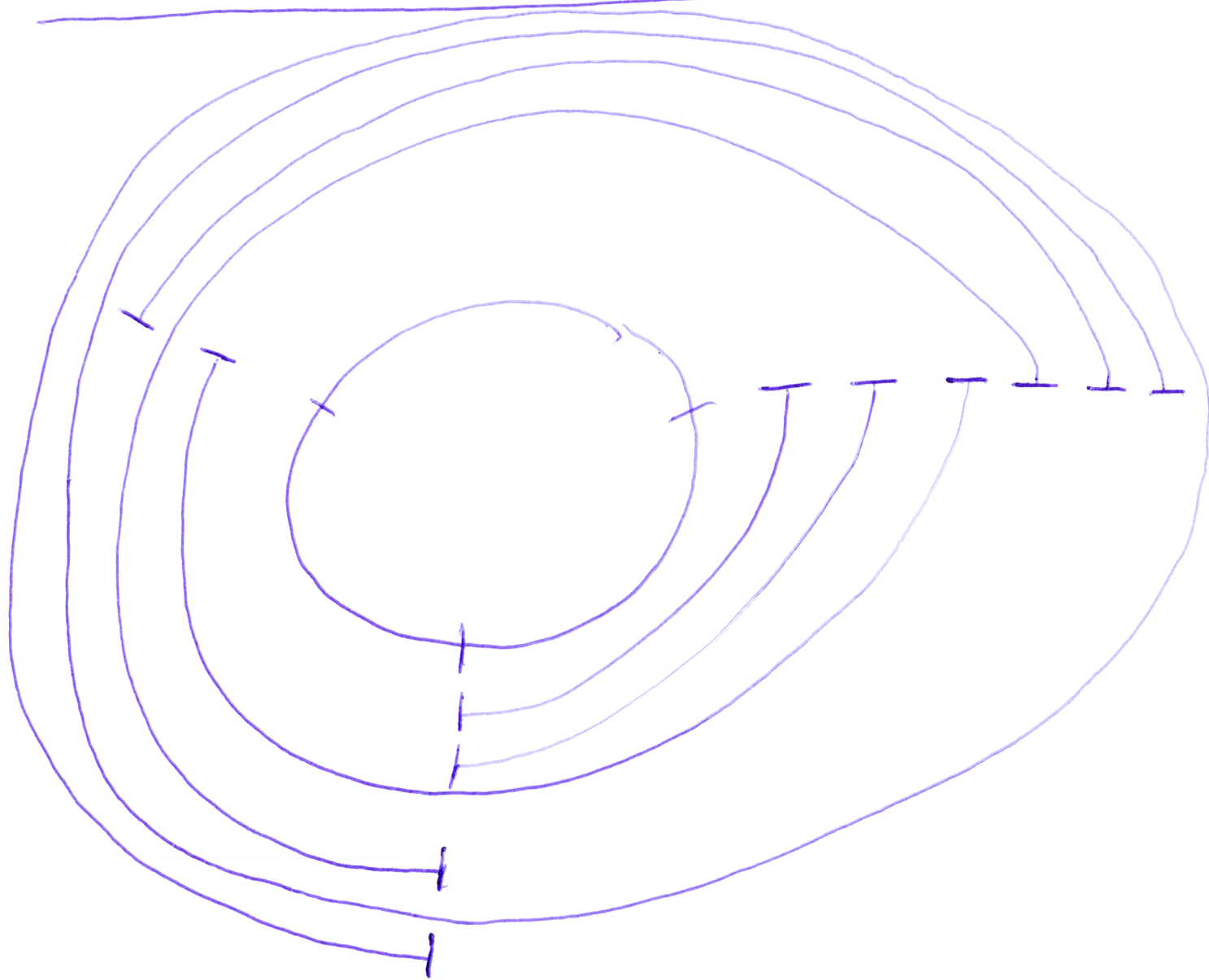
degree 2

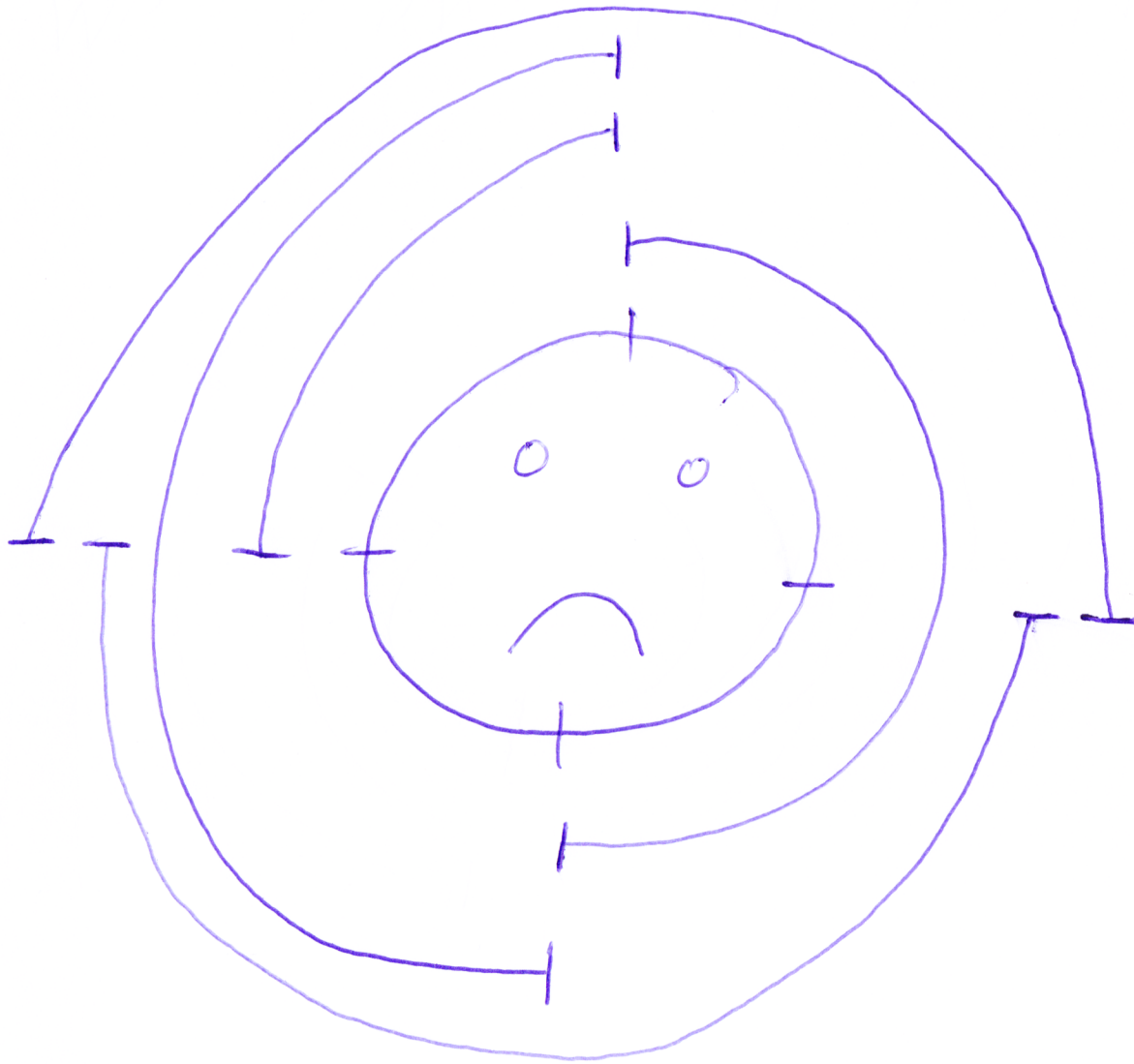


degree 4

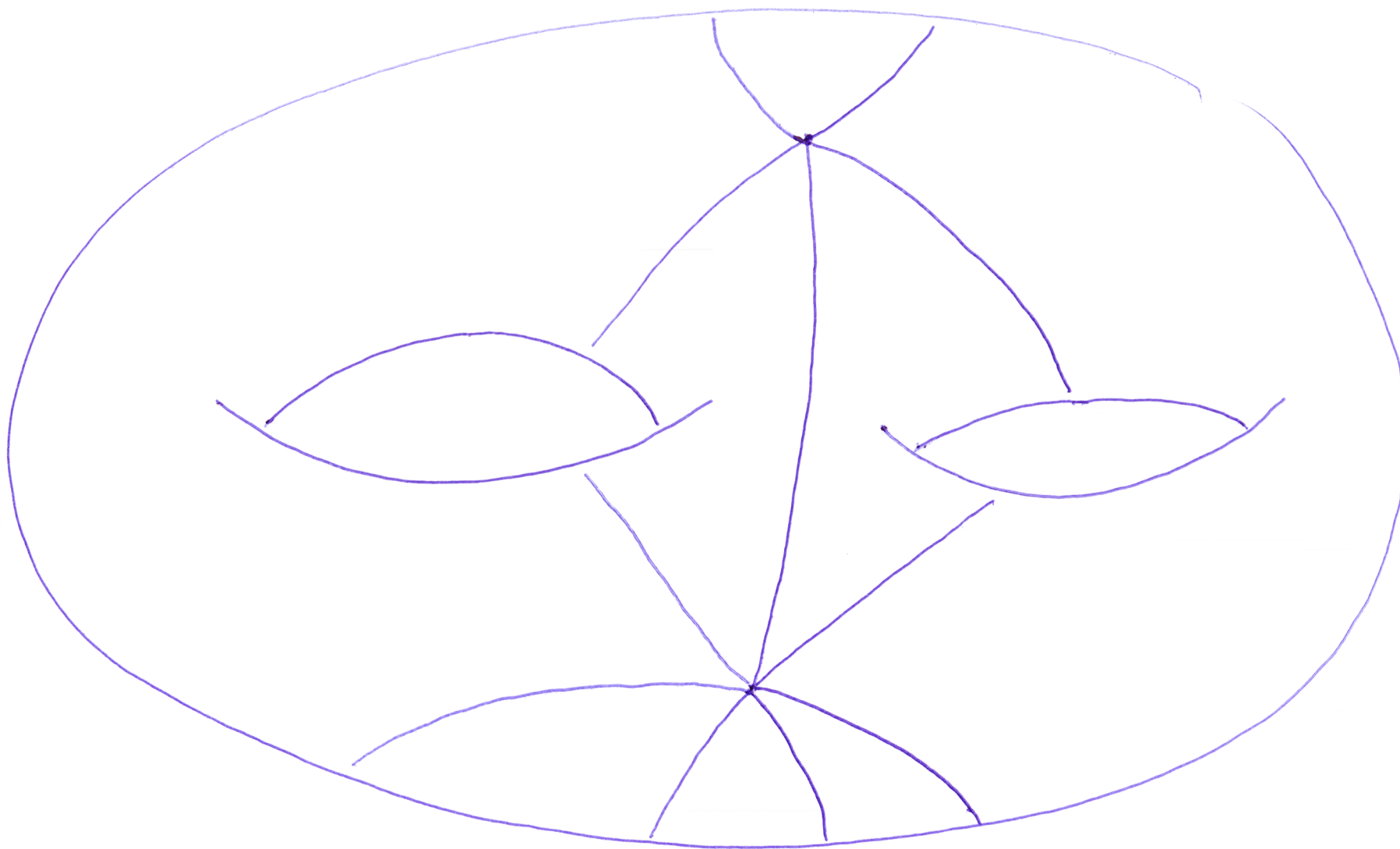
We can join immersed
1-manifolds with boundary
to form a cover of a
closed 1-manifold (a circle).

We can build more than one cover

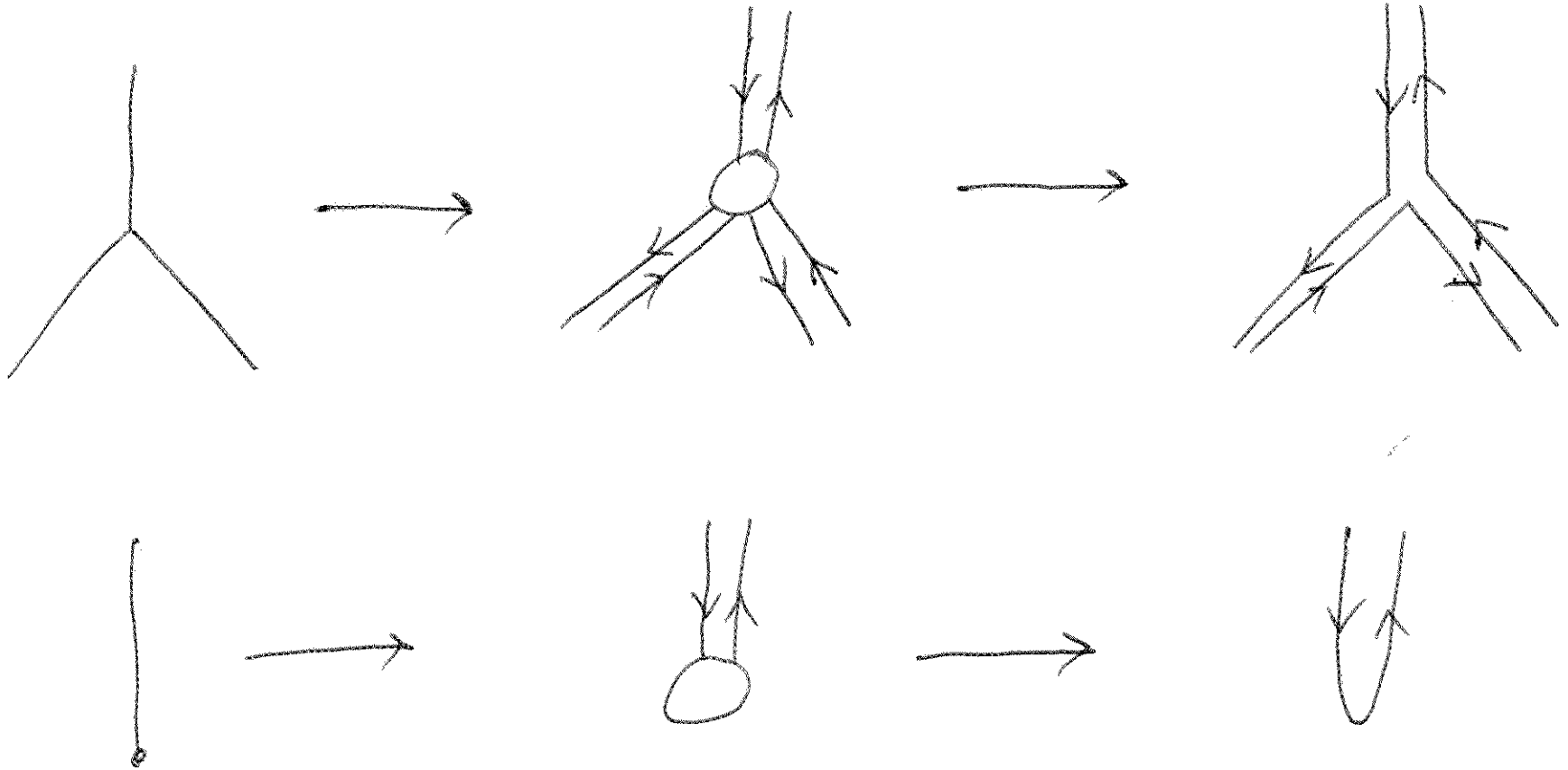


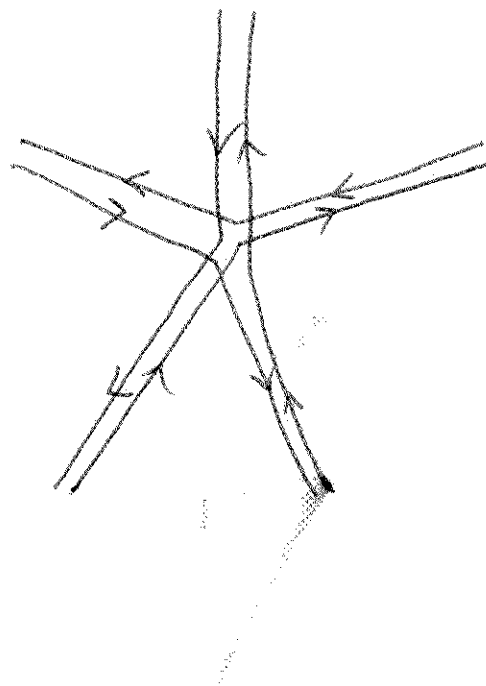
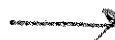
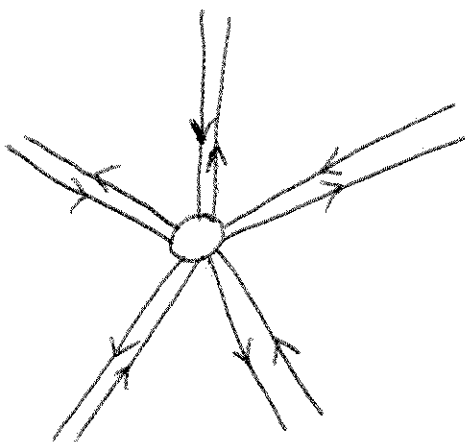
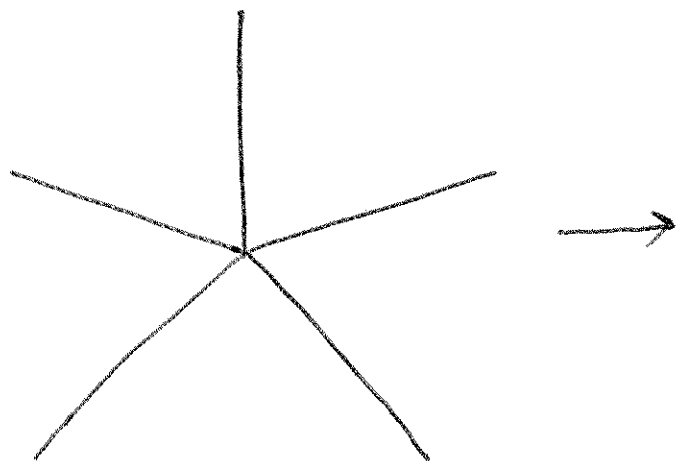
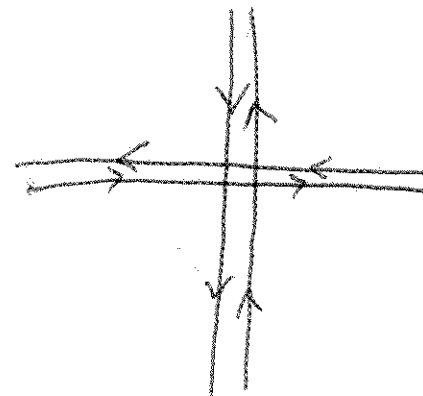
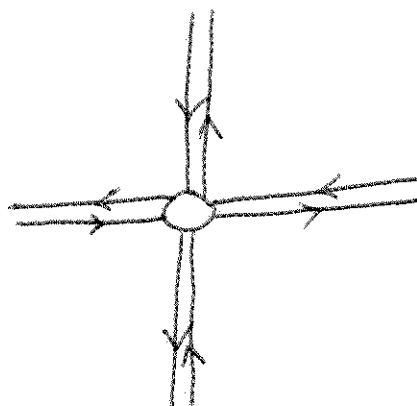
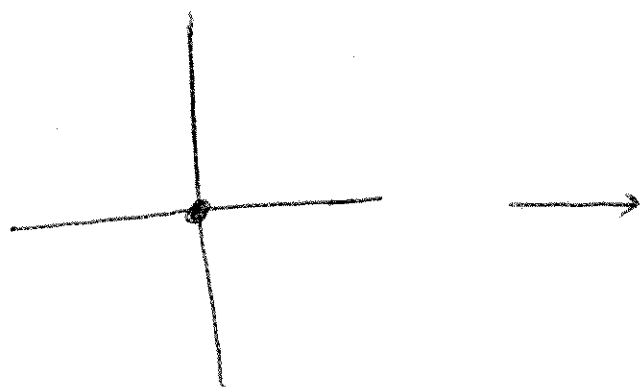


We can
fail to
build a
cover

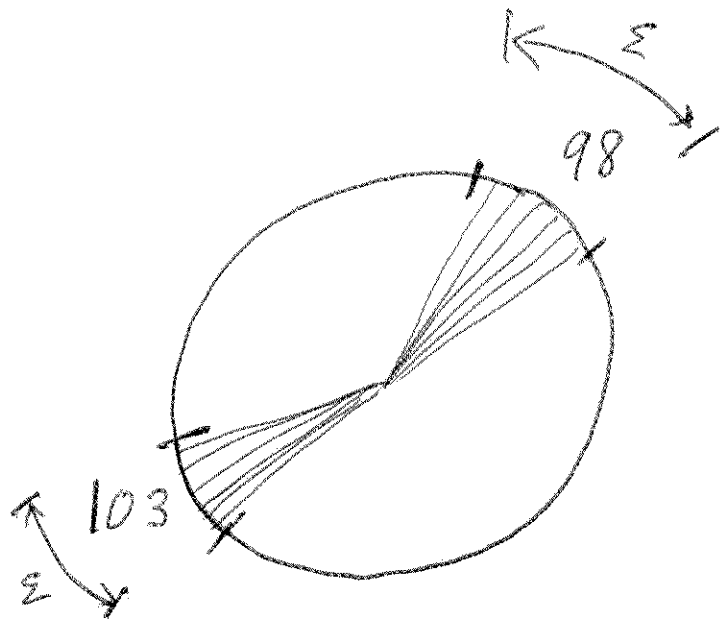


We can always build a piecewise
immersed 1-manifold in a closed
2-manifold if we use the
"doubling trick"



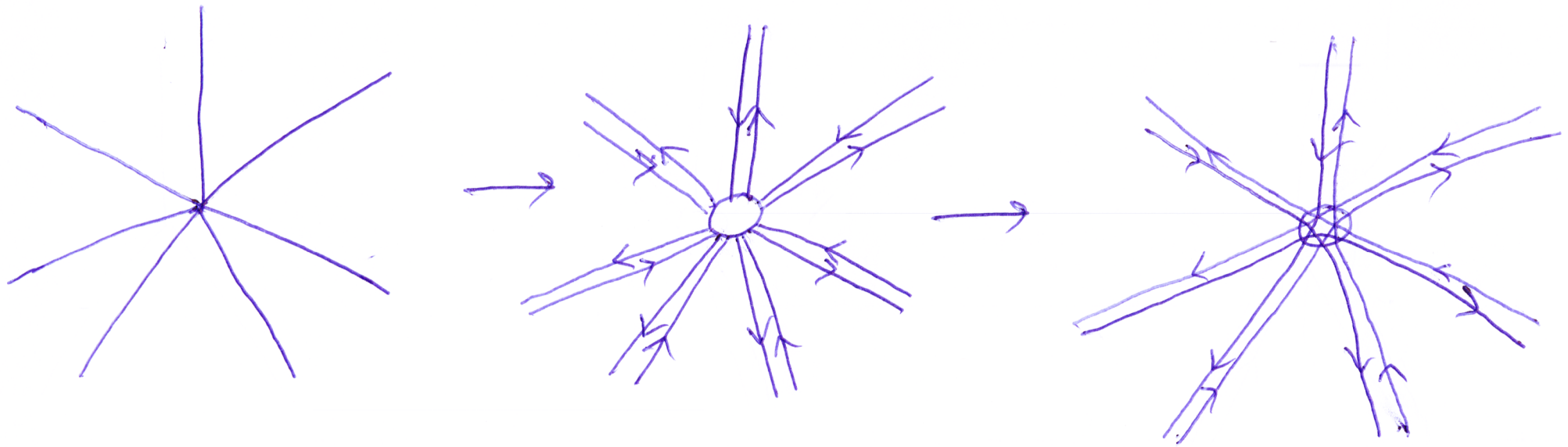


A set of points ~~of~~ on S' is
 "evenly distributed to the ~~set~~ scale ε "
 if every ε -interval of S' has
 the same number of points, ~~to~~ up
 to a factor of $1+\varepsilon$.

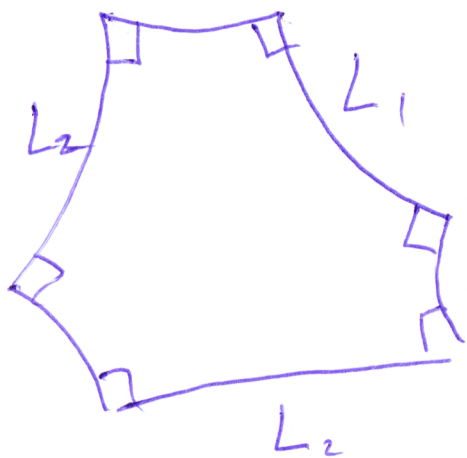


$$\frac{103}{98} < 1 + \varepsilon$$

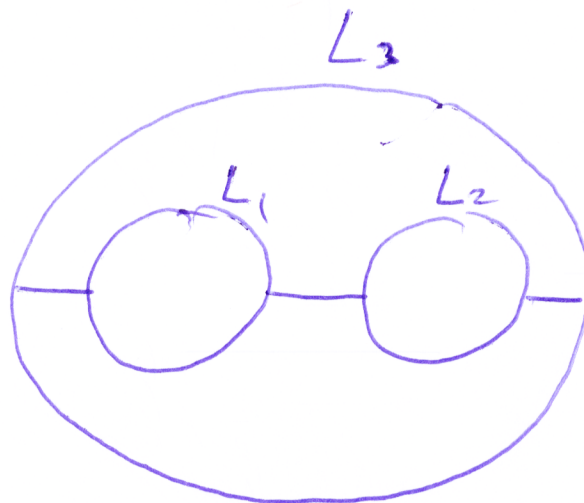
We can make the piecewise immersed 1-manifold nearly geodesic (small bending) if the segments are "evenly distributed" around each meeting point:



Hyperbolic Pairs of ~~Pants~~ Pants

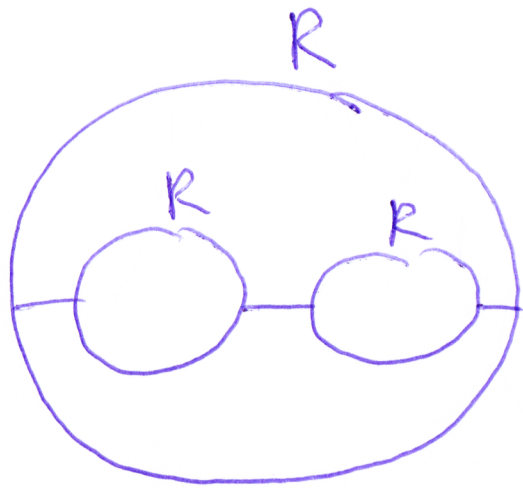


double
→

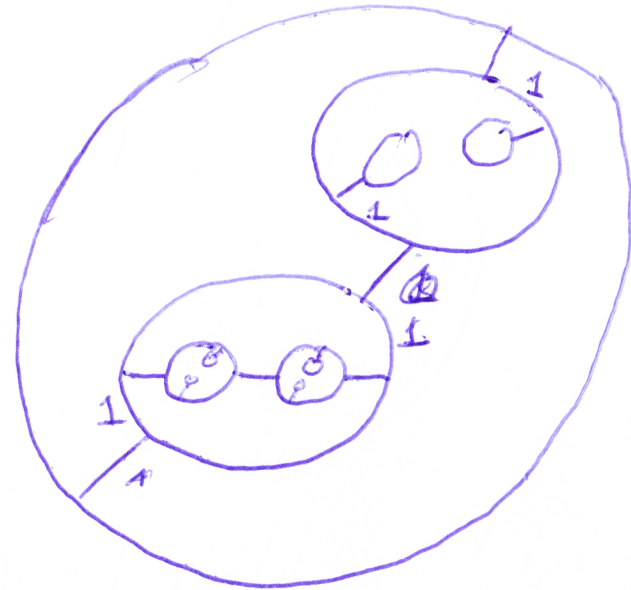


Half-lengths L_1, L_2, L_3

Perfect Pants and Perfect Surfaces



$$hl(c) = R$$

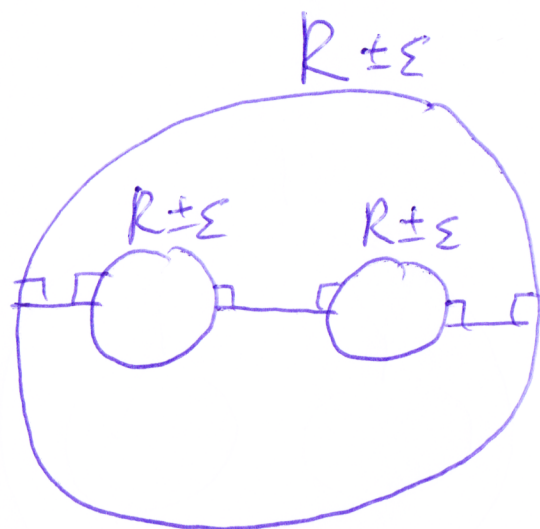


$$hl(c) = R$$

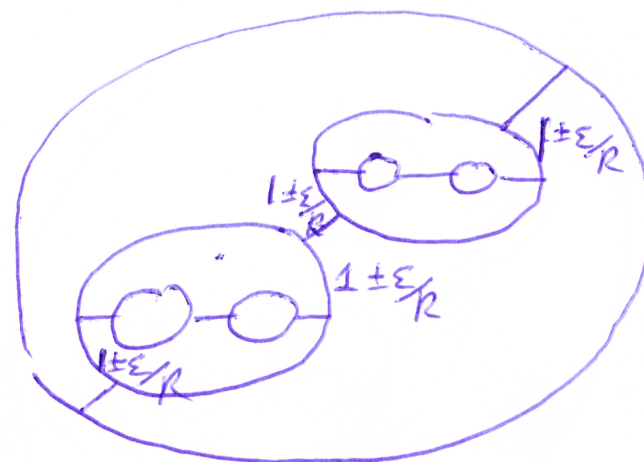
$$s(c) = 1$$

Theorem Any two ~~pairs~~ closed perfect surfaces have a common finite cover

Good Pants and Good Surfaces



$$|h(c) - R| < \varepsilon$$



$$|s(c) - 1| < \varepsilon/r$$

Theorem (KM) ~~Ann~~

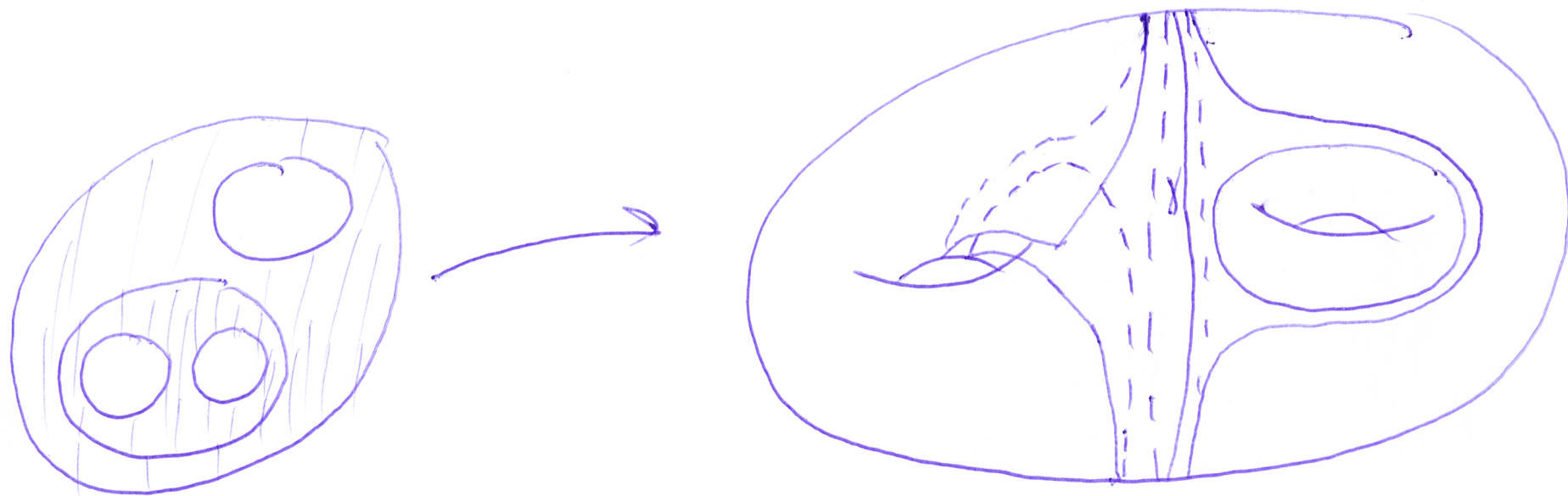
A good closed pants surface is close to a perfect surface.

An Immersed Pair of Pants



We can think of the pants as being isometrically immersed, and speak of the goodness of pants.

Gluing Together Immersed Pants

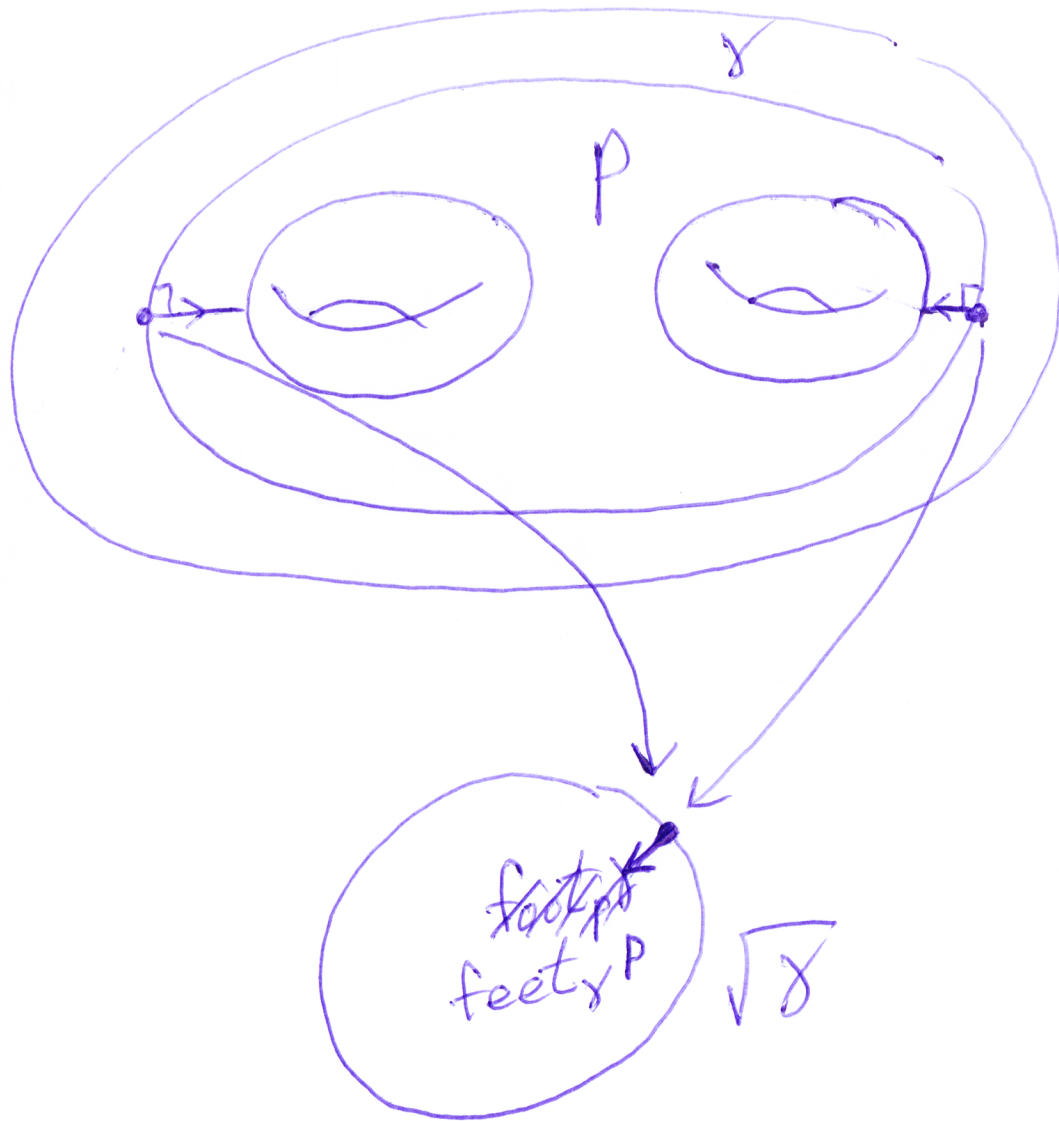


We can ~~ind in~~ identify cuffs of
the parametrizing pants whenever
the cuffs map to the same ~~ge~~-geodesic γ ,
and the pants are on opposite sides
of γ .

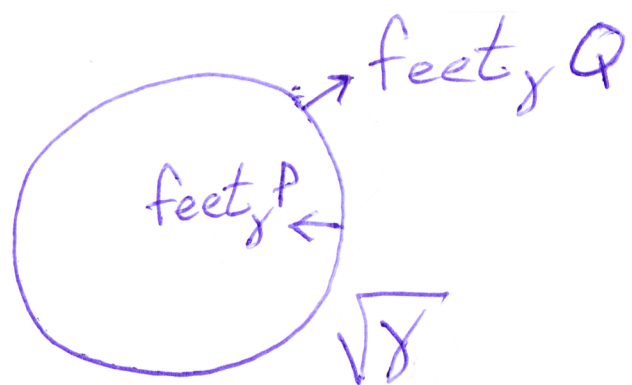
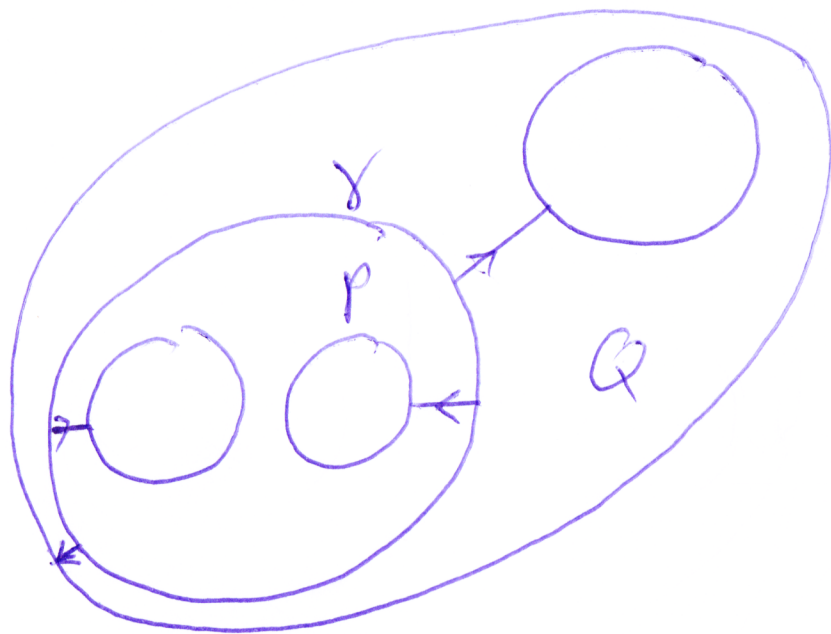
Easy Theorem

Let $\mathcal{Q}$ be a finite set of immersed pants, such that for every closed geodesic γ on S we have the same number of pants in $\mathcal{Q}$ on both sides of γ . (That number can be zero). Then we can assemble $\mathcal{Q}$ to form a finite cover of S .

The square root of a geodesic



Shear Coordinates and $N^4(\sqrt{x})$



The Equidistribution Theorem

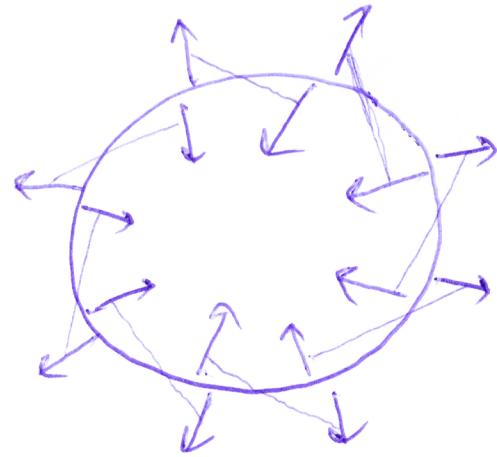
Let $\Pi_{\varepsilon, R}$ be the set of good pants. Then for every $\gamma \in \Gamma_{\varepsilon, R}$ (the set of good curves), the points

$$\{\text{feet}_\gamma P \mid P \in \Pi_{\varepsilon, R} \text{ \& } \gamma \in \partial P\} \subseteq N^4(\sqrt{\varepsilon})$$

are evenly distributed on $N^4(\sqrt{\varepsilon})$ to the scale e^{-gR} , for $g \neq g(S)$. $g \equiv g(S, \varepsilon)$.

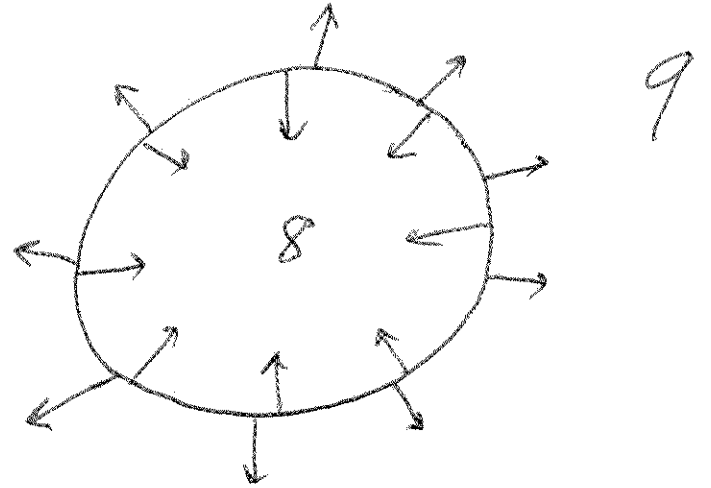
Equidistributed & Balanced $\Rightarrow$ Good Cover

If there were exactly the same number of pants on both sides of each geodesic γ , then we would be able to assemble the pants in $\mathbb{T}_{\varepsilon, R}$ to form a good cover of S :



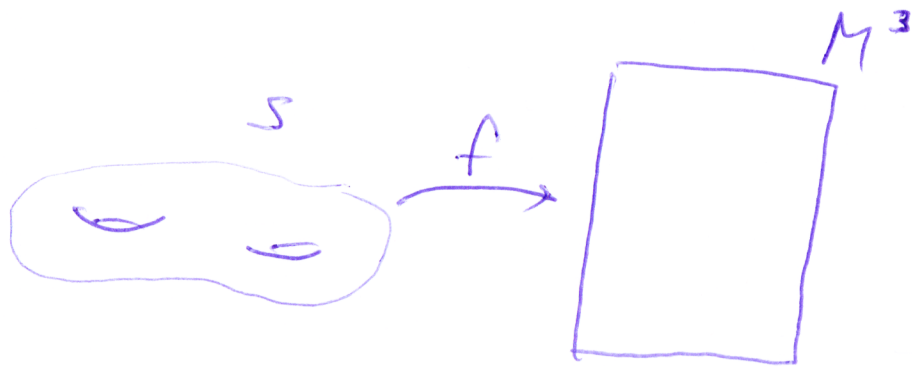
Equidistributed $\Rightarrow$ Nearly balanced

It follows from equidistribution that there are nearly the same number of points in $\Pi_{\varepsilon, R}$ on either side of a good geodesic γ



An Interlude in Three Dimensions

In a hyperbolic 3-manifold $N^1(\delta)$ (and $N^1(\sqrt{\delta})$) is connected, so we can use the "doubling trick" to build a nearly geodesic immersed surface (which is therefore essential).



$$\pi_1(f): \pi_1(S) \rightarrow \pi_1(M^3)$$

injective

The Idea of Self-Correction

We find ~~g~~ $g: \mathbb{Q}\Gamma_{\varepsilon, R} \rightarrow \mathbb{Q}\Pi_{\varepsilon, R}$

~~g~~ such that

1. $g(\alpha) = \alpha$ whenever $\alpha \equiv 0$ in $H_1(S)$

2. $\|g(\alpha)\|_{\infty} \leq P(R)e^{-R}\|\alpha\|_{\infty}$

(Thanks to Lewis Bowen for finding a clearer statement for condition 2).

Then let $\pi = \sum_{P \in \pi_{\varepsilon, R}} P$, and $\alpha = \partial \pi$.

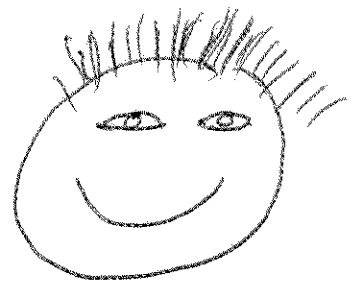
Then $\|\alpha\|_{\infty} \leq e^{(1-\varepsilon)R}$ by equidistribution

So $\|g(\alpha)\|_{\infty} \leq P(R)e^{-\varepsilon R} \ll 1$.

So $\pi - g(\alpha)$ is positive and equi-distributed

And $\partial(\pi - g(\alpha)) = \partial \pi - \partial g(\partial \pi) = \partial \pi - \partial \pi = 0$.

After clearing denominators, we can assemble the pants of $\pi - g(\alpha)$ to form a good cover!



The Good Pants Homology

We let $H_1^{\varepsilon, R}(S; \mathbb{Q}) = \mathbb{Q}\Gamma_{\varepsilon, R} / 2\mathbb{Q}\Pi_{\varepsilon, R}.$

We find a series of identities
leading to $H_1^{\varepsilon, R}(S; \mathbb{Q}) = H_1(S; \mathbb{Q}).$

The Algebraic Square Lemma

Under conditions of reasonable geometry,

$$\sum_{i,j=0,1} (-1)^{i+j} [A_i \cup B_j V] \equiv 0$$

in $H_1^{\varepsilon, R}$

(where $[X]$ denotes the closed
geodesic ^{freely} homotopic to X , for
 $X \in \pi_1(S, x)$).

We then define

$$A_T = \frac{1}{2}([TA\bar{T}B] - [T\bar{A}TB])$$

(which is independent of the choice
of B in $H_1^{\varepsilon, R}$)

and prove

$$(XY)_T = X_T + Y_T$$

in $H_1^{\varepsilon, R}$.