Bifurcation locus of cubic polynomial family

Hiroyuki Inou

Department of Mathematics, Kyoto University

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Critically marked cubic polynomial family

$$f_{a,b}(z) = z^3 - 3a^2 z + b.$$

- Critical points: $\pm a$.
- **Marked** critical point: $+a$.
- Co-critical points: $\mp 2a$ (i.e., $f_{a,b}(\mp 2a) = f_{a,b}(\pm a)$).

Two involutions:

- Exchange the role of critical points: $(a, b) \mapsto (-a, b)$.
- Half turn $-f_{a,b}(-z) = f_{-a,-b}(z)$: $(a, b) \mapsto (-a, -b)$.

Hence $\{f_{a,b}\}$ is regarded as the family of affine conjugacy classes of cubic polynomials with **marked critical point** and **Böttcher coordinate at ∞** (one can consider external rays).

Bifurcation measure

- $g_{a,b}(z) = \lim_{n\to\infty} \frac{1}{3^n} \log^+ |f^n_{a,b}(z)|$: the Green function.
- $G_\pm(a,b) = g_{a,b}(\pm a)$.
- $T_\pm = dd^c G_\pm$.
- $\mu_{\mathrm{bif}} = cT_+ \wedge T_-$: the **bifurcation measure**.

The periodic curves $\mathcal{S}_p$

For $p > 0$, let

$$\mathcal{S}_p = \{(a, b) \in \mathbb{C}^2 |\ f^k_{a,b}(a) \neq a \,(0 < k < p), f^p_{a,b}(a) = a\}.$$

Theorem 1 (Milnor)

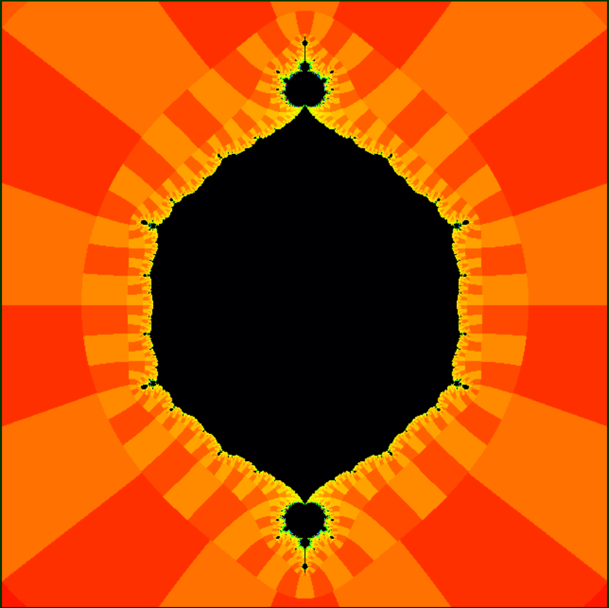
$\mathcal{S}_p$ is a smooth affine curve.

Milnor asked if the following conjecture is true:

Conjecture

$\mathcal{S}_p$ is connected.

$\mathcal{S}_1 = \{(a,b) \mid b = 2a^3 + a\} \cong \mathbb{C}$



$\mathcal{S}_2 \cong \mathbb{C}^*$

Normalizing $f_{a,b} \in \mathcal{S}_2$ so that the marked critical point is 0 and its forward orbit is

$$0 \mapsto 1 \mapsto 0.$$

Then it has the form

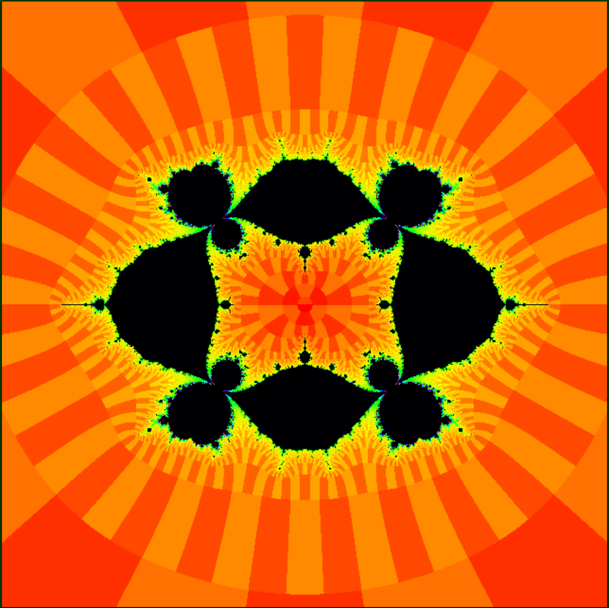
$$t^2 z^3 - (t^2 + 1) z^2 + 1$$

for some $t \in \mathbb{C}^*$ and the free critical point is $\frac{2(t^2+1)}{3t^2}$. Therefore,

$$(a, b) = \left(\frac{t^2 + 1}{3t}, -\frac{(t^2 - 2)(2t^4 - 8t^2 - 1)}{27t^3} \right).$$

gives a parametrization of $\mathcal{S}_2$.

$S_2 \cong \mathbb{C}^*$



$\mathcal{S}_3$: Torus with 8 punctures

Similarly, $f_{a,b} \in \mathcal{S}_3$ is affinely conjugate to

$$g_{\alpha,\beta}(z) = \alpha z^3 + \beta z^2 + 1$$

with

$$\alpha = -\frac{c^3 - c^2 + 1}{c^3 - c^2}, \qquad \beta = \frac{c^4 - c^3 + 1}{c^3 - c^2},$$

which has a period 3 critical orbit

$$0 \mapsto 1 \mapsto c \mapsto 0.$$

$\mathcal{S}_3$: Torus with 8 punctures

On the c-plane, one cannot take a univalent branch of $\gamma = \sqrt{\alpha}$, which is necessary to have Böttcher coordinates consistently. Hence one has to solve the following equation on (γ, c):

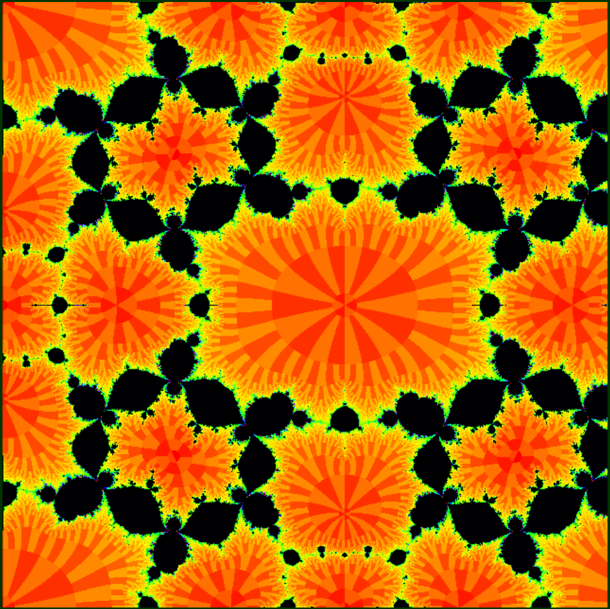
$$(c^3 - c^2)\gamma^2 = -c^3 + c^2 - 1,$$

which is equivalent to

$$\left(\frac{2ic\gamma}{c-1}\right)^2 = 4p^3 - g_2 p - g_3$$

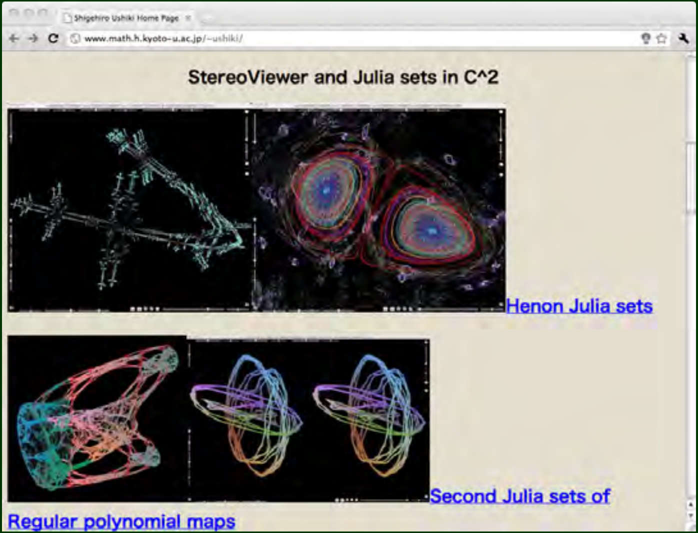
where $p = \frac{1}{c-1} + \frac{1}{3}$, $g_2 = -\frac{20}{3}$ and $g_3 = -\frac{44}{27}$. Therefore, $\mathcal{S}_3$ is a torus with punctures (indeed, there are 8 punctures) and it can be parametrized using a Weierstrass $\wp$-function.

$\mathcal{S}_3$: Torus with 8 punctures



Visualization of objects in $\mathbb{C}^2$

Ushiki has made a program (named StereoViewer) to visualize Julia sets of Hénon maps and so on:



Visualization of objects in $\mathbb{C}^2$

We've also made "real" objects for Ushiki's 60th birthday last year:

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Visualization of the cubic polynomial family

I (tried to) visualize the following:

- Bifurcation measure μ_{bif},
- the bifurcation locus of $\mathcal{S}_p$.

Visualization of the bifurcation measure

- For $f_{a,b}$ with $G_{a,b}(a) = G_{a,b}(-a) = r > 0$, the **critical portrait** of $f_{a,b}$ is the sets of external angles for the critical points.
- When $a \neq 0$ (i.e., both critical points are simple), it is equivalent to consider the external angles $(\theta_+, \theta_-) \in (\mathbb{R}/\mathbb{Z})^2$ of co-critical points.
- Let **Cb** be the set of all such critical portraits.
- Let $\mu_{\mathbf{Cb}}$ be the measure on Cb induced by the Lebesgue measure on $(\mathbb{R}/\mathbb{Z})^2$, (normalized so that the total mass is one).
- The **landing map** $\boldsymbol{e}$ **: Cb** $\rightarrow \mathbb{C}^{\mathbf{2}}$ is a measurable map defined by landing of stretching rays.

Theorem 2 (Dujardin-Favre)

$e_*\mu_{\mathrm{Cb}} = \mu_{\mathrm{bif}}$.

Therefore, by calculating stretching rays, one can numerically visualize the bifurcation measure.

Visualization of $\mathcal{S}_p$

- Direct calculation for $p = 1, 2, 3$.
- For any $p > 0$ (and other curves): Solve the Hamiltonian dynamics defined by the defining equation (under implementation).

Questions

- What do those pictures suggest?
- What should be drawn?
- What functionality should be implemented?