On the Symplectic Topology of Cotangent Bundles

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$$\left.\begin{array}{lll} M & \text{smooth} & \text{manifold} \\ \omega & \text{closed} & \text{2-form, } \omega^n \text{ volume form} \end{array}\right\} \begin{array}{l}\text{Symplectic} \\ \text{manifold}\end{array}$$

Key question of symplectic topology (Arnol'd):

Does $[\omega]$ determine everything?

Eliashberg, Gromov ~80: No!.

Sample Result: (Mclean, A-Seidel)

If $n \geq 3$ $\mathbb{R}^{2n}$ has uncountably many non-deformation equivalent "complete" symplectic structures.

<u>Today</u>. Simplest example beyond $\mathbb{R}^{2n}$.

Q smooth manifold

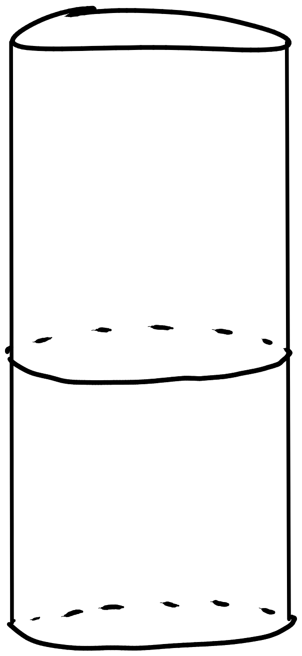
T^*Q cotangent bundle locally $(\underbrace{q_1, \ldots, q_n}_{\text{base}}, \underbrace{p_1, \ldots, p_n}_{\text{fibre}})$

λ Liouville form, $\sum p_i \, dq_i$

$d\lambda = \omega$ symplectic form

e.g.:

$Q = S^1$

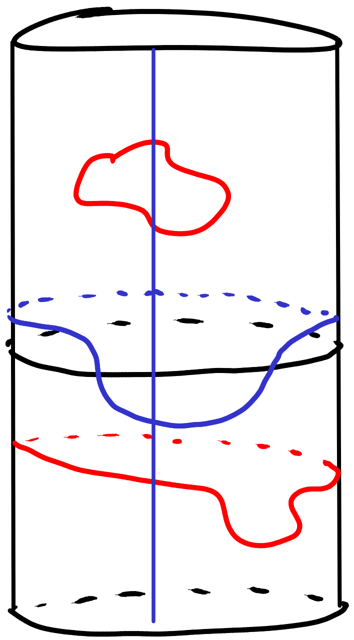


$T^*Q \cong S^1 \times \mathbb{R}$

Conj (Arnol'd): The symplectic structure on T^*Q determines the smooth structure on Q.

$L \subset T^*Q$ Exact Lagrangian Submanifold

$\dim L = \dim Q$ $\qquad [\lambda|_L] = 0 \in H^1(L, \mathbb{Z})$



e.g.:
- Zero section
- Cotangent fibres
- Graph of df.

Conjecture: (Arnol'd) Q is a closed smooth $L \subset T^*Q$ compact exact Lagrangian $\Longrightarrow$ L is isotopic to Q.

Theorem: (R. Hind) True for S^2.

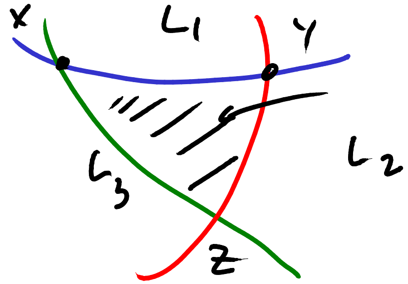
<u>Thm</u>: The inclusion of L is a homotopy equivalence.
(Extends results of Nadler, Fukaya-Seidel-Smith).

<u>Main tool</u>: Fukaya Category:

<u>Objects</u>: Exact Lagrangians (well-behaved at ∞) with local system.

<u>Morphisms</u>: Floer cohomology: "Categorification of intersection number"

<u>Composition</u>



holomorphic disc.

$x \cdot y = z$

<u>Key property</u>: If E is local systems on closed exact Lagrangian L

$$HF^\bullet((L,E),(L,E)) \cong H^\bullet(L, \mathrm{End}(E))$$

Strategy:

I Geometry:

Find some "family of Lagrangians" $\mathcal{L}_p$, parametrised by some space $\mathcal{P}$ such that

The moduli space of holomorphic discs with boundary on on $\mathcal{L}_p$ sweeps T^*Q with multiplicity 1.

Let $\mathcal{F}_{\mathcal{P}}$ denote Fukaya category with objects $\mathcal{L}_p$

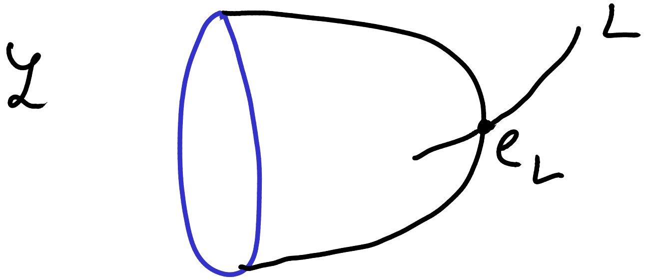
II Algebra:

Classify all modules of bounded homological dimension over $\mathcal{F}_{\mathcal{P}}$ with endomorphism algebra supported in degrees $[0,n]$.

Bridge from Geometry to Algebra: Cardy Relation

$\mathcal{L}$ consists of one Lagrangian. L any other Lagrangian.

Pick any pt $e_L \in L$



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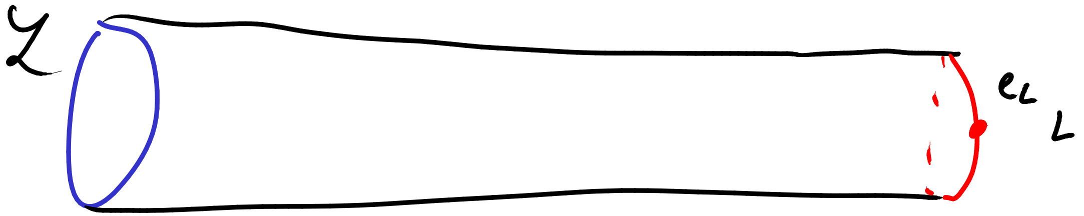
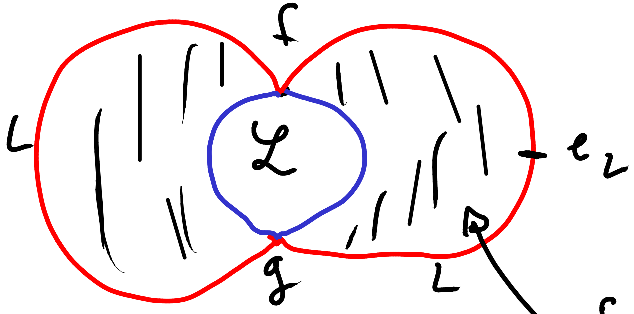
Constant disc.

Glue:



Send modular parameter to 0 :



$f \circ g = e_L \Rightarrow L$ is a summand of $\mathcal{L}$

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Back to $T^\vee Q$:

Let $\mathcal{P} = Q$, $\mathcal{L}_p = T^\vee_p Q$. Constant discs "sweep $T^\vee Q$" with multipicity $\underline{1}$.

What is Fukaya Category with objects $T^\vee_p Q$?

All fibres are isomorphic, endomorphism algebra

$C_{-*}(\Omega Q)$, equipped with Pontryagin product.

Structure theory for modules over $C_{-*}(\Omega Q)$

Every module Y with $\mathrm{Hom}_k(C_{-*}(\Omega Q), Y) = 0$ if $k \gg 0$, has a filtration with subquotients induced by "augmentation"

$C_{-*}(\Omega Q) \longrightarrow \mathbb{Z}[\pi_1 Q] \longrightarrow \mathrm{End}(V)$ For some vector space V.

Further consequences: The Fukaya category is part of "extended" TFT

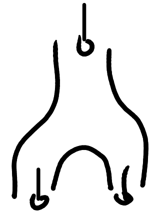
Viterbo, Abbondandolo-Schwarz, Cieliebak-Latschev ....

$$\begin{tikzcd} TFT_{Floer}(T^*Q) \arrow[r] \arrow[d] & TFT_{CY}(HF^*(T_q^*Q)) \arrow[d] \\ TFT_{String}(Q) \arrow[r] & TFT_{CY}(H_{-*}(\Omega_q Q)) \end{tikzcd}$$

Costello, Lurie

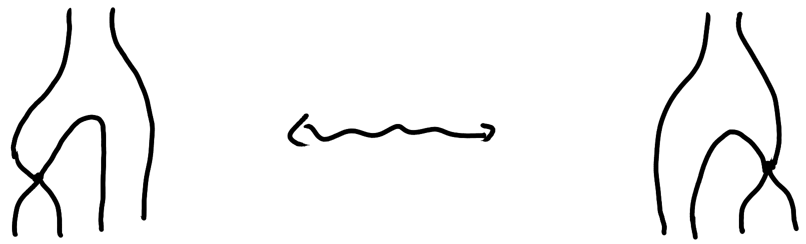
Not covered

Secondary operations: On $HF^*(L,L)$ there is a coproduct



Highly degenerate. Obtain secondary operation on

$Cone(H^*(L) \to HF^*(L,L))$



Thm (L. Ng): If $Q = S^3$, then coproduct & higher analogues applied to conormal bundles of $S^1 \hookrightarrow S^3$ detect unknot.

Smooth Structure:

<u>Thm:</u> If Σ is a htpy sphere which embeds as a Lagrangian in T^*S^{4k+1}, then Σ bounds a compact parallelisable manifold.

<u>Note:</u> ① h-cobordism implies $T^*\Sigma \overset{\text{diff}}{\simeq} T^*S^{4k+1}$

② $\exists\ \Sigma$ which do not bound parallelisable.

<u>Strategy:</u> Explicit construction of a bounding manifold from solutions to a PDE, using the Hopf fibration.

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New approach by S. Basu distinguishes smooth structures on lens spaces using string topology.