A Theorem on Subdivision Approximation and Its Application to Obtaining PTAS's for Geometric Optimization Problems

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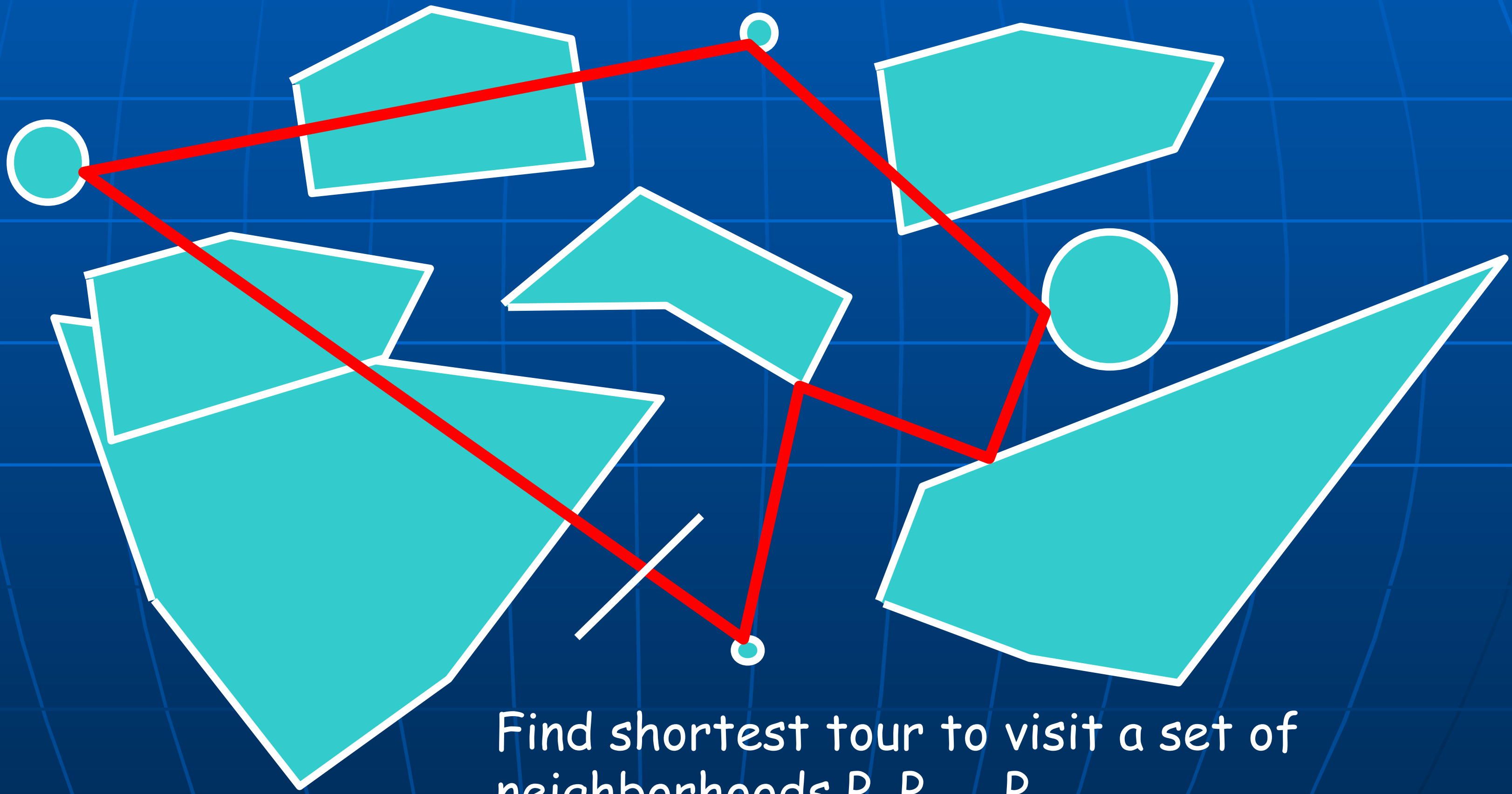
A Dedication

Happy 60th Birthday!!

Michael Ian Shamos:

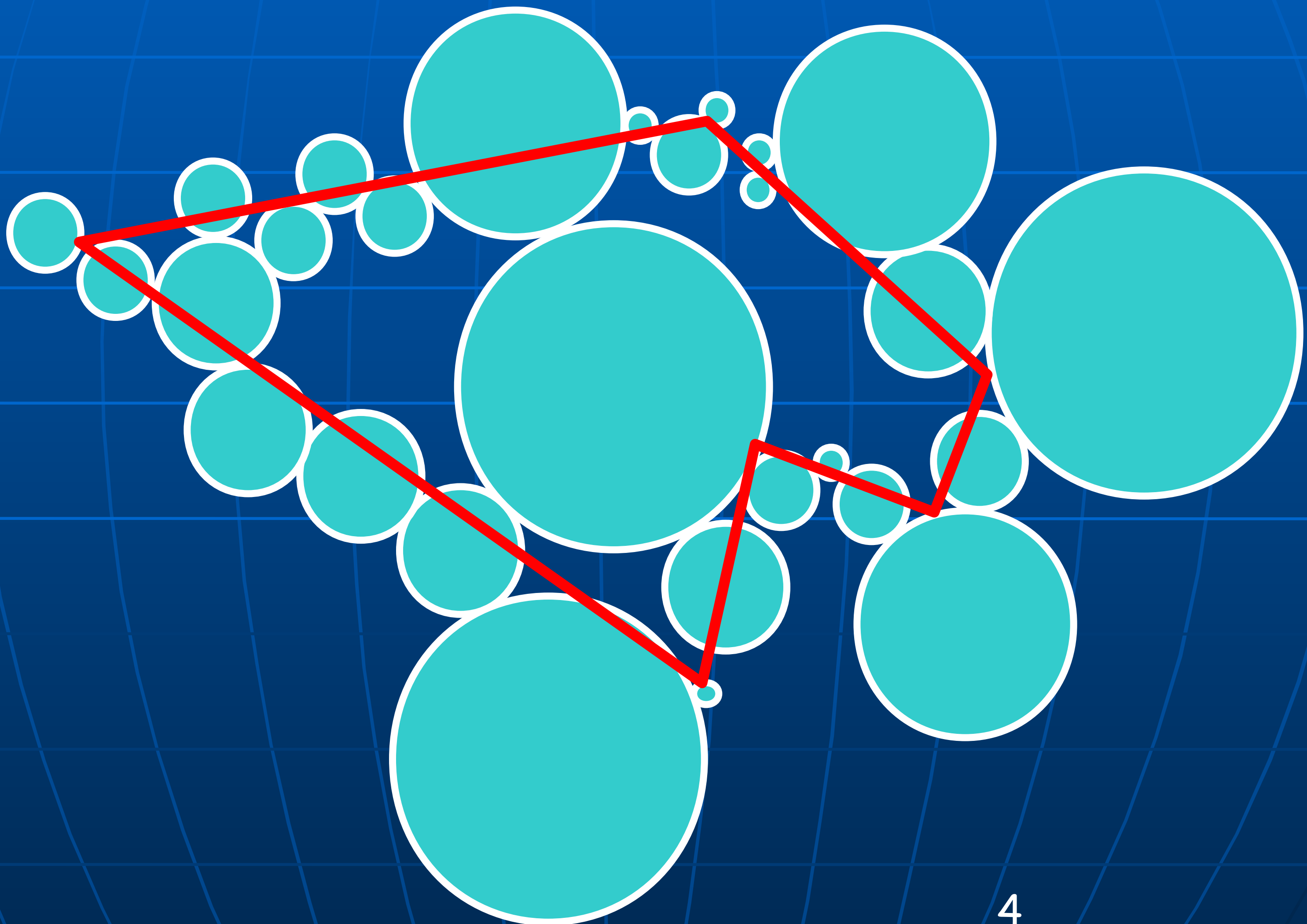
PhD thesis “Computational Geometry”, Yale University, 1978

Motivating Problem: TSP with Neighborhoods

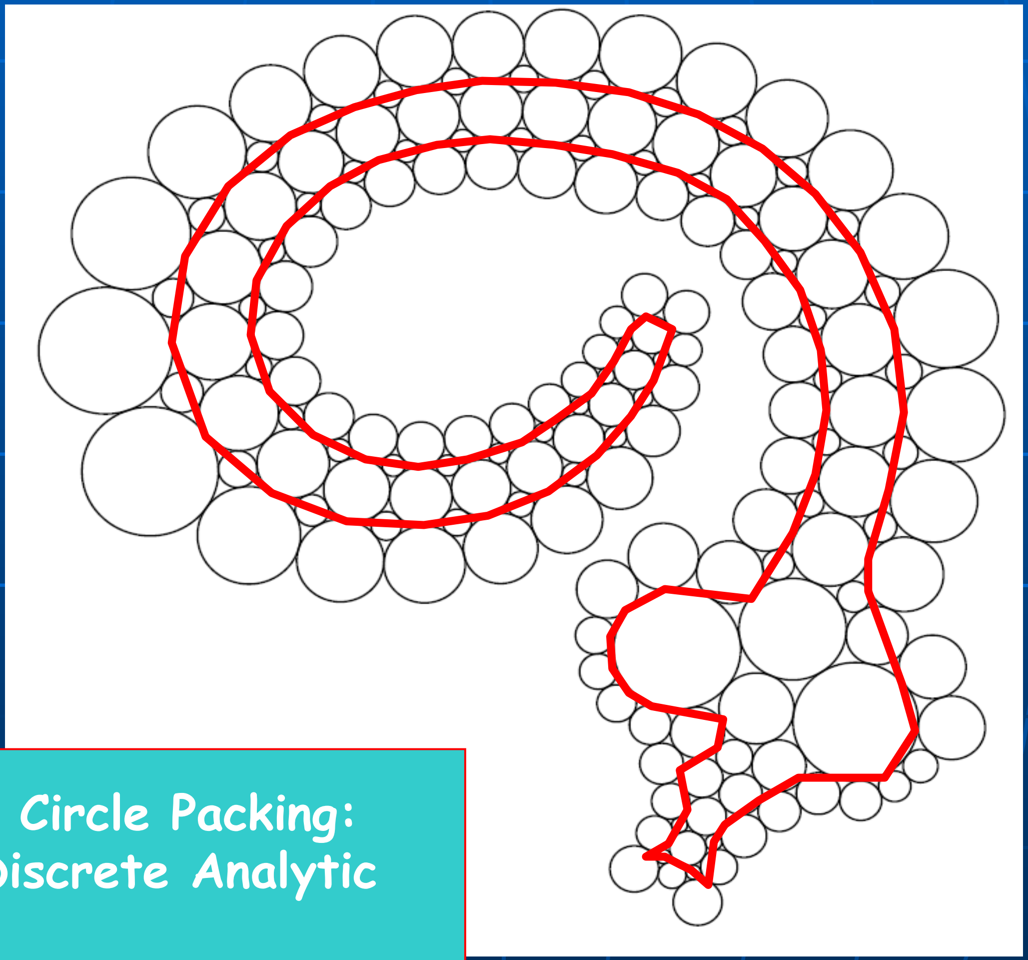


Find shortest tour to visit a set of neighborhoods $P_1, P_2, \ldots, P_n$

TSPN for Disk Packing

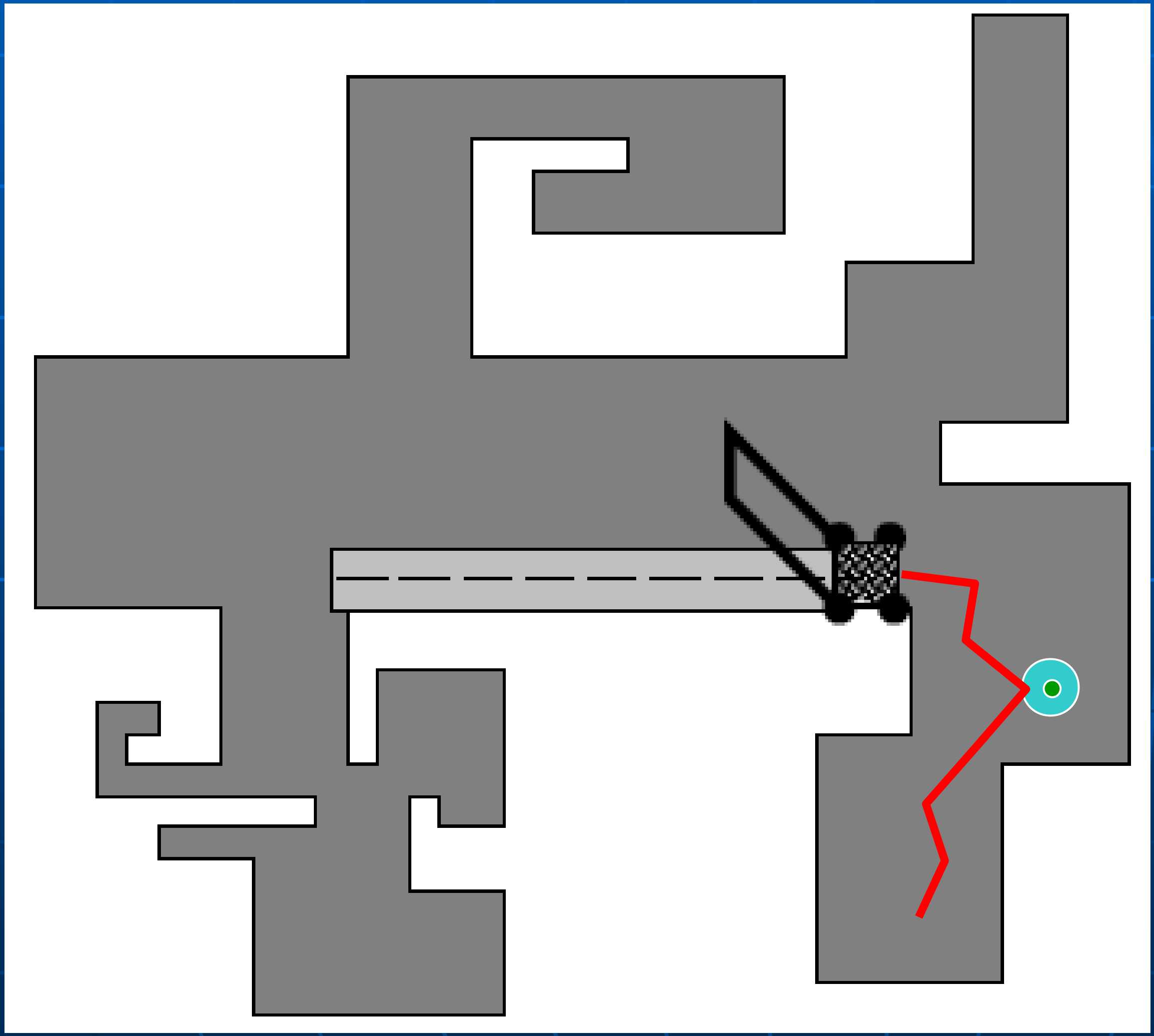


TSPN in a Circle Packing



"Introduction to Circle Packing: the Theory of Discrete Analytic functions"
by Ken Stephenson

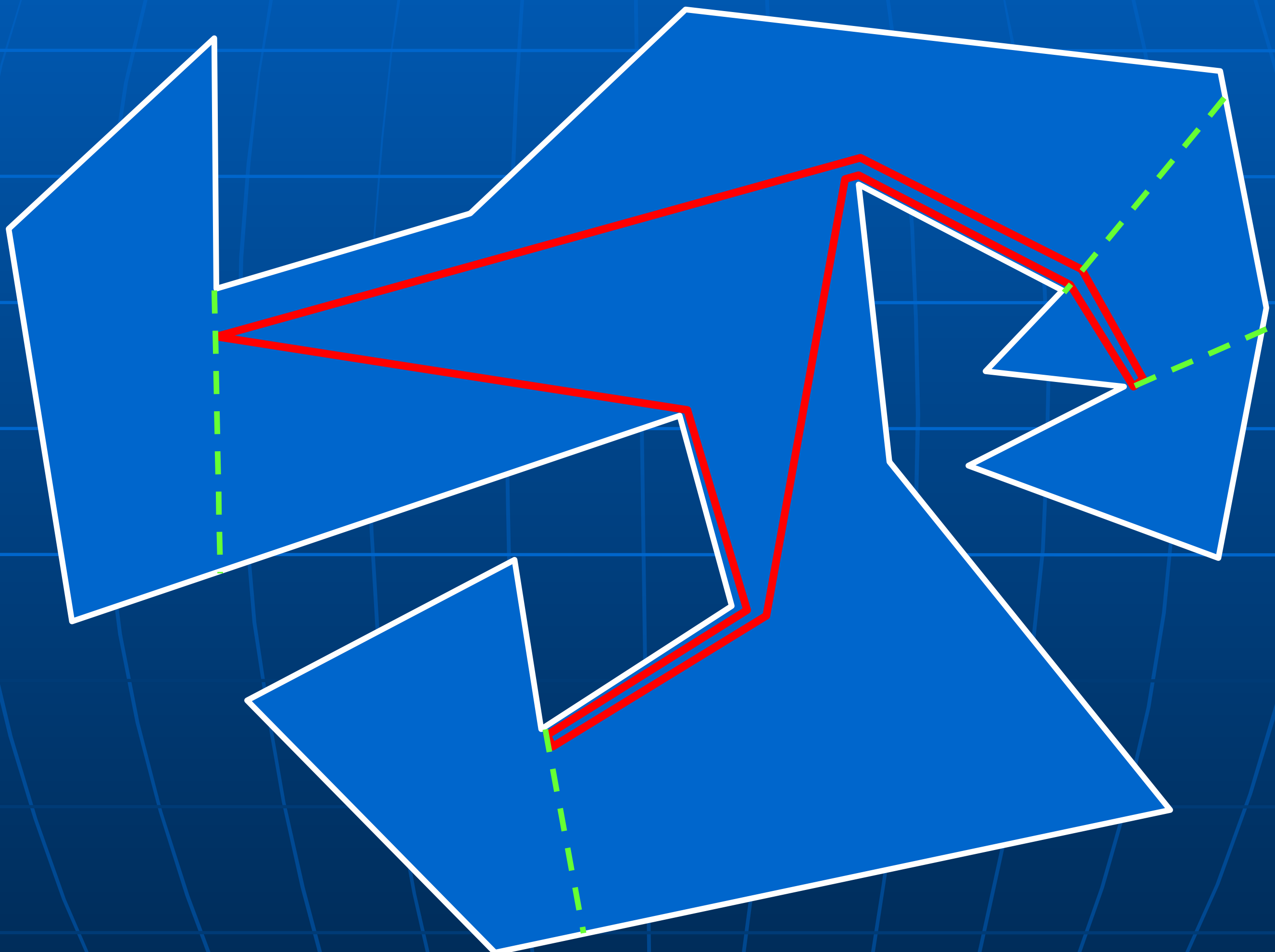
Another (Springtime) Motivation



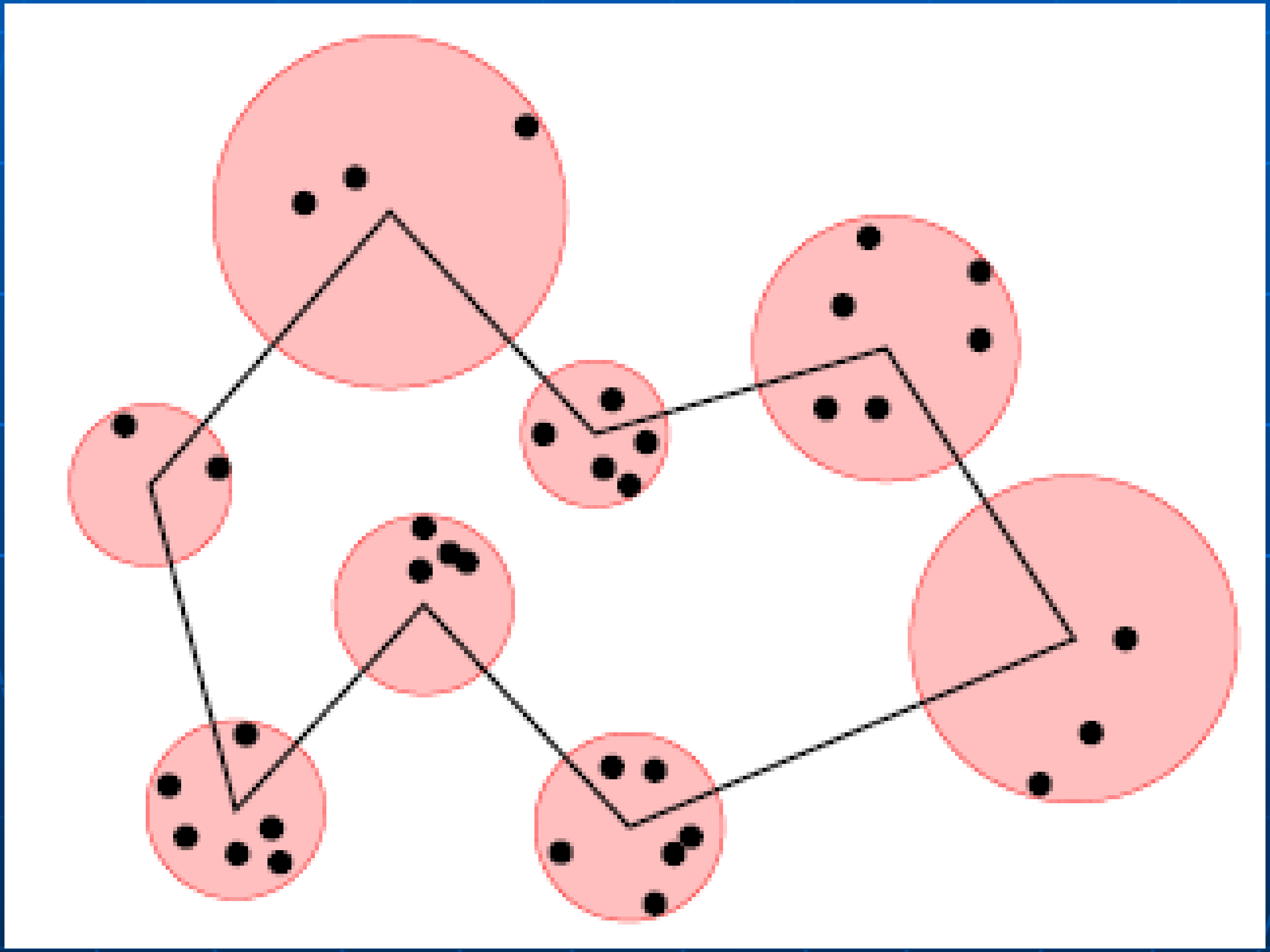
Best method of mowing the lawn?

TSPN: Visit the disk centered at each blade of grass

Watchman Route Problem



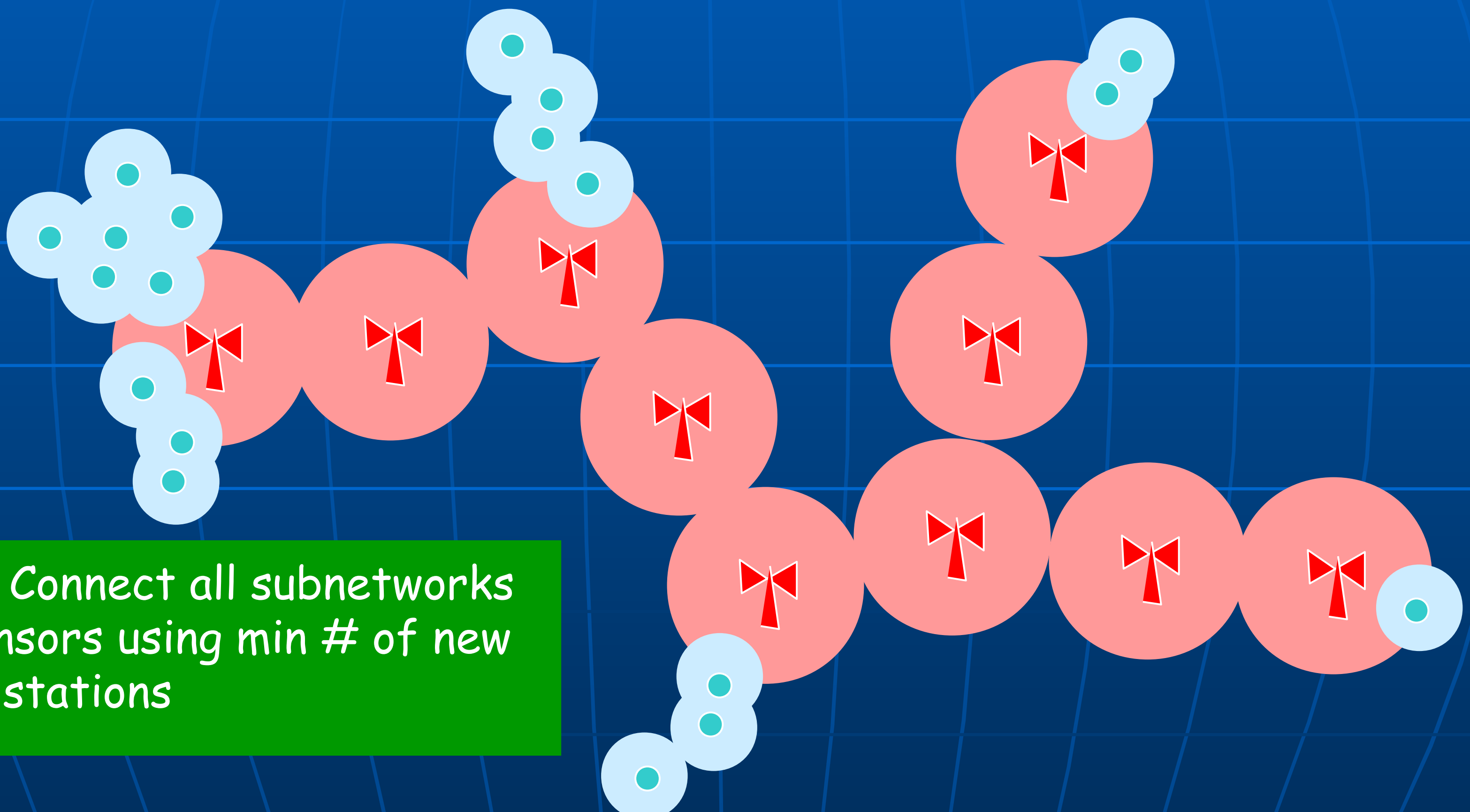
Sensor Network Application: Cover Tour Problem



Min: Tour length + C * (sum of radii)

Alt, Arkin, Bronnimann, Erickson, Fekete, Knauer, Lenchner, Mitchell, Whittlesey, SoCG'06

Sensor Network Application: Minimizing # Relay Stations

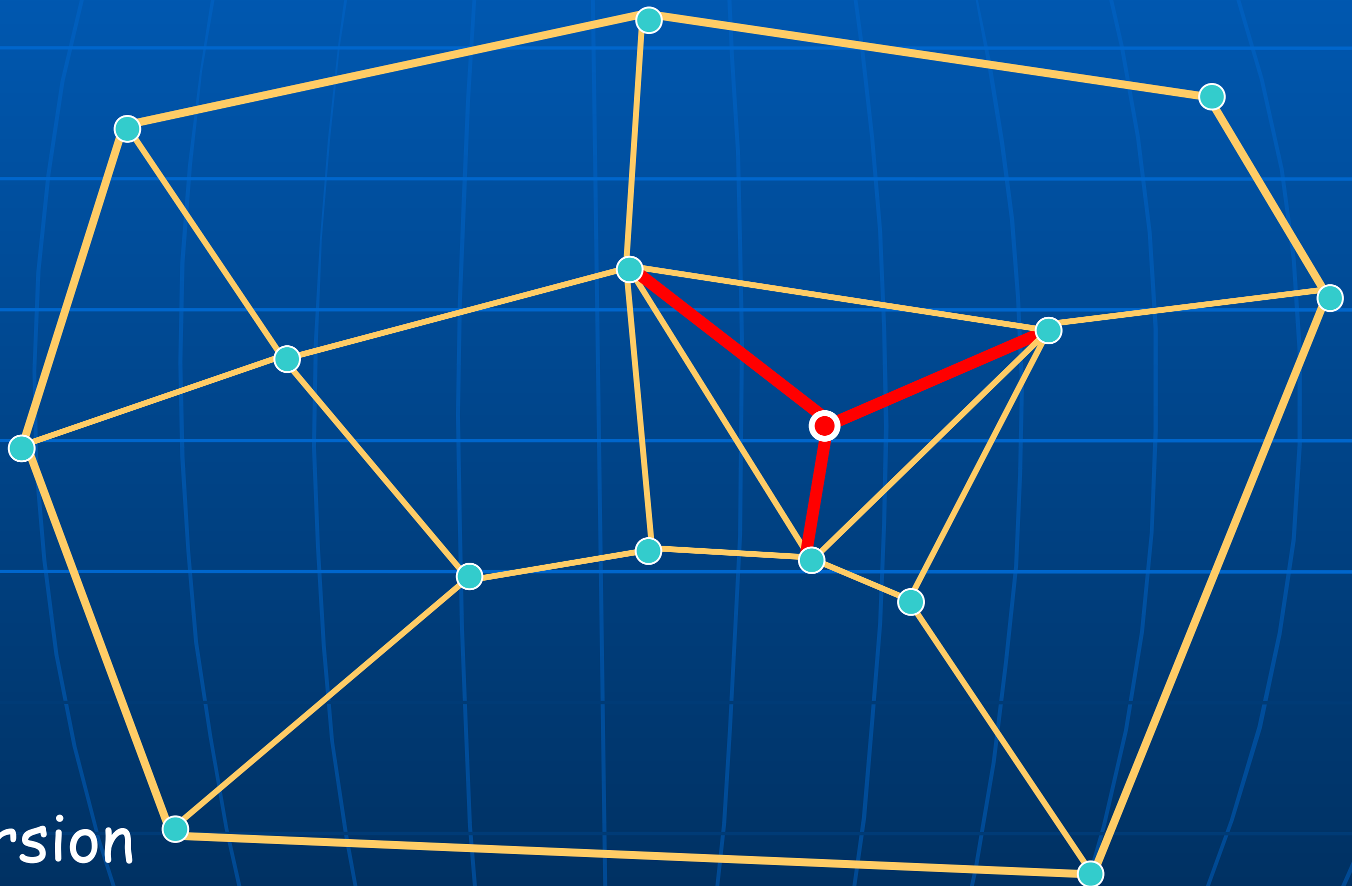


Goal: Connect all subnetworks of sensors using min # of new relay stations

New result: PTAS

Efrat, Fekete, Mitchell 2007

Min-Weight Convex Subdivision



Steiner version

Special Case: Min-weight (Steiner) triangulation

Approximation Algorithms

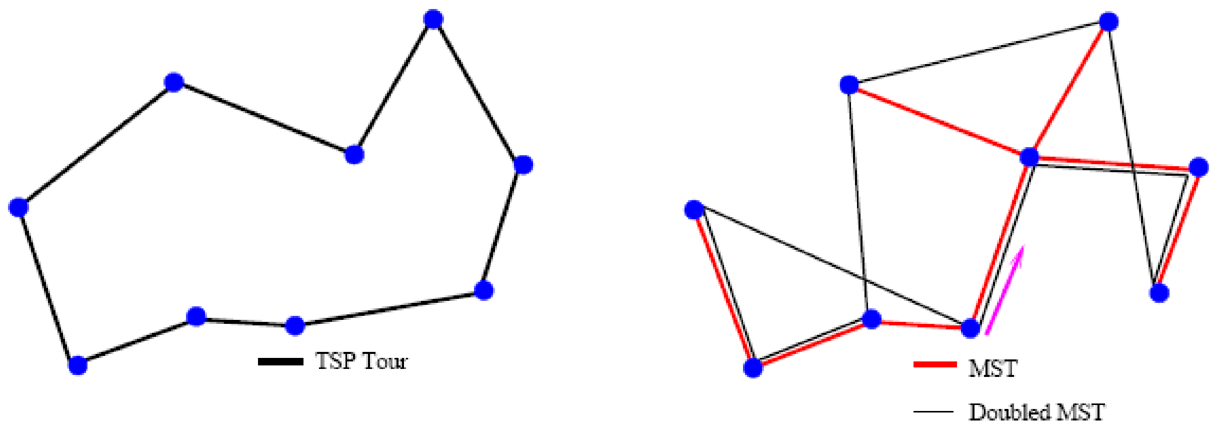
c-approximation: cost at most c times optimal, for a minimization problem ($c > 1$)

Polynomial Time Approximation Scheme (PTAS): method giving $(1 + \epsilon)$-approx to the optimal (minimum), in time polynomial in n, for *any* fixed $\epsilon > 0$.

Dependence on ϵ may be exponential in $(1/\epsilon)$; else FPTAS

Background on TSP

- S = set of n points in $\Re^d$
- NP-hard
- $n^{O(n^{1-1/d})}$ exact (subexponential) [SmWo98]
- Simple 2-approx: double the MST and shortcut (holds in metric spaces)



- Christofides: 1.5-approx

  (use MST $\cup$ min-weight matching on odd-degree nodes of MST)

PTAS for Geometric TSP

- $O(n^{O(1/\epsilon)})$ in $\Re^2$ [Ar96,Mi96]
- $O(n^{O(1)})$ in $\Re^2$ [Mi97]
- $O(n(\log n)^{(O(\frac{d}{\epsilon}))^{d-1}})$ expected $(O(n^{d+1} polylog)$ det.) [Ar97]
- $O(n \log n)$ deterministic [RaSm98]

  Idea: t-spanners and " t-banyons"
- NP-hard to get $(1+\epsilon)$-approx in $\Re^{O(\log n)}$, for some $\epsilon > 0$ [Tr97]
- MAX-SNP-hard in metric spaces

  No c-approx for $c < 129/128$ ($c < 41/40$, asym.) [PV99]

TSPN Recent Result [SODA'07]

- TSPN has a PTAS for regions/neighborhoods that are "**fat**", disjoint (or sufficiently disjoint) connected regions in the plane
- Applies also to "MST with neighborhoods", Steiner MSTN, and many related problems

PTAS = Polynomial-Time Approximation Scheme = $(1+\varepsilon)$-approx, any $\varepsilon>0$

Background on TSPN

Generalizes 2D Euclidean TSP (thus, NP-hard)

Introduced by [Arkin & Hassin, 1994]

- "obvious" heuristics do not work:
  - TSP approx on centroids (as representative points)
  - Greedy algorithms (Prim- or Kruskal-like)
- O(1)-approx, time O(n + k log k), for "nice" regions:
  - Parallel unit segments
  - Unit disks
  - Translates of a polygon P
- Combination Lemma

General Connected Regions

O(log k)-approx [Mata & M, SoCG'95]

Use guillotine rectangular subdivisions, DP

(*non* – disjoint: regions may overlap)

- $O(n^5)$ time

[Mata & M, SoCG'95]

- $O(n^2 \log n)$

[Gudmundsson & Levcopoulos, 1999]

k = # regions n = # vertices of all regions

O(1)-Approximations

- Unit disks, parallel unit segments, translates of *P*
  [Arkin & Hassin, 1993]
- Connected regions of comparable size
  [Dumitrescu & M, SODA'01]
- Disjoint **fat** regions of *any* size [de Berg, Gudmundsson, Katz, Levcopoulos, Overmars, van der Stappen, ESA'02]
- Discrete point sets within disjoint, fat, *non*-convex regions
  [Elbassioni, Fishkin, Mustafa, Sitters, ICALP'05]
- *Non* - disjoint, convex, fat, comparable size
  [Elbassioni, Fishkin, Sitters, ISAAC'06]

PTAS: $O(1+\varepsilon)$-Approximations

- Disjoint (or nearly disjoint) **fat** regions of comparable size

  [Dumitrescu & M, SODA'01]

- Point clusters within disjoint **fat** regions of comparable size in R^d

  [Feremans, Grigoriev, EWCG'05]

New: PTAS for disjoint (or nearly disjoint) **fat** regions of *arbitrary* sizes.

Def: P is **fat** if area(P) = Ω(diam^2(P))

Weaker notion than usual "fatness"

Related Work: APX-hardness

- General connected regions (overlapping):
  - No c-approx with c<391/390, unless P=NP

    [de Berg, Gudmundsson, Katz, Levcopoulos, Overmars, van der Stappen, ESA'02]

    (from MinVertexCover)
  - No c-approx with c<2, unless P j TIME ($n^{O(\log\log n)}$)

    [Safra, Schwartz, ESA'03]

    (from Hypergraph VertexCover)
- Line segments, comparable length

  [Elbassioni, Fishkin, Sitters, ISAAC'06]
- Pairs of points (disconnected) [Dror, Orlin, 2004]

Exact Poly-Time Solutions

TSPN for a set of infinite lines in 2D:

Is this the only nontrivial case exactly solvable in poly-time?

What about visiting planes in 3D?

Solved in $O(n^4 \log n)$ time using Watchman Route solution

[Dror, Efrat, Lubiw, M, STOC'03]

Recipe for PTAS

Structure Theorem

increasing length by $\leq (1+\varepsilon)$ factor

OPT → Network with special **recursive** structure

Use **dynamic programming** to compute shortest network with the required structure (connectivity, Eulerian subgraph, etc)

Optimal network with special structure → TSPN tour

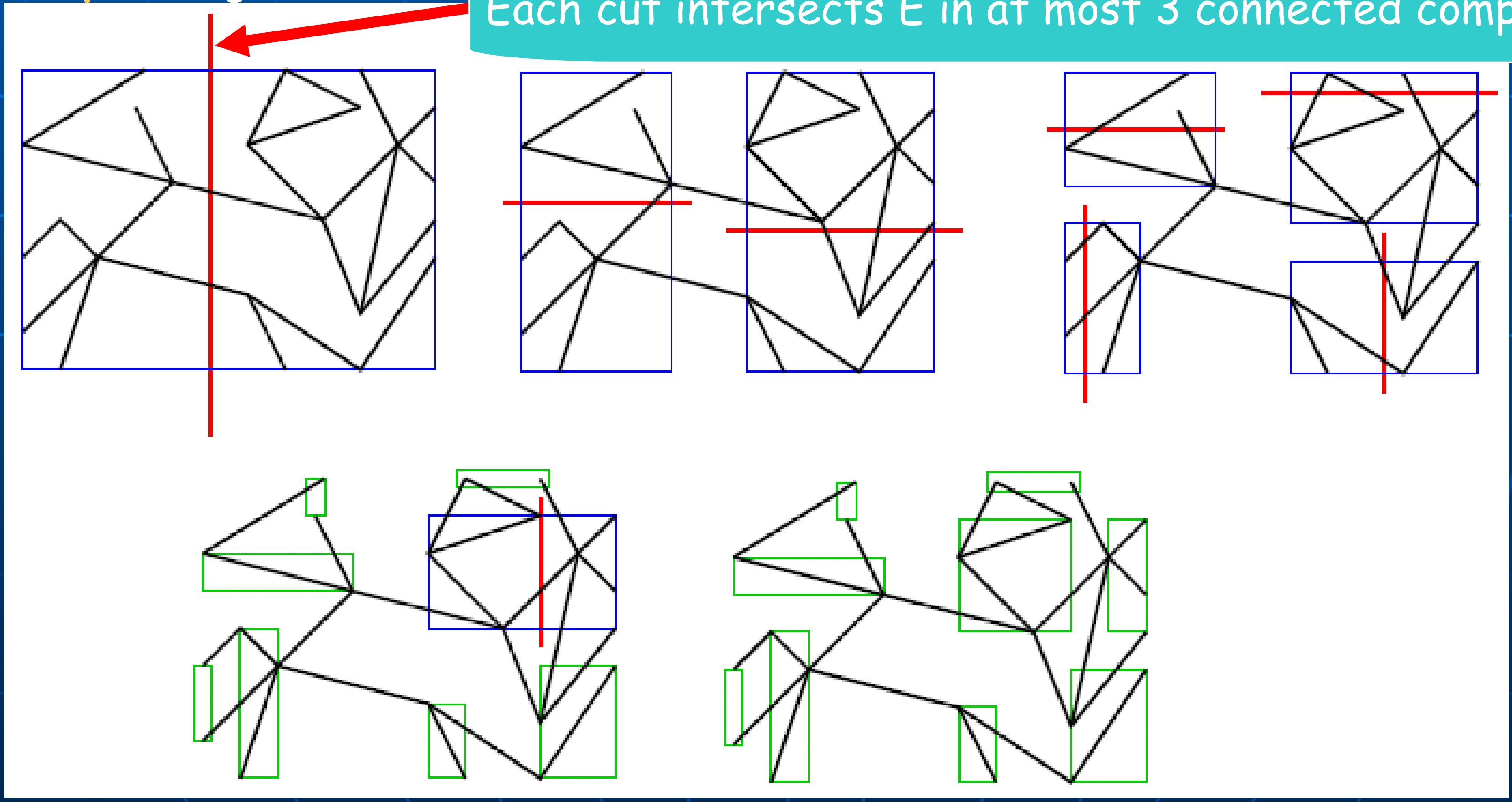
What should the special recursive structure be?

m-Guillotine Structure

Network edge set E is m-guillotine if it can be recursively partitioned by horiz/vertical cuts, each having small (O(m)) complexity wrt E

Example: 3-guillotine

Each cut intersects E in at most 3 connected components

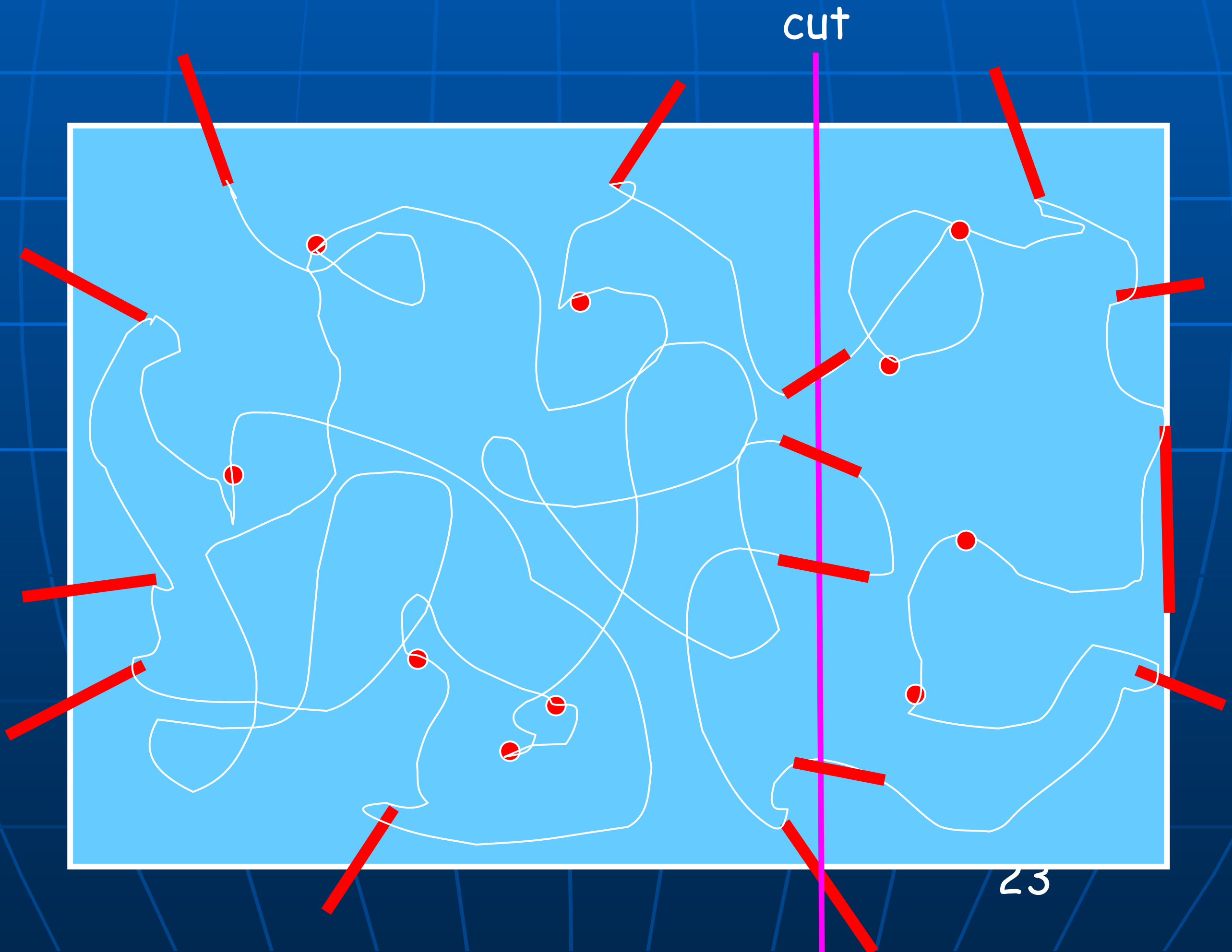


Desired Recursive Structure

Rectangular subproblem in dynamic program (recursion)

Constant (O(m))

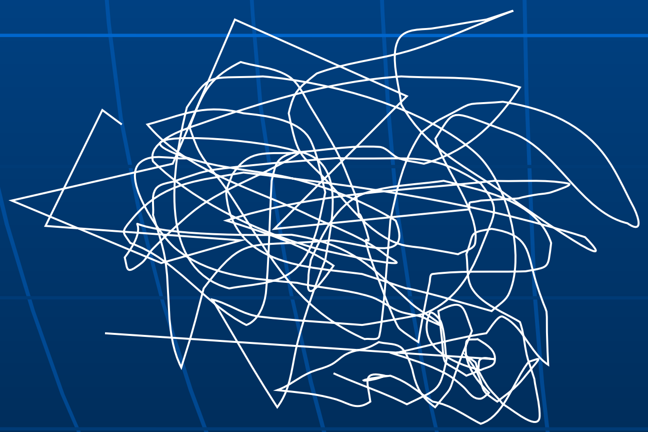
information flow across boundary



m-Guillotine Structure Theorem

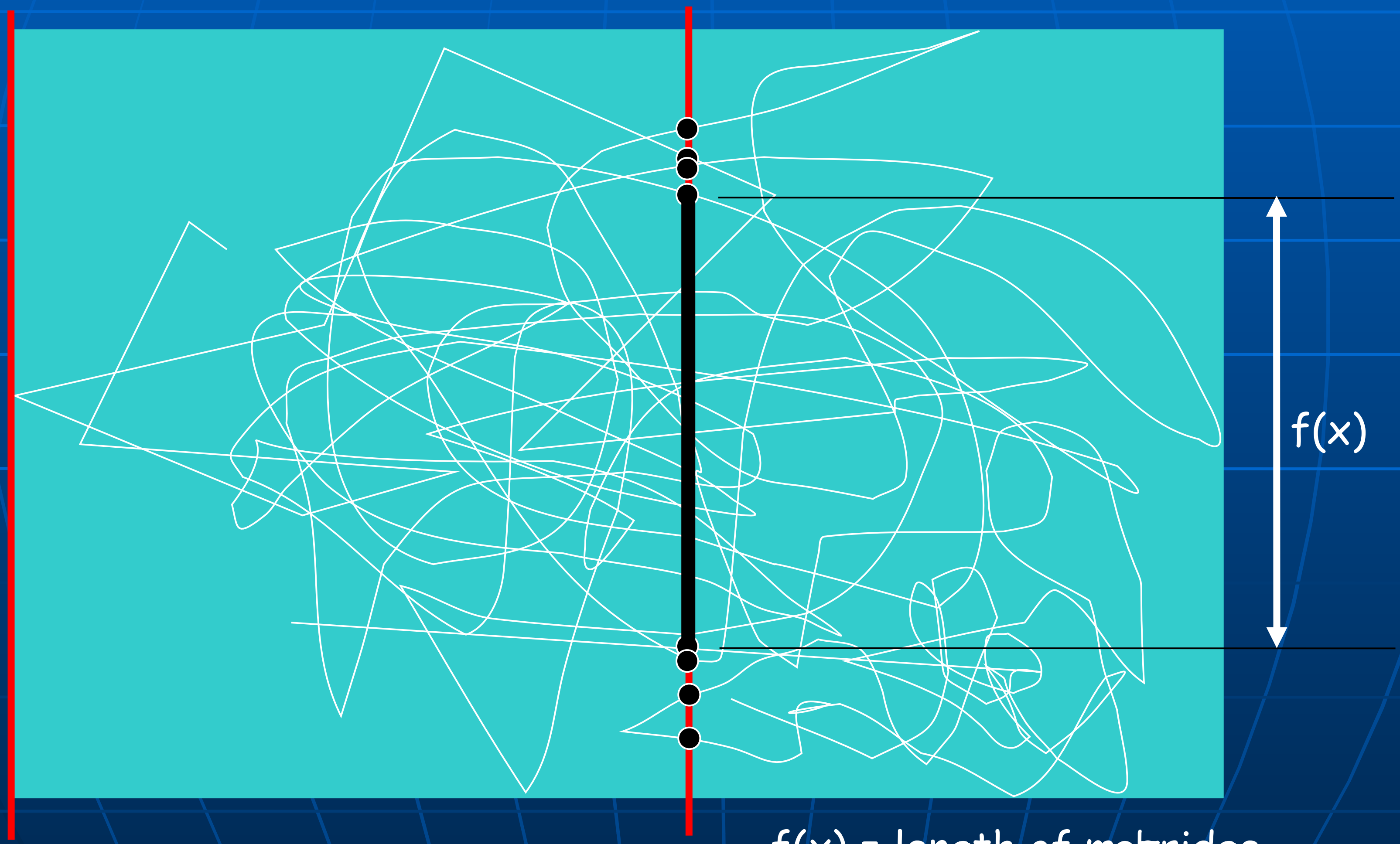
Any set E of edges of length L can be made to be m-guillotine by adding length O(L/m) to E, for any positive integer m.

Proof is based on a simple charging scheme.



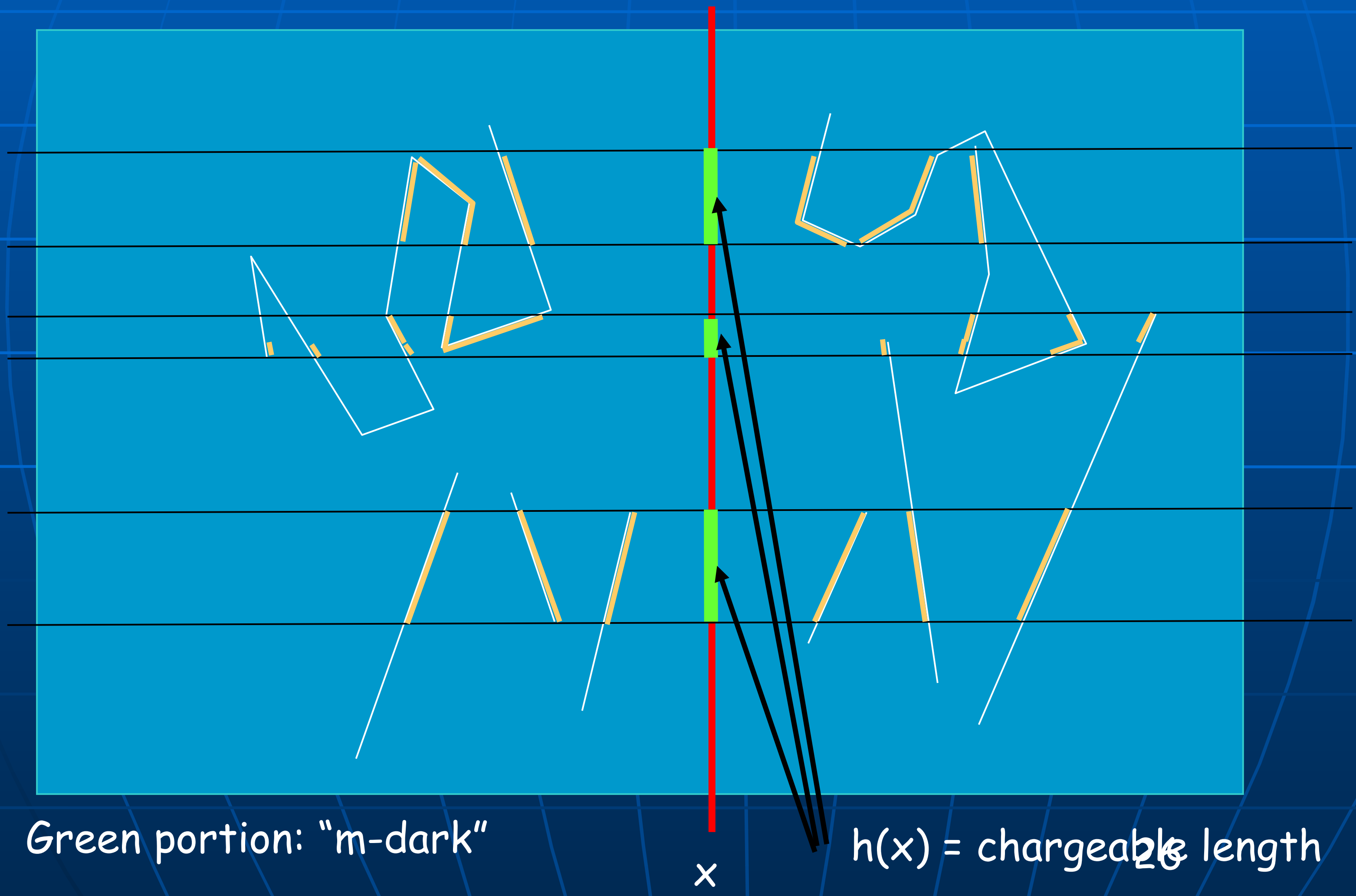
While this "scribble" may not be m-guillotine, it is "close" in that it can be made m-guillotine by adding only (1/m)th of its length

Possible Vertical Cuts



f(x) = length of my bridge
= cost of construction

Paying for the Bridge Construction: The Chargeable Length



Charging Scheme

- Let $f(x)$ = length of m-span of vertical line through x
  Let $g(y)$ = length of m-span of horizontal line through y

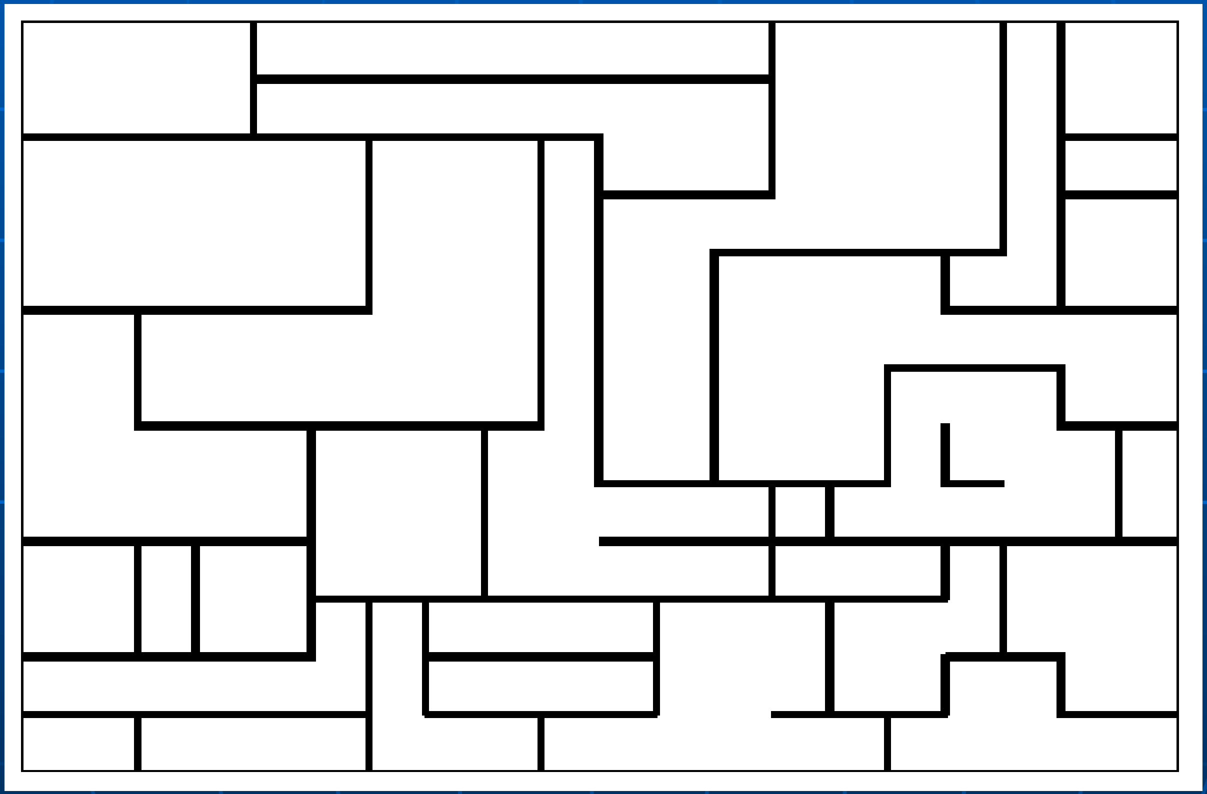
- Then,
  $$A_x = \int f(x)dx$$
  is simply the area of the "m-dark" (RED) region wrt horiz cuts
  Similarly,
  $$A_y = \int g(y)dy$$
  is the area of the "m-dark" (BLUE) region wrt vertical cuts

- Assume, WLOG, that $A_x \geq A_y$

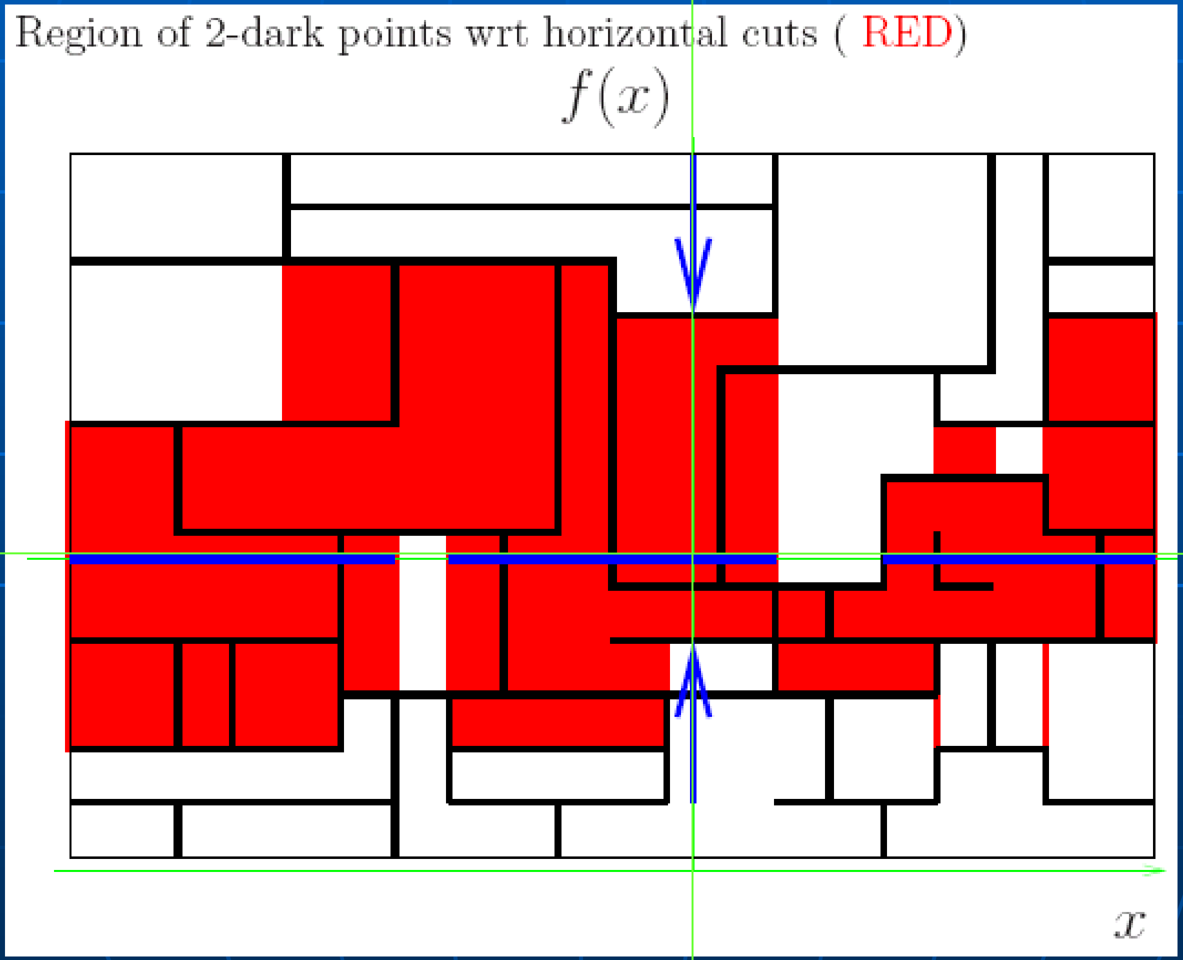
- Thus, for $h(y)$ = length of m-dark, for horiz line through y,
  $$A_x = \int h(y)dy \geq \int g(y)dy = A_y > 0$$
  So, $\exists y^*$ for which $h(y^*) \geq g(y^*)$;
  i.e., $\exists$ a horiz line through y^* whose m-dark portion $\geq$ m-span.

  (If $A_x \leq A_y$, then $\exists$ a vertical favorable cut.)

Charging Scheme



Charging Scheme



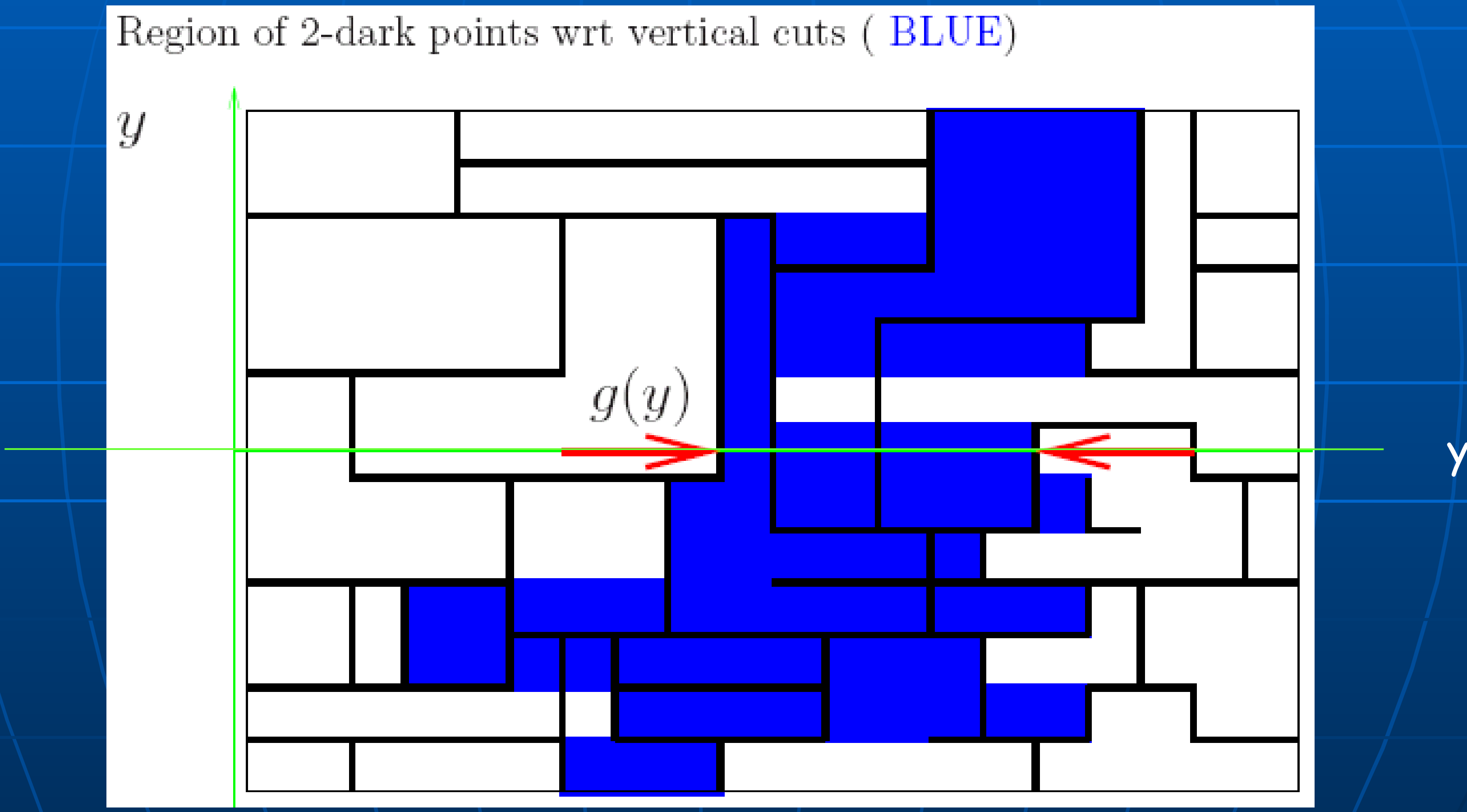
h(y) = chargeable length of horiz cut at y

y

Red area = $A_x = \int f(x)dx$ = $\int h(y)dy$

x   f(x) = cost of construction of vert cut at x

Charging Scheme



Blue area = $A_y = \int g(y)dy$

Charging Scheme

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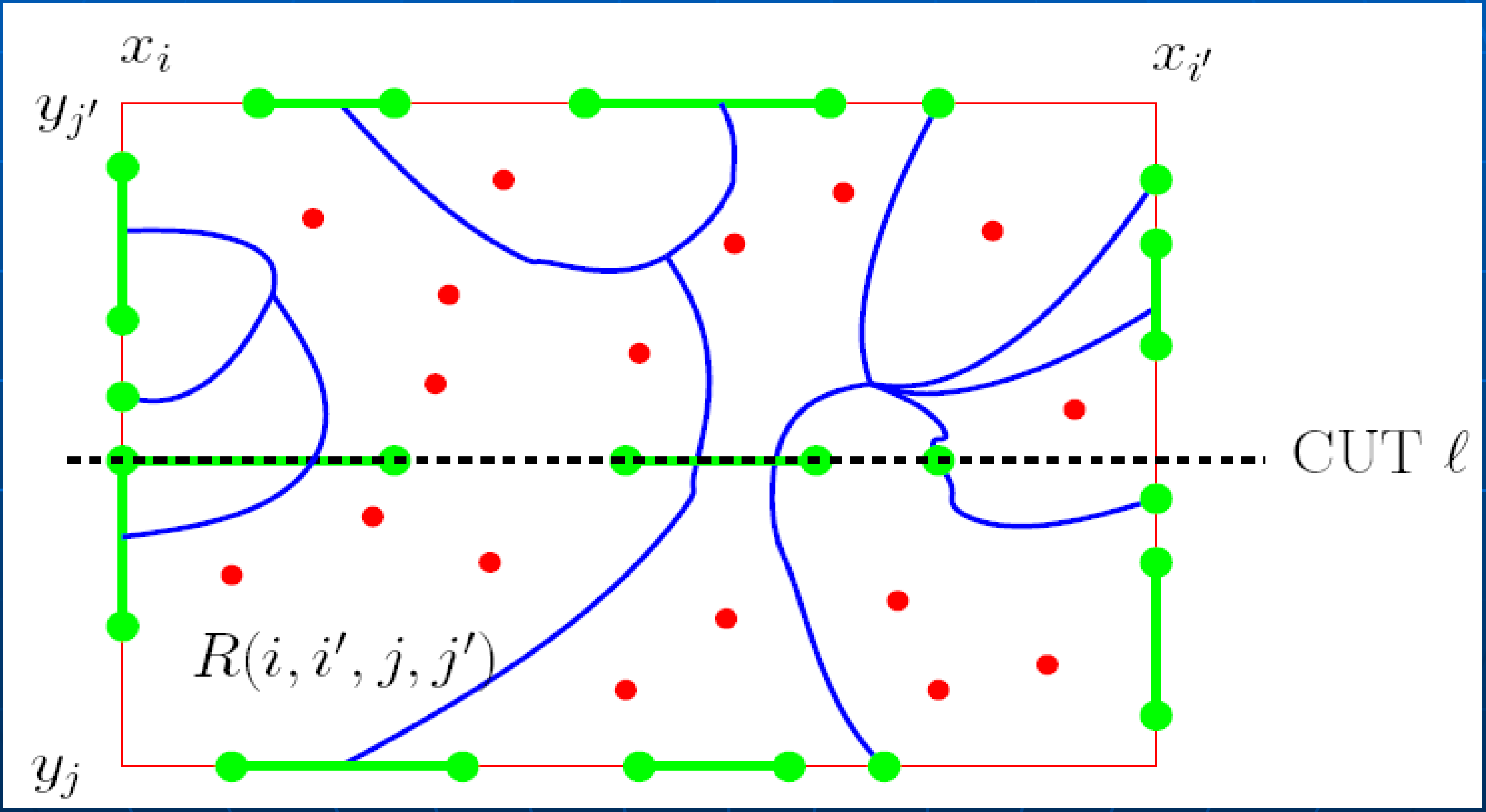
Subproblem



Dynamic Program: Min Steiner Tree

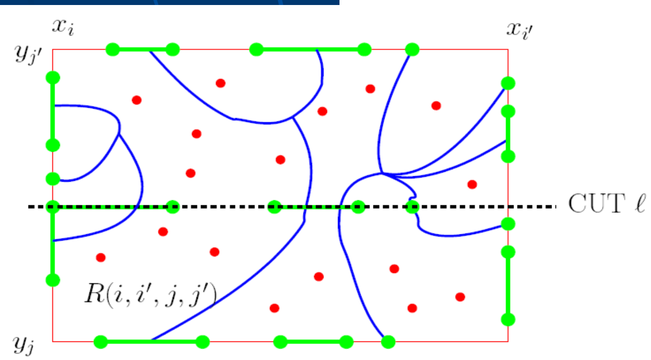
Sorted x-coord: $x_1 < x_2 < \cdots$ (for P, and grid lines)
Sorted y-coord: $y_1 < y_2 < \cdots$

Subproblem: $O(n^4 \cdot (n^{2m})^4) = O(n^{8m+4})$ choices

Input:

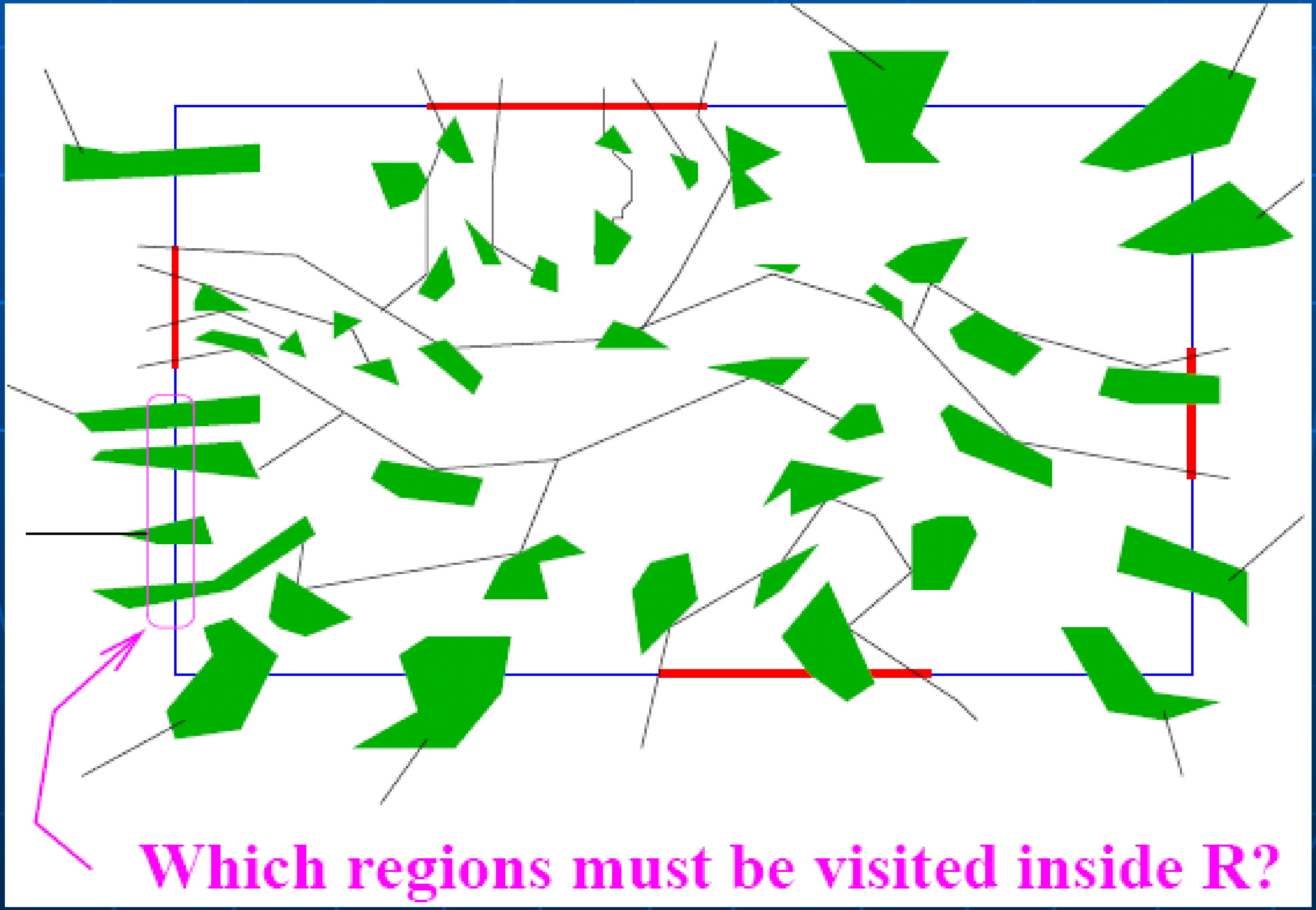
1. a rectangle $R(i, i', j, j')$, defined by x_i, $x_{i'}$, y_j, $y_{j'}$ $O(n^4)$
2. four sets of "boundary information", Σ_l, Σ_r, Σ_b, and Σ_t, determined by $\leq 2m$ endpoints on each side $O((n^{2m})^4)$
3. a partition, $\mathcal{P}$, of $\cup_\alpha \Sigma_\alpha$, giving required connectivity among boundary pieces $O(1)$

Objective: Find min-length m-guillotine subdivision, S_G^* (edges E_G^* interior to $R(i, i', j, j')$), such that E_G^* covers P and E_G^* connects the boundary pieces, according to partition $\mathcal{P}$.



Difficulty in Applying TSP Methods to TSPN / MSTN

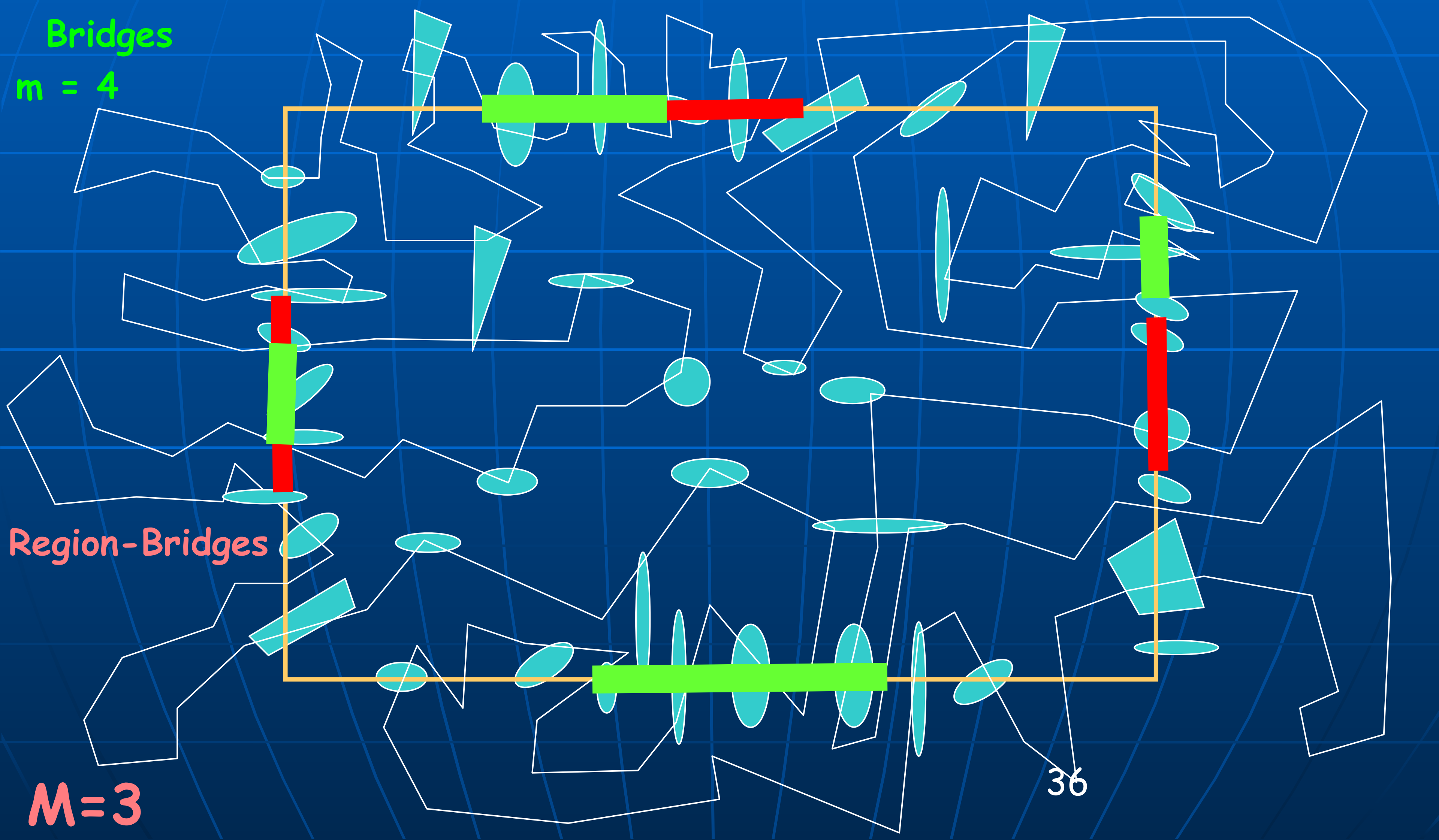
Consider a subproblem (rectangle):



New Structure

- Build region-bridges in order to encode succinctly which regions are the "responsibility" of a subproblem
- Cannot afford to build m-region-bridges for $m = O(1/\varepsilon)$, constant wrt n.
- But *can* afford to build M-region-bridges, with $M=O((1/\varepsilon)\log n)$ and this is "just right", since the remaining M bridges that are not part of the bridge can be specified in the subproblem: $2^M = 2^{O(\log n)}$ is $poly(n)$

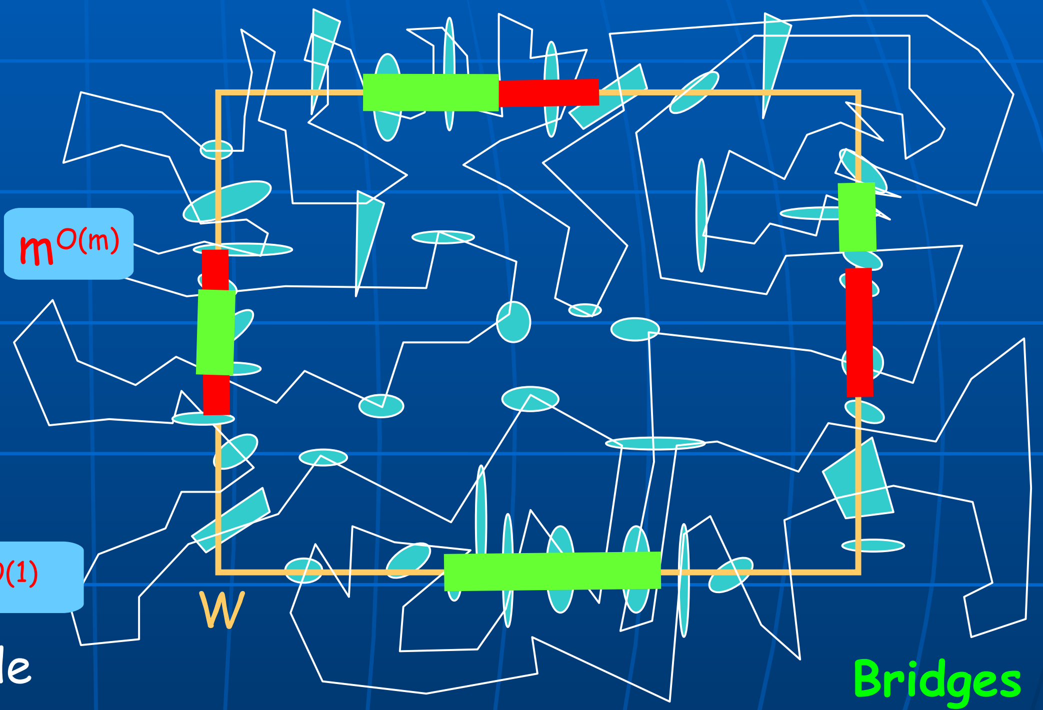
Subproblem: A Window into OPT



Subproblem Optimization

Specification of a subproblem:

1. Window W n^4
2. $\leq$ 4 Bridges, $\leq$ 2m segs/side of W $m^{O(m)}$
3. $\leq$ 4 Region-Bridges, one bit per $\leq$ 8M non-bridged crossing region: Is the subproblem responsible to visit? $n^{O(1)}+2^{O((1/\varepsilon)\log n)} = n^{O(m)}$
4. Protruding regions not in R_{W_0} $n^{O(1)}$
   specified by $\leq$ 2 sequences per side
5. Connection pattern among the O(m) segs crossing into W $m^{O(m)}$



$m = 1/\varepsilon$

$M = (1/\varepsilon) \log n$

Total # subproblems = $n^{O(m)}$

and $n^{O(m)}$ choices for the best horiz/vertical cut, in DP optimization

(m,M)-Guillotine Structure

Definition: Network edge set E is (m,M)-guillotine if it can be recursively partitioned by horiz/vertical cuts, each containing the "m-span" (bridge) of E and the "M-region-span" (region-bridge) of the set of regions.

What the DP Computes

Minimum-length network that is

1. (m,M)-guillotine wrt window W_0 and regions R_{W_0}
2. Connected
3. Containing an Eulerian spanning subnetwork
4. Spanning (visits all regions)

Main Idea of PTAS

Use m-guillotine PTAS method, with new structure to address difficulty with TSPN

OPT → (m,M)-guillotine network with special structure

Structure Theorem

increasing length by $\leq (1+\varepsilon)$ factor

Use **dynamic programming** to compute shortest (m,M)-guillotine network with the required structure (connectivity, Eulerian subgraph, etc)

Optimal (m,M)-guillotine network with structure → TSPN tour

New Structure Theorem

Theorem: Let E be a connected set of edges of length L, spanning all regions. Then, for any positive integers m and M, there is a superset, E', of E, of length at most $L + (\mathrm{sqrt}(2)/m)\, L + (\mathrm{sqrt}(2)/M)\, \lambda(R_{W_0})$.

Sum of region diameters

We pick $M = (1/\varepsilon) \log n$, and $m = 1/\varepsilon$

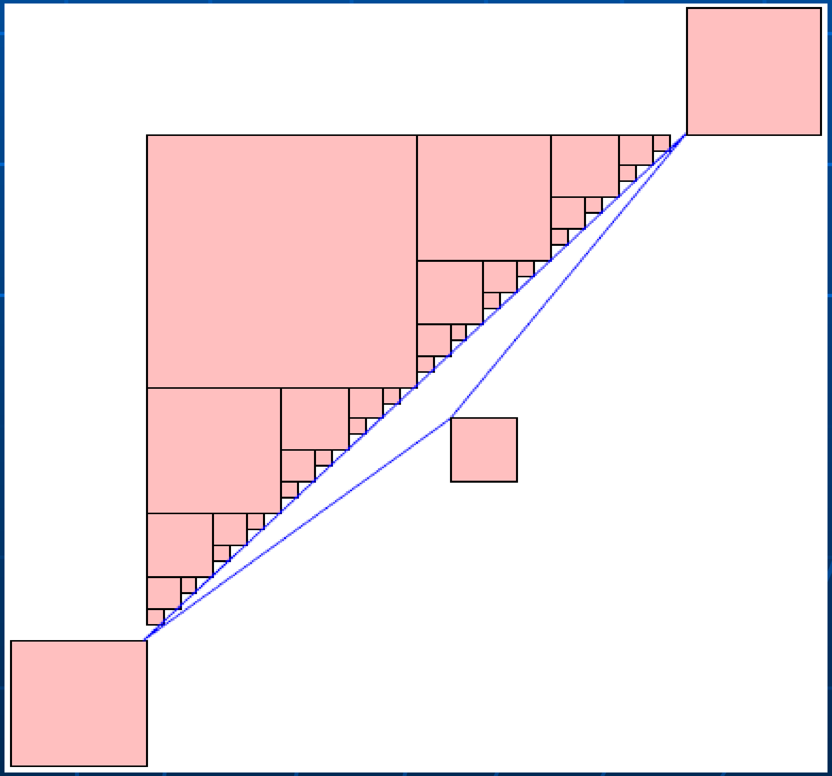
Then, by the Key Lemma, we see that T^* can be converted to be (m,M)-guillotine, adding length $O(T^*/\varepsilon)$

Key Lemma: $L^* \geq C\ \lambda(R_{W_0}) / \log n$

Key Geometric Observation

The sum of the perimeters of a set of n disjoint **fat** regions that are visited by a path of length L is at most O(L log n)

Uses PACKING argument



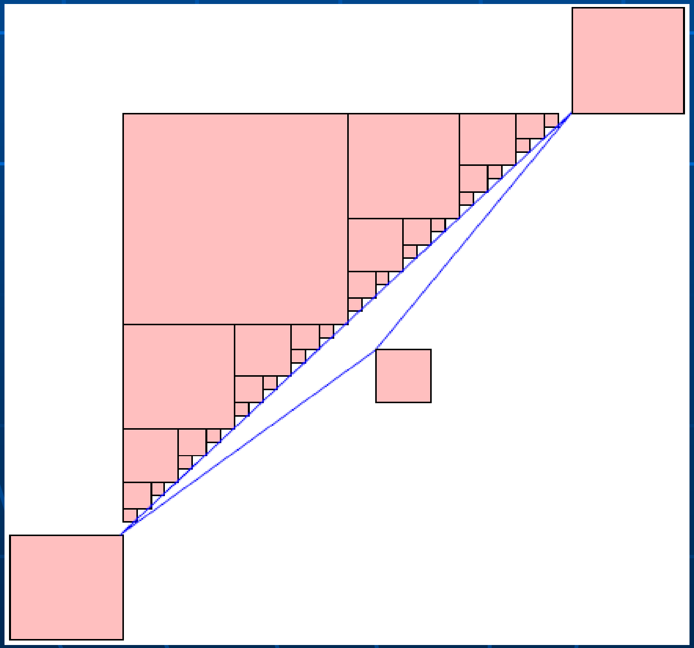
Ex: Bound is tight

Key Lemma: Lower Bound on OPT

Key Lemma: $L^* \geq C \;\; \lambda(R_{W_0}) \,/\, \log n$

Relates tour length of OPT, L^*, to sum of diameters, $\lambda(R_{W_0})$

Ex: Tight



$L^* = \Omega(\, \lambda(R_{W_0}) \,/\, \log n)$

Proof of Key Lemma

Cluster regions by size (diameter), into log (n/ε) classes

There are n_i regions with diameter in range $(d_i/2, d_i)$

Area (packing) argument:

Let A_i = area($(T^* \oplus B(d_i)) \cap W_0$)

This is where fatness and disjointness are used!

Minkowski sum with ball of radius d_i

By **fatness**, $A_i \geq C_0\, d_i^2\, n_i$, for some constant C_0

Thus, by Claim below, $C_0\, d_i^2\, n_i \leq 2d_i\, L^*$, or $L^* \geq (C_0/2)\, d_i\, n_i$

Summing on i, we get $L^* \geq C\, \lambda(R_{W_0}) / \log n$

Claim: $A_i \leq 2d_i\, L^*$

Bucketing Regions by Size

- Consider each minimal covering AAB, W_0, snapped to grid (only $O(n^4)$ of them)
- R_{W_0} = regions fully within W_0 (each has diameter $O(D)$)
- Partition R_{W_0} into $K=O(\log(D/\delta))=O(\log(n/\varepsilon))$ classes according to the diameters falling in ranges:

$(0,\delta)$ $(\delta,2\delta)$ $(2\delta,4\delta)$ $(4\delta,8\delta)$ … $(2^{i-1}\delta,2^i\delta)$ … $(2^{K-2}\delta,2^{K-1}\delta)$

Class i

$d_i=2^i\delta$

$\delta = \varepsilon D/n$

Can shrink to single grid points, by Grid Lemma

Proof of Claim

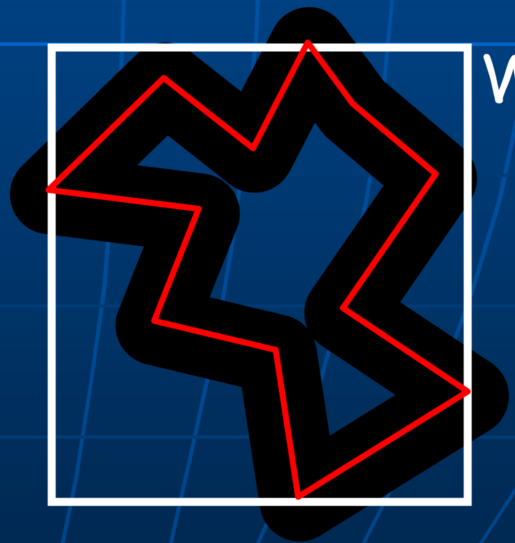
Claim: $A_i = \text{area}((T^* \oplus B(d_i)) \cap W_0) \leq 2d_i L^*$

Minkowski sum with ball of radius d_i

Proof:

$\text{area}(T^* \oplus B(d_i)) \leq 2d_i L^* + \pi d_i^2$

That portion within W_0 does not include (at least) the area, πd_i^2



W_0=BB(T^*)

Main Result

Theorem: TSPN for disjoint fat regions has a PTAS.

PTAS also for the case of nondisjoint regions, if there are disjoint disks $\beta_1,\ldots,\beta_n$, with β_i ¿ P_i and diam(P_i)/diam(β_i) < C.

Improve running time to $O(n^C)$, with C independent of $1/\varepsilon$: use grid-rounded guillotine subdivisions.

Generalizations/Extensions

- Disconnected regions: sets of points/regions that are within a “nice” set of regions
- k-TSPN
- Steiner MST with Neighborhoods
- MST with Neighborhoods (MSTN)
- k-MSTN

Approximation of 2D TSPN: Connected Regions

	Fat Regions	non-Fat Regions
Comparable sizes	Disjoint PTAS Newest PTAS O(1) Non-Disjoint	Disjoint O(1) O(1) APX-hard Non-Disjoint
Arbitrary size	Disjoint New ~~O(1)~~ PTAS Newest O(1), PTAS? O(log n) Non-Disjoint	Disjoint O(log n) O(log n) APX-hard Non-Disjoint

Conjecture: PTAS for all

Conjecture: O(1) for all

Laundry List of Problems

- Know PTAS
  - TSP, k-TSP, Steiner MST, k-MST
  - Red-blue separation
  - Min-weight convex subdivision
  - TSPN, fat regions
  - Orienteering problem
  - Lawnmowing problem
- OPEN: PTAS?
  - TSPN, disjoint regions in 2D
  - Vehicle routing; min-weight cover with k-tours
  - Deg-3, deg-4 spanning trees
  - Min-weight triangulation
  - Watchman route problem
  - Min-area triangulated surface; special case: terrain