# Expanding Thurston maps

Mario Bonk

UCLA

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# Branched covering maps

Let $S^2$ be a topological 2-sphere. A map $f\colon S^2 \to S^2$ is a *branched covering map* iff

- it is continuous and orientation-preserving,
- near each point $p \in S^2$, it can be written in the form $z \mapsto z^d$, $d \in \mathbb{N}$, in suitable complex coordinates.

$d = \deg_f(p)$ *local degree* of $f$ at $p$.

$C_f = \{p \in S^2 : \deg_f(p) \geq 2\}$ set of *critical points* of $f$.

**Remark:** Every rational map $R\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ on the Riemann sphere $\widehat{\mathbb{C}}$ is a branched covering map.

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# The postcritical set

If $f\colon S^2 \to S^2$ is a branched covering map, then

$$P_f = \bigcup_{n\in\mathbb{N}} f^n(C_f)$$

is called the *postcritical set* of $f$. Here $f^n$ is the $n$th-iterate of $f$.

**Remarks:** Points in $P_f$ are obstructions to taking inverse branches of $f^n$. Each iterate $f^n$ is a covering map over $S^2 \setminus P_f$.

# Thurston maps

A map $f\colon S^2 \to S^2$ is called a *Thurston map* iff

- it is a branched covering map,
- it has a finite postcritical set $P_f$.

Different viewpoints on Thurston maps:

- $f$ well-defined only up to isotopy relative to $P_f$ (one studies dynamics on isotopy classes of curves etc.), or
- $f$ pointwise defined (one studies pointwise dynamics under iteration etc.).

Often one wants to find a "good representative" $f$ in a given isotopy class.

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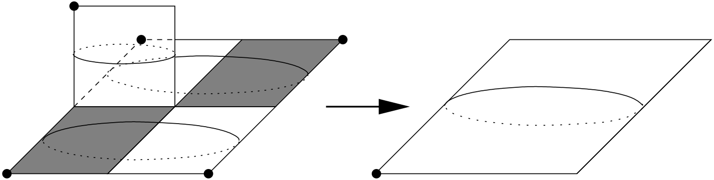
# Example of a Thurston map I



- $\#C_f = 6$,
- $\#P_f = 4$,
- Subdivision rule: Combinatorial data specifying how the two level-0 tiles are subdivided by 6 and 4 level-1 tiles, respectively.

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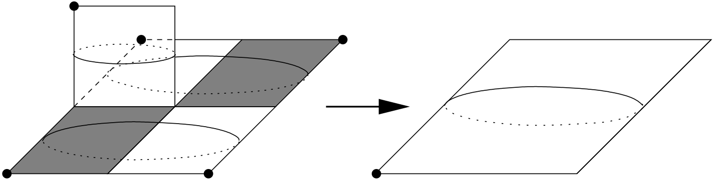

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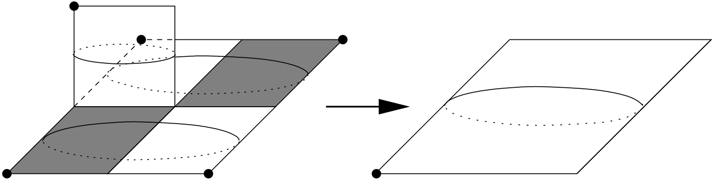

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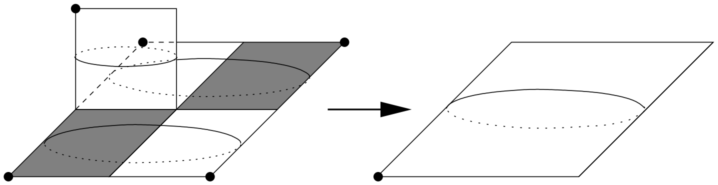

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# A basic problem

When is an expanding Thurston map $f$ conjugate to a rational map? So when is there a homeomorphism $\phi\colon S^2 \to \widehat{\mathbb{C}}$ and a rational map $R\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ s.t.

$$\begin{array}{ccc} S^2 & \stackrel{\phi}{\longleftrightarrow} & \widehat{\mathbb{C}} \\ \big\downarrow f & & \big\downarrow R \\ S^2 & \stackrel{\phi}{\longleftrightarrow} & \widehat{\mathbb{C}} \end{array}$$

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# Tiles

Let $n \in \mathbb{N}_0$, $f\colon S^2 \to S^2$ be a Thurston map, and $J \subseteq S^2$ be a Jordan curve with $P_f \subseteq J$. Then a *tile of level $n$* or *$n$-tile* is the closure of a complementary component of $f^{-n}(J)$.

- tiles are topological 2-cells (=closed Jordan regions),
- tiles of a given level $n$ form a cell decomposition $\mathcal{D}^n$ of $S^2$.
- the cell decompositions $\mathcal{D}^n$ for different levels $n$ are usually not compatible (only if $J$ is invariant, i.e., $f(J) \subseteq J$ equiv. $J \subseteq f^{-1}(J)$).

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# Example of a Thurston map II

$$f(z) = 1 + \frac{\omega - 1}{z^3}, \qquad \omega = e^{4\pi i/3}.$$

$C_f = \{0, \infty\}$. Orbits of critical points: $0 \mapsto \infty \mapsto 1 \mapsto \omega \mapsto \omega$.
$P_f = \{1, \omega, \infty\}$, $J =$ line through 1, $\omega$, $\infty$.

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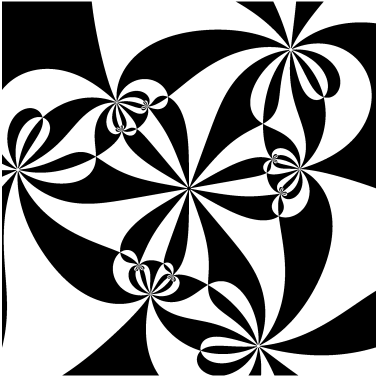

Tiles of level 4

# Expanding Thurston maps

A Thurston map $f\colon S^2 \to S^2$ is *expanding* if the size of $n$-tiles goes to 0 uniformly as $n \to \infty$; so we require

$$\lim_{n\to\infty} \max_{n\text{-tile } X^n} \operatorname{diam}(X^n) = 0.$$

This is:

- independent of Jordan curve $J$,
- independent of the underlying base metric on $S^2$.

**Remark:** A rational Thurston map $R$ is expanding iff $R$ has no periodic critical points iff $\mathcal{J}(R) = \widehat{\mathbb{C}}$ for its Julia set.

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**Theorem.** (B.-Meyer) Let $f$ be an expanding Thurston map. Then for each sufficiently high iterate $f^n$ there exists a (forward-)invariant quasicircle $\mathcal{C} \subseteq S^2$ with $P_f = P_{f^n} \subseteq \mathcal{C}$.

**Corollary.** (B.-Meyer, Cannon-Floyd-Parry) Let $f$ be an expanding Thurston map. Then every sufficiently high iterate $f^n$ is described by a subdivision rule.

**Remark:** If $J \subseteq S^2$ is an arbitrary Jordan curve with $P_f \subseteq J$, then there exists $n$, and a quasicircle $\mathcal{C}$ isotopic to $J$ rel. $P_f$ s.t. $f^n(\mathcal{C}) \subseteq \mathcal{C}$.

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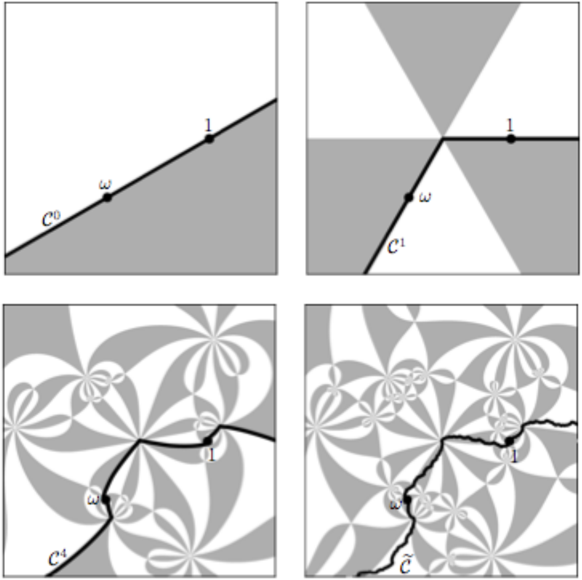
# Iterative construction of invariant curves



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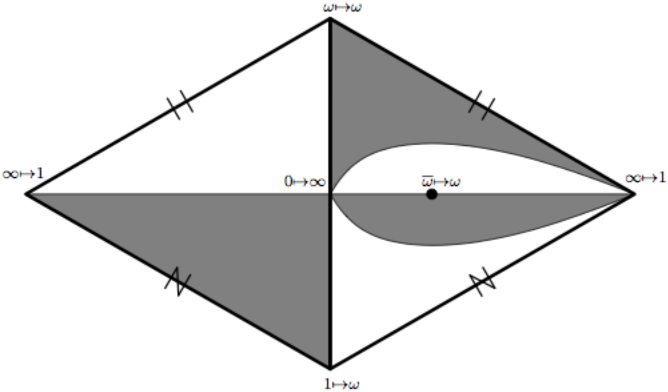

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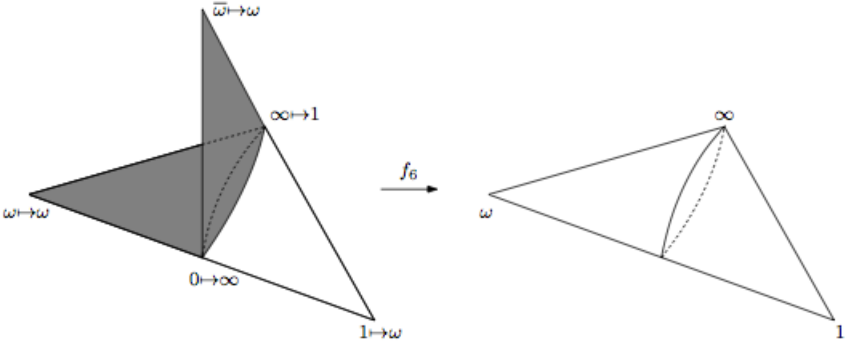

**Proposition.** Let $f$ be an expanding Thurston map. Then there exists a metric $d$ on $S^2$ unique up to snowflake equivalence s.t. for all $n$-tiles $X^n$,

$$d\text{-diam}(X^n) \simeq \Lambda^{-n},$$

where $\Lambda > 1$.

Two metrics $d_1$ and $d_2$ are *snowflake equivalent* iff there ex. $\alpha > 0$ s.t.

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# The metric gauge of a Thurston map

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**Theorem.** (B.-Meyer, Pilgrim-Haïssinsky)
Let $f\colon S^2 \to S^2$ be an expanding Thurston map, and $d$ a metric in the canonical snowflake gauge.
Then $f$ is conjugate to a rational map if and only if $f$ has no periodic crititical points and $(S^2, d)$ is quasisymmetrically equivalent to the standard sphere $\mathbb{S}^2$.

# Quasisymmetric maps

A homeomorphism $f: X \to Y$ between metric spaces is *(weakly-) quasisymmetric* (=qs) if there exists $H \geq 1$ s.t.

$$|x-y| \leq |x-z| \Rightarrow |f(x)-f(y)| \leq H|f(x)-f(z)|$$

for all $x, y, z \in X$.

- $f$ is quasisymmetric if it maps balls to "roundish" sets of uniformly controlled eccentricity.
- Quasisymmetry global version of quasiconformality.
- bi-Lipschitz $\Rightarrow$ qs $\Rightarrow$ qc.
- In $\mathbb{R}^n$, $n \geq 2$: qs $\Leftrightarrow$ qc.
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# Cannon's conjecture

**Version I:** Suppose $G$ is a Gromov hyperbolic group with $\partial_\infty G \approx \mathbb{S}^2$. Then $G$ admits an action on hyperbolic 3-space $\mathbb{H}^3$ that is discrete, cocompact, and isometric.

If true, the conjecture would give a characterization of fundamental groups $\pi_1(M)$ of closed hyperbolic 3-orbifolds $M$ from the point of view of geometric group theory.

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This is equivalent to:

**Version II:** Suppose $G$ is a Gromov hyperbolic group with $\partial_\infty G \approx \mathbb{S}^2$. Then $\partial_\infty G$ is qs-equivalent to $\mathbb{S}^2$.

If true, the conjecture would give a characterization of fundamental groups $\pi_1(M)$ of closed hyperbolic 3-orbifolds $M$ from the point of view of geometric group theory.

# The quasisymmetric uniformization problem

Suppose $X$ is a metric space homeomorphic to a "standard" metric space $Y$. When is $X$ qs-equivalent to $Y$?

- Precise meaning of "standard" metric space depends on context.
- Examples: $Y = \mathbb{R}^n$, $\mathbb{S}^n$, standard 1/3-Cantor set $C$, etc.
- Case $Y = \mathbb{S}^2$ particularly interesting in view of Cannon's conjecture and the characterization of rational Thurston maps.

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# Linear local contractibility

A metric space $X$ is *linearly locally contractible* iff there exists a constant $L \geq 1$ s.t. the inclusion map

$$B(a, R) \hookrightarrow B(a, LR)$$

is homotopic to a constant map whenever $a \in X$ and $R \leq \text{diam}(X)/L$.

Rules out cusps!

Linear local contractibility is a qs-invariant.

# Ahlfors regularity

A metric space $X$ is called *Ahlfors $Q$-regular*, $Q > 0$, if

$$\mathcal{H}^Q(\overline{B}(a, R)) \simeq R^Q$$

for all closed balls $\overline{B}(a, R) \subseteq X$ with $R \leq \operatorname{diam}(X)$.

$\mathcal{H}^Q$ is $Q$-dimensional Hausdorff measure.

A $Q$-regular space has Hausdorff dimension $Q$.

**Theorem.** (B., Kleiner 2002) Let $S$ be a metric 2-sphere. If $S$ is Ahlfors 2-regular and linearly locally contractible, then $S$ is qs-equivalent to $\mathbb{S}^2$.

# Qs-parametrization of 2-spheres

**Theorem.** (B., Kleiner 2002) Let $S$ be a metric 2-sphere. If $S$ is Ahlfors 2-regular and linearly locally contractible, then $S$ is qs-equivalent to $\mathbb{S}^2$.

**Remark:** This has recently been applied to find a "combinatorial characterization" of Lattès maps (Qian Yin, Ph.D. thesis, 2011).

# Further directions

- What are the special properties of subdivison rules associated with rational Thurston maps?
- Can one reprove Thurston's characterization of rational maps using the combinatorial approach?
- An expanding Thurston map need not have an invariant Jordan curve containing the postcritical set $P_f$. Does there always exist an invariant graph $G \supseteq P_f$?
- Can one extend the theory of expanding Thurston maps to Thurston maps that are only expanding on their "Julia sets"? (Analog of subhyperbolic rational maps).

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