Algebraic patterns for dynamical systems

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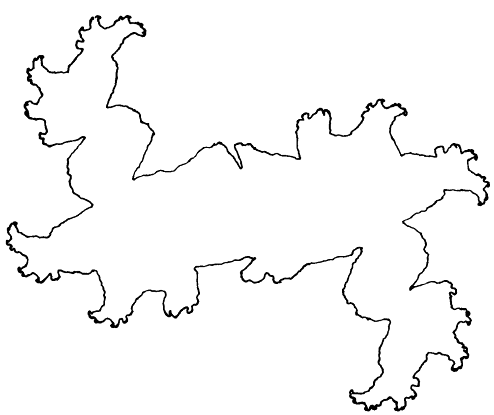
Quasiconformal homeomorphisms and dynamics
I. Solution of the Fatou-Julia problem on wandering domains

By Dennis Sullivan

Introduction

If one perturbs the analytic dynamical system $z \overset{R}{\to} z^2$ on the Riemann sphere $\overline{\mathbf{C}}$ to $z \overset{R_a}{\to} z^2 + az$ for small a, the following happens: Before perturbation the round unit circle C is invariant under iteration of R and R is expanding on C ($|R'(z)| > 1$), R has dense orbits, and is even ergodic on C relative to linear measure. After perturbation R_a now preserves a unique Jordan curve C_a close to C and again R is expanding and has dense orbits on C_a. Now C_a is not a rectifiable curve. It is a fractal curve with Hausdorff dimension > 1 which increases with $|a|$. (Figure 1). The intricacies of C are of a self-similar nature



These Poincaré deformations (1883) have been t

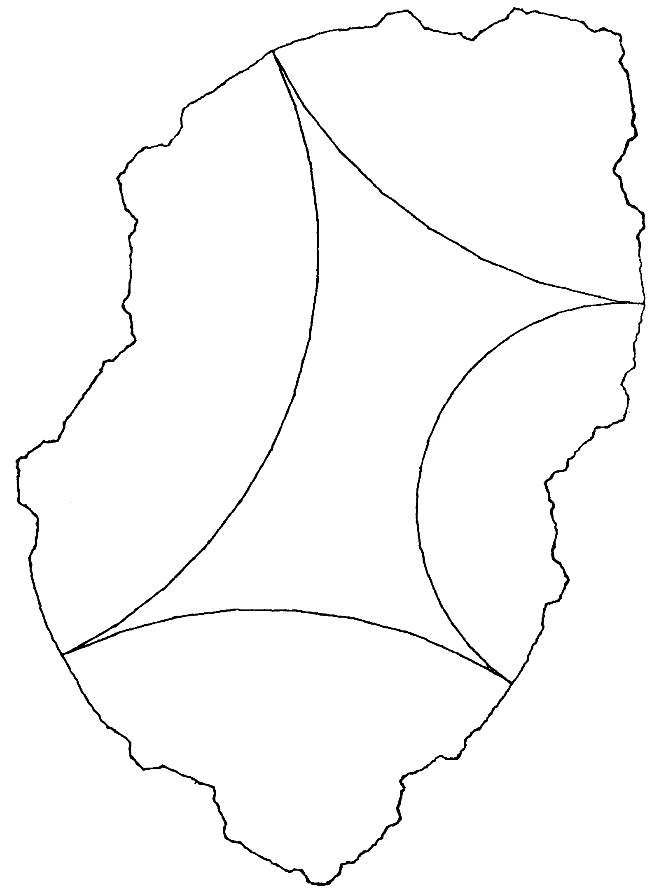


Figure 2

The Dictionary

Sullivan, 1970/80s

We close with a sample of the dictionary between analytic iteration and crete subgroups of PSL(2, **C**) which lies behind this series of papers.

Complex analytic iteration	*Discrete subgroups of* PSL(2, **C**)
entire mapping	arbitrary Kleinian group
Blaschke product	arbitrary Fuchsian group
rational mapping, R	finitely generated Kleinian group, Γ
degree of mapping, d	number of generators, n
$(2d-2)$ analytic parameters	$(3n-3)$ analytic parameters
$(2d-2)$ critical points	(?) ends of hyperbolic 3 manifolds
Fatou-Julia limit set ([8], [11])	Poincaré limit set (1880)

Carpets

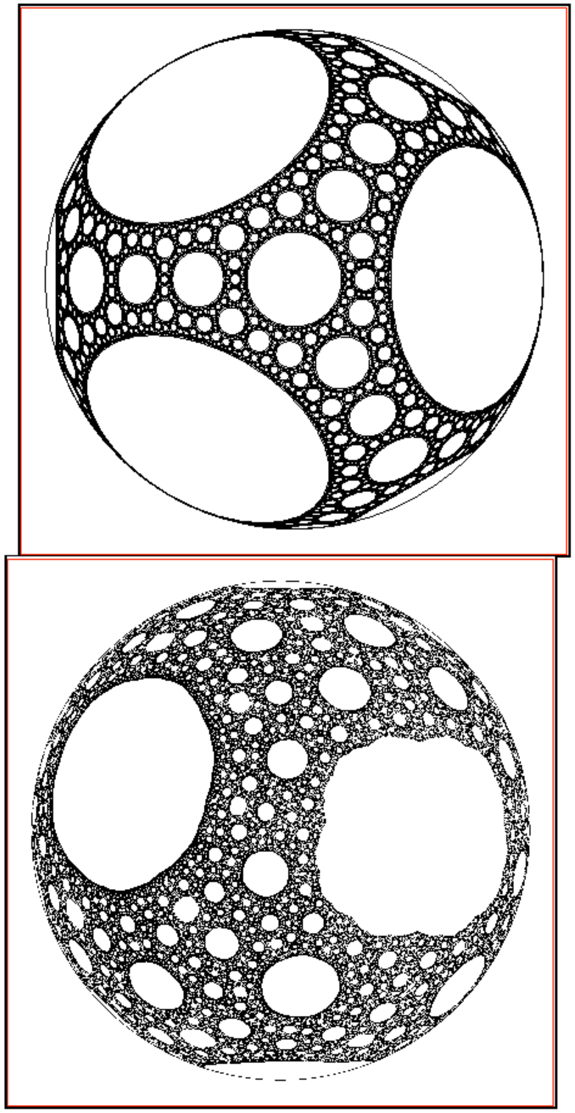
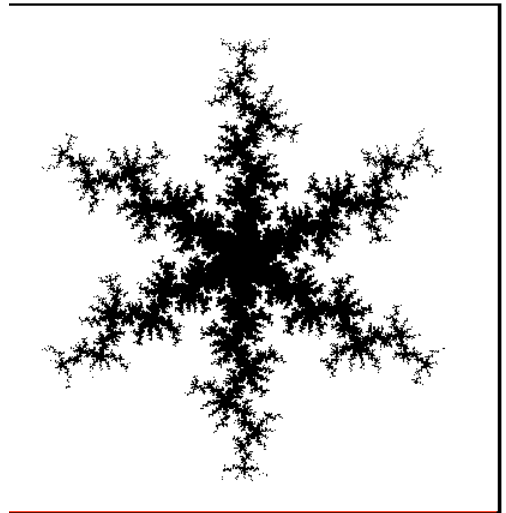
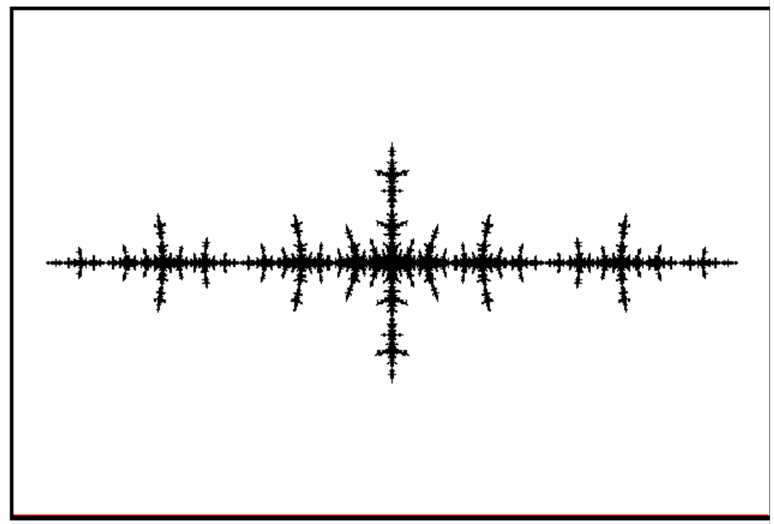
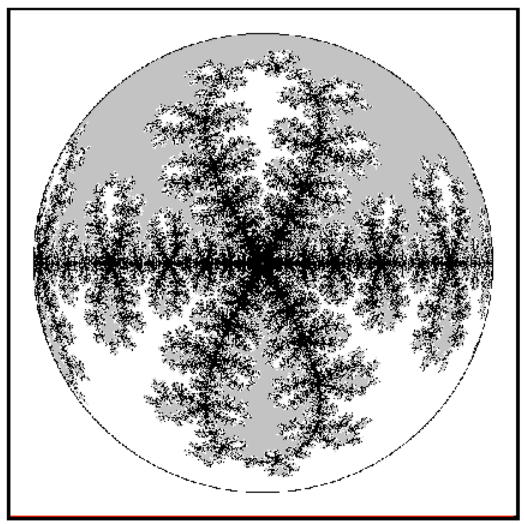


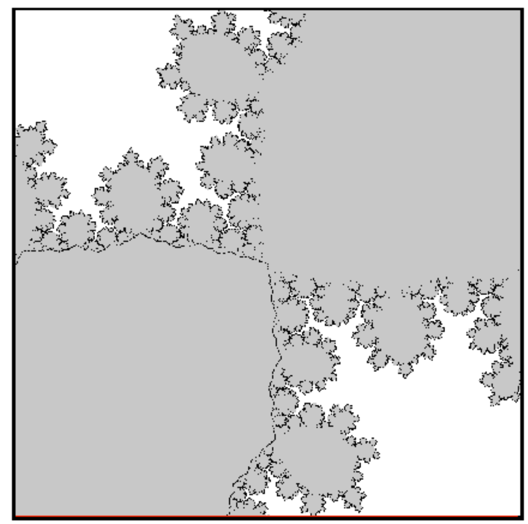
Figure 6. Sierpiński curves.

Renormalization









Hyperbolic manifolds	Interval maps	Siegel disks/ Circle maps
Discrete surface group $\Gamma \subset PSL_2(\mathbb{C})$ $M = \mathbb{H}^3/\Gamma$	$\mathbb{R}$-quadratic polynomial $f(z) = z^2 + c$	Nonlinear rotation $f(z) = \lambda z + z^2$ or $\lambda z^2(z-3)/(1-3z)$
Representation $\rho : \pi_1(S) \to \Gamma$	Quadratic-like map $f : U \to V$	Holomorphic commutin pair (f, g)
Ending lamination $\epsilon(M) \in \mathcal{GL}(S)$	Tuning invariant $\tau(f) = \langle \sigma(\mathcal{R}^n(f)) \rangle$	Continued fraction $\theta = [a_1, a_2, \cdots]$, $\lambda = e^{2\pi}$
Inj. radius$(M) > r > 0$	Bounded combinatorics	Bounded type
Cut points in Λ $= \bigcup_1^\infty$ (Cantor sets)	Postcritical set $P(f) = \overline{\bigcup f^n(c)}$, $f'(c) = 0$ = (Cantor set)	= (circle or quasi-circle
($\mathbb{R}$-tree of $\epsilon(M), \pi_1(S)$)	$(\varprojlim \mathbb{Z}/p_i, x \mapsto x+1)$	$(\mathbb{R}/\mathbb{Z}, x \mapsto x + \theta)$
$\Lambda(\Gamma)$ is locally connected	$J(f)$ is locally connected	$J(f)$ is locally connected
area $\Lambda(\Gamma) = 0$	area$(J(f)) = 0$?	
Inj. radius $\in [r, R]$ in core(M)	$(\mathcal{F}(f), J(f))$ is uniformly twisting	
Mapping class $\psi \in \mathrm{Mod}(S)$	Kneading permutation	Automorphism $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $\mathbb{Z}^2$
Renormalization Operators $\mathcal{R}(\rho) = \rho \circ \psi^{-1}$	$\mathcal{R}(f) = f^p(z)$	$\mathcal{R}(f,g) = (f^a g^b, f^c g^d)$
Stable Manifold of Renormalization M = asymptotic fiber	f = limit of doublings	θ = golden ratio
Elliptic points deep in $\Lambda(\Gamma)$	Critical point $c_0(f)$ deep in $J(f)$ or $K(f)$	
$\rho \circ \psi^{-n}$, $n = 1, 2, 3 \ldots$	f^n, $n = 1, 2, 4, 8, 16, \ldots$	f^n, $n = 1, 2, 3, 5, 8, \ldots$
Geometric limit of $\mathcal{R}^n(\rho)$	Quadratic-like tower $\langle f_i : i \in \mathbb{Z} \rangle$; $f_{i+1} = f_i \circ f_i$	Tower of commuting pairs
Hyperbolic 3-manifold $S \times [0,1]/\psi$ fibering over the circle	Fixed-points of Renormalization	
Conformal structure is $C^{1+\alpha}$-rigid at deep points $\Longrightarrow$ Renormalization converges exponentially fast M is asymptotically rigid	$J(f)$ is self-similar at the critical point $c_0(f)$	

Parallels	
Infinitely renormalizable map $f(z) = z^2 + c$, $c \in \mathbb{R}$	Siegel linearizable r $f(z) = e^{2\pi i\theta} z + z^2$, θ
Tuning invariant	Continued fractic
$c = s_1 * s_2 * s_3 \cdots$	$\theta = [a_1, a_2, \cdots]$
Bounded combinatorics	Bounded type
$P(f)$ = quasi-Cantor set	$P(f)$ = quasi-circ
Quadratic-like map	Holomorphic pai
Feigenbaum polynomial	Golden mean polyno
$(\mathbb{Z}_2, x+1)$	$(S^1, x+\theta)$
f^n, $n = 1, 2, 4, 8, 16, \ldots$	f^n, $n = 1, 2, 3, 5, 8, 1$
$(\mathcal{F}(f), J(f))$ is uniformly twisting	
The critical point of f is a deep point of $K(f)$	
Conjugacies are $C^{1+\alpha}$ on $P(f)$	

Table 2.

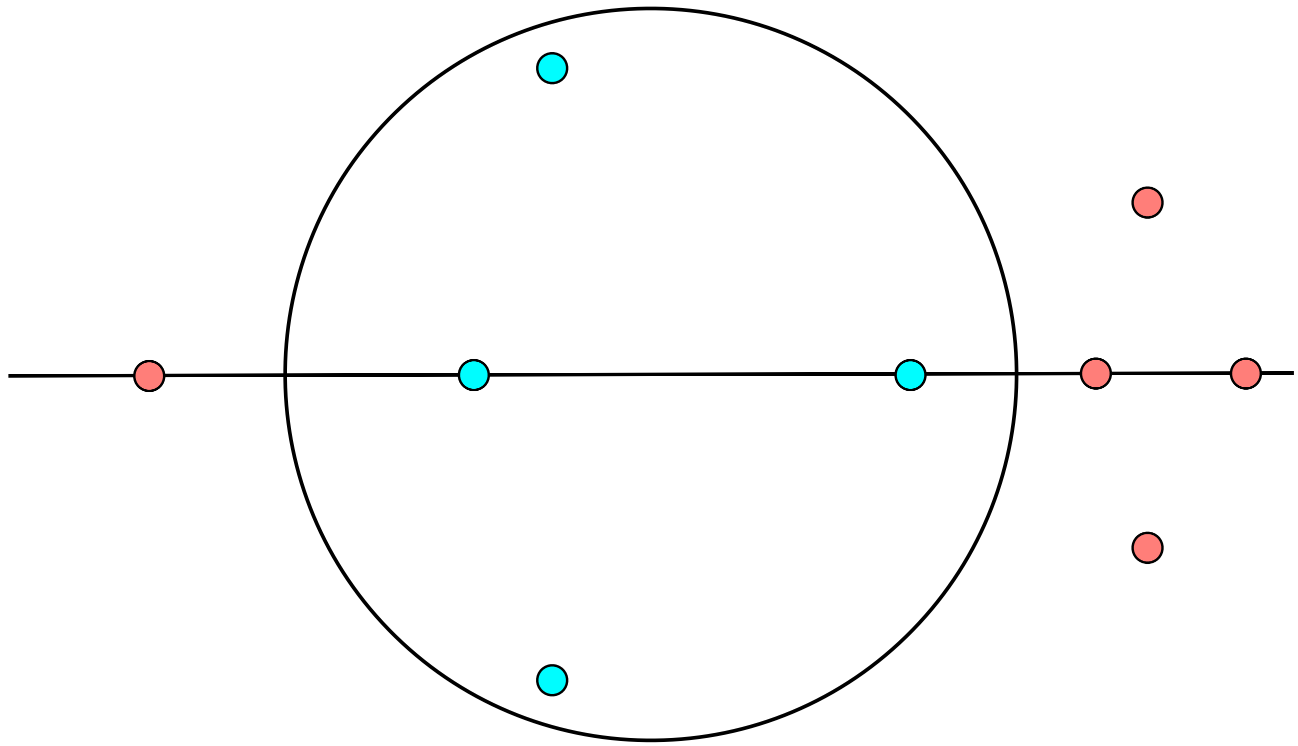
Riemann surfaces	D
Fuchsian group $G \subset \mathrm{Aut}(\Delta)$	Blaschke p
Quasifuchsian group Γ	Mating $F(z)$ of
Unit tangent bundle $\mathrm{T}_1(X)$	Riemann su
Geodesic flow	Susp
Closed geodesic γ	Periodi
Length of γ	Log of the r
Length of a random geodesic	Growth of (f^n)
Weil-Petersson metric on $\mathcal{T}_g$	Metric

Table 1.

Dictionary	
Kleinian group $\Gamma \cong \pi_1(S)$	Quadratic-like map $f : U$ –
Limit set $\Lambda(\Gamma)$	Julia set $J(f)$
Bers slice B_Y	Mandelbrot set M
Mapping class $\psi : S \to S$	Kneading permutation
$\psi : AH(S) \to AH(S)$	Renormalization operator
Cusps in ∂B_Y	Parabolic bifurcations in
Totally degenerate group Γ	Infinitely renormalizable polynomial $f(z) = z^2 +$
Ending lamination	Tuning invariant
Fixed point of ψ	Fixed point of R_p
Hyperbolic structure on $M^3 \to S^1$	Solution to Cvitanović-Feigen equation $f^p(z) = \alpha^{-1} f(\alpha$

Algebraic integers

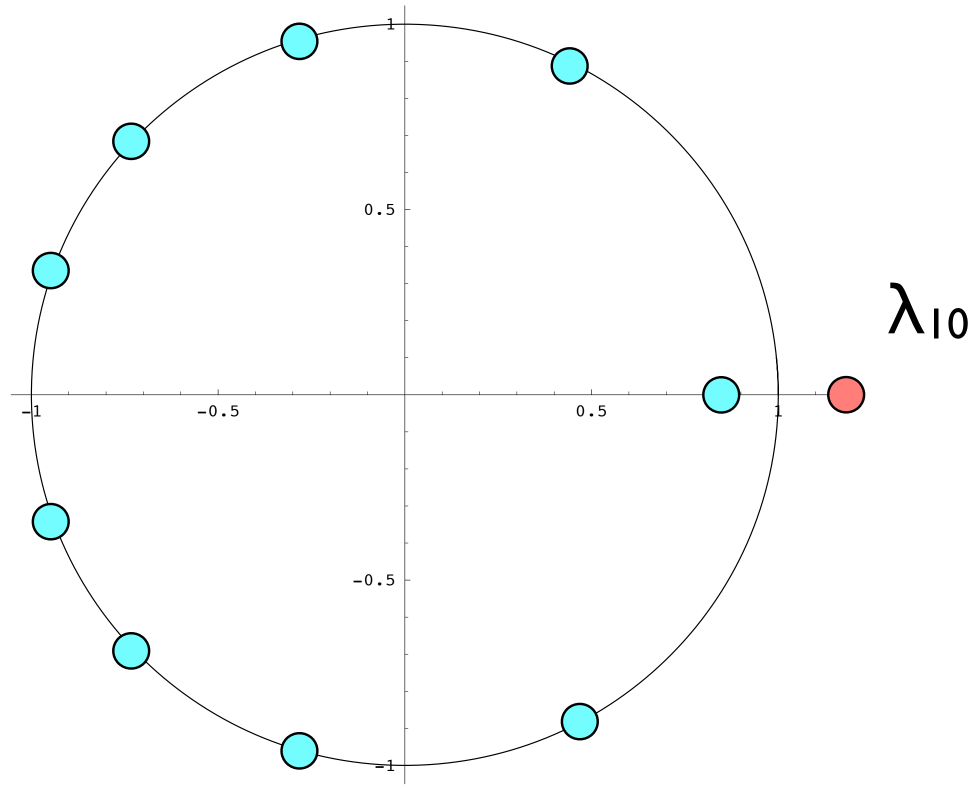
What is the smallest integer λ > 1?



Mahler measure

M(λ) = product of conjugates with |λ_i| > 1

Lehmer's Number



$$\lambda_{10} = 1.176280...$$

$$P_{10}(x) = x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1$$

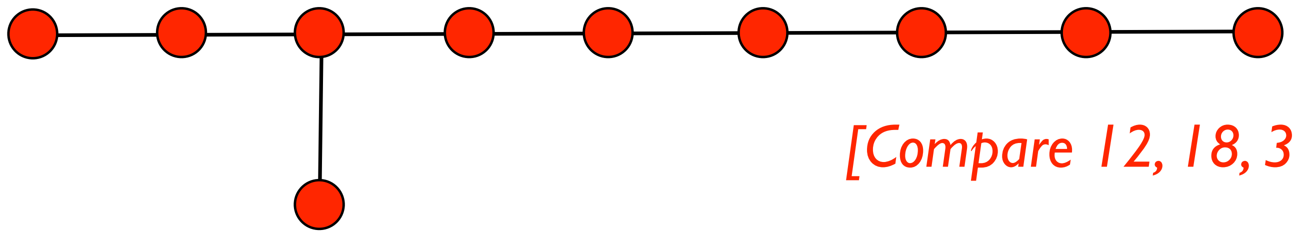
Smallest Salem Numbers, by Degree

		$P_d(x)$
λ_2	2.61803398	$x^2 - 3x + 1$
λ_4	1.72208380	$x^4 - x^3 - x^2 - x + 1$
λ_6	1.40126836	$x^6 - x^4 - x^3 - x^2 + 1$
λ_8	1.28063815	$x^8 - x^5 - x^4 - x^3 + 1$
λ_{10}	1.17628081	$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$
λ_{12}	1.24072642	$x^{12} - x^{11} + x^{10} - x^9 - x^6 - x^3 + x^2 - x + 1$
λ_{14}	1.20002652	$x^{14} - x^{11} - x^{10} + x^7 - x^4 - x^3 + 1$

γ^2

Conjecture (Lehmer) $\lambda_{10} = \inf M(\alpha)$ over all algebraic integers with $M(\alpha)>0$.

Lehmer's polynomial = det(xI-w) for E_{10}

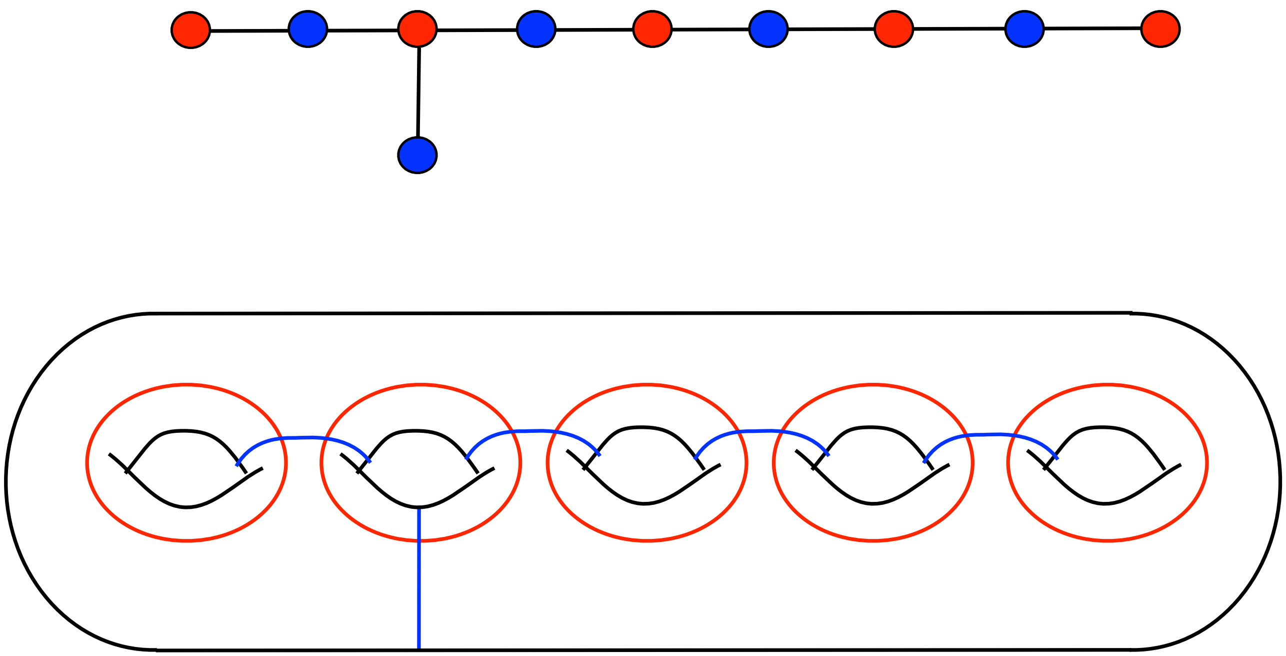


[Compare 12, 18, 30, ∞]

Theorem(2002). *The spectral radius of any w in any Coxeter group satisfies* $r(w) = 1$ *or* $r(w) \geq \lambda_{10} > 1$.

Proof: uses Hilbert metric on the Tits cone.

Lehmer's number for topologists



$\psi = \tau_A \tau_B$ in the mapping-class group for genus 5

$h(\psi) = \log \lambda_{10}$

Entropy

Entropy of English = h = about log 3 (or less)

Schneier, Applied Cryptography, 1996

Number of possible English books with N characters is about 3^N (not 26^N)

X compact, $f : X \to X$ continuous

$h(f) = \log \lambda \Leftrightarrow$

$|\{\text{orbit patterns of length } N\}| \sim \lambda^N$.

Torus examples

X = torus R^n/Z^n

f : X → X linear map induced by A in $GL_n(Z)$

h(f) = log (product of eigenvalues of A with $|\lambda| > 1$)

= log [spectral radius of $f^* \mid H^*(X)$]

Lehmer's conjecture ⇔

$h(f) \geq \log \lambda_{10}$ for torus maps.

Entropy on Complex Surfaces

X = smooth projective surface over $\mathbb{C}$

$f : X \to X$ holomorphic

Q. What small values can h(f) assume?

(dim=1 $\Rightarrow$ zero entropy)

Entropy and Salem numbers

cf. Shub's entropy conjecture

Theorem (Gromov, Yomdin)

$$h(f) = \log [\text{spectral radius of } f|H^*(X)] \,.$$

Corollary. *For projective surfaces,*

$$h(f) = \log [\text{a Salem number } \lambda]$$
$$= \text{spectral radius on } H^2(X)$$

Flavors of Projective Surfaces

Theorem (Cantat) *A surface X admits an automorphism $f : X \to X$ with positive entropy only if X is birational to:*

4	• *a complex torus $\mathbb{C}^2/\Lambda$,*	$\log(\lambda_4)$
22	• *a K3 surface*, or*	$\log(\lambda_{10})$
∞	• *the projective plane $\mathbb{P}^2$.*	

(*or Enriques)

Q. What is the minimum of h(f) for each type?

A. It is the minimum consistent with Lehmer's conjecture.

Theorem (Sullivan, 1971)

The mapping-class group
of a simply-connected compact manifold X
is an arithmetic group.

Synthesis Problem:

Salem number

⇒ automorphism of Hodge theory

⇒ projective surface + map

Abelian varieties $\mathbb{C}^2/\Lambda$

Theorem. *For a projective torus, one can achieve* $h(f) = \log(\lambda_4)$ *and this is optimal.* $(\lambda_4 = 1.722..)$

Synthesis: $f|H^1(X,\mathbb{Z}) \simeq \mathbb{Z}^4 \Rightarrow \Lambda \subset \mathbb{C}^2 \Rightarrow X = \mathbb{C}^2/\Lambda$

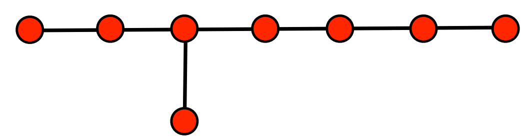
$$f = \begin{bmatrix} 0 & \omega \\ 1 & 1 \end{bmatrix} : E\times E \to E\times E \qquad E = \mathbb{C}/\mathbb{Z}[\omega]$$

Rational Surfaces

X = blowup of $\mathbb{P}^2$ at n points

$H^2(X,\mathbb{Z}) \simeq \mathbb{Z}^{1,n} \supset K_X^{\perp} \simeq$ [E_n lattice]

$K_X = (-3,1,1,\ldots,1)$

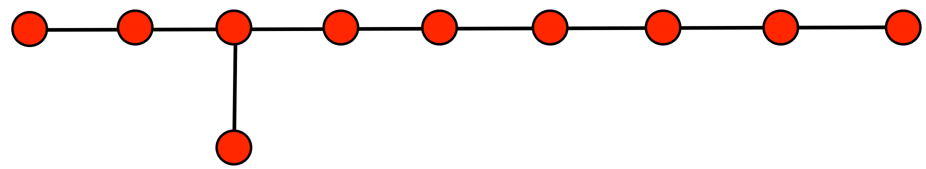


$Aut(X) \subset W(E_n)$ *(Nagata)*

Theorem. *The Coxeter automorphism of E_n can be realized by an automorphism $F_n : X_n \to X_n$ of $\mathbb{P}^2$ blown up at n suitable points.*

Lehmer's automorphism

$F_{10} : X_{10} \to X_{10}$



First case where $h(F_n) > 0$

Theorem (2005). The map F_{10} has minimal positive entropy among all surface automorphisms, namely

$$h(F_{10}) = \log(\lambda_{10}).$$

Rational Surfaces: Synthesis

X = blowup of n points on a cuspidal cubic C in $\mathbb{P}^2$

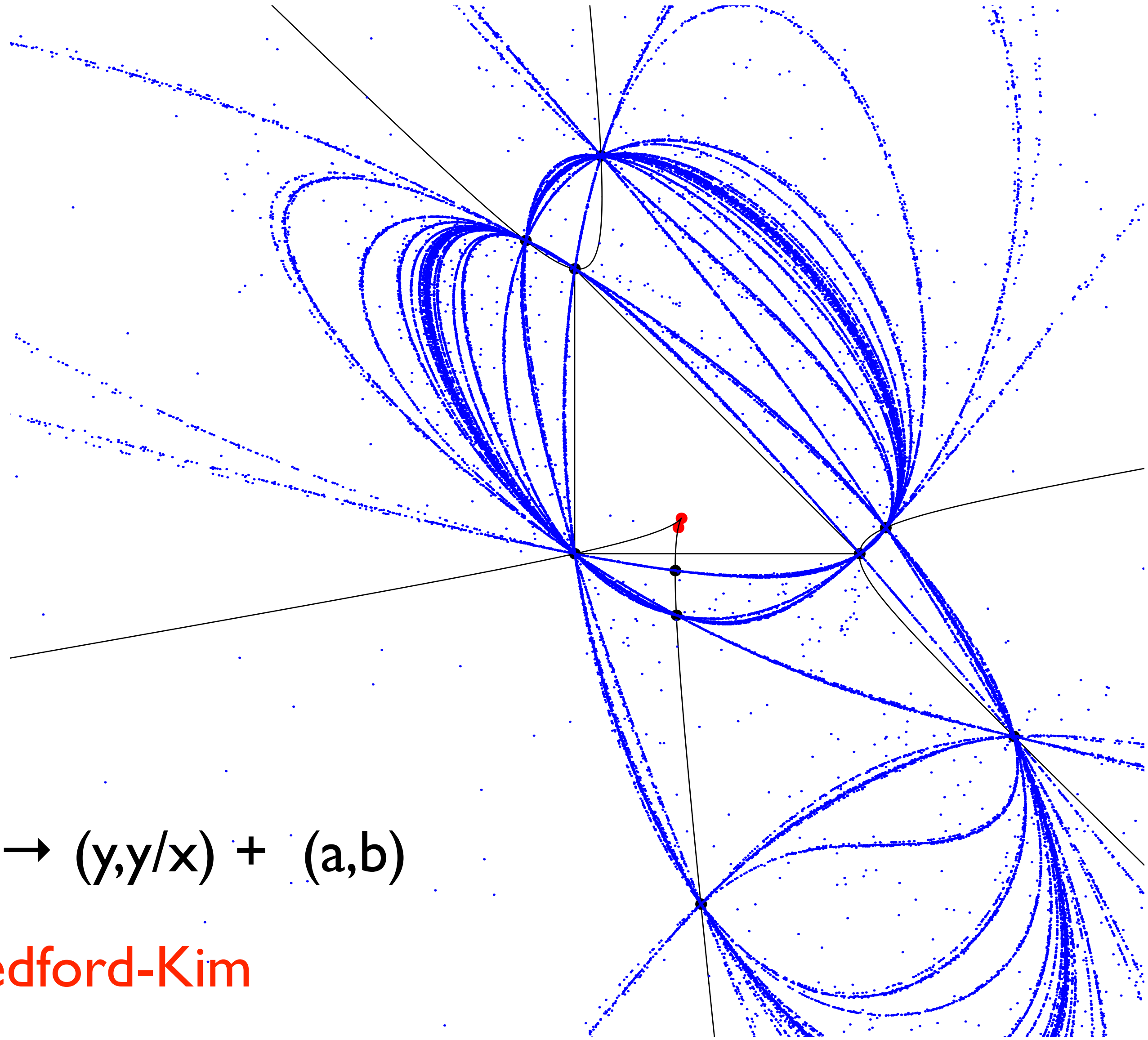
$$[E_n \text{ lattice}] \simeq \mathrm{Pic}^0(X_n) \rightarrow \mathrm{Pic}^0(C) \simeq \mathbb{C}$$

$\hookrightarrow$ Coxeter element w

$\hookrightarrow$ Eigenvalue λ of w

λ eigenvector of w ⇒ positions of n points on C

10 points on a cuspidal cubic



$(x,y) \to (y, y/x) + (a,b)$

Bedford-Kim

K3 surfaces/$\mathbb{R}$

Mazur

$X \subset \mathbb{R}^3$ defined by

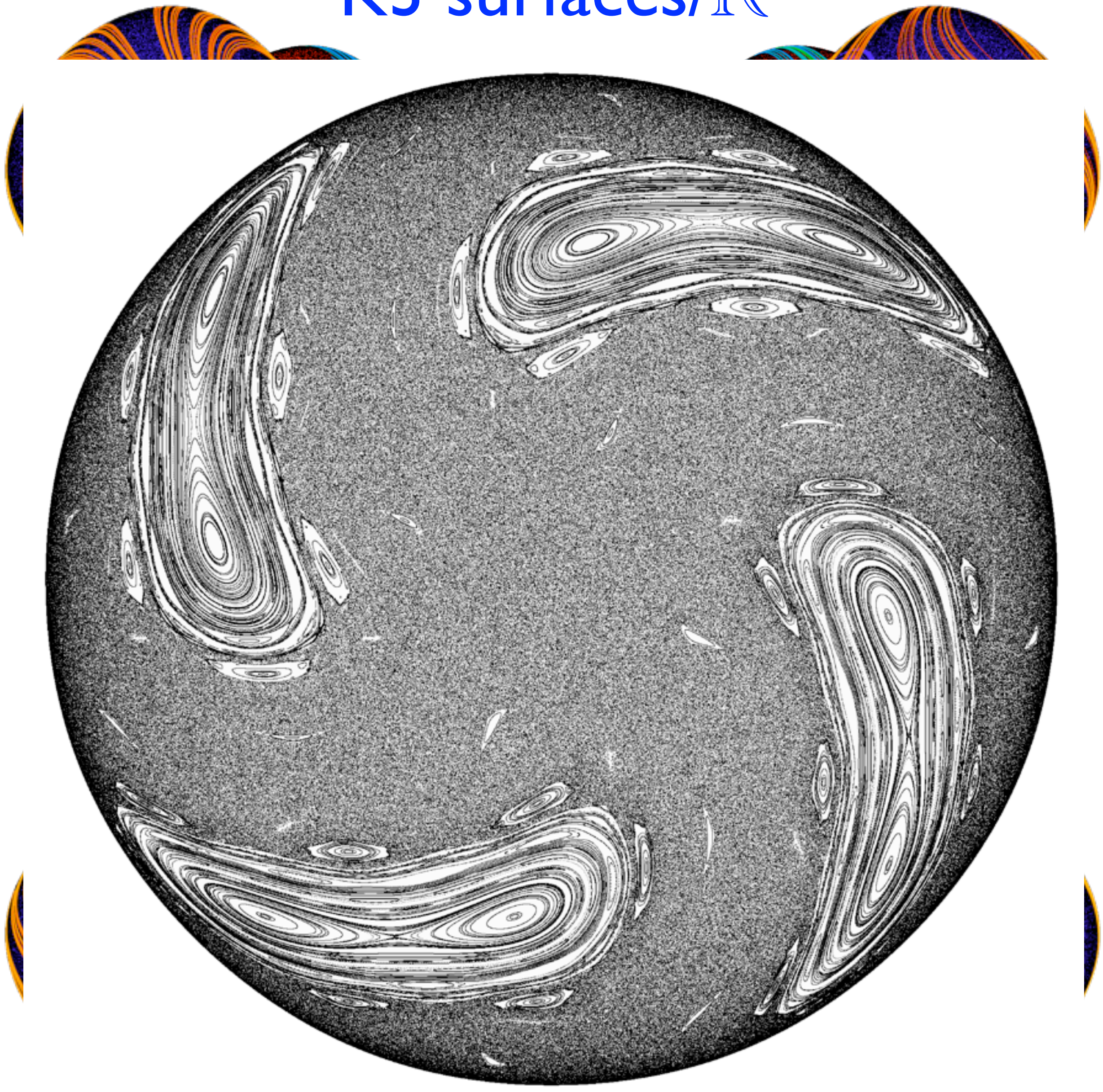
$$(1+x^2)(1+y^2)(1+z^2) + A\,xyz = 2$$

$f: X \to X$ defined by

$$f = I_x \circ I_y \circ I_z$$

The map f is area-preserving!

K3 surfaces/$\mathbb{R}$



K3 Surfaces: Synthesis

Gross-M, 2002

Input: Degree 22 Salem polynomial with $|P(\pm 1)|=1$.

$$P(t) = 1+t-t^3-2t^4-3t^5-3t^6-2t^7+2t^9+4t^{10}+5t^{11} +4t^{12}+2t^{13}-2t^{15}-3t^{16}-3t^{17}-2t^{18}-t^{19}+t^{21}+t^{22}$$

Output: K3 surface X and $f : X \to X$, with $\det(tI-f|H^2(X)) = P(t)$.

X is not projective!

Islands over $\mathbb{C}$

Theorem. *There exists a K3 surface automorphism $f: X \to X$ with positive entropy and an invariant island -- a Siegel disk. Any such example is non-projective.*

Theorem (Oguiso, 2003). *Blowing up $X x X$ gives a simply-connected 4-dimensional counterexample to the Kodaira conjecture.*

h(f) = log [a degree 22 Salem number]

Pic(X) = 0

cf. Voisin

K3 surfaces and glue

Theorem (2009) *There exists a K3 surface automorphism with* $h(f) = \log(\lambda_{10})$, *and this is optimal*.

(Gross-M - deg(λ)=22)

(Oguiso - $\lambda_{14} = 1.2002...$*)*

($\lambda_{10} = 1.176...$)

K3 automorphism with $h(f) = \log \lambda_{10}$

Coxeter automorphism of $E_{10}(a)$
$(3,7)$

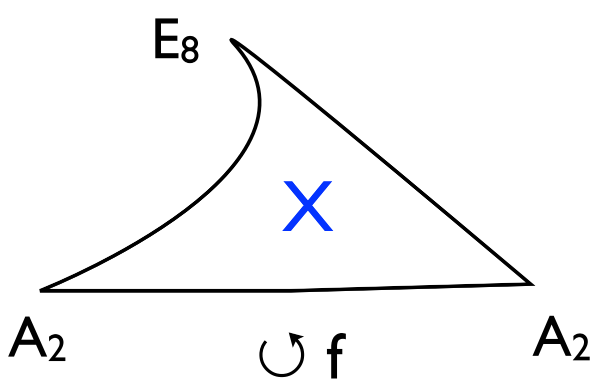
$\mathbb{F}_3^2$

Positive automorphism of $A_2 \oplus A_2$
$(0,4)$

Identity factor E_8
$(0,8)$

$H^{2,0}+H^{0,2}$
+transcendental cycles

$\circlearrowleft$



Pic(X)

Signature (0,12)
determinant 9
blows down to 3 points

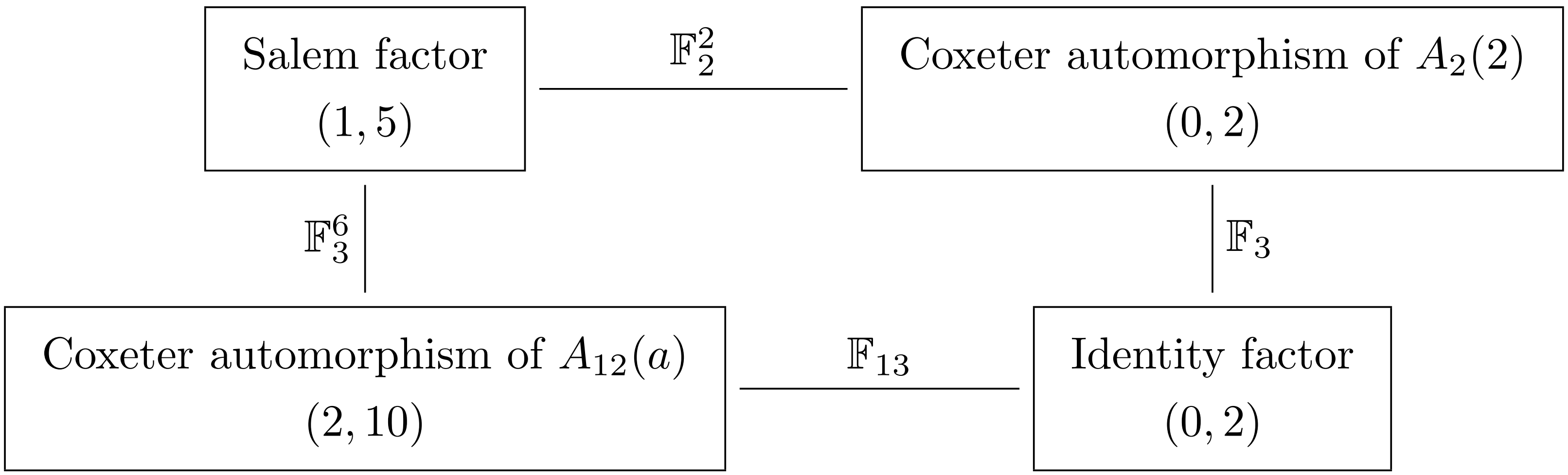
Projective K3 surfaces

Theorem (2009). *For projective K3 surfaces, one can achieve* $h(f) = \log(\lambda_6)$.

Theorem (2011). *In fact, one can achieve* $h(f) = \log(\lambda_{10})$, *and this is optimal.*

$(\lambda_6 = 1.401\ldots >> \lambda_{10} = 1.176)$

Projective K3 -- entropy log λ_6



$H^{2,0}+H^{0,2}$

+transcendental cycles

Pic(X): signature (1,9), determinant $9477 = 3^6 \cdot 13$

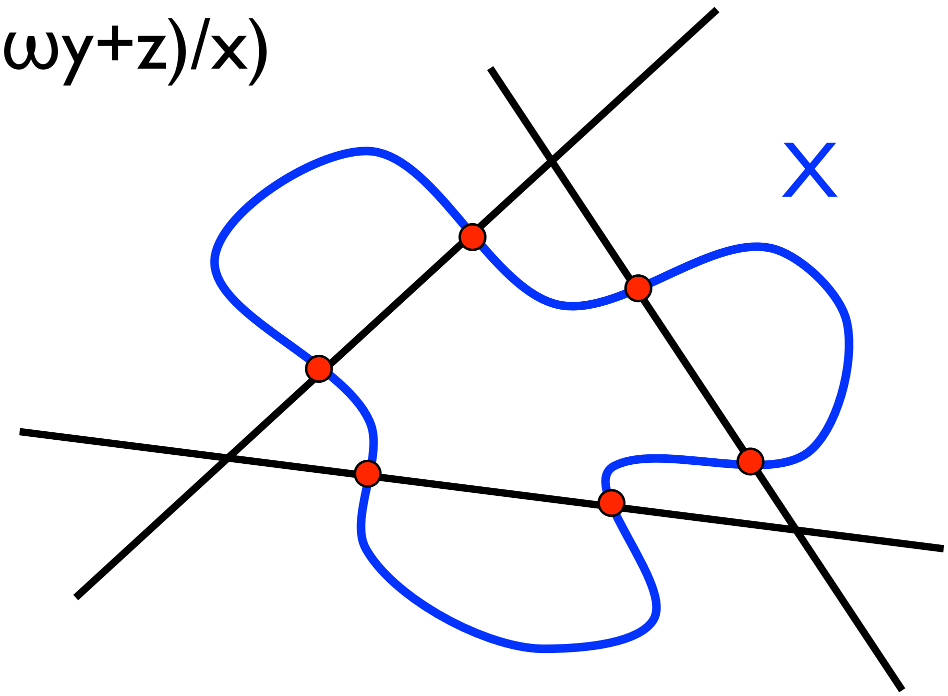
Projective K3 with h(f) = log(λ_8)

Bedford-Kim, 2011

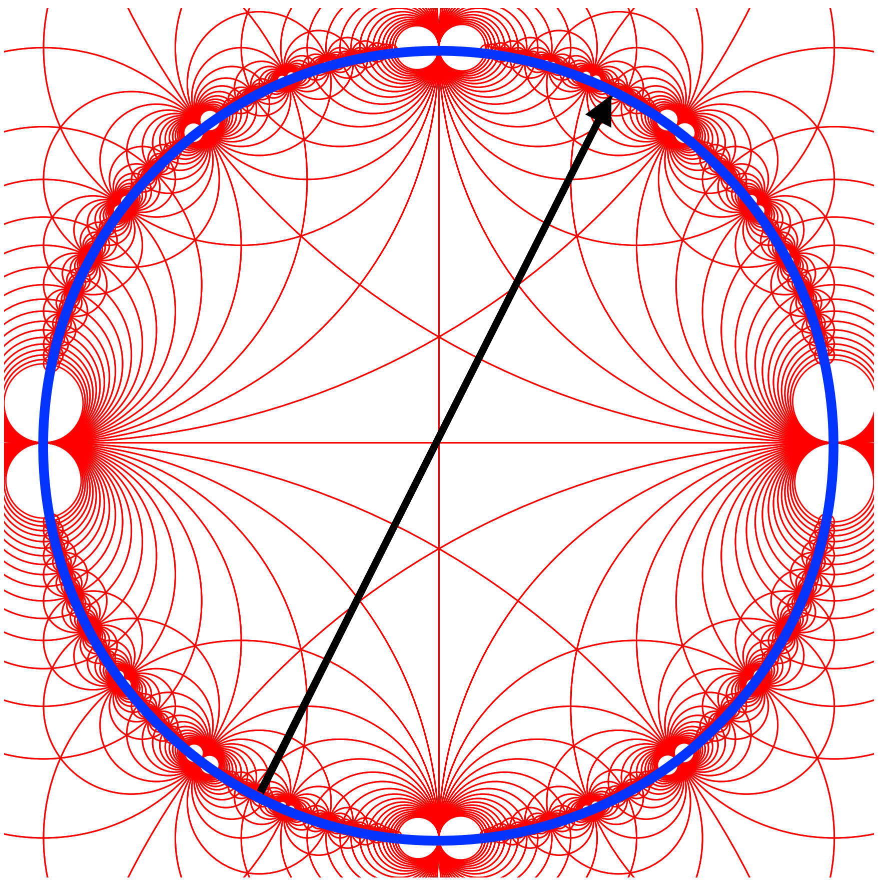
$$f : \mathbb{P}^3 \rightarrow \mathbb{P}^3$$

$$(x, y, z) \rightarrow (y, z, (a+\omega y+z)/x)$$

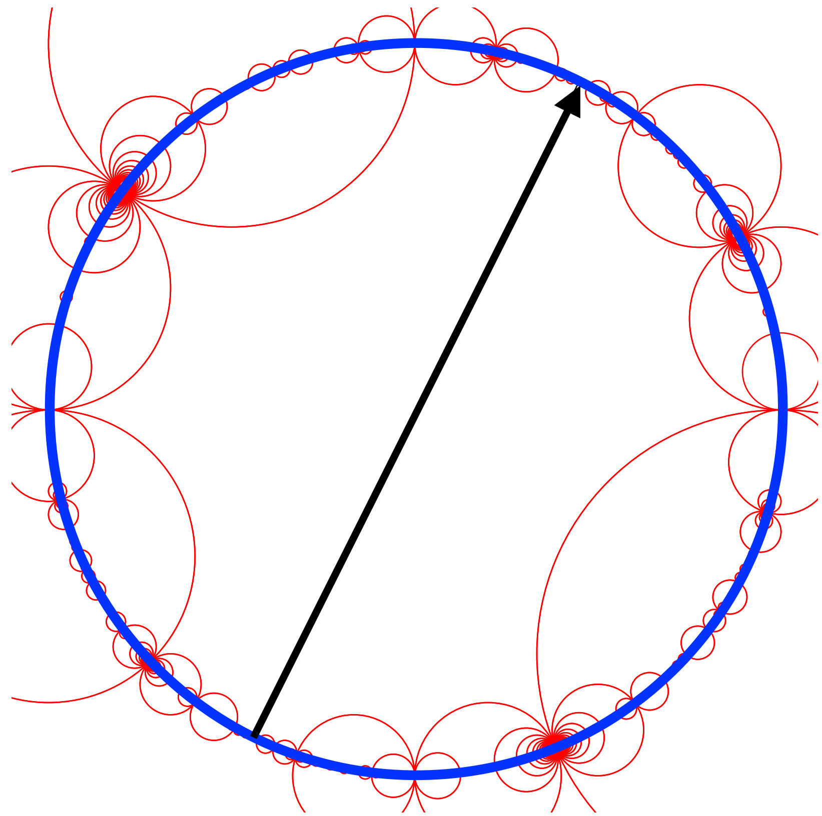
Preserves a pencil of quartic = K3 surfaces in projective space



Opening the Kähler cone



Obstructed



Conservative solution *Liberal solution*