

Riemann maps, Welding, Axes and Categorization of 2D Shapes

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(work with Matt Feiszli)

- I. Background from computer vision
- II. Fast welding via the Hilbert transform
- III. Boundary derivative of the Riemann map
- IV. Diffeomorphisms and measured laminations
- V. Cell decompositions and \mathcal{S} vs. $\text{Diff}(S^1)$
- VI. Welding gives a multi-valued composition law on \mathcal{S}

I. Background from Computer Vision

- Let \mathcal{S} = space of simple closed curves $C = \partial R$ in \mathbb{R}^2 .
Arguably the prototypical nonlinear inf.diml.manifold.
Won't specify how smooth C should be.
- Cx. Analysis: the most general domain R where thm X holds; or wild domains invariant by Kleinian gps.
- Comp.Vis.: domains = contours of objects in images, p/w smooth boundaries
- 3 CV questions:
 1. What does R_1 similar to R_2 mean? Need metric on \mathcal{S}
 2. How to construct prob.measures on \mathcal{S} to represent likely deformations of a shape
 3. Cell decomposition of \mathcal{S} to reflect shape categories.

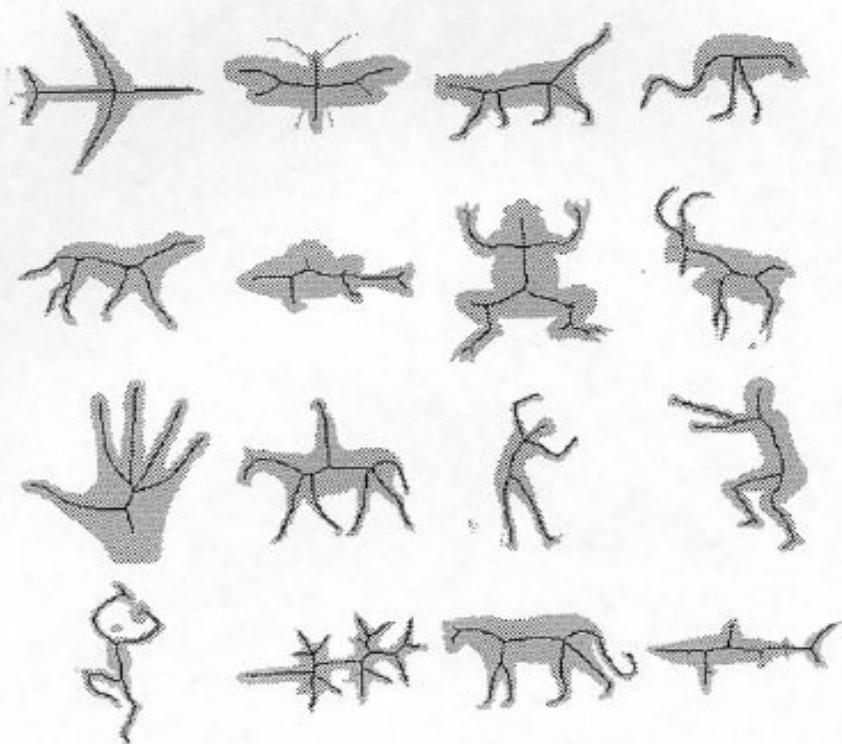
Variants of \mathcal{S}

- C as image of embeddings of S^1 vs. degree 1 immersions of S^1 .
- \mathcal{S} or \mathcal{S} /transl. or \mathcal{S} /transl+rot or \mathcal{S} /transl+scaling or \mathcal{S} /transl,rot,scaling.
- degree of smoothness (\sim Sobolev spaces, k L^p derivatives)
- Submersion approach, $\text{Imm}(S^1, \mathbb{R}^2) \rightarrow \mathcal{S}$ or $\text{Diff}(\mathbb{R}^2) \rightarrow \mathcal{S}$ vs. conformal, potential-theoretic approach

Medial Axes

The usual axis for shape recognition is Blum's '*medial axis*': the locus of centers of bitangent circles:

Going along with the axis is the pairing of points on the boundary and the lamination of R (here pseudo animals from a stochastic model of Zhu)



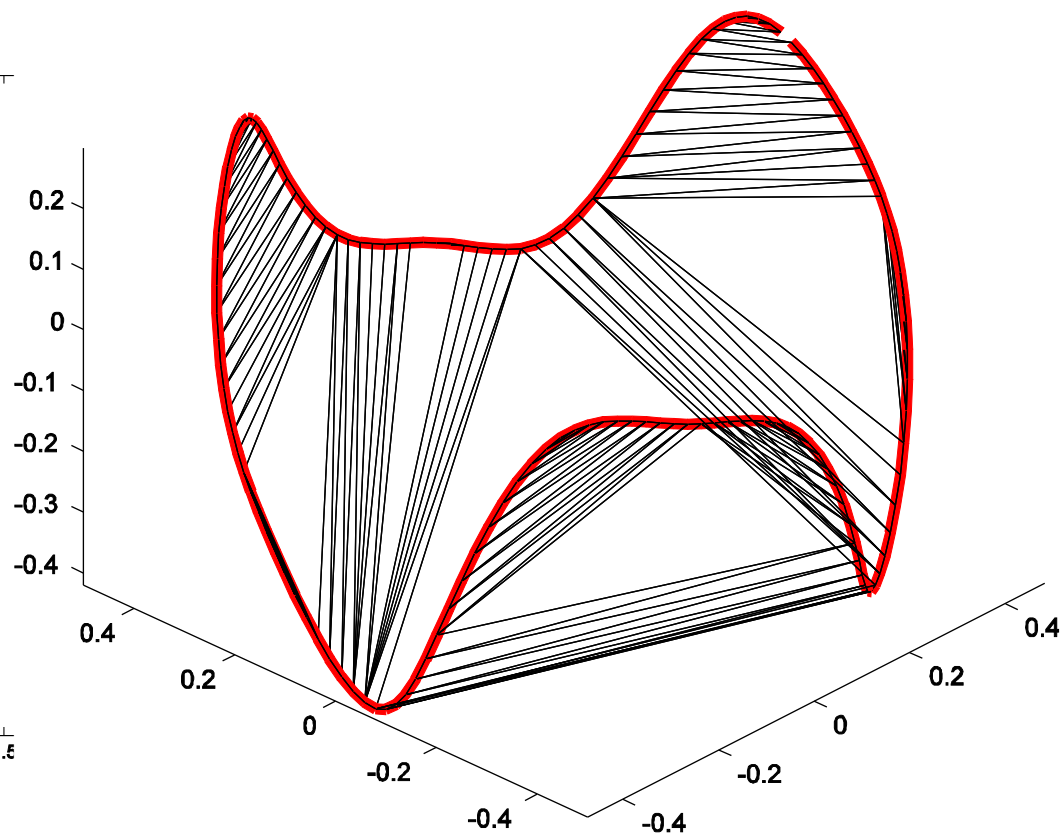
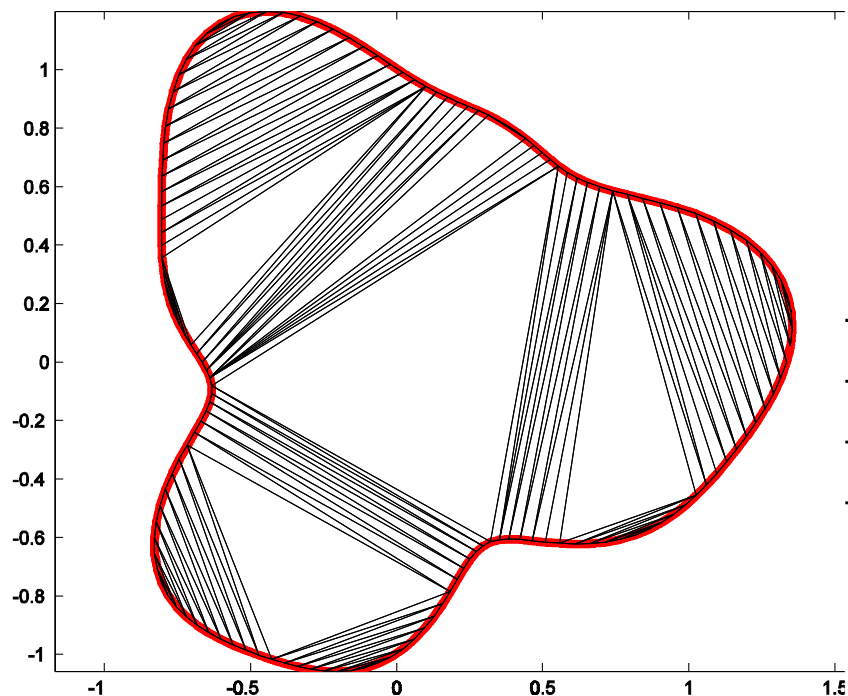
Thurston showed that the medial axis was dual to a convex hull construction:

Euclidean version: Lift the region R to a sphere by stereographic projection. Planes supporting it from below meet the sphere in circles contained in R , so the following correspond:

- Pts of axis;
- bitangent circles;
- bottom supporting planes;
- rulings of the bottom surface of the convex hull

Hyperbolic version: consider R as in the boundary of upper half space \mathbb{H}^3 and take the envelope of the hemispheres over the bitangent circles in R : *Thurston's dome*(R), a hyperbolically developable surface whose rulings are 1:1 with the pts of the axis.

An example of the medial axis via the Euclidean convex hull



Players from complex analysis

- Riemann map φ_- from D to interior R of shape C
- Same φ_+ from exterior of D to exterior of R
- ‘Fingerprint’ or ‘welding map’ $\psi = \varphi_+^{-1} \circ \varphi_- : S^1 \rightarrow S^1$
associated to C (creating a bijection Diff/SL_2 with $\mathcal{S} \bmod \text{transl., scaling}$)
- The lamination of R by circular arcs perpendicular to C and Thurston’s μ map from D to R (1:1 if D is smooth) which is an isometry on this lamination of D and integrates infinitesimal elliptic maps (see below).

II. Fast welding via the Hilbert transform

Given $\psi \in \text{Diff}(S^1)$, seek $f_- = f_+ \circ \psi$ where

$$f_-(z) = c_1 z + c_2 z^2 + \dots, \text{ holo. in } D$$

$$f_+(z) = z + a_1 z^{-1} + a_2 z^{-2} + \dots, \text{ holo. in } \hat{\mathcal{E}} - D$$

Use the periodic Hilbert transform:

$$H\left(\sum a_n e^{in\theta}\right) = \cotn(\theta / 2) * \sum a_n e^{in\theta} = \sum_{n \neq 0} i \cdot \text{sign}(n) a_n e^{in\theta}$$

Then $iH(f_+ \circ \psi) = iH(f_-) = -f_- = -f_+ \circ \psi$ and

$$iH(f_+) = +f_+ - 2e^{i\theta}, \text{ hence}$$

$$iH(f_+ \circ \psi) \circ \psi^{-1} - iH(f_+) = -2f_+ + 2e^{i\theta}$$

But $K(f) = H(f \circ \psi) \circ \psi^{-1} - H(f)$ is conv. with a smooth kernel

$$f_+ = \left(I + iK / 2 \right)^{-1} e^{i\theta}$$

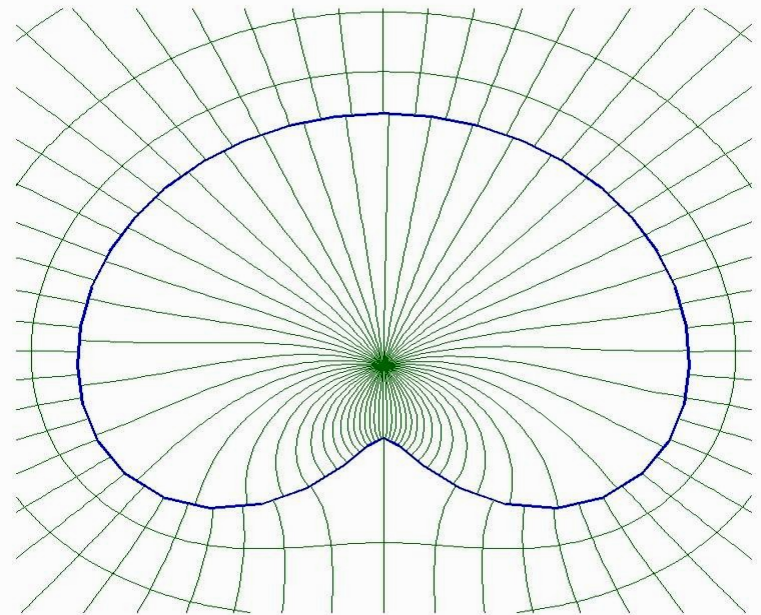
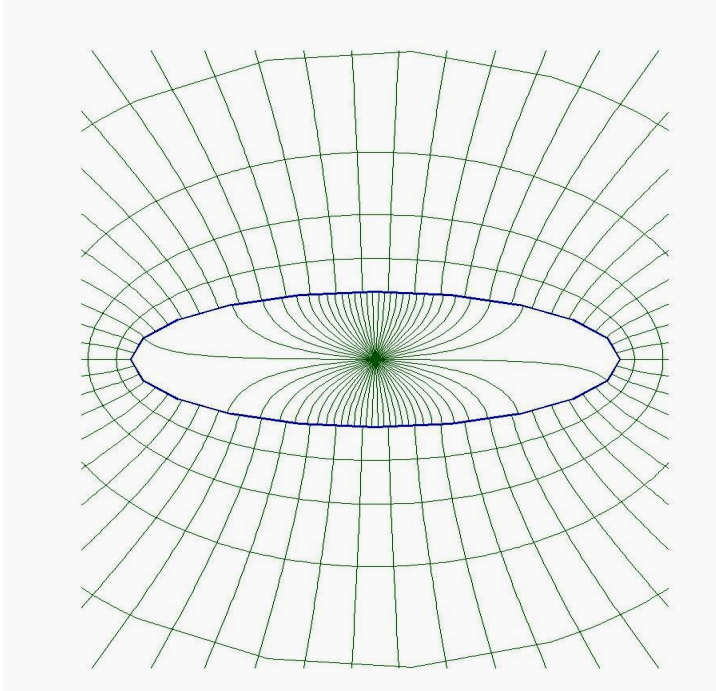
MatLab code for this:

```
n = size(phi1,1);  
% interpolate to half grid points  
phi1x = [(phi1(n)-2*pi); phi1; (phi1(1)+2*pi); (phi1(2)+2*pi)];  
phi1mid = (-phi1x(1:n) + 9*phi1 + 9*phi1x(3:n+2) - phi1x(4:n+3))/16;  
phi2x = [(phi2(n)-2*pi); phi2; (phi2(1)+2*pi); (phi2(2)+2*pi)];  
phi2mid = (-phi2x(1:n) + 9*phi2 + 9*phi2x(3:n+2) - phi2x(4:n+3))/16;  
% Set up the integral equation  
L1 = abs(sin((phi1*ones(1,n)-ones(n,1)*phi1mid')/2));  
L2 = abs(sin((phi2*ones(1,n)-ones(n,1)*phi2mid')/2));  
K = log((L1(:,[n 1:n-1]).*L2) ./ (L1.*L2(:,[n 1:n-1])));  
% Solve it!  
f = (eye(n)+i*K/(2*pi))\(\exp(i*phi1));
```

Feiszli: compute the Riemann map similarly?

A map $\varphi : S^1 \rightarrow C$ extends to a holo map on interior only if $\arg_{\varphi(\theta)}(C) - \theta$, $\log \varphi'$ are Hilbert transforms.

III. Boundary derivatives of the Riemann map



$j : \bar{D} \rightarrow \bar{R}$, then if $j(0) = P_0$, $j(e^{iJ}) = P$

$|j'(e^{iJ})|$ large if P "far" from P_0 and/or $k_C(P)$ large pos.

$|j'(e^{iJ})|$ small if P "near" P_0 and/or $k_C(P)$ large neg.

Bounds on $|\phi'(e^{i\theta})|$ (M. Feiszli)

Assume $\phi(0) \in$ medial axis, $\xi \in C$,

C_ξ = bitgt circle in R thru ξ , radius d , center P_ξ

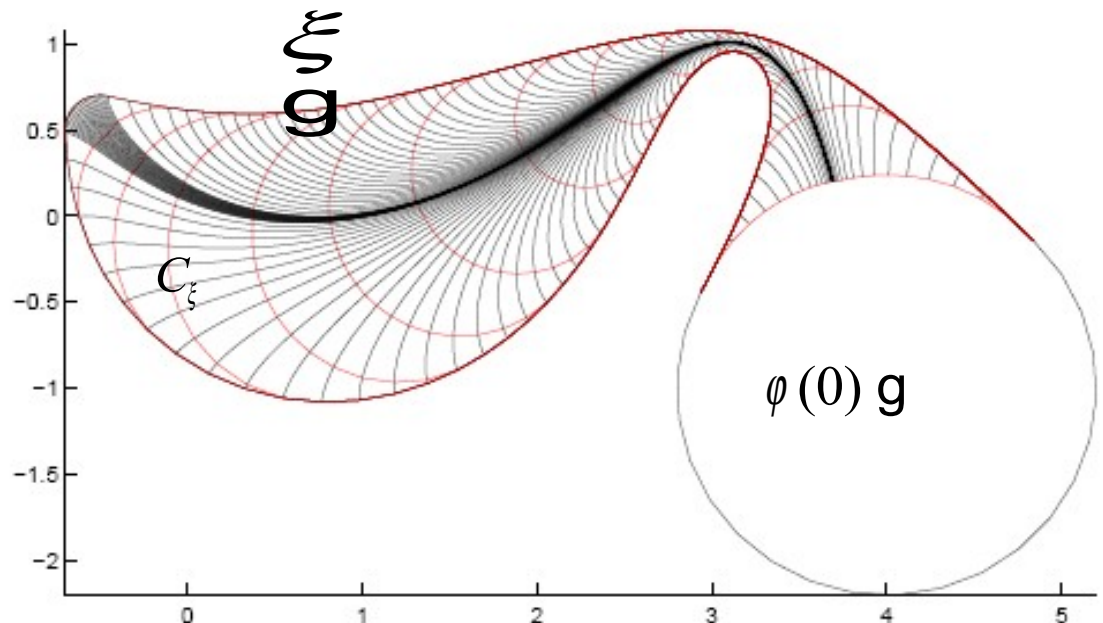
then factor $\phi = \psi \circ A$, A Mobius, $\psi(0) = P_\xi$

$$|\phi'(\phi^{-1}\xi)| = |A'(\phi^{-1}\xi)| \cdot |\psi'(\psi^{-1}\xi)|,$$

(i) $|A'(\phi^{-1}\xi)|$ estimated by 'dist.' from $\phi(0)$ to P_ξ ,

$\log |A'(\phi^{-1}\xi)| \approx c \int_\Gamma ds / r$, Γ medial axis connecting $\phi(0), P_\xi$

Proof via
Thurston's \square map
and Sullivan's
theorem



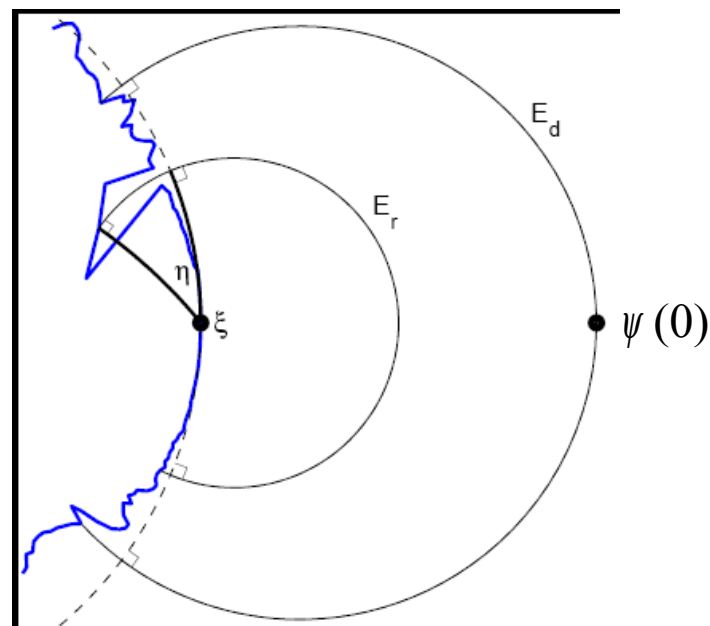
Theorem 1. *Let ξ be a point on C at which curvature exists and is equal to some $\kappa \in \mathbb{R}$. Let $\psi(0)$ be the center of the bitangent circle at ξ , so $\psi(0) = \xi + d\vec{n}$ for some $d \in \mathbb{R}$. Then*

$$d \geq |\psi' \circ \psi^{-1}(\xi)| \geq \frac{\pi}{8} \left(\frac{d}{2 - \kappa d} \right) \exp \left\{ - \int_0^d \frac{2}{2 - \kappa r} \frac{\eta(r)}{\pi + \eta(r)} \frac{dr}{r} \right\}$$

Remark: When the boundary is an arc of a circle, the error term vanishes and theorem 1 gives us:

$$|\psi' \circ \psi^{-1}(\xi)| \geq \frac{\pi}{8} \left(\frac{d}{2 - \kappa d} \right)$$

Up to the constant $\pi/8$ this formula is correct for a map of the disc to one component of the complement of a circle.



The Interior Conformal Axis

- Suppose we consider all $(\phi \circ A)$, A Möbius, so as to discount $\phi(0)$. What is left?

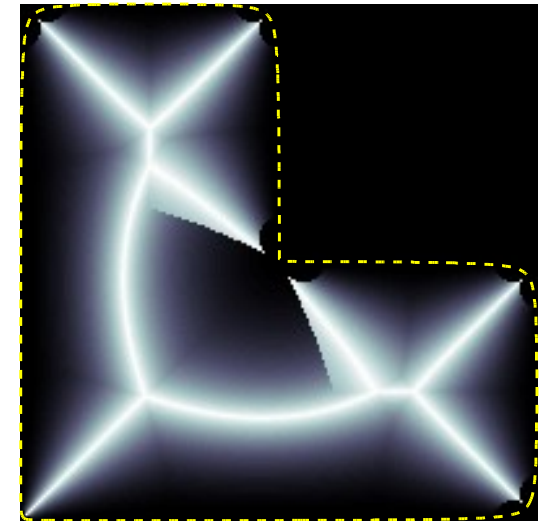
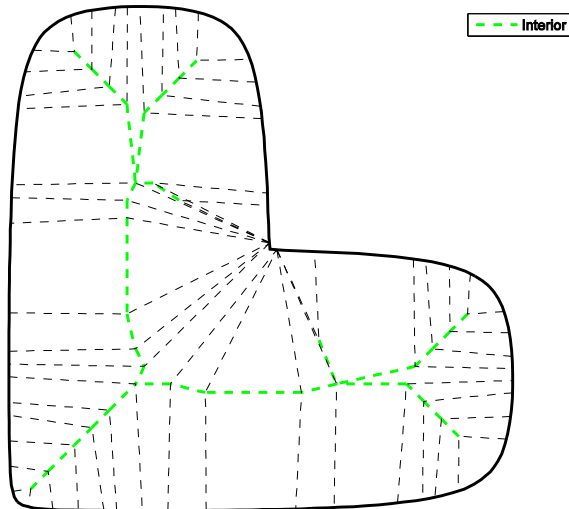
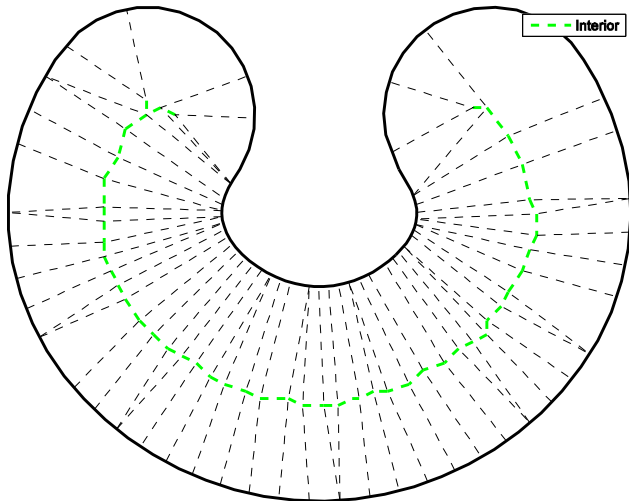
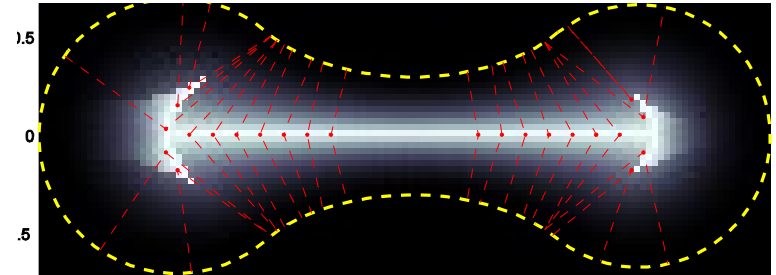
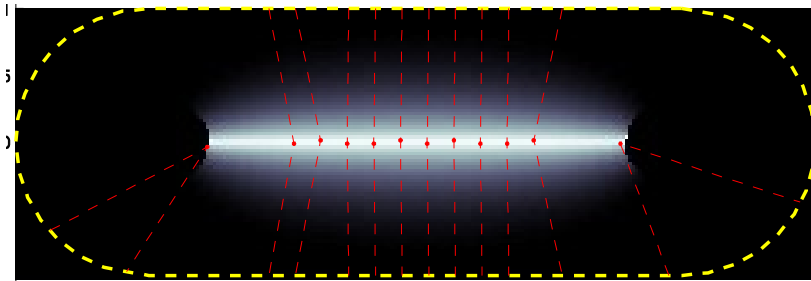
$$\text{Ax}(R) = \left\{ P \hat{=} R \mid \text{if } j(0) = P, \text{ then } |j(e^{iJ})| \text{ has 2 equal global min.} \right\}$$

- This generalizes to \mathbb{R}^n :

$$\text{Ax}(R) = \left\{ Q \hat{=} R \mid \left| \frac{\int K(P, Q)}{\int n_P} \right|_{P \hat{=} R} \text{ has 2 equal minima in } P \right\},$$

where $K(P, Q) =$ Dirichlet-Laplacian Green's kernel of R

Examples of the Interior Conformal Axis (*Feiszli*)



Conformal axis via convex hulls: define

$$\tilde{C} \subset \text{the cone } \left\{ z = \sqrt{x^2 + y^2} \right\}$$

by the map $\vartheta \mapsto |\varphi'(e^{i\vartheta})| \cdot (\cos \vartheta, \sin \vartheta, 1)$

and take its convex hull in the interior of the cone. WHY?

lemma: $|(\varphi \circ A)'(A^{-1}(e^{i\vartheta}))| = |\varphi'(e^{i\vartheta})| \cdot (\lambda \cos \vartheta + \mu \sin \vartheta + \nu)$,
 where $A \leftrightarrow (\lambda, \mu, \nu)$ is a Mobius map of D

Rulings of the bottom surface of the convex hull are defined by supporting hyperplanes below \tilde{C} meeting it at 2 points and these correspond to A for which $|(\varphi \circ A)'(e^{i\vartheta})|$ has a double minimum, hence to points of the conformal axis....

THIS WAY OF CONSTRUCTING AXES GENERALIZES AND
 GIVES A THIRD WAY TO DEFINE THE MEDIAL AXIS

IV. Diffeomorphisms and Measured Laminations

Thurston's 'earthquake theorem' gives a bijection:

$$SL_2 \setminus \text{Diff}(S^1) \cong ML = \text{measured laminations on } D$$

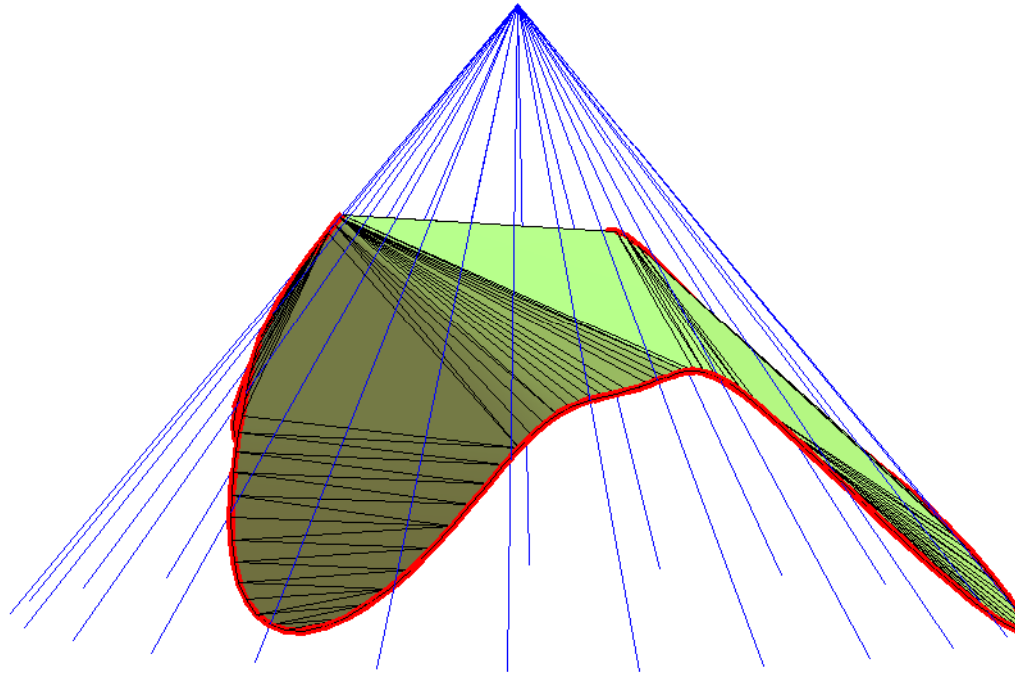
The conformal axis construction generalizes to a similar theorem using bending instead of earthquakes:

$$\textit{Theorem: } \text{Rot} \setminus \text{Diff}(S^1) \cong D \times ML$$

$\forall \phi$, form lower convex hull of $\tilde{C} = \{ \phi'(\vartheta).(\cos \vartheta, \sin \vartheta, 1) \}$,
take tgt plane to hull over 0 plus its bending lamination

The idea is that we can recover this hull from its bending lamination by integrating an ODE. The tangent planes to the hull are space-like planes whose normals are points of D , hence (in the smooth case) we get a map from the set of lamina to D . Developability means the image only moves normal to the lamina and its speed is given by the measure.

An attempt to visualize a cone hull



At each point, one has a ruling and a tangent plane. As you move, the ruling rotates infinitesimally in the plane, as determined by the lamination, and then the plane rotates infinitesimally about the ruling, as determined by the measure.

Link between the 2 types of axis

(Feiszli, work in progress)

Given any $\leftarrow:D\rightarrowtail R$, we get a diffeo of S^1 :

$$f : S^1 \xrightarrow{\text{res } \psi} \partial R \xrightarrow{\text{scaled arc length}} S^1$$

hence a lamination from the double min of $f \circ A$, hence an axis as the locus of points $\leftarrow \leftarrow \curvearrowright \leftarrow \leftarrow \rightarrow \rightarrow$ and a measure on the lamination.

For $\leftarrow \sqrt{\cdot}$ Riemann map, we get the conformal axis,

For $\parallel \dashv$ Thurston's isometry $D \rightarrow dome(R)$, we get the medial axis and bending of the dome and the cone hull give the same measure.

V. Cell decompositions and \mathcal{L} vs. $\text{Diff}(S^1)$

Measured laminations can be broken up naturally into cells:

$$ML \supset_{\text{open,dense}} ML_{\text{gen}} = \coprod_{\gamma} ML_{\gamma}$$

ML_{gen} = laminations with 1D lamina plus fin.# triangles

ML_{γ} = generic laminations of graph type γ

OR

$$ML_{\gamma}^{\varepsilon} = \text{laminations } \Lambda \text{ with } \Lambda \supset \Lambda' \in ML_{\gamma}, \mu(\Lambda - \Lambda') < \varepsilon$$

(I believe the latter cover ML .)

The boundary derivative of the Riemann map defines an *injection* so that the cell decomposition can be carried over to \mathcal{S}

$$\mathcal{S}/\text{transl,scaling,rot.} \subset \rightarrow \text{Rot} \setminus \text{Diff}(S^1) / SL_2$$

$$C \mapsto (\varphi : D \rightarrow R) \mapsto (f = (\text{scaled arc len.}) \circ \varphi|_{S^1})$$

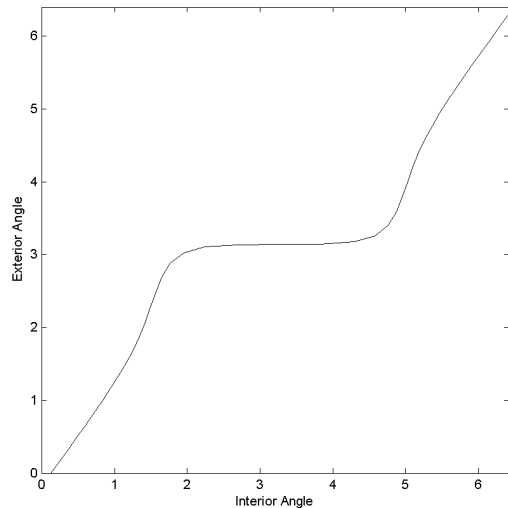
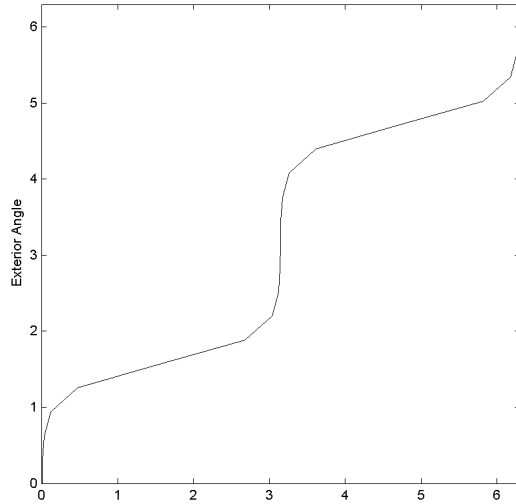
injective because $\log f'(\vartheta) = \text{cnst.Re}(\log \varphi')(e^{i\vartheta})$

But much better is to use the *welding* map which gives a *bijection* so that we can get coords on the cells from Diff

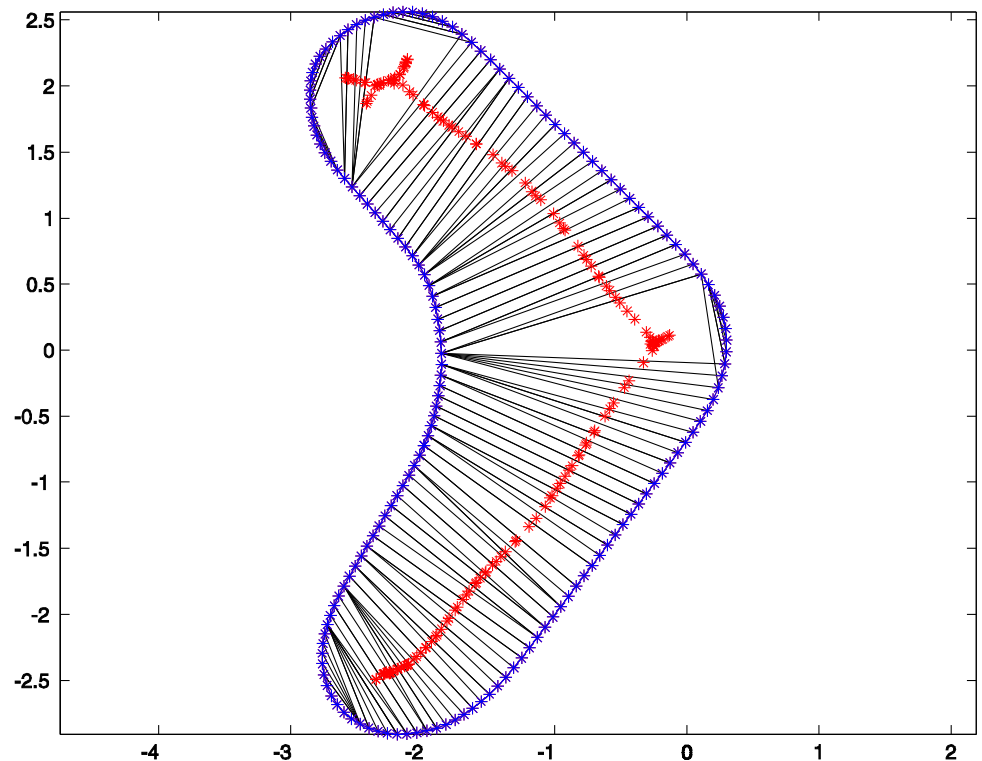
$$\mathcal{S}/\text{transl,scaling,rot.} \xrightarrow[\prec \dots]{\sim} \text{Rot} \setminus \text{Diff}(S^1) / SL_2$$

$$C \mapsto \left(\begin{array}{c} \varphi_0 : D \rightarrow R \\ \varphi_\infty : \hat{\mathcal{L}} - D \rightarrow \hat{\mathcal{L}} - R \end{array} \right) \mapsto (f = \varphi_\infty^{-1} \circ \varphi_0|_{S^1})$$

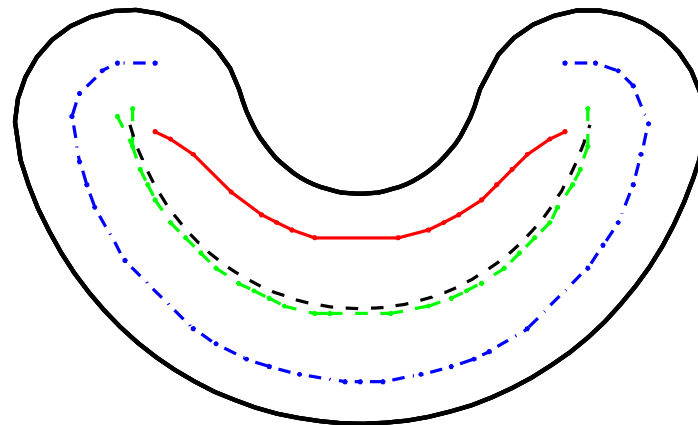
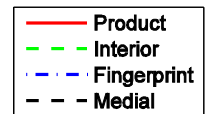
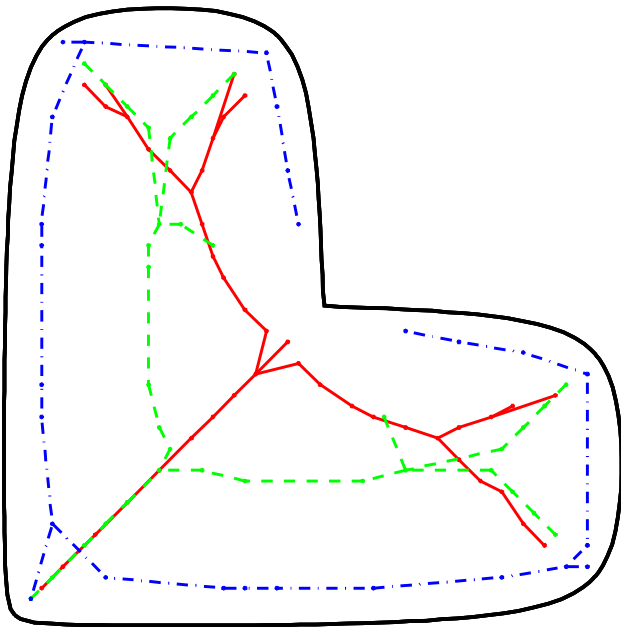
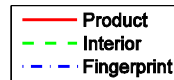
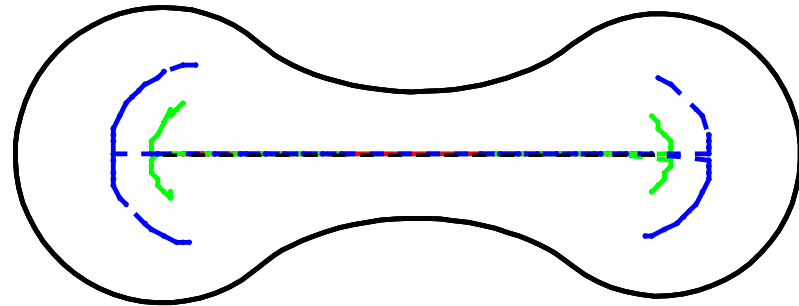
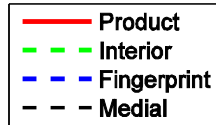
Examples with welding map $f(e^{iJ}) = j_{\mathbb{H}}^{-1} \left(j_0 \left(e^{iJ} \right) \right)$



Top left: f for an ellipse; bottom left: f for a kidney shaped region; below: a welding axis



Comparison of Axes from Interior Riemann Map; Medial axis; Welding Map; and product of derivatives of the interior/exterior Riemann maps



Kimia's cell
decomposition
via axes and
graphs to
classify fishes
– just to prove
it is used in
CV

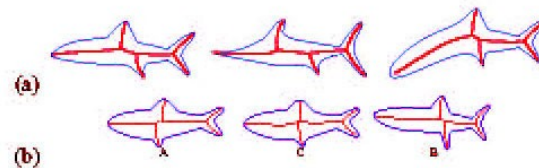


Figure 4: (a) A few examples of shapes belonging to the same shape cell, i.e., having the same shock graph topology. (b) This figure illustrates one of the transitions (contract). A and B are the original shapes, and the C is the degenerate shape at the transition.

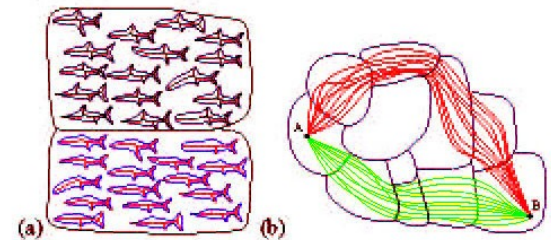
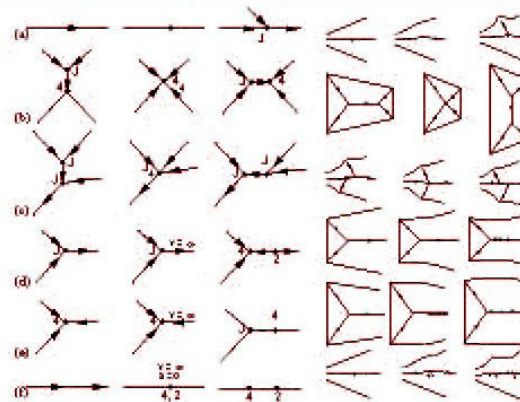
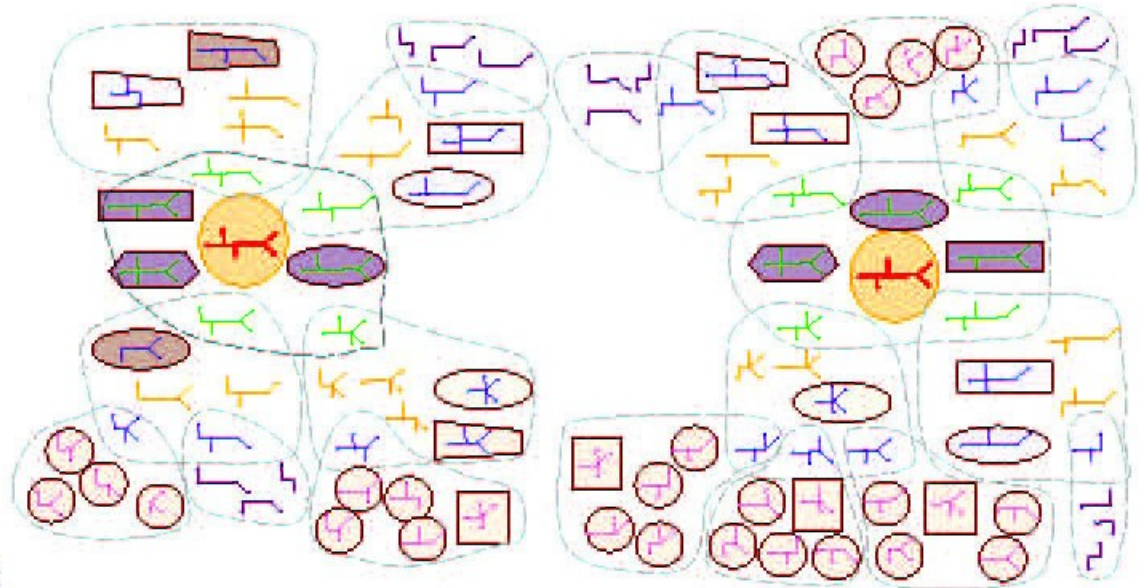


Figure 6: (a) Gaps in the topology of the shape space created by shock transitions. (b) This figure sketches two deformation path bundles between shapes A and B . Each of the two groups of deformations represent two distinct bundles of equivalent deformations.



VI. Welding gives a multi-valued composition law on \mathcal{S}

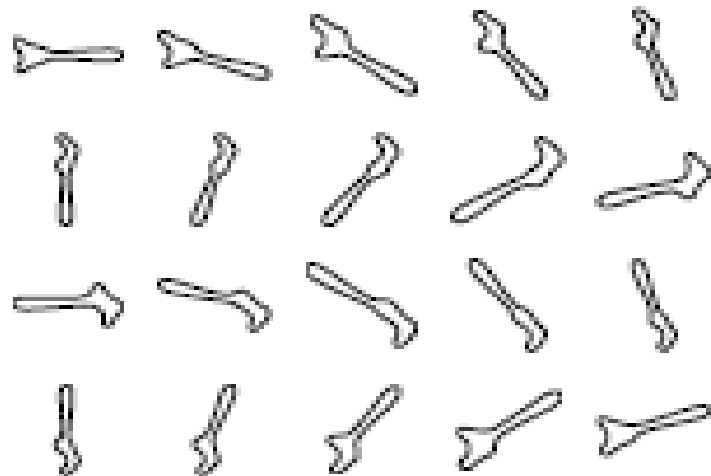
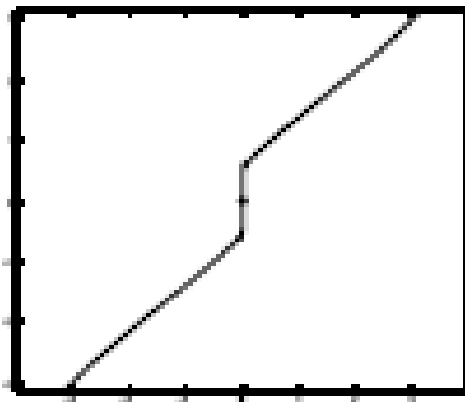
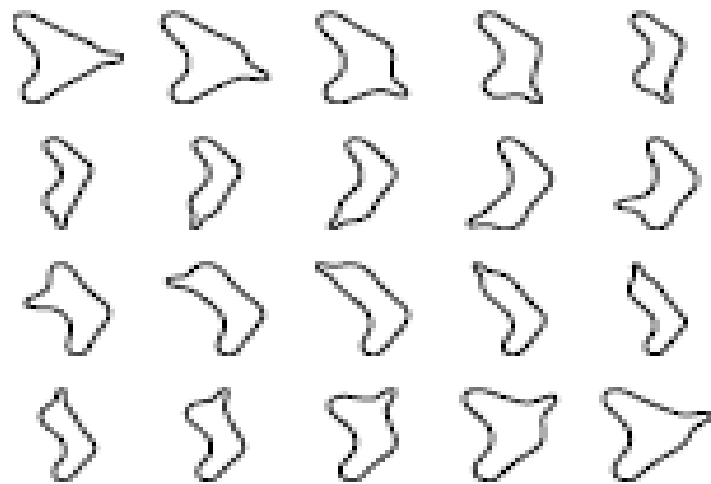
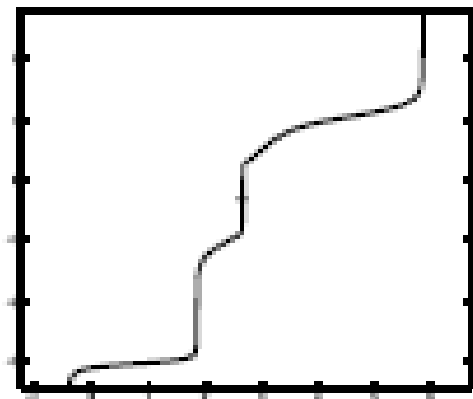
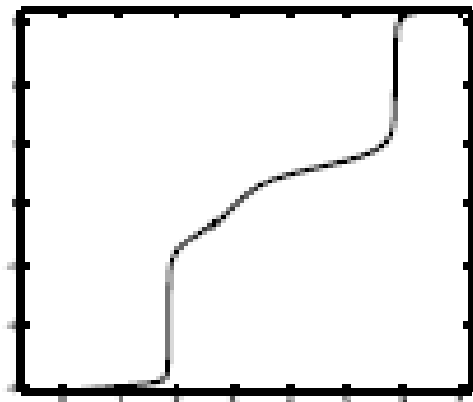
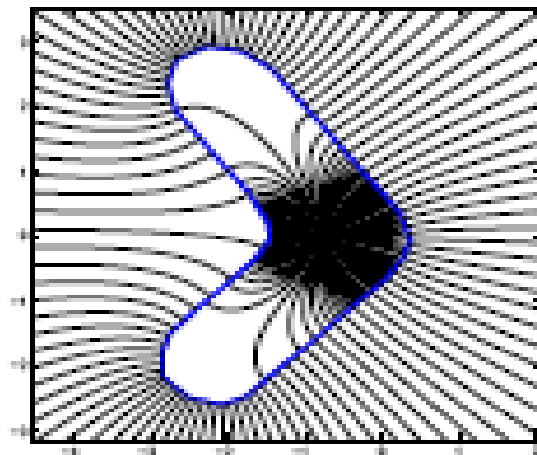
$$\mathcal{S}^+ = \{ C, P \mid P \in \text{Int}(C) \} / \text{transl, scaling, rot} \xrightarrow{\sim} \text{Rot} \backslash \text{Diff} / \text{Rot}$$

hence

1. $\forall \bar{A}, \bar{B} \in \text{Rot} \backslash \text{Diff} / \text{Rot}$,
get the set of products $\{ \overline{A.R_\vartheta.B} \}$
2. $\forall P \in \text{Int}C_1$, plus a rel.orientation of C_1, C_2 ,
get a unique $C_1 \circ C_2$

Perhaps a mathematical analog of how human perception divides shapes into pieces

What
composing
2 diffeos
can do to
the shape.
Decompos-
ing a shape
into parts is
essential to
perception.



Conclusion:

complex analysis gives a large number of tools for the analysis of everyday smooth shapes. We have focused on measured laminations, but the Weil-Peterson metric is equally promising, esp because it makes S into a homogeneous manifold.