Opportunities in Map-Making

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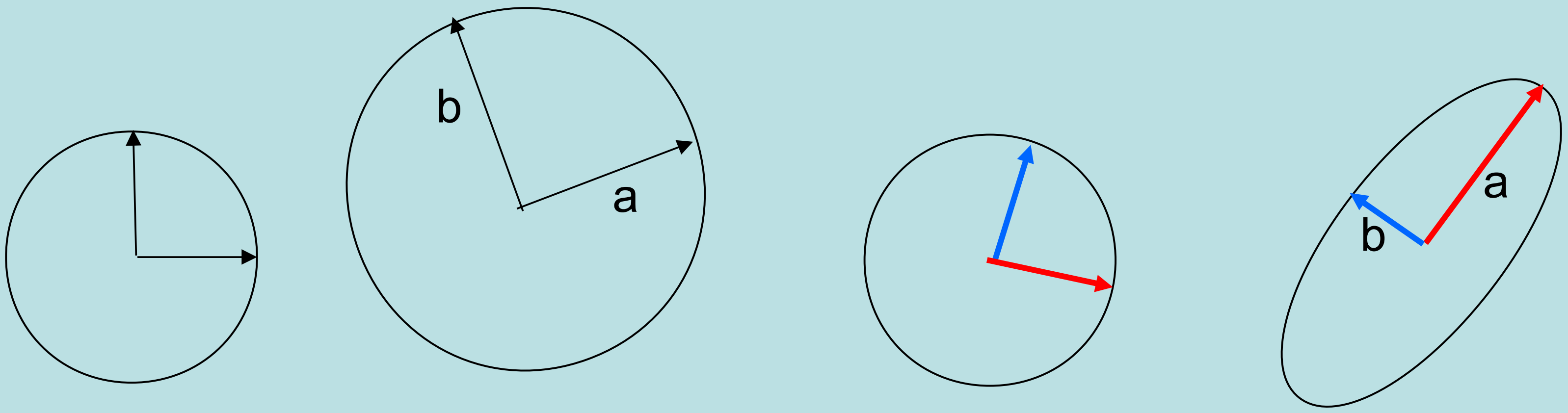
Cartographers can make maps that:

- Preserve all angles (conformal), or
- Preserve all areas (authalic),
- But they cannot do both simultaneously
- They can argue about which is better, preserving angles or preserving areas[1]
- They can try to preserve one of the two (angles or areas) while, at the same time, trying reduce distortion of the other

[1]Interesting story about the Peters projection, ask me later…

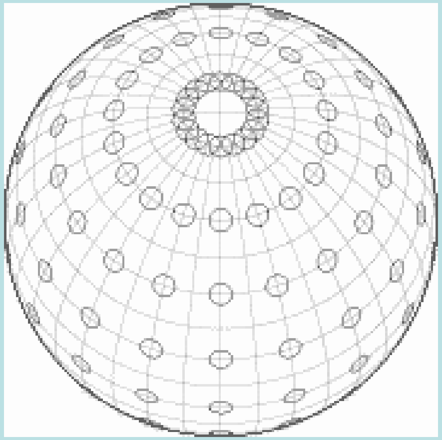
Conformal vs Authalic

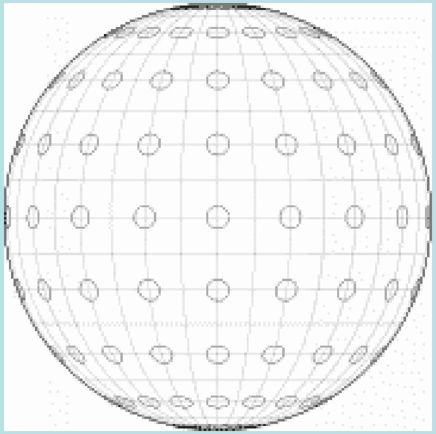


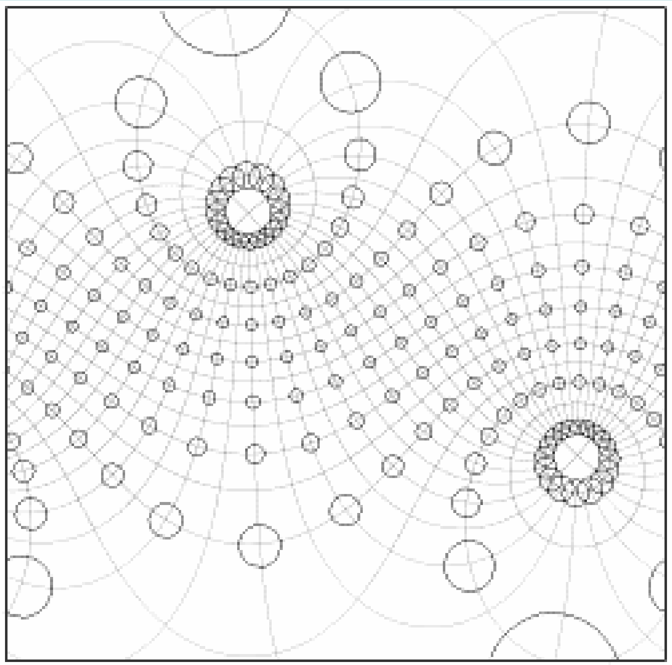
a = b

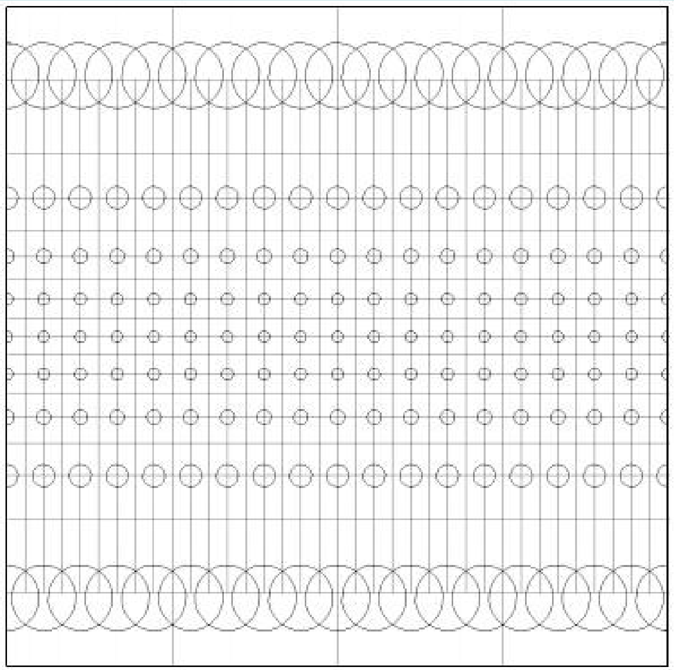
a = 1/b

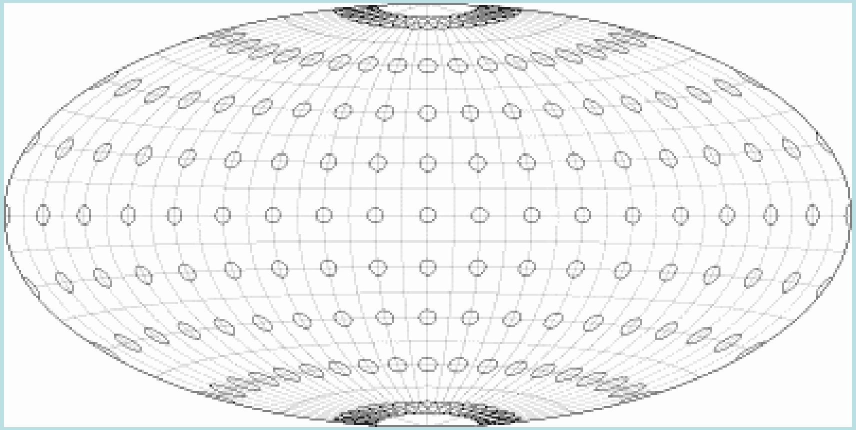
Tissot Indicatrix

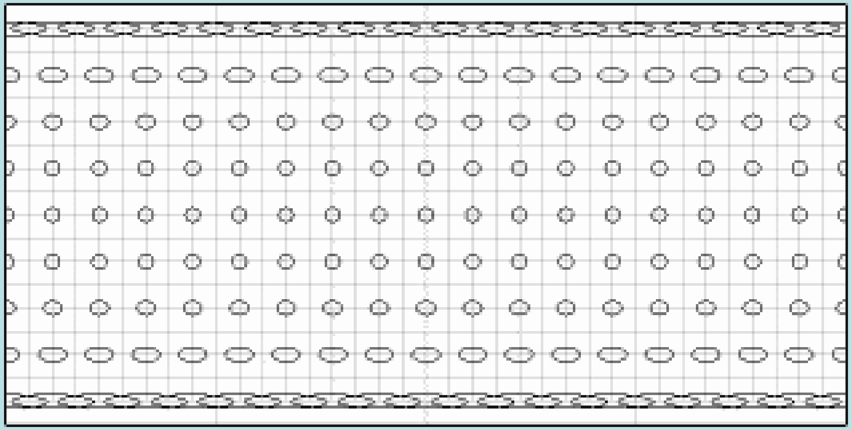












My agenda today

- Many of you have looked into discretizing conformality via triangulations and circles.
- I will try to describe and illustrate some analogous attempts at discretizing area-management via triangulations and ellipses.
- NOT ellipse-packing, just as distortion indicators

Extending functions

- Conformal extension of a simple closed curve to its interior or the entire plane.
- Piecewise linear (affine) extension of a point map to a map of the hull of the domain point set.
  - Can make the PL extension a homeomorphism.
- Piecewise linear extension of a map between polygon boundaries to their interiors.
- Piecewise linear extension of an edge set map to a map of the hull of the domain edge set.
  - Can make the extension have the same area scale within each polygon.
  - Can also adjust areas.

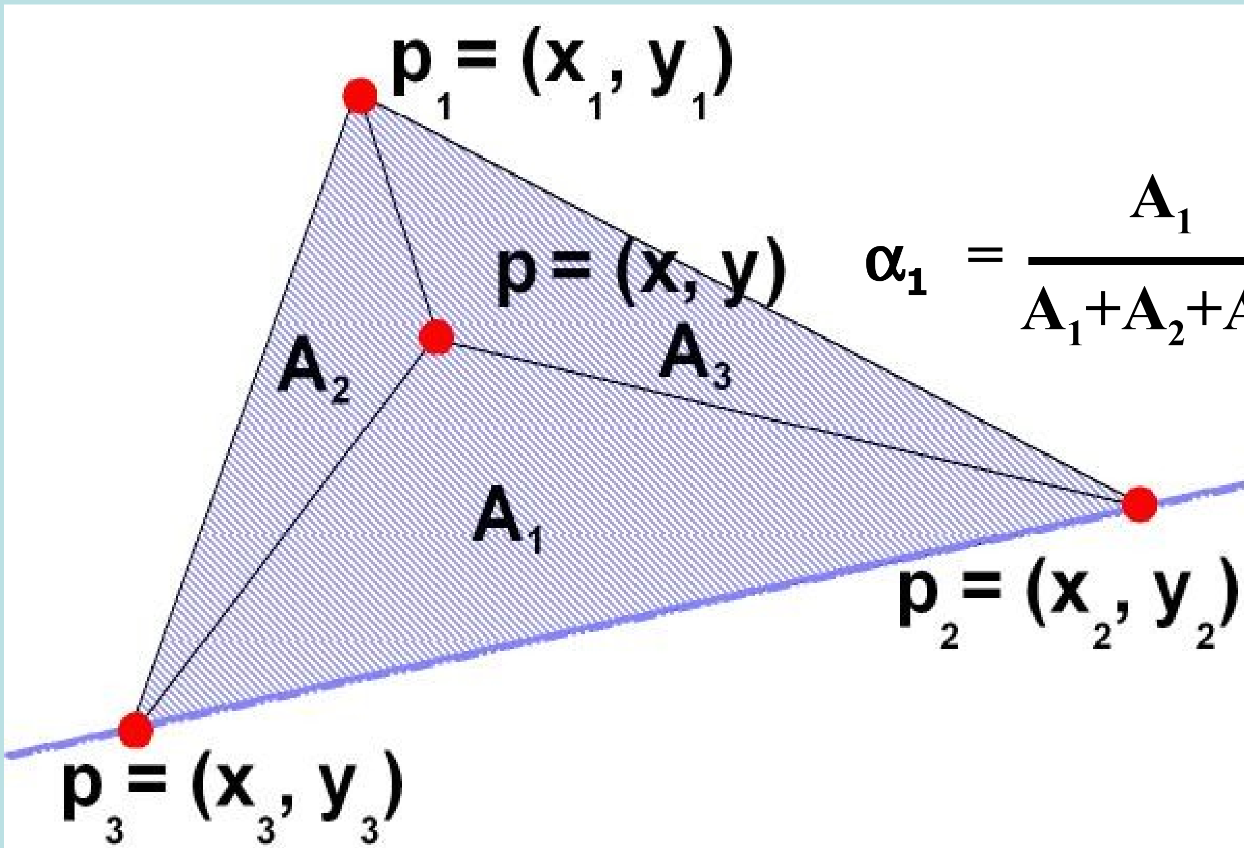
Affine transformations of the plane

- An affine function has the form:

  $$(x,y) \mapsto (ax+by+c,\ dx+ey+f)$$

- An affine function is a linear transformation followed by a translation
- The action on any 3 non-collinear points completely determines the affine map
- A triangle on three vertices maps to the triangle on the three image vertices.
- Lines go to lines with length ratios preserved
- Parallel lines go to parallel lines

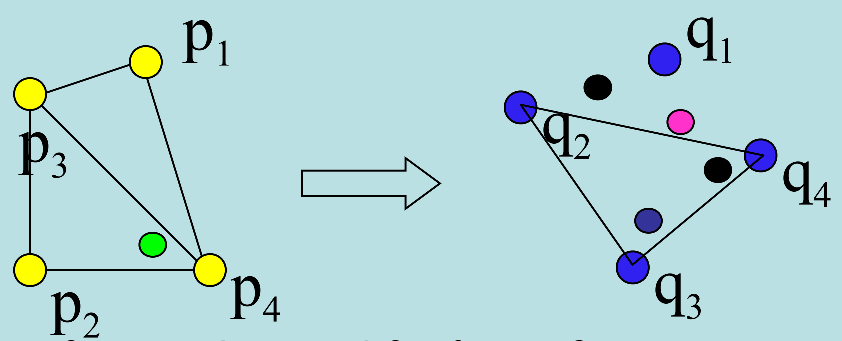
Barycentric Coordinates are just Relative Triangle Areas



$$\alpha_1 = \frac{A_1}{A_1+A_2+A_3} = \frac{\begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}$$

Describing Piecewise Affine Maps

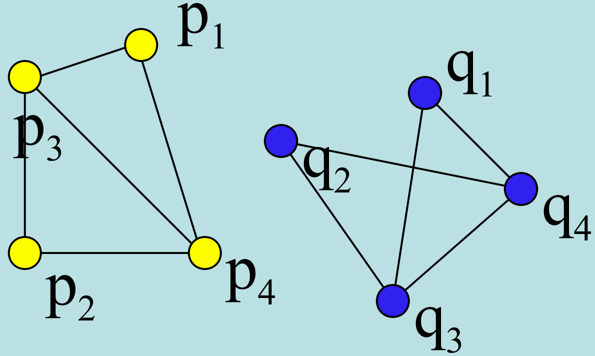
- Piecewise affine interpolation is uniquely and implicitly defined BY SIMPLY **DRAWING** THE TRIANGULATION IN THE DOMAIN SPACE AND **LABELING** THE TRIANGLE VERTICES IN BOTH THE DOMAIN AND RANGE SPACES TO SHOW THE ASSOCIATION:



Piecewise affine transformations

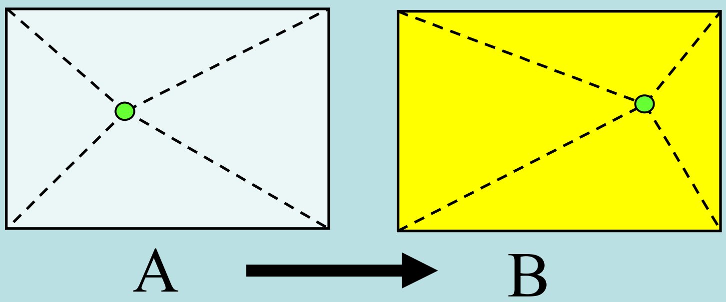
- The interpolating function that preserves barycentric coordinates is precisely the affine transformation from the domain triangle onto the range triangle
- Barycentric coordinates can be computed for the domain (before the function itself is specified), then may be "evaluated at run time" (after the range values of the triangle vertices are specified)

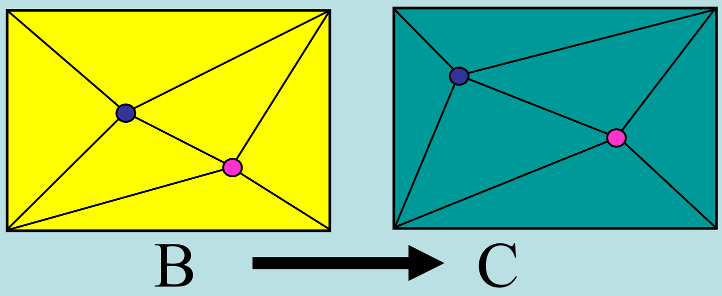
Triangulation maps: Piecewise Linear Maps

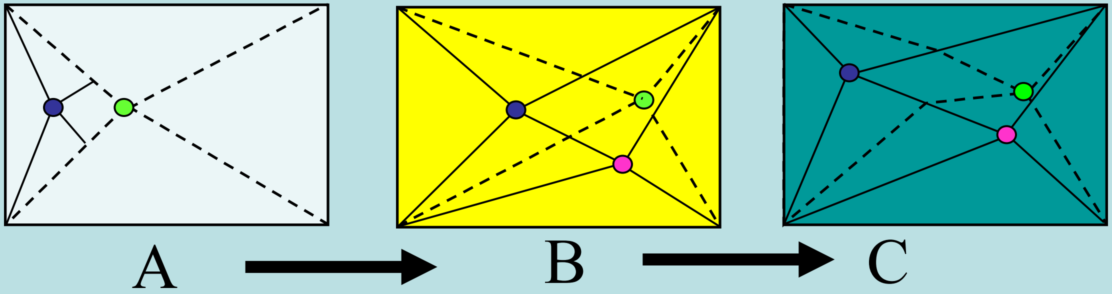


- PL maps are described **fully** by their action on triangles
- PL maps are described fully by their action on vertices after domain triangles have been specified
- PL maps agree on shared edges that are straight line segments
- But we REALLY prefer our functions to be invertible (homeomorphisms)

The composition of two Piecewise Affine maps is a Piecewise Affine map



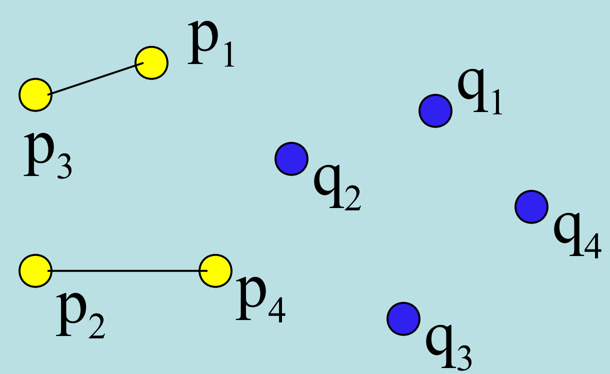




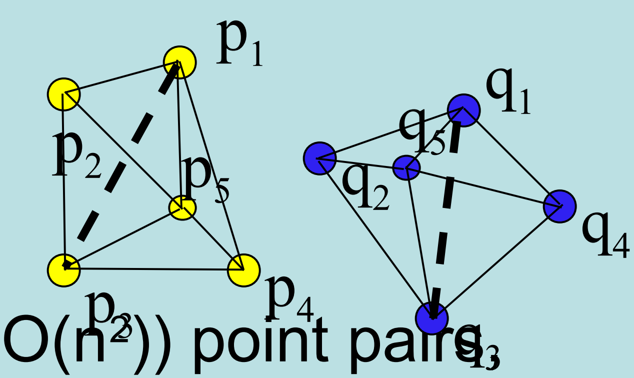
Invertible piecewise affine maps are also called piecewise linear homeomorphisms (PLH maps).

Isomorphic triangulations: Piecewise Linear Homeomorphisms

- Non-existence results
  - Sometimes there just isn't any simultaneous triangulation on the given point sets

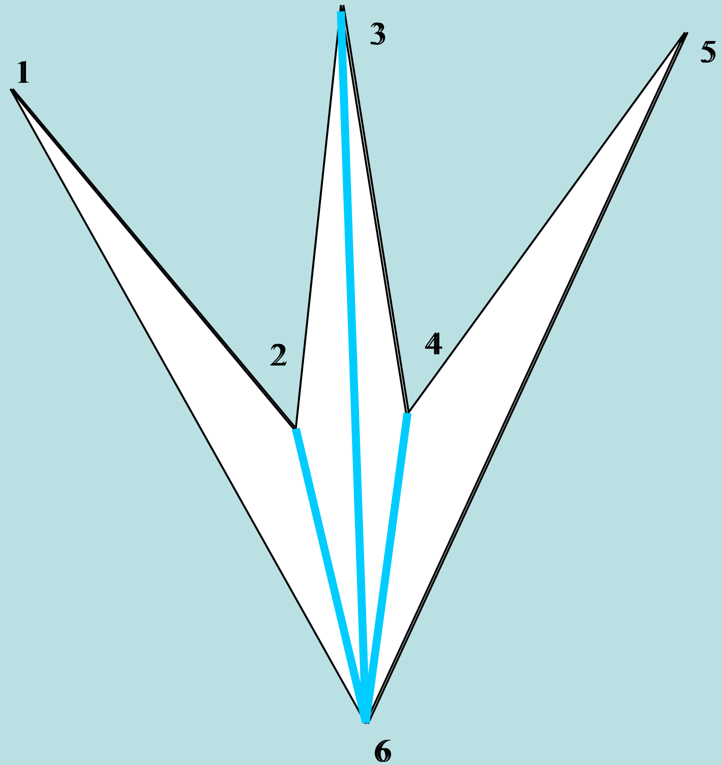


- Existence and complexity results
  - Sometimes all triangulations work
  - Sometimes some triangulations work, others do not
  - If we allow ourselves to add some ($O(n^2)$) point pairs, we can always find a simultaneous triangulation



Any simple polygon can be mapped to any other simple polygon by a PLH map with pre-specified PLH boundary behavior

- Any **n**-sided polygon can be triangulated.
- Any triangulation of an **n**-sided polygon is also a triangulation of a similarly labeled regular **n**-gon.
- Every **n**-gon can be mapped by an *invertible* PLH map onto a regular **n**-gon.


3
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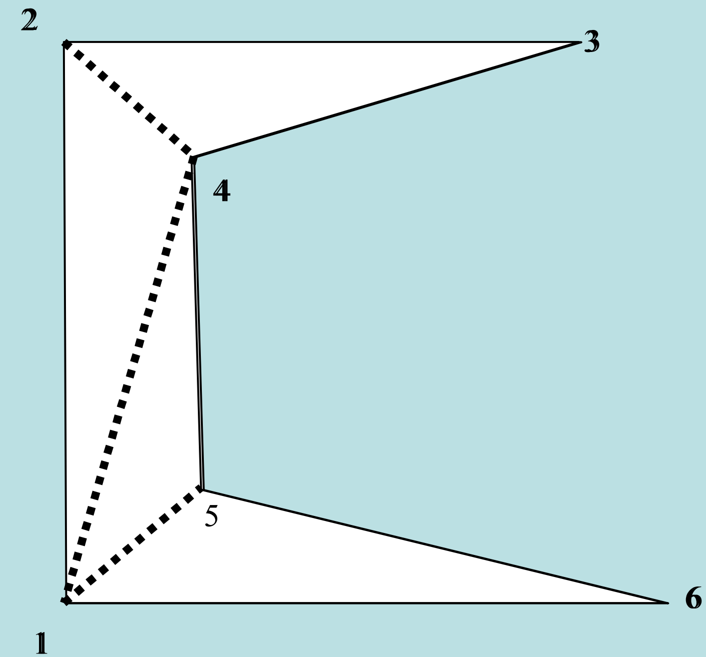
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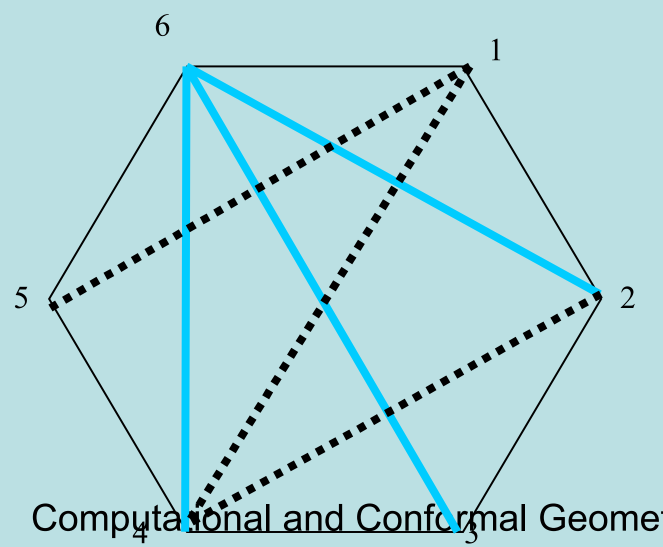
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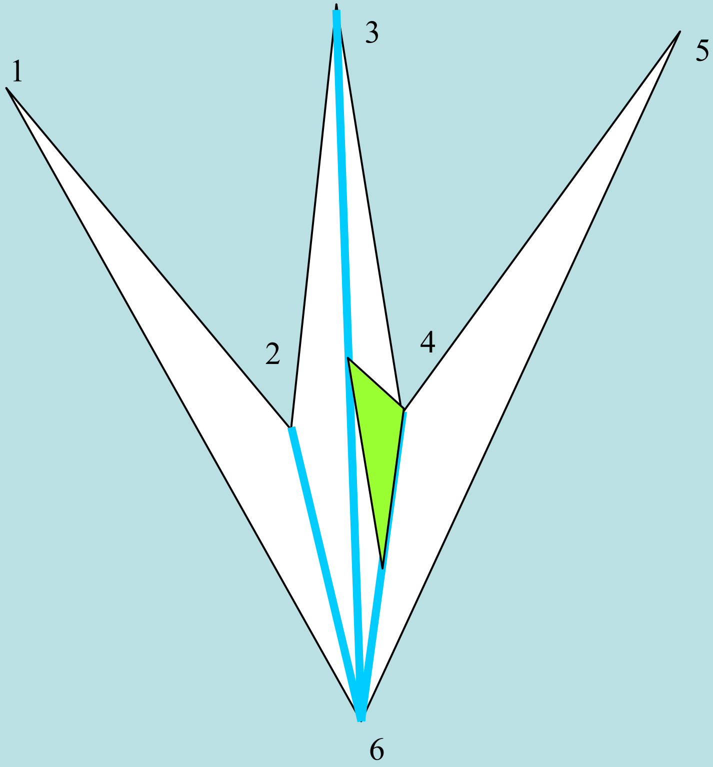
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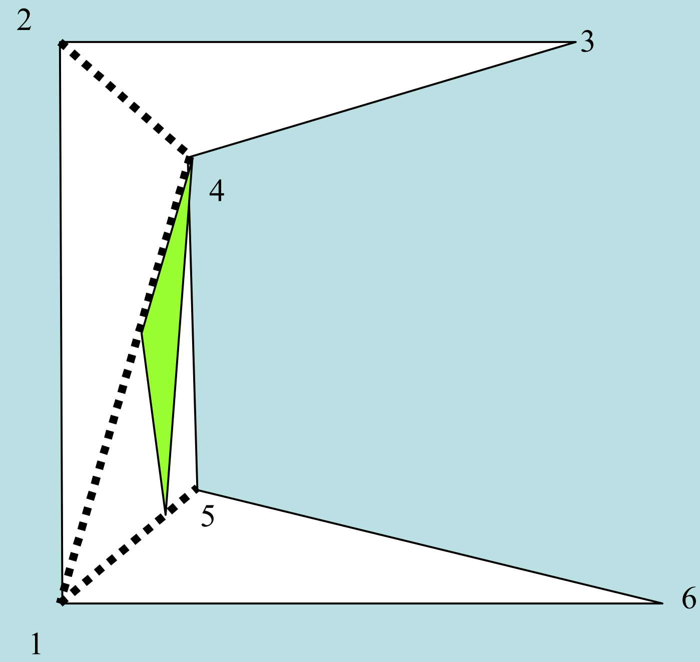
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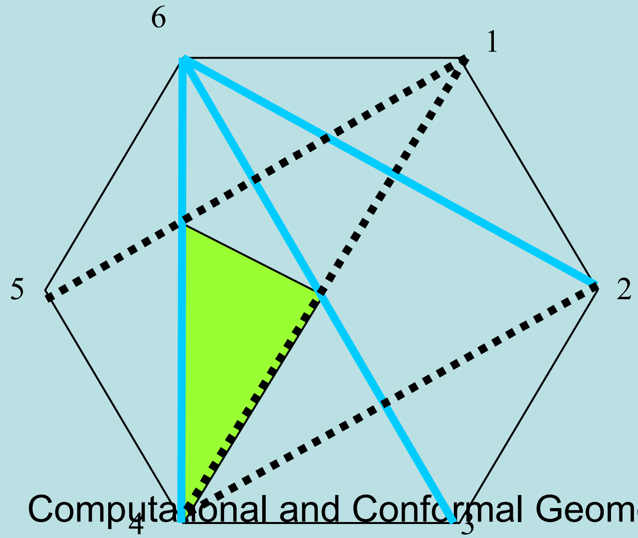
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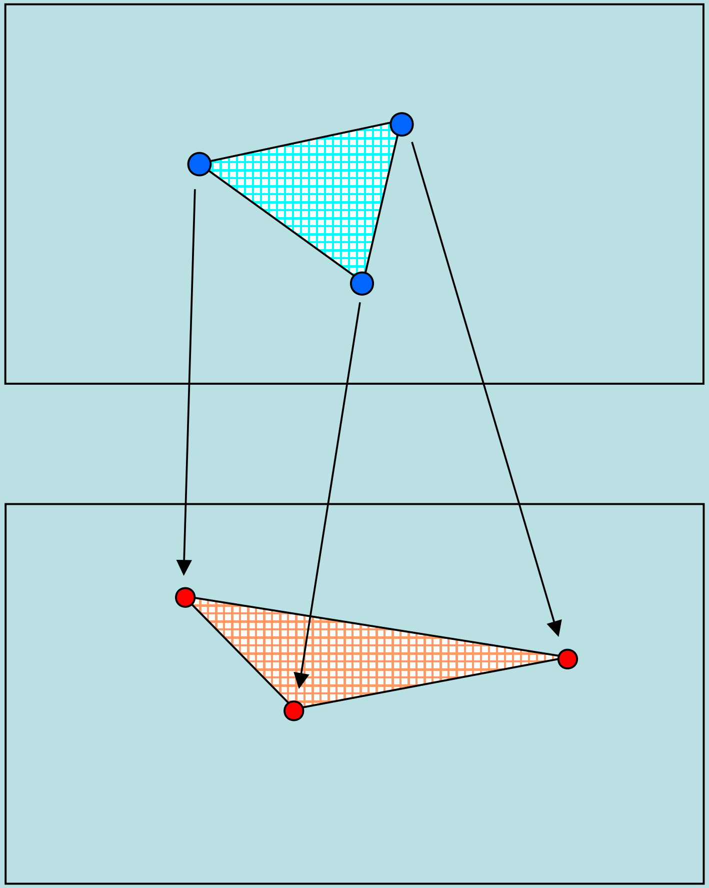
Can we control distortion more?

- Are there area-preserving PLH maps?
  - Can we construct them to extend PLH maps on the boundary? YES! It simply means getting corresponding triangles to have the same area
  - Can we find 1-continuous-parameter families of transformations that are area-preserving PLH maps for every parameter value? YES! These are zero-compression deformations.

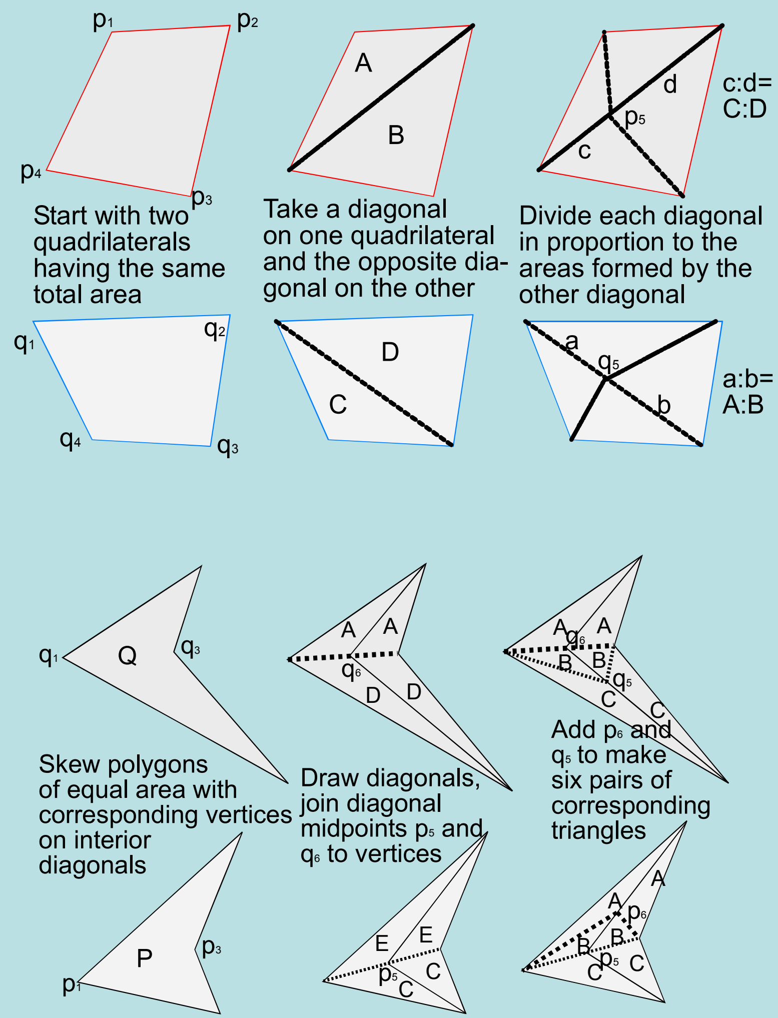
Piecewise-Linear Authalic

- The affine map that sends three points in the plane to any other three points in the plane is authalic if and only if the two triangles on the sets of three points have the same area.



Piecewise-Linear Authalic

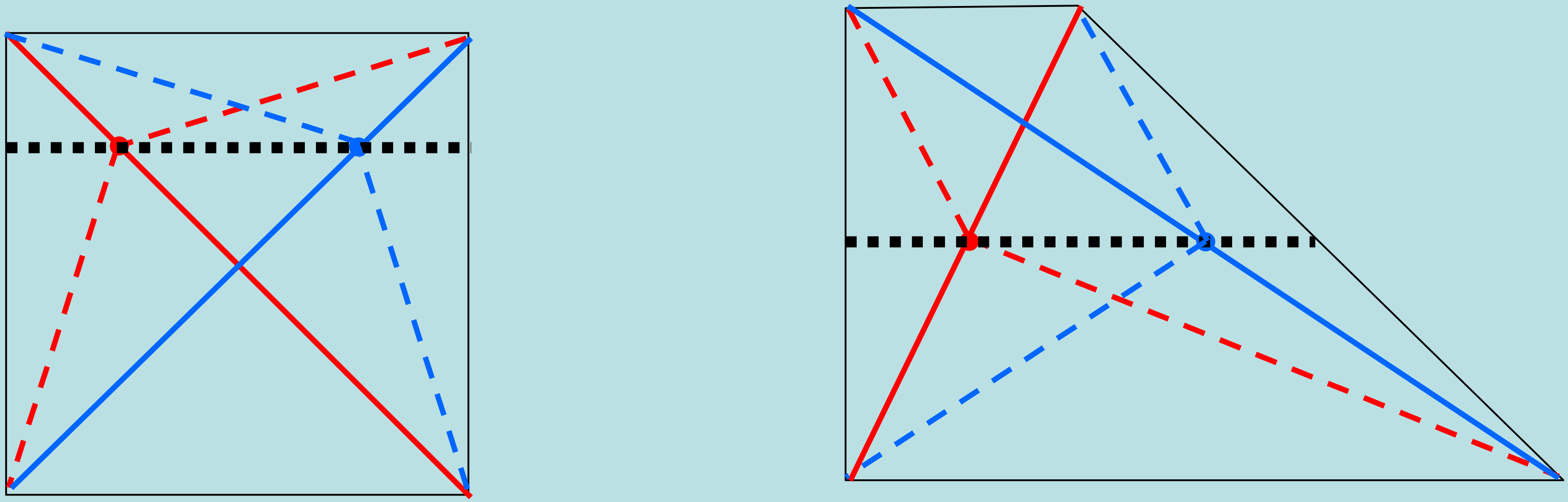
- Two polygons of the same area may be jointly triangulated (with Steiner points) so that corresponding triangle pairs have equal area. The resulting triangulation map is authalic.



Piecewise-Linear Authalic

- Given that two polygons of the same area may be jointly triangulated (with Steiner points) so that the resulting triangulation map is authalic, one may want to know:
  - What is the fewest number of Steiner points (i.e., triangles) needed?
  - Can we find the triangulation of fewest triangles with the least flattening/stretching local scale difference?

P-L Authalic vs Quasiconformal



Possible Extensions/Applications

- 3D
- "Best" quasi-conformal area-preserving piecewise affine transformation using no more than ***k*** Steiner points
- Zero-compression morphing

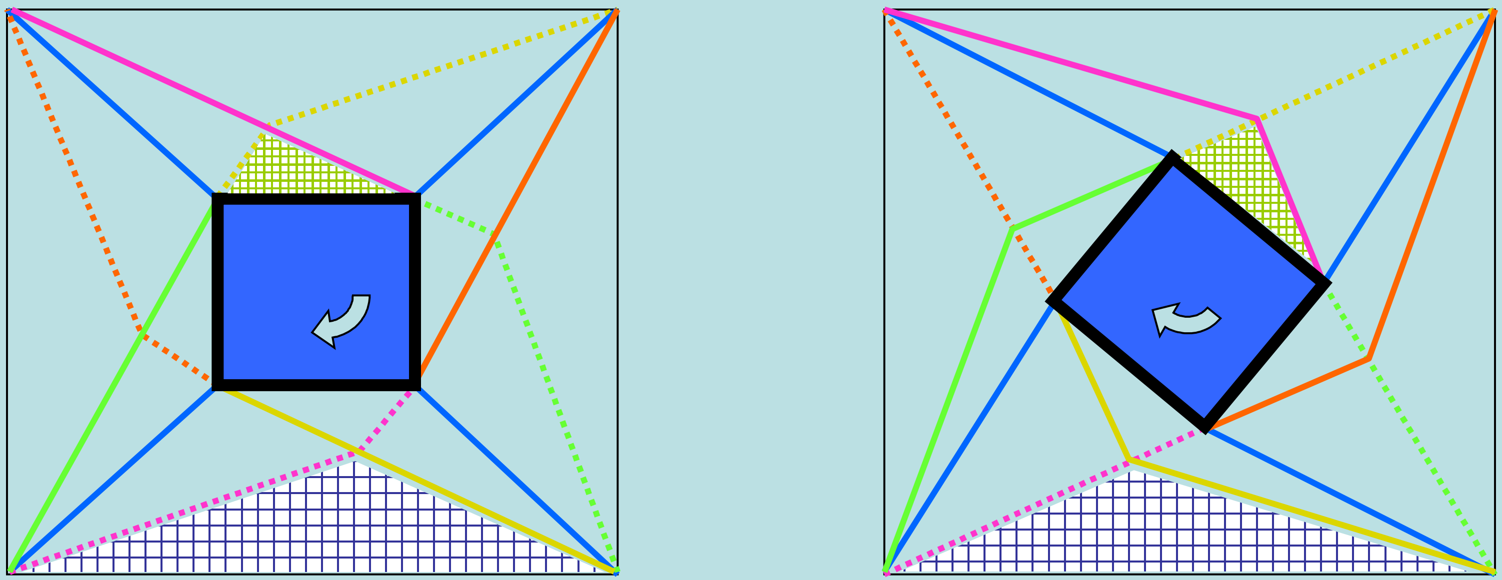
Morphing

- Triangulation maps: $\{p_i\}, \{q_i\}, T_{\{p_i\}}$, $p_i \mapsto q_i$
- Homotopies of triangulation maps: $\{p_i\} \times [0,1] \mapsto R$
  - $(p_i,t) \mapsto q_i(t)$; $(p_i,0) \mapsto q_i(0)=p_i$, $(p_i,1) \mapsto q_i(1)=q_i$
  - Cheap (faceted) morphing
  - Barycentric coordinates pre-computed
- Homotopies in both domain and range
  - $p_i(t) \mapsto q_i(t)$
  - Domain triangles and range triangles change in unison, maintaining relative size throughout
  - At every stage, the transformation is area-preserving
  - “Area-preserving” guarantees “no singularities”

What is Zero-Compression Morphing?

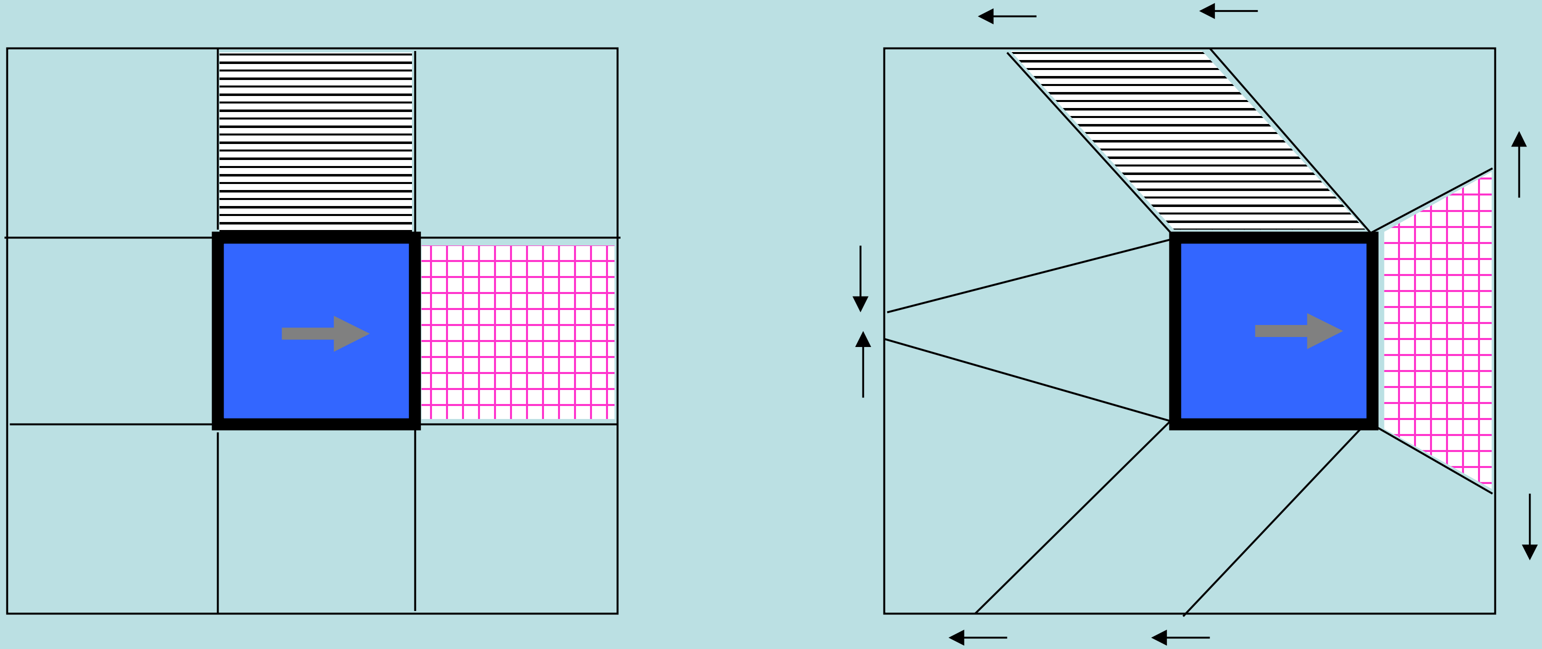
It is continuous deformation that preserves area (volume) everywhere at all times. If we can identify corresponding pairs of simultaneously changing quadrilaterals that have matching areas at each stage throughout the deformation, then we may add Steiner points and triangulate to produce area-preservation everywhere.

Zero-Compression Morphing



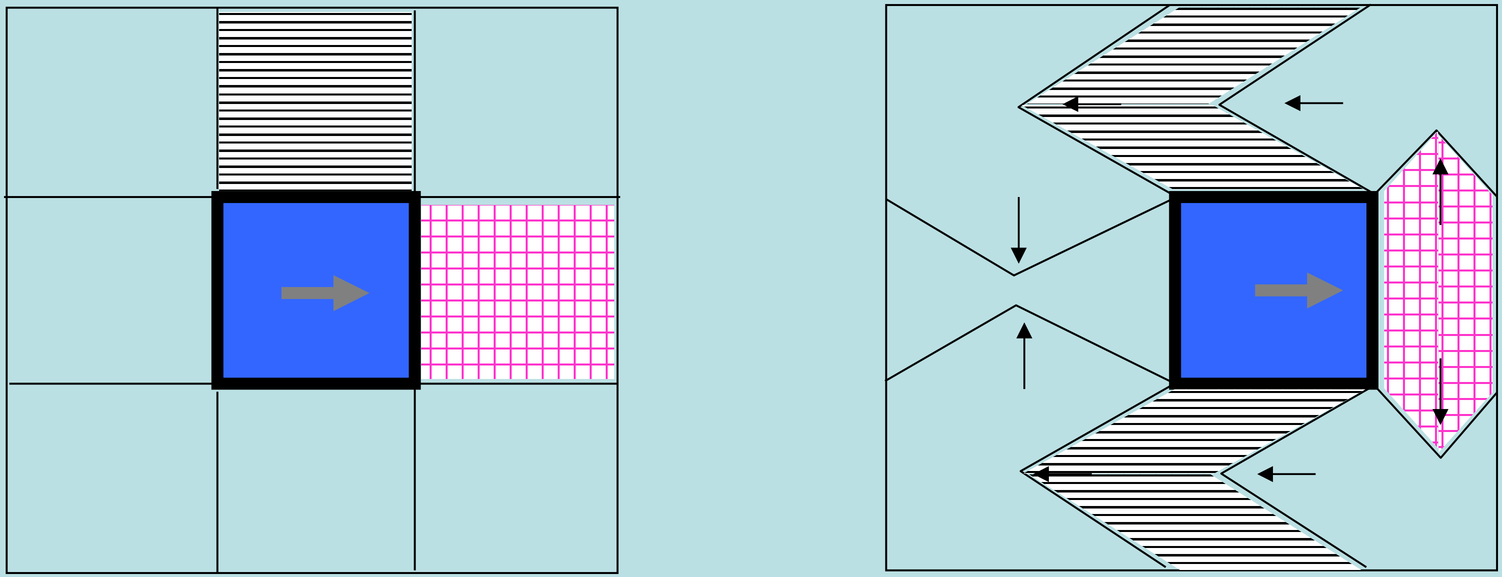
Example 1: A Rotating Raft

Zero-Compression Morphing



Example 2: A Sliding Raft

Zero-Compression Morphing

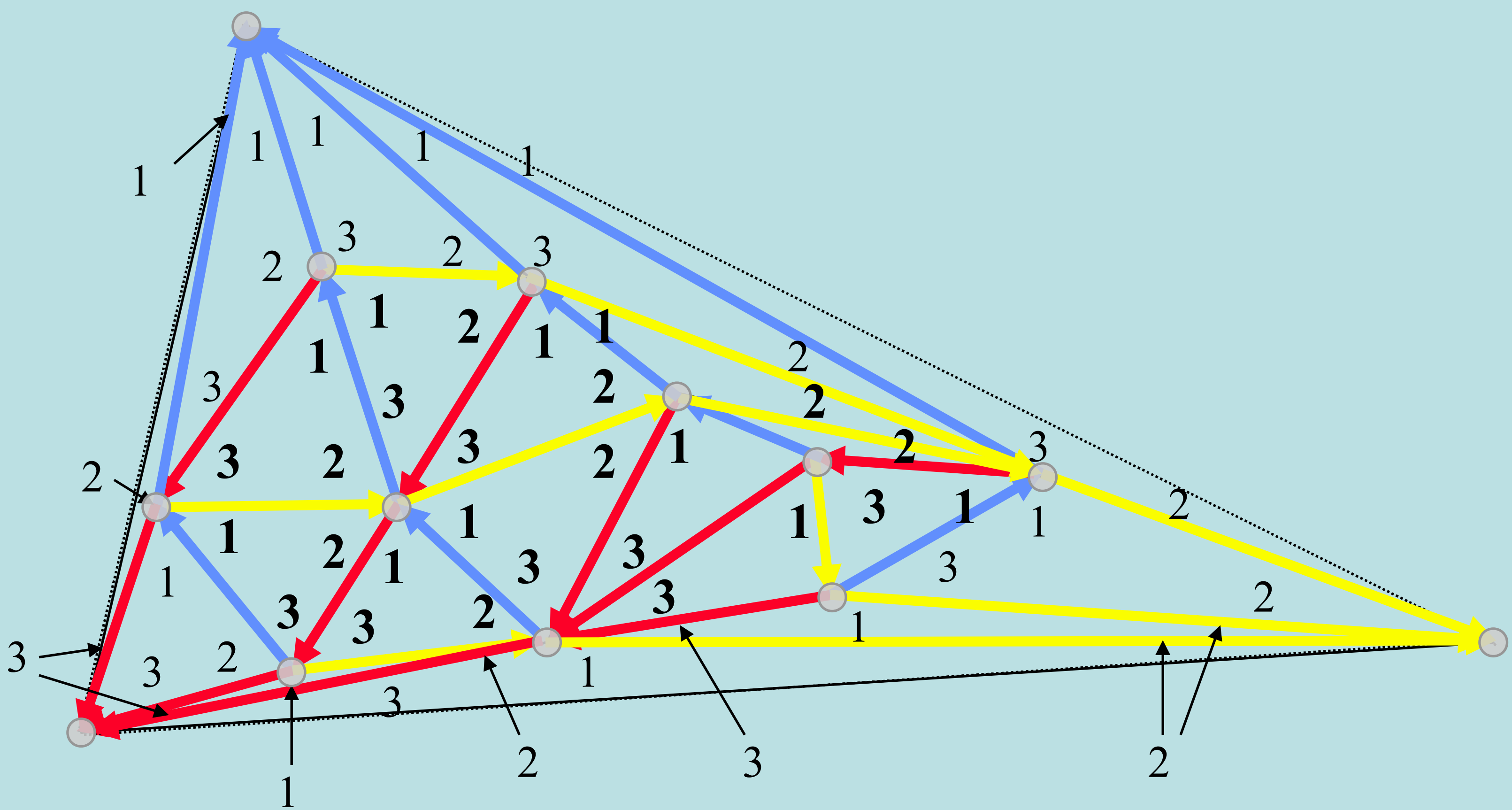


Example 3: A Sliding Raft with No Distortion at Surface/Media Contact Boundaries

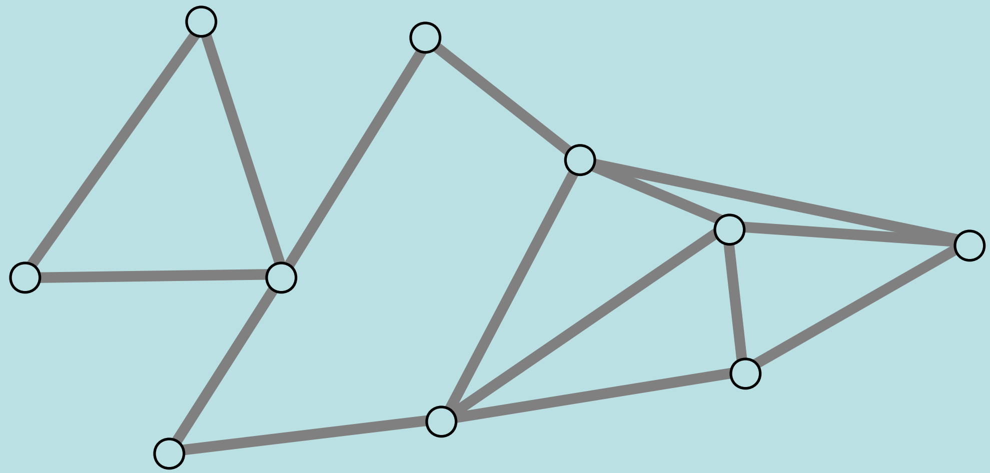
Summary

- Easy-to-evaluate piecewise affine maps are fully described by a triangulation of the domain space and an assignment of an image to each vertex in the domain triangulation
- We can build a PLH map from one simple polygon onto any other simple polygon that has the same area-scale everywhere
- We may even build continuous 1-parameter families (homotopies) of PLH same-area-scale transformations.

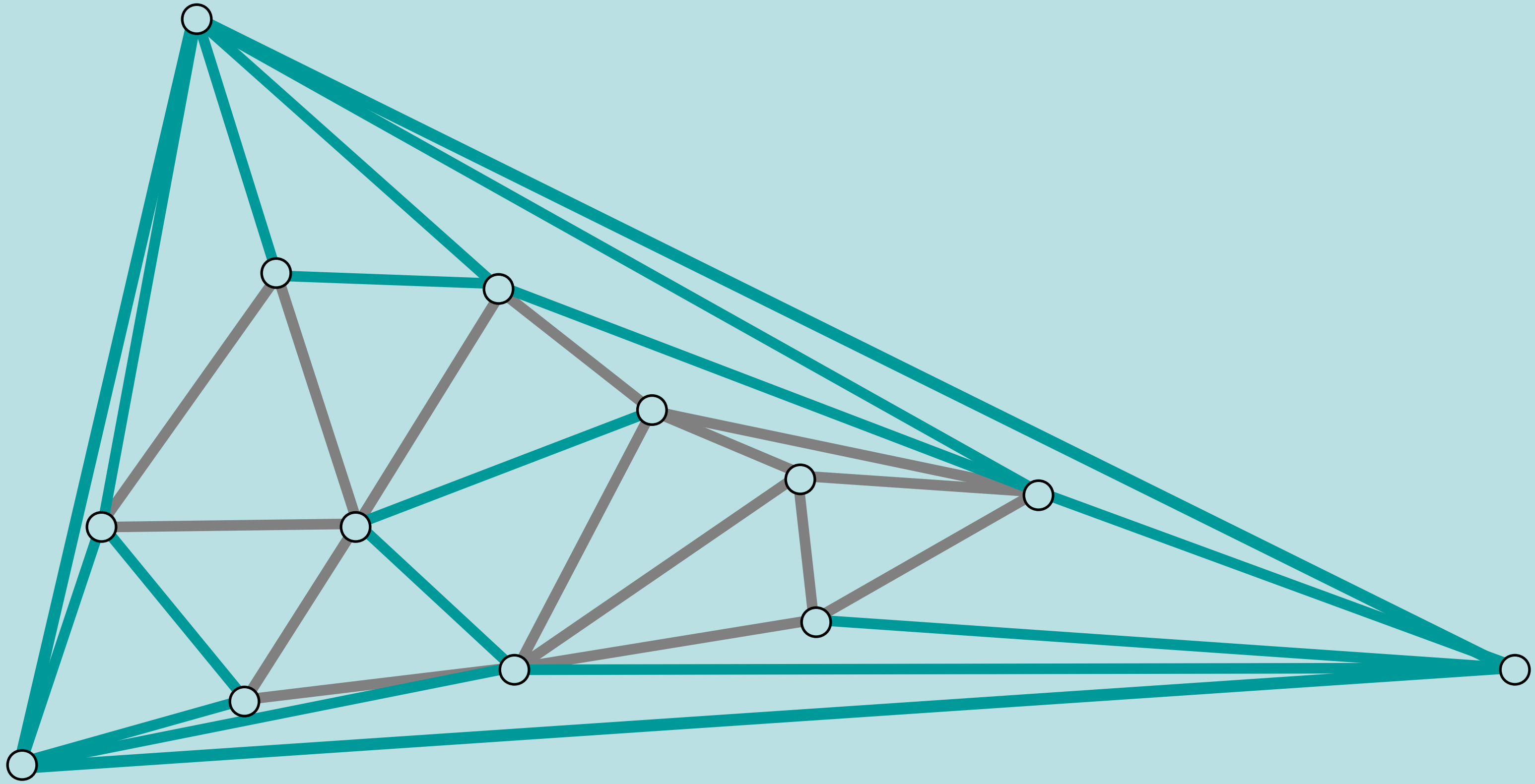
Part II: Discrete topological coordinates and discrete constructions of cartograms (based on work by Walter Schnyder)



Start with a Plane Graph



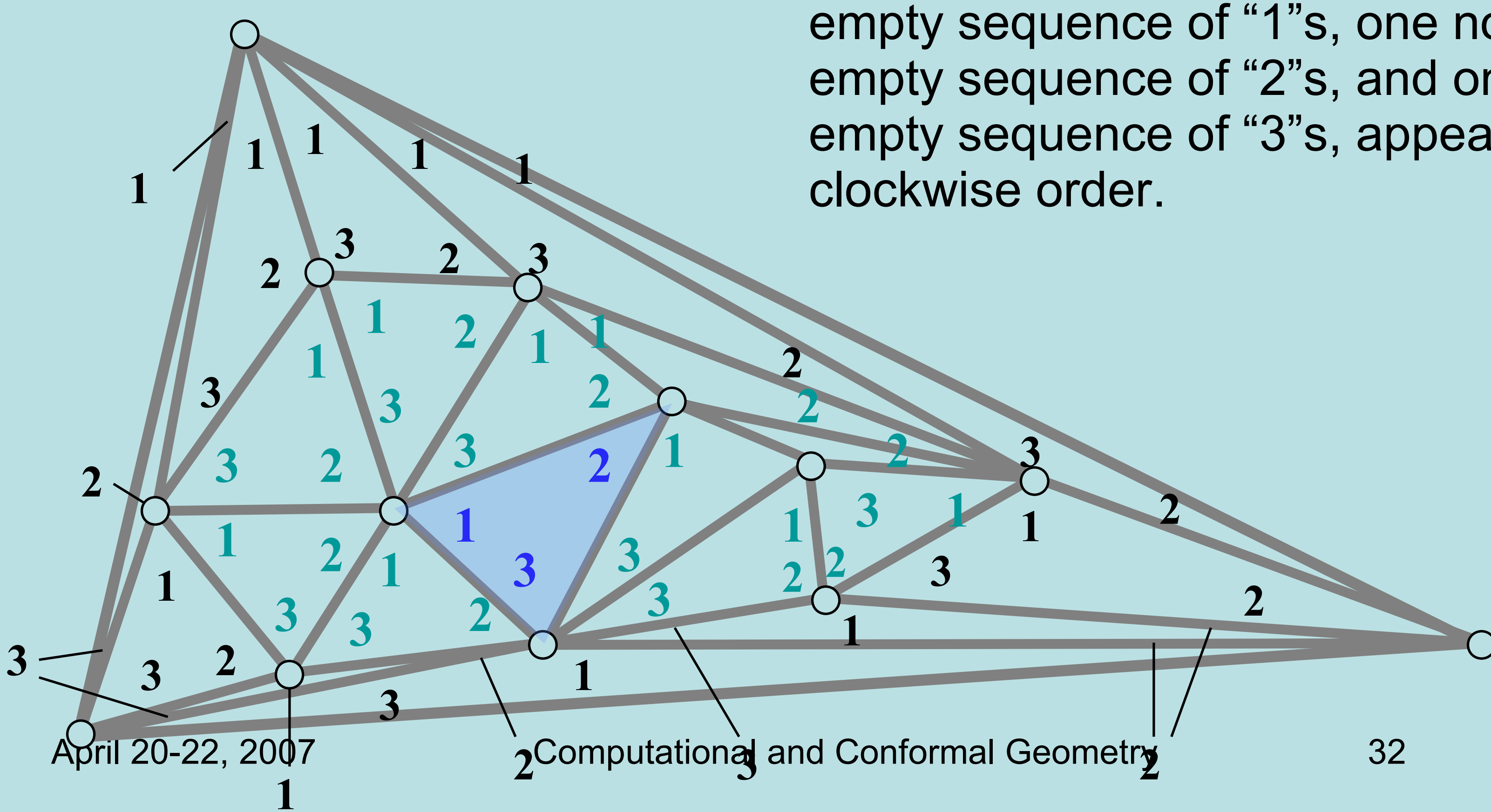
Embed the plane graph into a Triangular graph (Outer-planar triangulation)



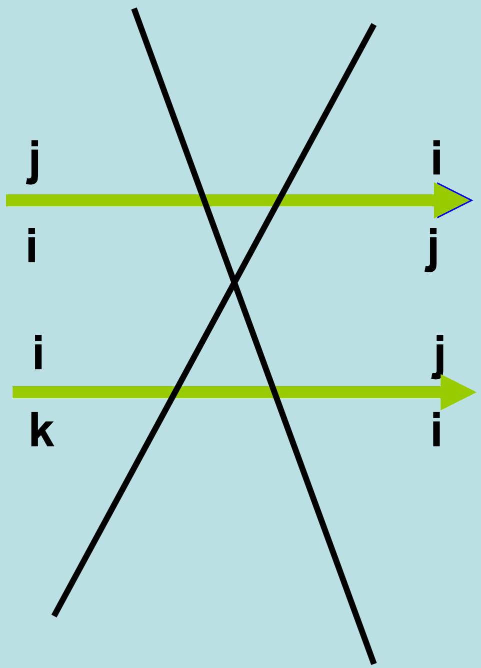
Normal Angle Labeling
(Existence proof by Schnyder)

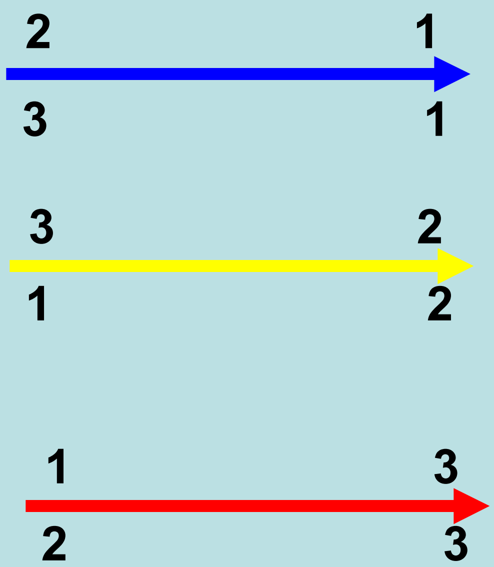
I. Every triangle has all 3 labels appearing in clockwise order.

II. Every interior vertex has one non-empty sequence of “1”s, one non-empty sequence of “2”s, and one non-empty sequence of “3”s, appearing in clockwise order.

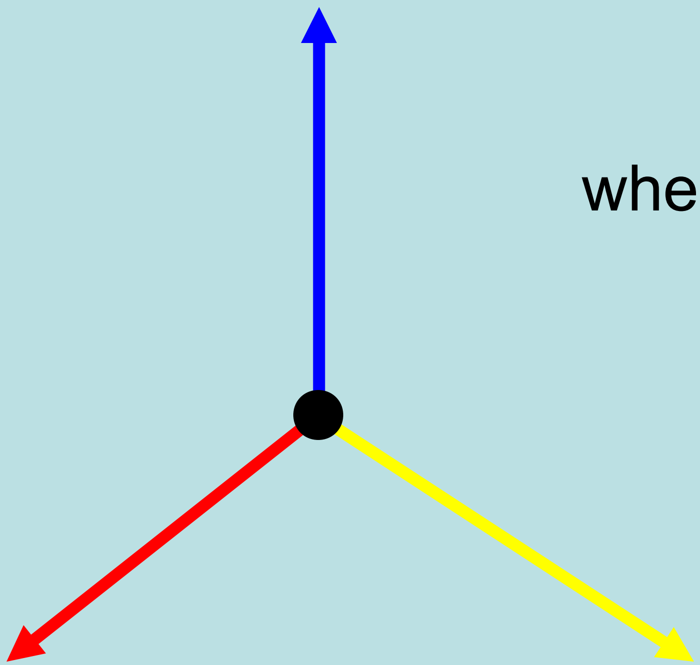


There are only 3 kinds of lines

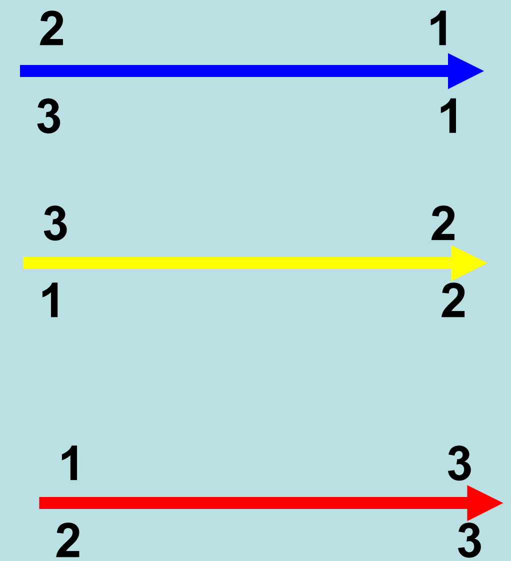


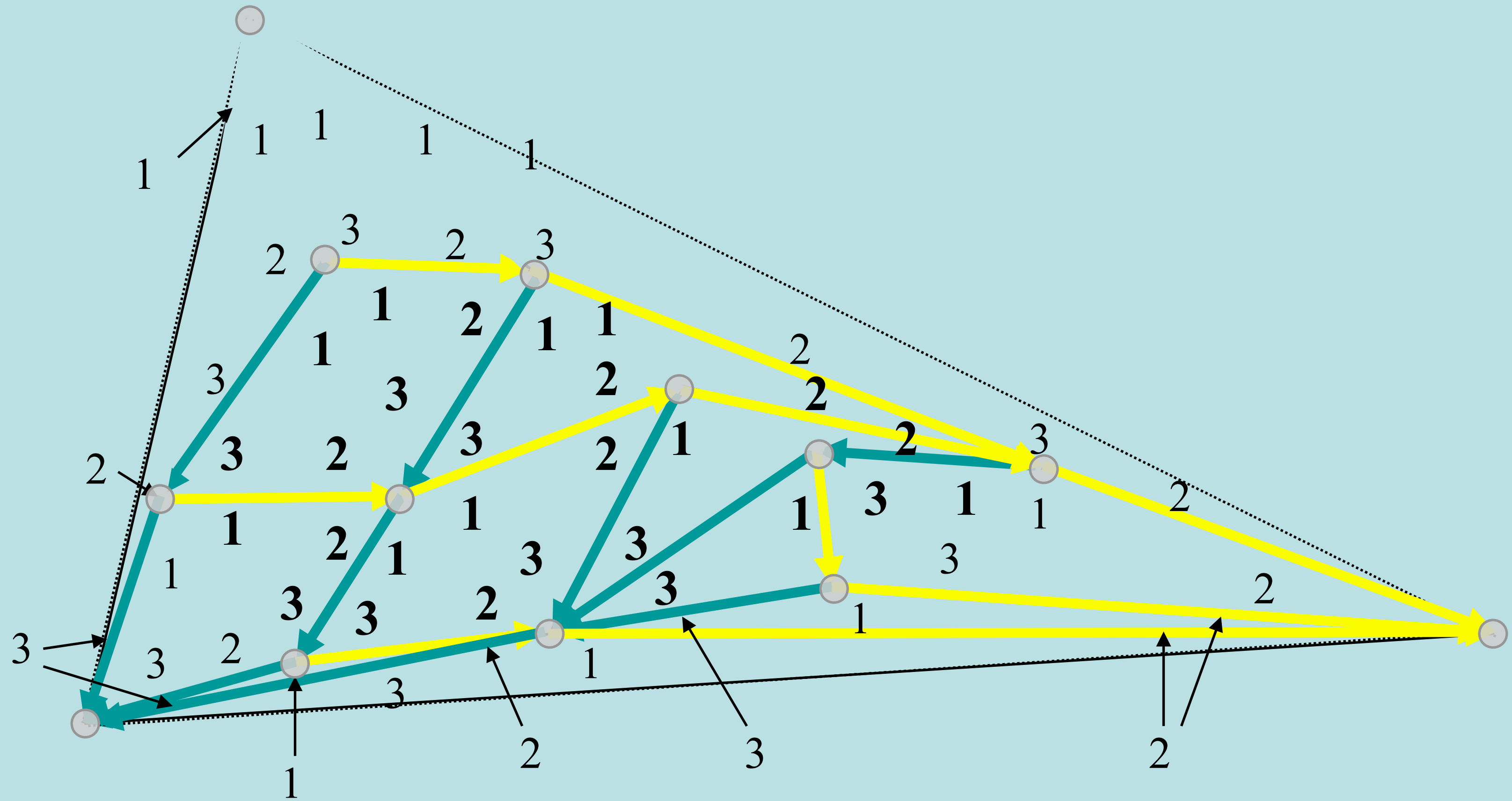


Every interior vertex has outdegree 3



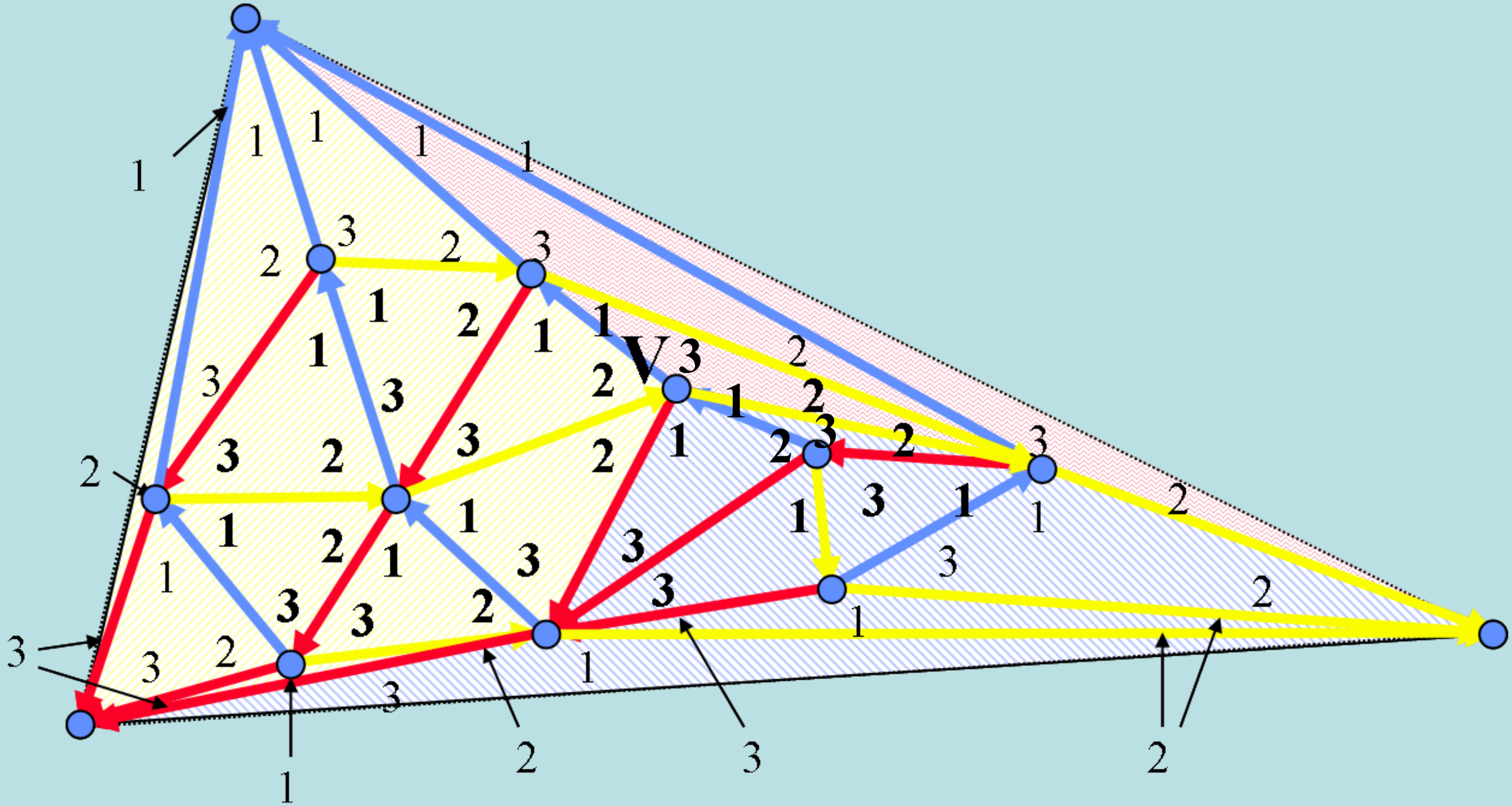
where

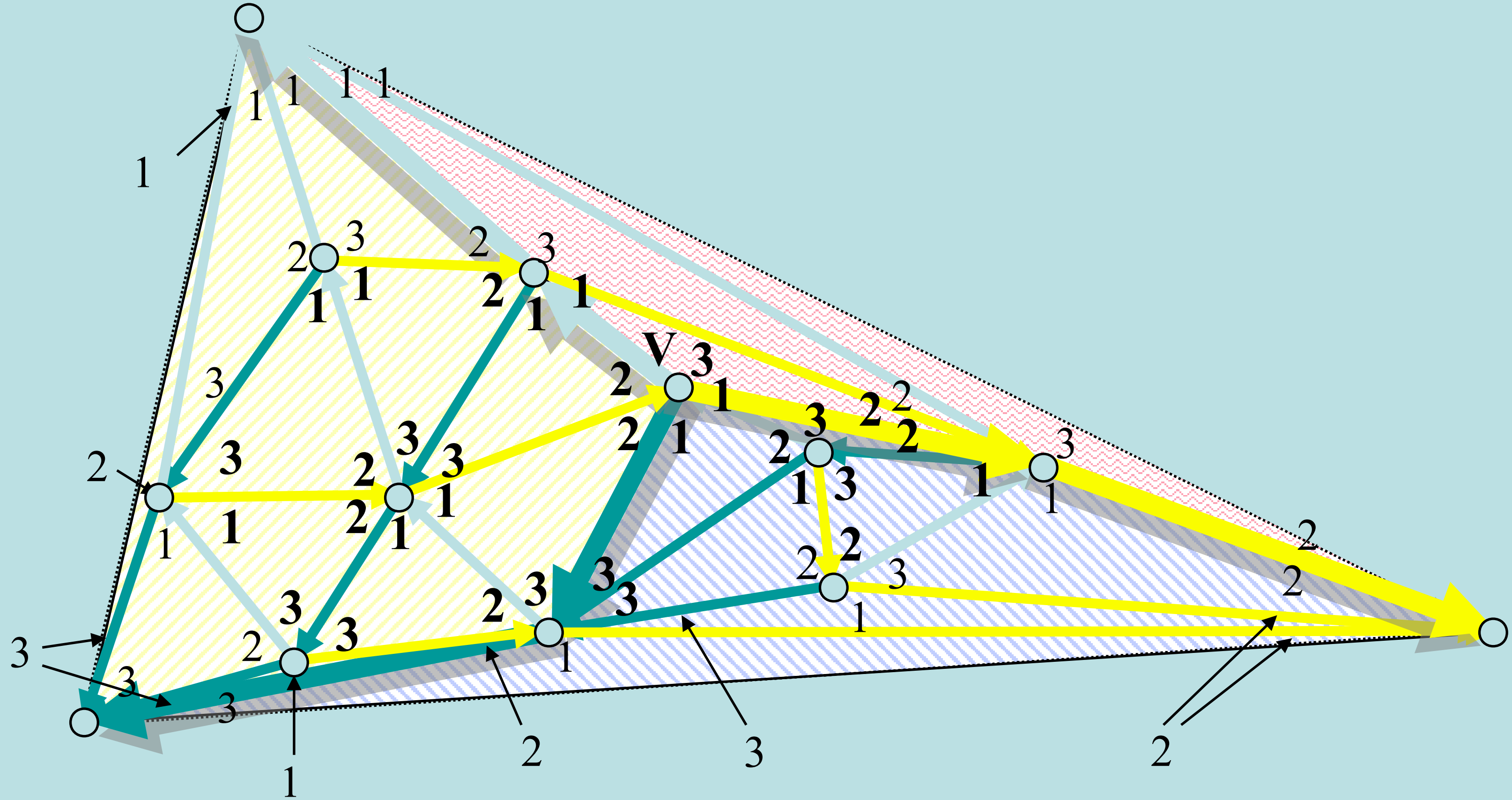




The edges of any one type (color) form a rooted tree that contains all of the interior vertices

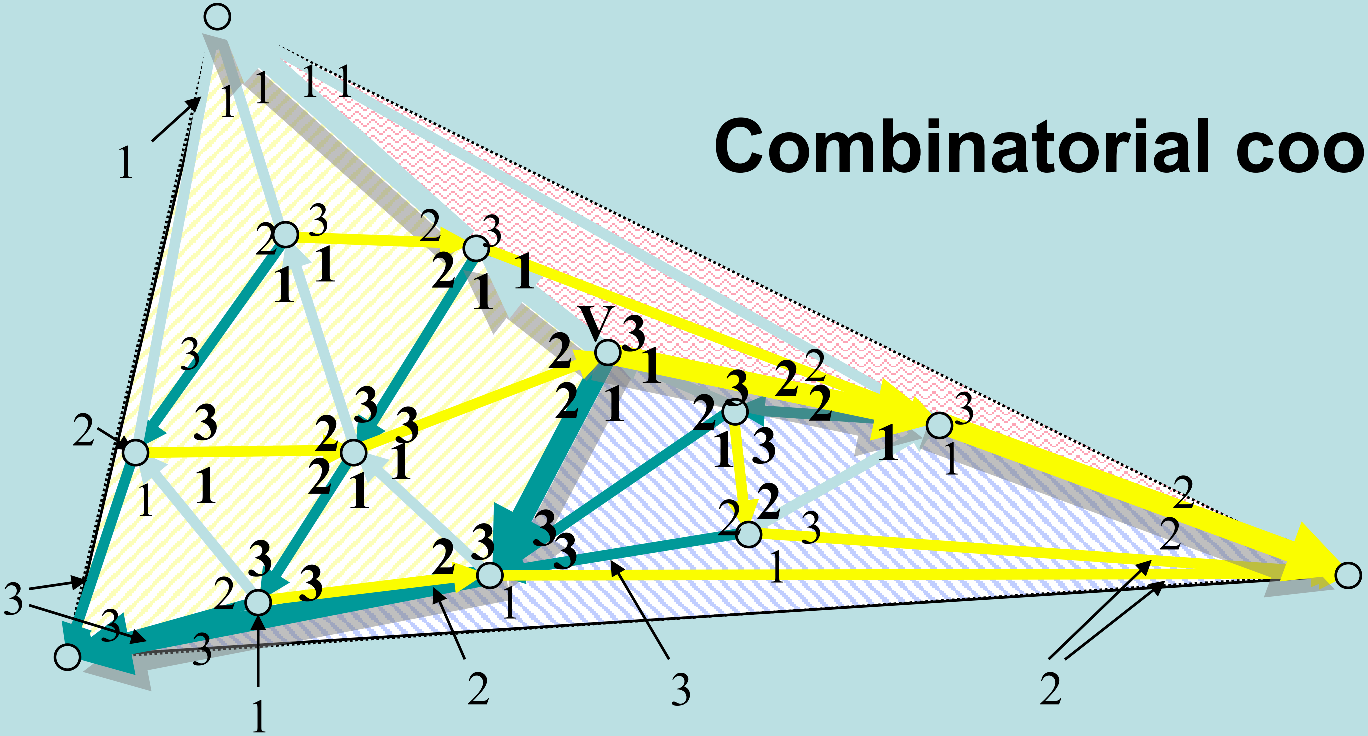
Three-tree decomposition of interior edges (Schnyder)





The three different colored paths from V to the three extreme triangle vertices partition the triangle set into three disjoint subsets of the [2n–5] interior triangles

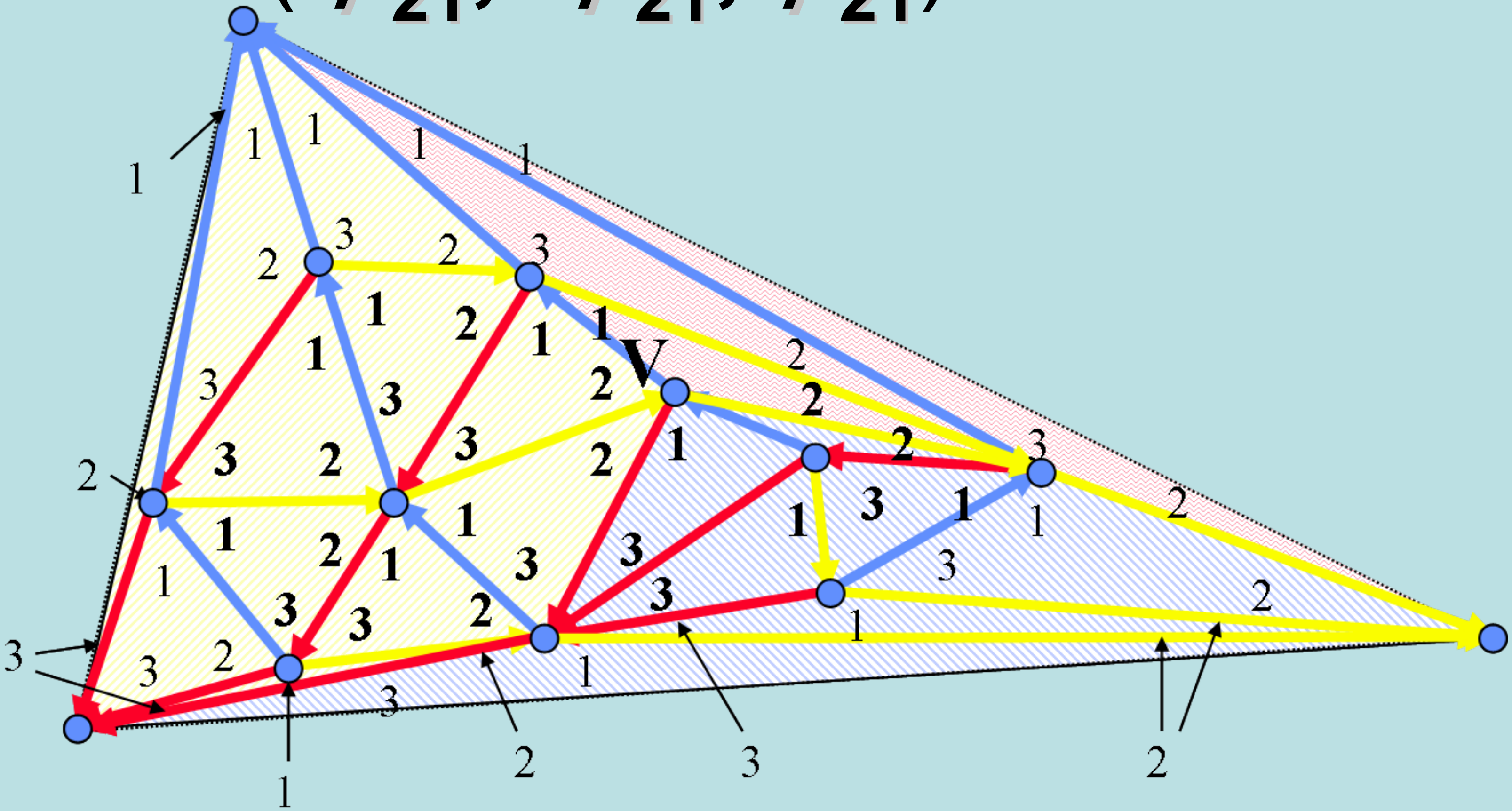
Combinatorial coordinates?



Count triangles in each partition of the 3-partition of the triangulation, then divide the count by the total number of internal triangles, and interpret the value as barycentric coordinates on three affinely independent points.

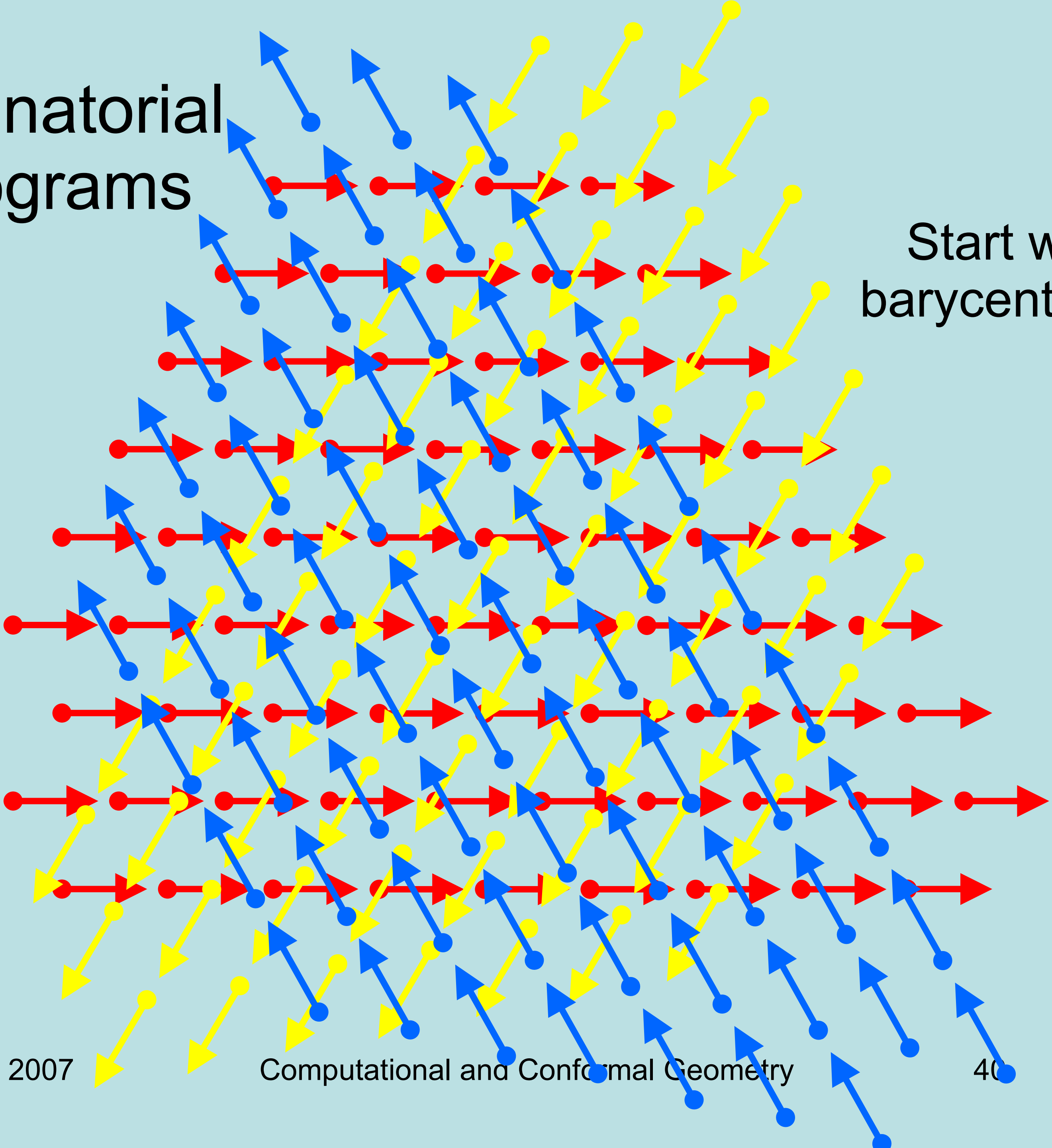
Triangle-count barycentric coordinates:

$$V = \left({}^{7}/_{21}, {}^{11}/_{21}, {}^{3}/_{21}\right)$$

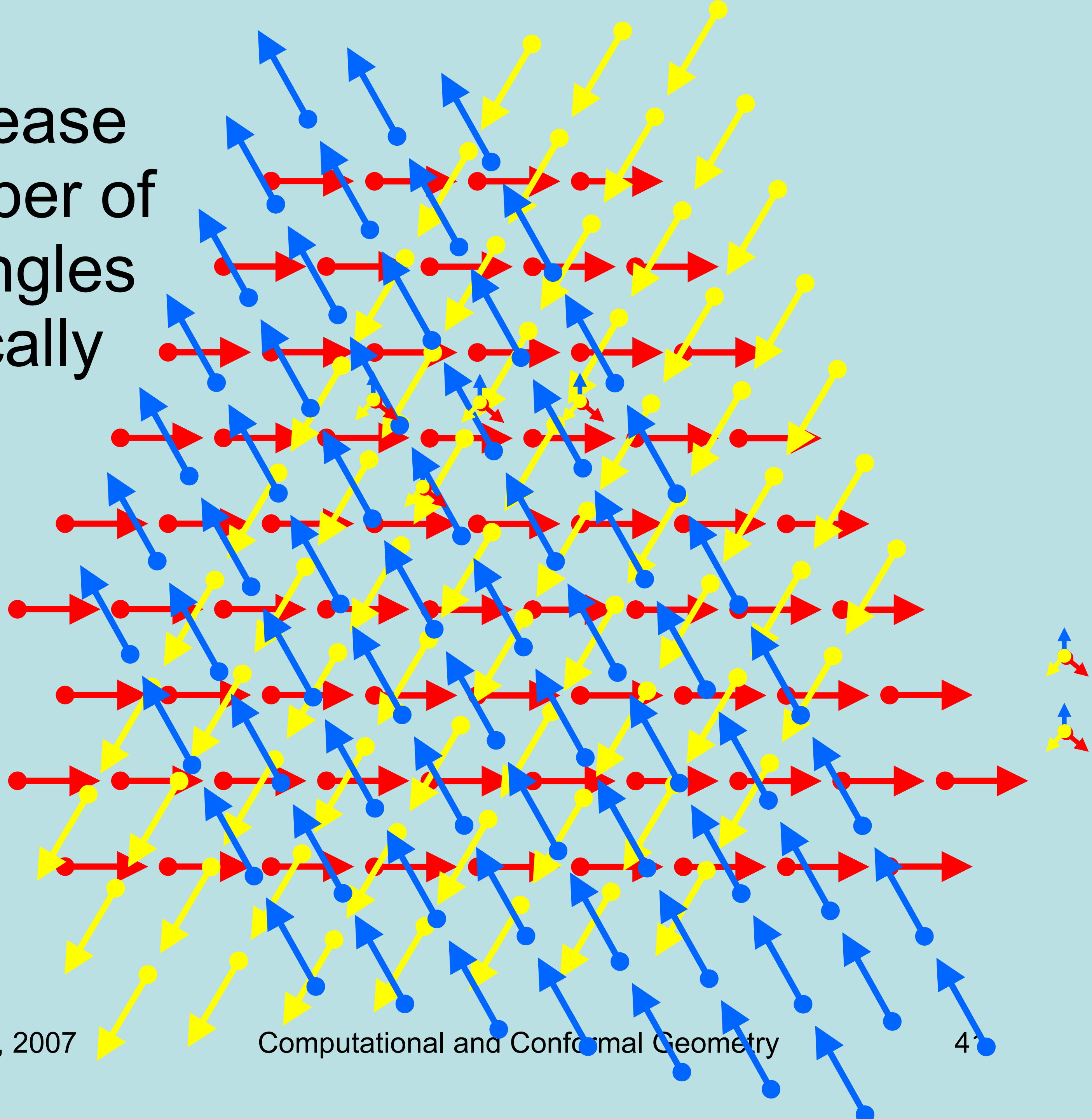


Combinatorial Cartograms

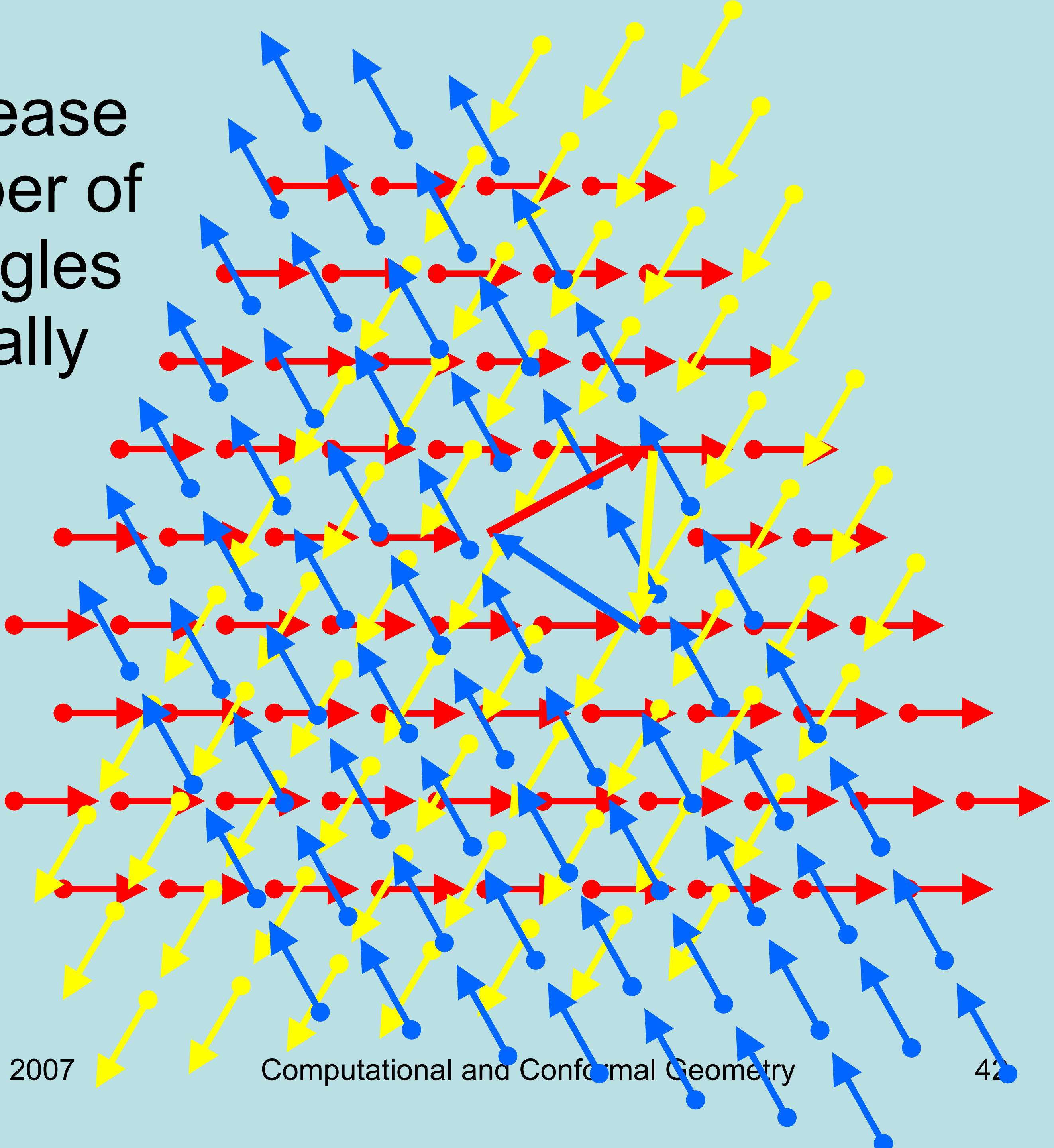
Start with a barycentric grid



Increase Number of Triangles Locally



Decrease Number of Triangles Locally



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