How Dennis and I intersected

Jim Stasheff

UNC-CH and U Penn

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Intersection

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Primary intersection: Rational homotopy theory and ∞-algebra

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Dennis' minimalist/computational
Quillen's 'maximalist'/categorical.

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Two elegant approaches:
Dennis' minimalist/computational
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The secret? L_∞-algebra in Dennis' *minimal* models.

OR
Symmetrising the cup product over the integers replaces associativity by ∞-homotopy associativity.

Transverse intersection

$$\chi$$

Manifolds?

Manifolds of the homotopy type of (non-Lie) groups

PD spaces

A point of intersection - Poincaré duality spaces

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Larry Taylor's special Massey products

$$< x, ?, z >$$

PD theorem

Theorem

For simply connected rational Poincaré duality spaces Y with fundamental class $\mu \in H^N$, there is a dg Lie algebra model $\mathcal{L}(H(Y))$ with

$$d(\mu) = 1/2\ \Sigma[x_i, x^i],$$

where $\{x_i\}$ is a basis for $H(Y)$ in degrees $k:\ 0 < k < N$ and $\{x^i\}$ is a dual basis.

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where $\{x_i\}$ is a basis for $H(Y)$ in degrees $k:\ 0 < k < N$ and $\{x^i\}$ is a dual basis.

Equivalently $Y = X \cup e^N$ where e^N is attached by ordinary Whitehead products (not iterated) with respect to some basis of the rational homotopy groups of X.

Work with Steve Halperin

Filtered dgca models and perturbations

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Filtered dgca models and perturbations

Given a rationally nilpotent space or dgca (A, d_A) and an isomorphism $\phi : \mathcal{H} \to H((A, d_A))$, we perturbed the minimal model for the cohomology algebra $\mathcal{H}$ to create a canonically filtered Sullivan model for (A, d_A).

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That minimal model is a purely algebraic construct closely related to a multiplicative resolution of $\mathcal{H}$ by free graded commutative algebras, sometimes called the Koszul-Tate resolution, (and extended to the graded case by Jozefiak).

Perturbations

Definition

For a filtered complex (C, d) with d of degree 1, a *perturbation* is a linear map $p : C \to C$ of degree 1 such that p lowers filtration by at least 1 and

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There is an intimate relation between perturbations and deformations.

Obstructions

Given a cohomology isomorphism f, Steve and I used the filtered models to construct a sequence of obstructions $O_n(f)$ (of classical type in algebraic topology) to the realization of f by a map of models.

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The next obvious question:

"How different can rational homotopy types be if the cohomology algebras agree?"

Intrinsic formality

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If $\mathcal{H}$ is $(n-1)$- connected and $\mathcal{H}^i = 0$ for $i \geq 3n-1$, then $\mathcal{H}$ is intrinsically formal, i.e., there is only one homotopy type with the given cohomology algebra.

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For PD spaces, the hypothesis can be extended to $\mathcal{H}^i = 0$ for $i \geq 4n-1$.

Work with Mike Schlessinger

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The Koszul-Tate resolution of

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A particular *non*-minimal Sullivan model of the formal space with a given cohomology algebra $\mathcal{H}$, using the adjointness between dgcas and dg Lie *co*algebras, cf. Quillen and John Moore.

Quillen's approach to rational homotopy theory

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For a space X, there is a dgL $\mathcal{L}_X$ such that $H(\mathcal{L}_X)$ is isomorphic as graded Lie algebra to the rational homotopy groups of the loop space of X with respect to the Samelson product.

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Chains are more natural than cochains, hence the use of differential graded coalgebras.
Compare also equivalent work of John Moore.

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This construction defines a rational homotopy space A which is manifestly formal.

Given a rational homotopy space (A, d_A) and an isomorphism

$$\phi : \mathcal{H} \to H(A),$$

perturb $d_\mathcal{A}$ to a model for (A, d_A).

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Dg Lie algebras for perturbations

Perturbations of $\mathcal{A}$ sit naturally in a sub-dg Lie algebra of $Der(\mathcal{A}(s\mathcal{L}^c\mathcal{H}))$.
Perturbations of $\mathcal{L}^c\mathcal{H}$ sit naturally in a sub-dg Lie algebra of $Coder(\mathcal{L}^c\mathcal{H})$.

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For our model $\mathcal{A}$, the total degree minus resolution degree is called the *weight* and similarly for $\mathcal{L}^c\mathcal{H}$.

Definition

Denote by $Pert\mathcal{A}(s\mathcal{L}^c\mathcal{H}))$ *the dg Lie algebra of weight decreasing derivations of* $\mathcal{A}(s\mathcal{L}^c(\mathcal{H})))$.

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Main Homotopy Theorem

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Let $\mathcal{H}$ be a simply connected graded commutative algebra of finite type. The set of augmented homotopy types of dgca's

$$(A, d_A, \phi : \mathcal{H} \approx H(A))$$

is in 1–1 correspondence with the path components of $\hat{\mathcal{C}}(Pert\mathcal{A}(s\mathcal{L}^c(\mathcal{H})))$.

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also known as a *gauge transformation*.

The hard part is to go from a homotopy to such an equivalence.

Comparison theorems

Theorem

For simply connected $\mathcal{H}$ of finite type, the natural dg Lie map $Pert(\mathcal{L}^c(\mathcal{H})) \to Pert(\mathcal{A}(s\mathcal{L}^c(\mathcal{H})))$ is a homology isomorphism.

The space of homotopy types

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To go from the formal to the non-formal type is known as a *jump deformation*.

L_∞-structure on $H(L)$

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The definitive treatment in the dg Lie case (which is more subtle) is due to Huebschmann.

For $L = Pert(\mathcal{L}^c(\mathcal{H}))$, these higher order brackets can often be related to Massey products and attaching maps.

Examples and computations

$Pert(\mathcal{L}^c(\mathcal{H}))$ can be identified with a subspace of $Hom\,(\mathcal{L}^c(\mathcal{H}), \mathcal{H})$ and hence each element as a sum of elements of $Hom\,(\bar{\mathcal{H}}^{\otimes k+2}, \mathcal{H})$ which lowers the internal $\mathcal{H}$-degree by k.

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Attaching a cell by an ordinary Whitehead product $[S^p, S^q]$ means the cell carries the product cohomology class. Massey (and Uehara) introduced Massey products in order to detect cells attached by iterated Whitehead products such as $[S^p, [S^q, S^r]]$.

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In terms of cells, this means we cannot attach simultaneously both e^8 to realize $\langle x_1, x_1, x_2\rangle$ **and** attach e^{13} to realize $\langle x_2, x_1, x_8\rangle$.

Continuous moduli

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For

$$\mathcal{H} = H(S^3 \vee S^3 \vee S^{12}),$$

the attaching map of the 12-cell is in

$$\pi_{11}(S^3 \vee S^3) \otimes \mathbb{Q}$$

of dimension 6, while

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Alternatively, the space of 5–fold Massey products $\mathcal{H}^{\otimes 5} \to \mathcal{H}$ is of dimension 6.

Extension to fibrations

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Mike and me: Classification in terms of homotopy classes of maps $\mathcal{C} \to \mathcal{C}(\mathcal{L})$ of a dg commutative coalgebra (base space) into $\mathcal{C}(\mathcal{L})$ for an appropriate dgLie algebra $\mathcal{L}$.

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The essential idea is to work with fibrations as twisted tensor products of Sullivan models.

Algebraic model of a fibration

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such that $p^{-1}(*) = \mathcal{F}$ and p satisfies the homotopy lifting property.

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Under reasonable assumptions, there is a B–derivation D on $B \otimes F$ and an equivalence between

$$E \text{ and } (B \otimes F, D).$$

Doing the twist

The algebra structure and the differential may be twisted.

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The *twisting term* $\tau \in Der(F, \bar{B} \otimes F)$, the sub-dgL of $Der(B \otimes F)$ consisting of those derivations of $B \otimes F$ which vanish on B and reduce to 0 on F via the augmentation.

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Strong equivalence classes of fibrations correspond to the quotient by the action of automorphisms θ of $B \otimes F$ which are the identity on B and reduce to the identity on F via augmentation.

Classification

Denote by $\mathcal{L}(B, F) \subset Der\,(F, \bar{B} \otimes F)$ the analog of *Pert*.
Dualize with impunity and consider

$$\mathcal{A}(s\mathcal{L}(B, F))$$

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Theorem

For connected B and reasonable F, free as gca and of finite type, the set of strong fibre homtopy equivalence classes of fibrations

$$B \to E \to F$$

is in bijection with the set of homotopy classes of maps $\mathcal{A}(s\mathcal{L}(B, F)) \to B$.

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$\mathcal{A}(s\mathcal{L}(B, F))$ is very much like a model for the classifying space, but...

The problem is that it has terms of negative degree, so presto changeo we

A string of intersections

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String topology is a chain or homology theory.

String field theory is a cochain or form or cohomology theory.

Comparison

String topology works with chains and intersection algebra structures on a space of strings in a manifold.

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String field theory ∞-convolution algebras involve integration over appropriate moduli spaces.

Compactified configuration and moduli spaces

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WARNING: We need a Linneaus to organize the zoo.

Example: OCHA = Open Closed Homotopy Algebra

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Basic idea: An L_∞-algebra L with an ∞-action via ∞-derivations on an A_∞-algebra A.

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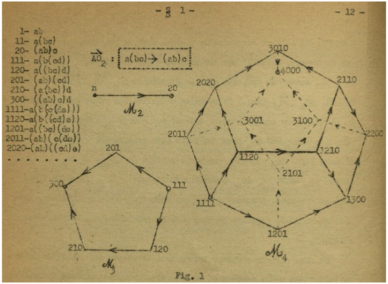
there are 'horrible' maps

$$L^{\otimes p} \longrightarrow A;$$

the operation $L \longrightarrow A$ corresponds to closing an open string.

The associahedra



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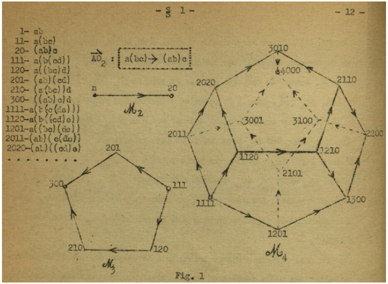

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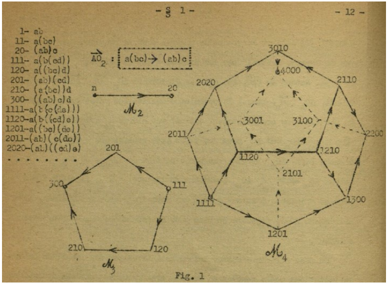

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Realization as convex polytopes, even with integer coefficients.

A la prochaine

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Dennis:

Best wishes for many happy years ahead and fruitful interactions/intersections.