Complex Projective Varieties with a Large Group of Holomorphic Diffeomorphisms

Joint work with Abdelghani Zeghib

Using ideas from
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— I —

From Linear Representations,

to Actions by Diffeomorphisms

- Almost **simple Lie groups**

$$\mathrm{SL}_n(\mathbf{R}),\ \mathrm{SL}_n(\mathbf{C}),\ \mathrm{SO}_{p,q}(\mathbf{R}),\ \ldots$$

- **Rank** of a Lie group

$$\mathrm{rk}_{\mathbf{R}}(\mathrm{SL}_n(\mathbf{R})) = \mathrm{rk}_{\mathbf{R}}(\mathrm{SL}_n(\mathbf{C})) = n-1,$$

$$\mathrm{rk}_{\mathbf{R}}(\mathrm{SO}_{p,q}(\mathbf{R})) = \min\{p,q\},\ \ldots$$

- **Maximal Torus** $= A \subset G$:

$$\text{diagonal matrices in } \mathrm{SL}_n(\mathbf{R})$$

- Example : $\mathrm{SL}_2(\mathbf{R})$; $A =$ diagonal subgroup $\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$.
- Action on $\mathbf{R}^2$.
- Action on homogenous polynomials of degree n.
- **Weights** are given by

$$x^k y^{n-k} \mapsto a^{2k-n} x^k y^{n-k}.$$

$$-n, -n+2, -n+4, \dots, n-4, n-2, n.$$

The highest weight determines the irreducible representation.

- **Lattices** in a Lie group $G =$

  discrete subgroups $\Gamma \subset G$ such that $Haar(G/\Gamma) < \infty$.

- Examples :

$$\begin{array}{rcll} \Gamma & = & \mathrm{SL}_n(\mathbf{Z}) & \text{in } G = \mathrm{SL}_n(\mathbf{R}) \\ \Gamma & = & \mathrm{SL}_n(\mathbf{Z}[\sqrt{-1}]) & \text{in } G = \mathrm{SL}_n(\mathbf{C}) \end{array}$$

Theorem [Margulis].—
Let Γ *be a lattice in a simple Lie group* G*. If* $\mathrm{rk}_{\mathbf{R}}(G) \geq 2$*, then* Γ *is almost simple: All normal subgroups are finite or cofinite.*

Theorem [Margulis].—
Let Γ *be a lattice in a simple Lie group* G*. If* $\mathrm{rk}_{\mathbf{R}}(G) \geq 2$*, then all finite dimensional linear representations of* Γ *are built from :*

- *restrictions of linear representations of* G *;*
- *unitary representations.*

- $M =$ a compact manifold.
- If G acts on M by diffeomorphisms then

$$\dim(M) \geq \mathrm{rk}_{\mathbf{R}}(G).$$

- **Example**: $\mathrm{SL}_3(\mathbf{R})$ does not act on $\mathbb{S}^1$.

Zimmer's Conjecture.—
Let Γ be a lattice in a simple Lie group G. If Γ acts faithfully on a compact manifold M by diffeomorphisms, then

$$\dim(M) \geq \mathrm{rk}_{\mathbf{R}}(G).$$

Theorem (Ghys, Burger-Monod).—
Zimmer's conjecture has a positive answer if M is the circle.

Theorem (Kaimanovich - Masur).—
If $\mathrm{rk}_{\mathbf{R}}(G) \geq 2$, all morphisms

$$\Gamma \longrightarrow \quad \text{Mapping Class Group of genus} \;\; g$$

have finite image.

— II —

Automorphisms of Projective Varieties and Zimmer's Conjecture

- M = smooth, connected, complex manifold.
- $\mathrm{Aut}(M)$ = group of automorphisms of M
  = group of holomorphic diffeomorphisms.

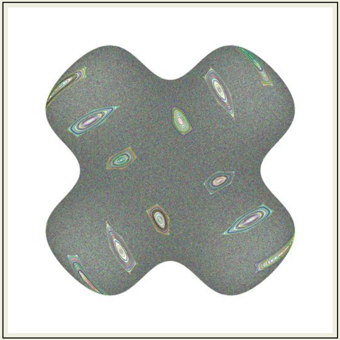
Theorem [Bochner-Montgomery].—
If M is a compact complex manifold, then $\mathrm{Aut}(M)$ is a complex Lie group.

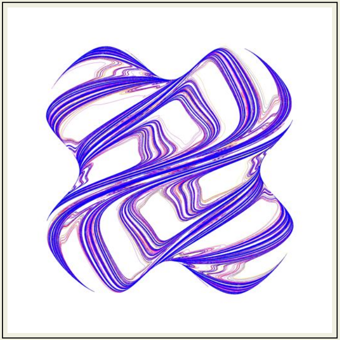
- $\mathrm{Aut}(\mathbb{P}^n(\mathbf{C})) = \mathrm{PGL}_{n+1}(\mathbf{C})$.

- $\mathrm{Aut}(M)$ may have an **infinite number of connected components**.

- $M \subset \mathbb{P}^1(\mathbf{C}) \times \mathbb{P}^1(\mathbf{C}) \times \mathbb{P}^1(\mathbf{C})$ surface with $\deg(M) = (2, 2, 2)$.

$$\mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z} \subset \mathrm{Aut}(M)$$





- E = elliptic curve $\mathbf{C}/\mathbf{Z}[\sqrt{-1}]$.

$$T = E^n = \mathbf{C}^n/\mathbf{Z}[\sqrt{-1}]^n.$$

Then

$$\mathrm{Aut}(T) = T \rtimes \mathrm{GL}_n(\mathbf{Z}[\sqrt{-1}])$$

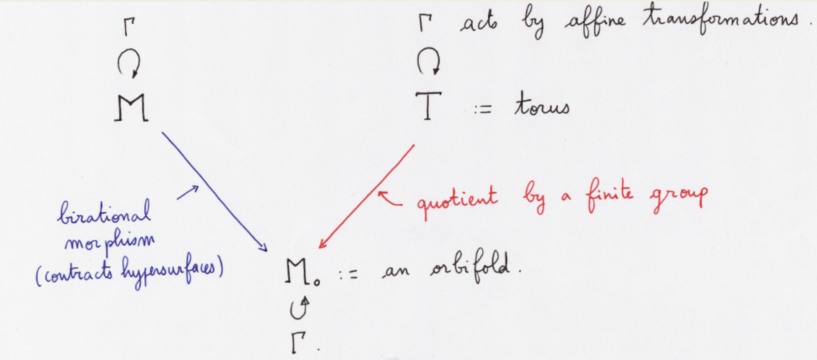
- $\mathrm{GL}_n(\mathbf{Z}[\sqrt{-1}])$ commutes to the multiplication by $\sqrt{-1}$.

$$M_0 = T/\{\sqrt{-1}\} \text{ is an orbifold}$$

Blow up the singularities

$$\mathrm{PGL}_n(\mathbf{Z}[\sqrt{-1}]) \subset \mathrm{Aut}(M).$$

Definition (Kummer examples) .—
A pair (M, Γ) *with* M *a compact complex manifold and* $\Gamma \subset \mathrm{Aut}(M)$ *is a* **Kummer example** *if one has a commutative diagram :*



Theorem (Cantat, Zeghib) .—
Let Γ *be a lattice in a simple Lie group* G *with* $\mathrm{rk}_{\mathbf{R}}(G) \geq 2$. *Assume* Γ *embeds into* $\mathrm{Aut}(M)$, *with* M *compact, Kähler, and connected. Then*

(1) $\dim_{\mathbf{C}}(M) \geq \mathrm{rk}_{\mathbf{R}}(G)$;

(2) *if* $\dim_{\mathbf{C}}(M) = \mathrm{rk}_{\mathbf{R}}(G)$ *then* $M = \mathbb{P}^n(\mathbf{C})$ *and* $G = \mathrm{PSL}_{n+1}(\mathbf{R})$ *or* $\mathrm{PSL}_{n+1}(\mathbf{C})$;

(3) *if* $\dim_{\mathbf{C}}(M) = \mathrm{rk}_{\mathbf{R}}(G) + 1$ *then either* G *acts on* M *or* (M, Γ) *is, up to finite index, a Kummer example.*

- **Remark**.— One can list all examples in (3).

— III —

Hodge Theory, Linear Representations, and Complex Geometry

Theorem (A. Fujiki, D. Lieberman) .— *If M is a compact Kähler manifold, the connected component of the identity* $\mathrm{Aut}(M)^0$ *has finite index in the kernel of the morphism*

$$\left\{ \begin{array}{ccc} \mathrm{Aut}(M) & \to & \mathrm{GL}(H^*(M, \mathbf{Z})) \\ f & \mapsto & f^* \end{array} \right.$$

Alternative. (Fujiki-Lieberman + Margulis) —

(i) Γ acts (almost) trivially on the cohomology, and G embeds into $\mathrm{Aut}(M)^0$;

(ii) Γ acts (almost) faithfully on $H^*(M, \mathbf{Z})$ and this action extends to a linear representation of G itself.

- Assume (ii) in what follows: $\rho : G \to \mathrm{GL}(H^*(M, \mathbf{R}))$

Borel Density Theorem .— *Lattices* $\Gamma \subset G$ *are Zariski dense.*

• Assume the action of Γ on $H^*(M, \mathbf{Z})$ extends to a faithful representation of G.

(a) G preserves the Hodge decomposition $H^k(M, \mathbf{C}) = \oplus_{p+q=k} H^{p,q}(M, \mathbf{C})$.

(b) The cup product is G-equivariant.

(c) The representation on $H^k(M, \mathbf{R})$ is dual to $H^{2n-k}(M, \mathbf{R})$.

- $\kappa \in H^{1,1}(M, \mathbf{R})$ = class of a Kähler form.
- Quadratic form

$$Q_\kappa(u, v) := \int_M u \wedge v \wedge \kappa^{n-2}$$

- **Primitive subspace** $\mathcal{P}_\kappa$ = orthogonal complement of κ:

$$\mathcal{P}_\kappa = \left\{ u \in H^{1,1}(M, \mathbf{R}) \mid \quad \int_M u \wedge \kappa \wedge \kappa^{n-2} = 0 \right\}.$$

Hodge Index Theorem .— *Q_k is negative definite on the hyperplane $\mathcal{P}_\kappa$.*

- **Consequence**.— If $u \wedge u = u \wedge v = v \wedge v = 0$ then $u = c^{ste}\, v$.

$$\dim_{\mathbf{C}}(\mathbf{M}) = \mathbf{3} \;;\; \mathbf{W} = \mathbf{H}^{1,1}(\mathbf{M}, \mathbf{R}).$$

$\star \;\star\; \star\star$

- **Easy Fact**: $\rho : G \longrightarrow \mathrm{GL}(W)$ is faithful.
- Cup product and duality

$$\begin{cases} W \times W & \rightarrow & H^{2,2}(M, \mathbf{R}) = W^{dual} \\ (u, v) & \mapsto & u \wedge v \end{cases}$$

- $H \subset G$, a copy of $\mathrm{SL}_2(\mathbf{R})$.
- $\rho : H \to \mathrm{GL}(W)$ induced linear representation.
- Maximal torus of $H =$ diagonal group

$$A_a = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}.$$

- $m =$ Highest weight of H in W : There exists u in $W \setminus \{0\}$ such that

$$\rho(A_a) \cdot u = a^m u \quad \forall a \neq 0$$

- m is also the highest weight on W^{dual}

Lemma (Restriction on the weights) .—
The highest weight m is at most 4 ; its multiplicity is 1.

Proof

(1) $u \mapsto a^m u$ and $v \mapsto a^{m-2} v$

(2) By equivariance of $\wedge$
$u \wedge u \mapsto a^{2m} u \wedge u$,
$u \wedge v \mapsto a^{2m-2} u \wedge v$,
$v \wedge v \mapsto a^{2m-4} v \wedge v$

(3) By duality the highest weight on W^{dual} is at most m

(4) If $m > 4$ we have $2m - 4 > m$

(5) (2), (3) and (4) imply $u \wedge u = u \wedge v = v \wedge v = 0$

(6) (5) contradicts Hodge Index Theorem $\square$

Lemma .—
The group G does not contain any copy of $\mathrm{SL}_2(\mathbf{R}) \times \mathrm{SL}_2(\mathbf{R})$.

Proposition (Possible Lie Groups) .—
If the rank of G is ≥ 2 then G is isogenous to $\mathrm{SL}_3(\mathbf{R})$ *or* $\mathrm{SL}_3(\mathbf{C})$.

- $c_1(M)$ and $c_2(M) =$ **Chern classes** of M
- $\kappa =$ a Kähler class.

Yau's Theorem.— *Let M be a compact Kähler manifold, and κ a Kähler class on M. If*

$$\kappa^{n-1} \wedge c_1(M) = \kappa^{n-2} \wedge c_2(M) = 0,$$

then M is covered by a torus $\mathbf{C}^n/\Lambda$.

- $\dim(M) = 3$; $G = \mathrm{SL}_3(\mathbf{R})$.
- diagonal subgroup: $A_{\mathbf{t}} = \begin{pmatrix} e^{t_1} & 0 & 0 \\ 0 & e^{t_2} & 0 \\ 0 & 0 & e^{t_3} \end{pmatrix}$, $t_1 + t_2 + t_3 = 0$.
- $\mathcal{K} \subset W$ the **Kähler cone**.
- Assume that $\mathcal{K}$ is G-invariant.

Proposition (Perron-Frobenius) .—
There exist $u, v \in \overline{\mathcal{K}}$ eigenvectors for A:

$$\forall\, \mathbf{t} = (t_1, t_2, t_3) \quad \begin{cases} \rho(A_{\mathbf{t}}) \cdot u & = & \exp(\alpha(\mathbf{t}))u \\ \rho(A_{\mathbf{t}}) \cdot v & = & \exp(\beta(\mathbf{t}))v \end{cases}$$

with α and β linearly independant.

- Assume $w = u + v$ is a Kähler class.
- For all $\mathbf{t} = (t_1, t_2, t_3)$

$$\rho(A_{\mathbf{t}}) \cdot (u \wedge c_2(M)) = \exp(\alpha(t))(u \wedge c_2(M)) = u \wedge c_2(M)$$

Thus

$$\begin{aligned} u \wedge c_2(M) &= 0, \\ v \wedge c_2(M) &= 0, \\ w \wedge c_2(M) &= 0 \end{aligned}$$

- Similarly $w \wedge w \wedge c_1(M) = 0.$
- By Yau's Theorem, M is covered by a torus.

Theorem (Demailly, Paun) .—
If $w \in \overline{\mathcal{K}}$ is not Kähler then $\exists S \subset M$ analytic such that

$$\int_S w^{\dim(S)} = 0$$

- $u+v$ not Kähler: $\exists S \subset M$ a proper analytic Γ-invariant subset.
- $\dim(S) < \dim(M)$; recursion on $\dim(M)$:

  $S = \mathbb{P}^{n-1}(\mathbf{C})$ and it can be blown down to a quotient singularity.