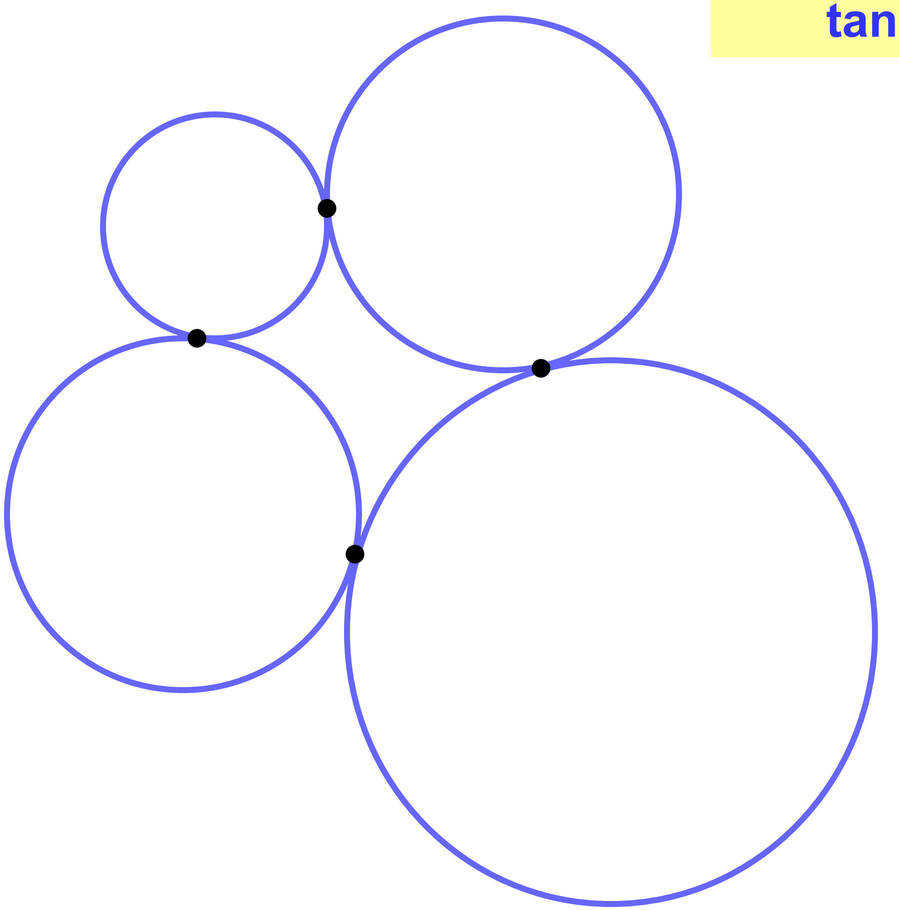
Three Applications of Disk Packing with Four-Sided Gaps

Marshall Bern

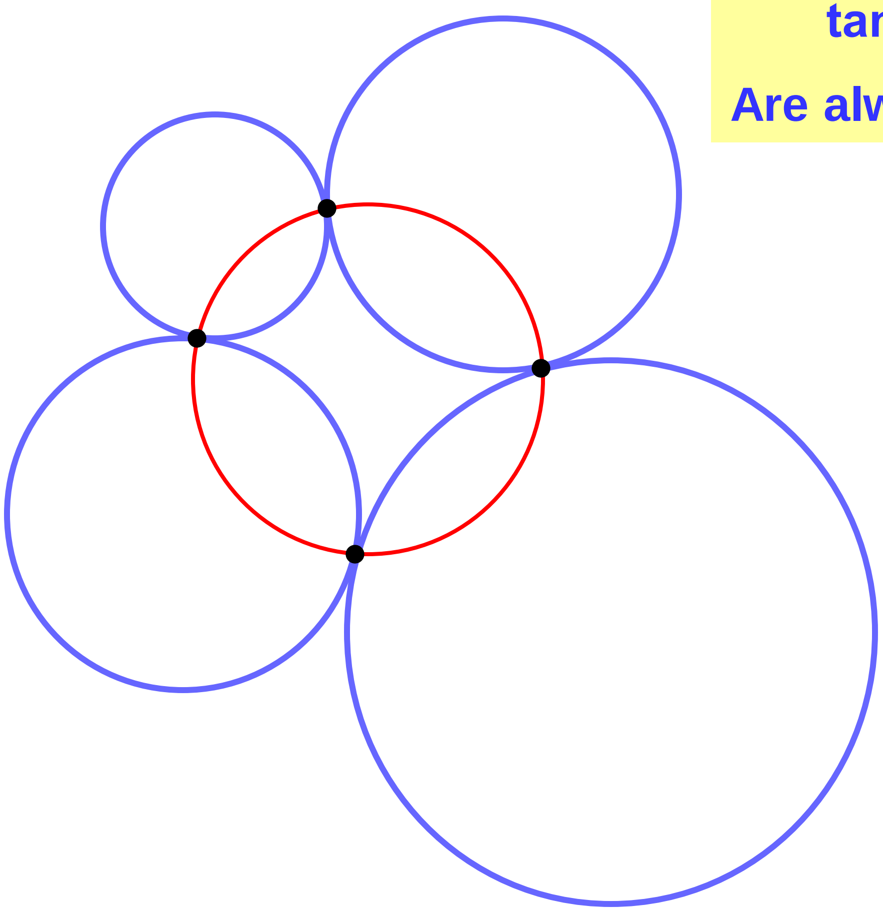
Palo Alto Research Center

Circle Magic

The points of tangency of four disks,
tangent in a cycle, …



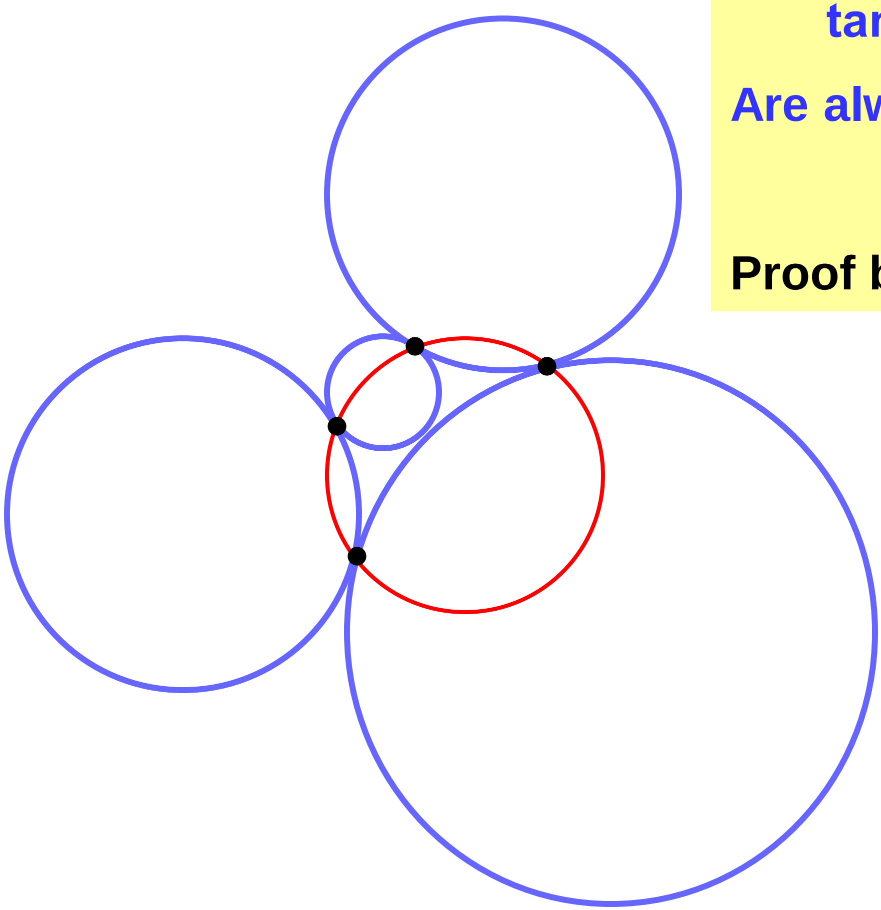
Circle Magic



The points of tangency of four disks, tangent in a cycle, …

Are always cocircular!

Circle Magic



The points of tangency of four disks, tangent in a cycle, …

Are always cocircular!

Proof by PowerPoint ☺

Outline

1) **Disk packing of a polygon**

Basic Technique

1) **Nonobtuse triangulation of a polygon**

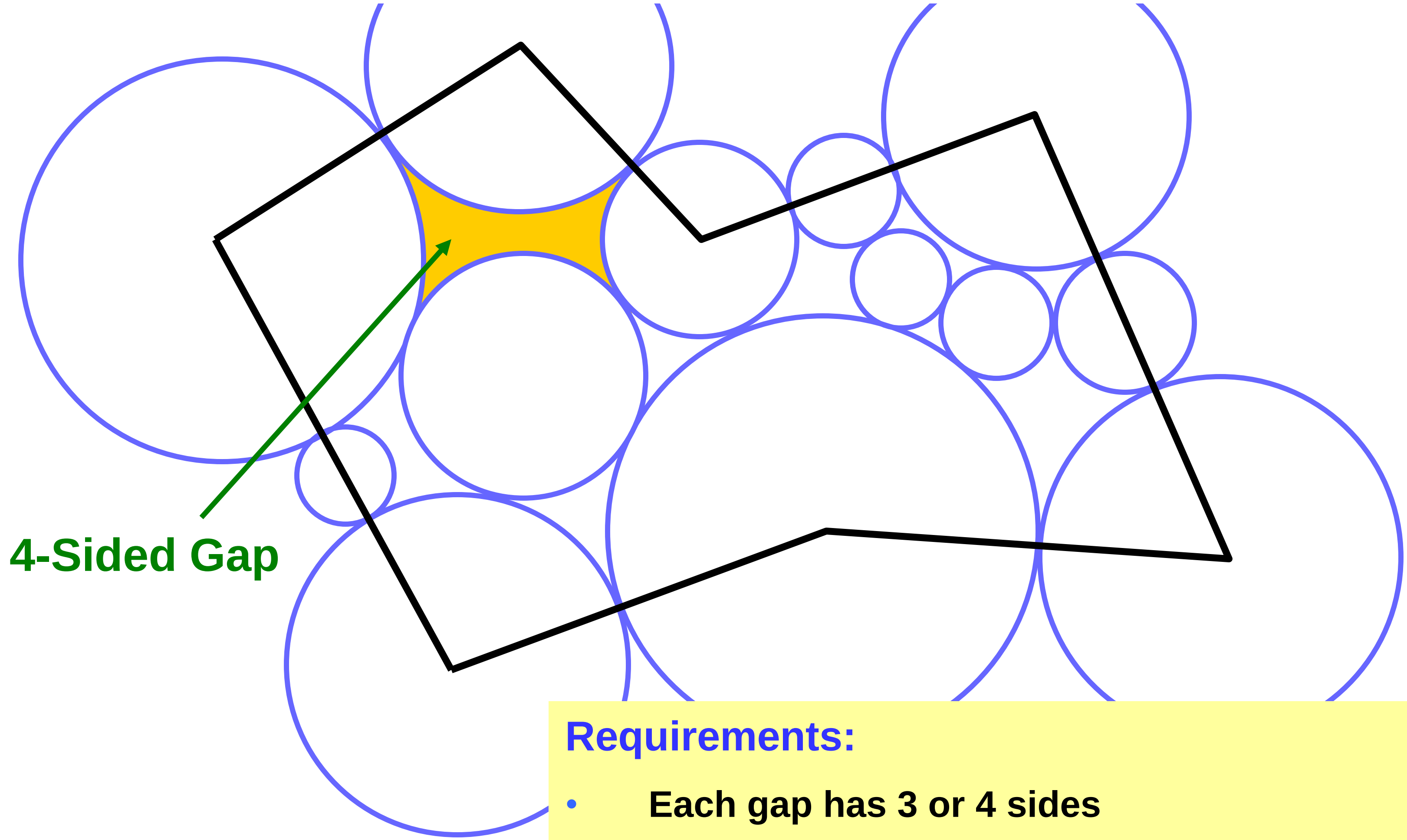
1) **Origami magic trick**

Applications

1) **Origami embedding of Euclidean Piecewise-Linear 2-manifolds**

Disk Packing of a Polygon

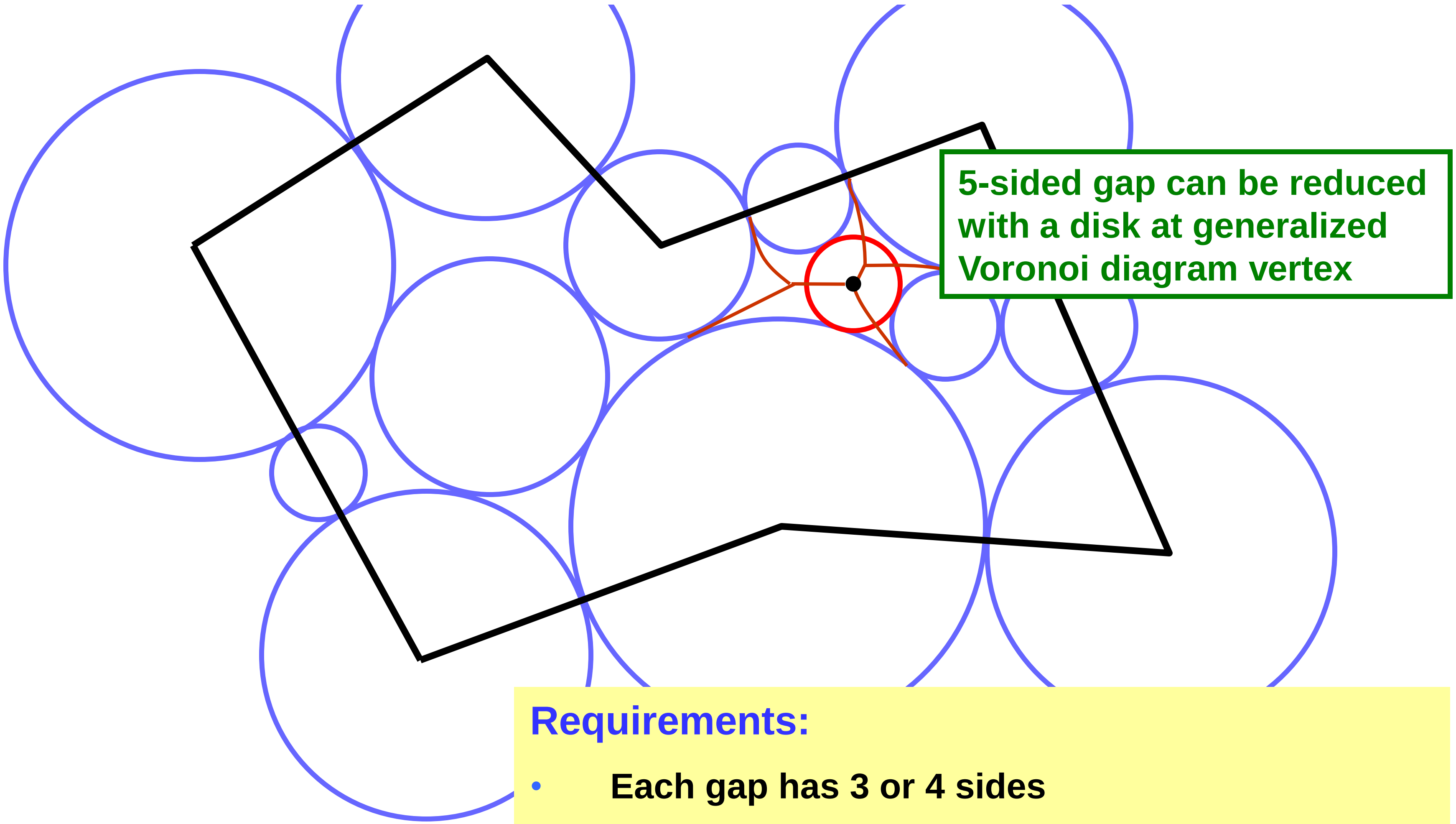
[Bern – Scott Mitchell – Ruppert, 1994]



Requirements:

- **Each gap has 3 or 4 sides**
- **A disk is centered on each vertex**
- **Each side of the polygon is a union of radii**

Does such a packing always exist?



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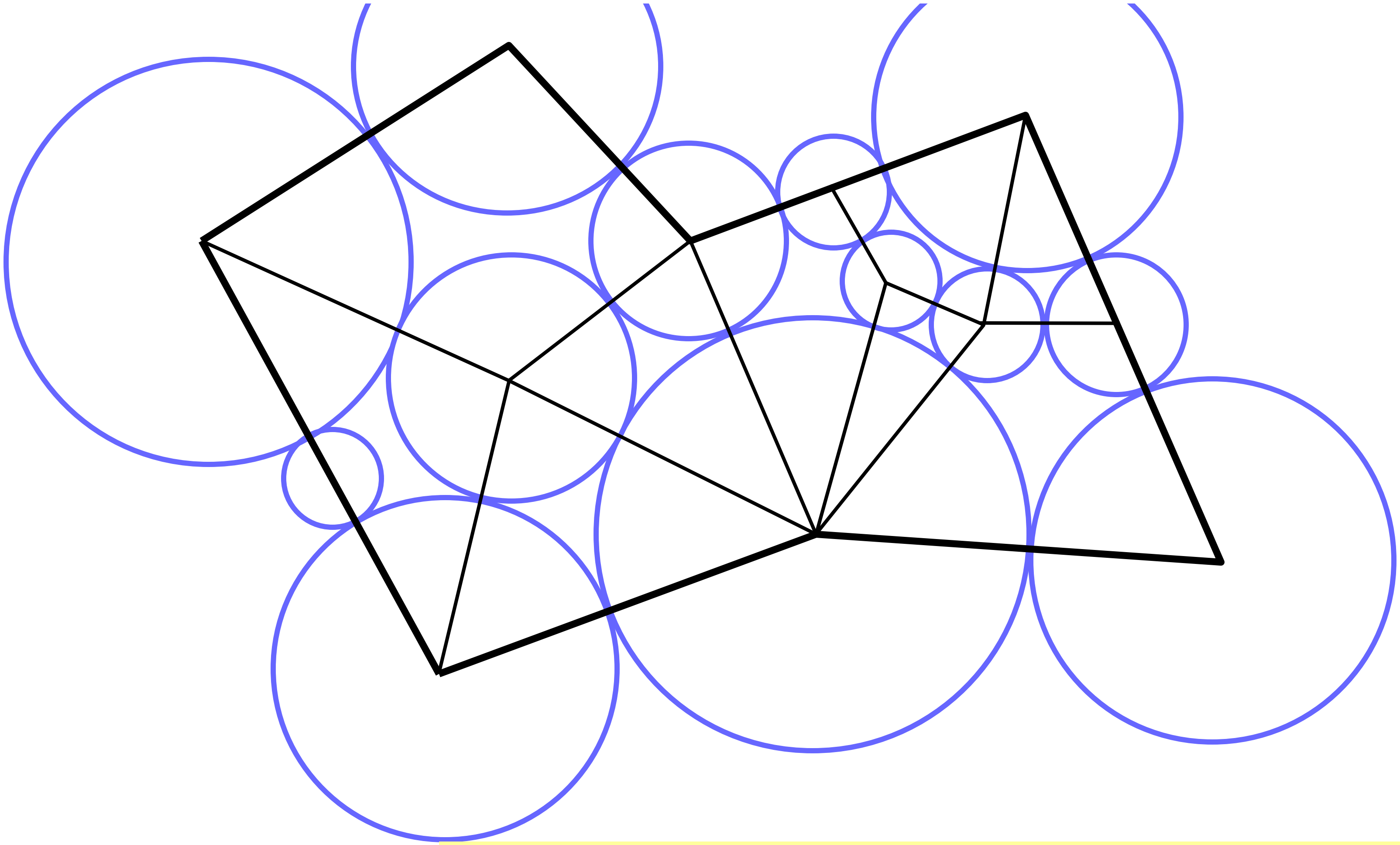
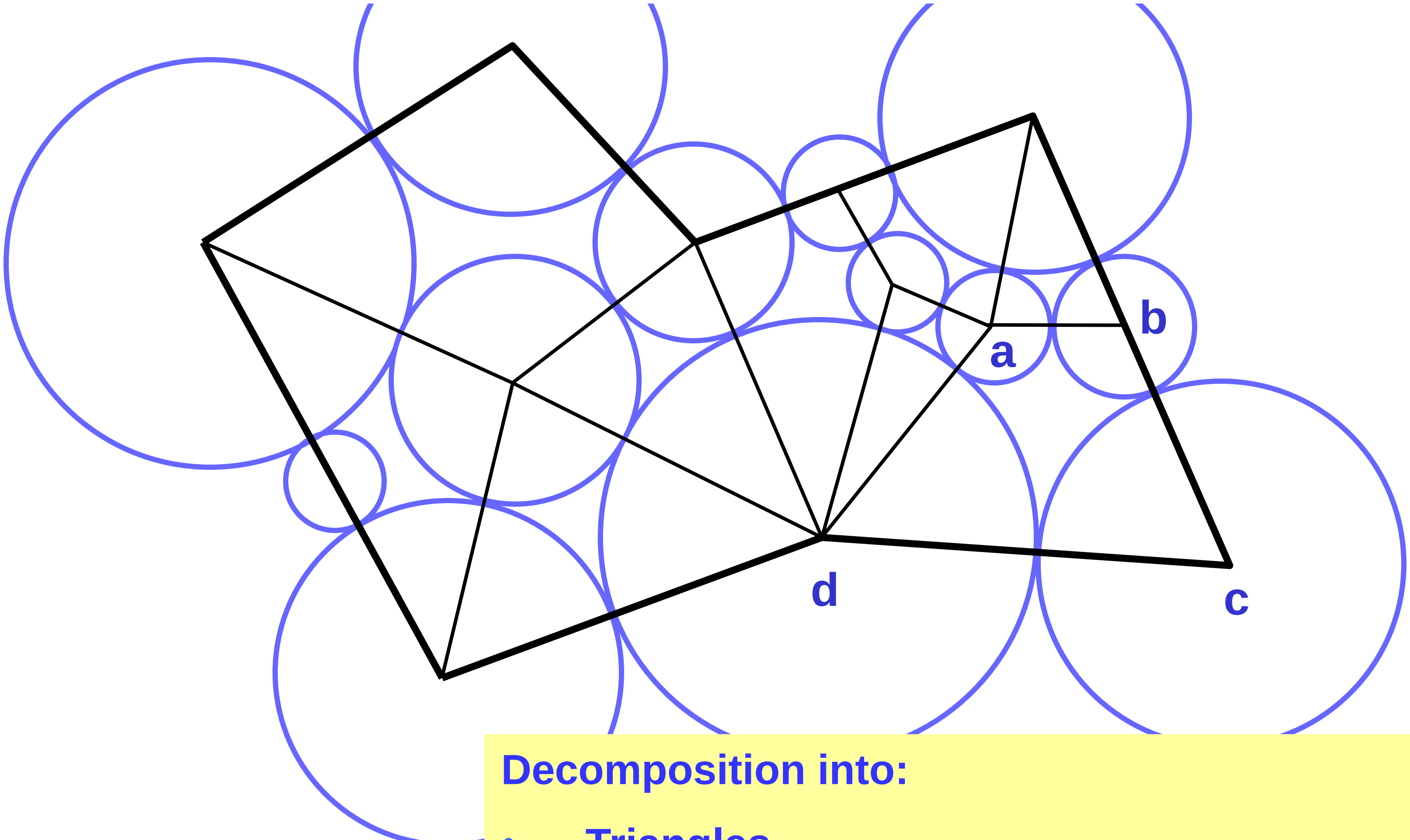
Disk Packing Induces Decomposition



Connect the centers of each pair of tangent disks

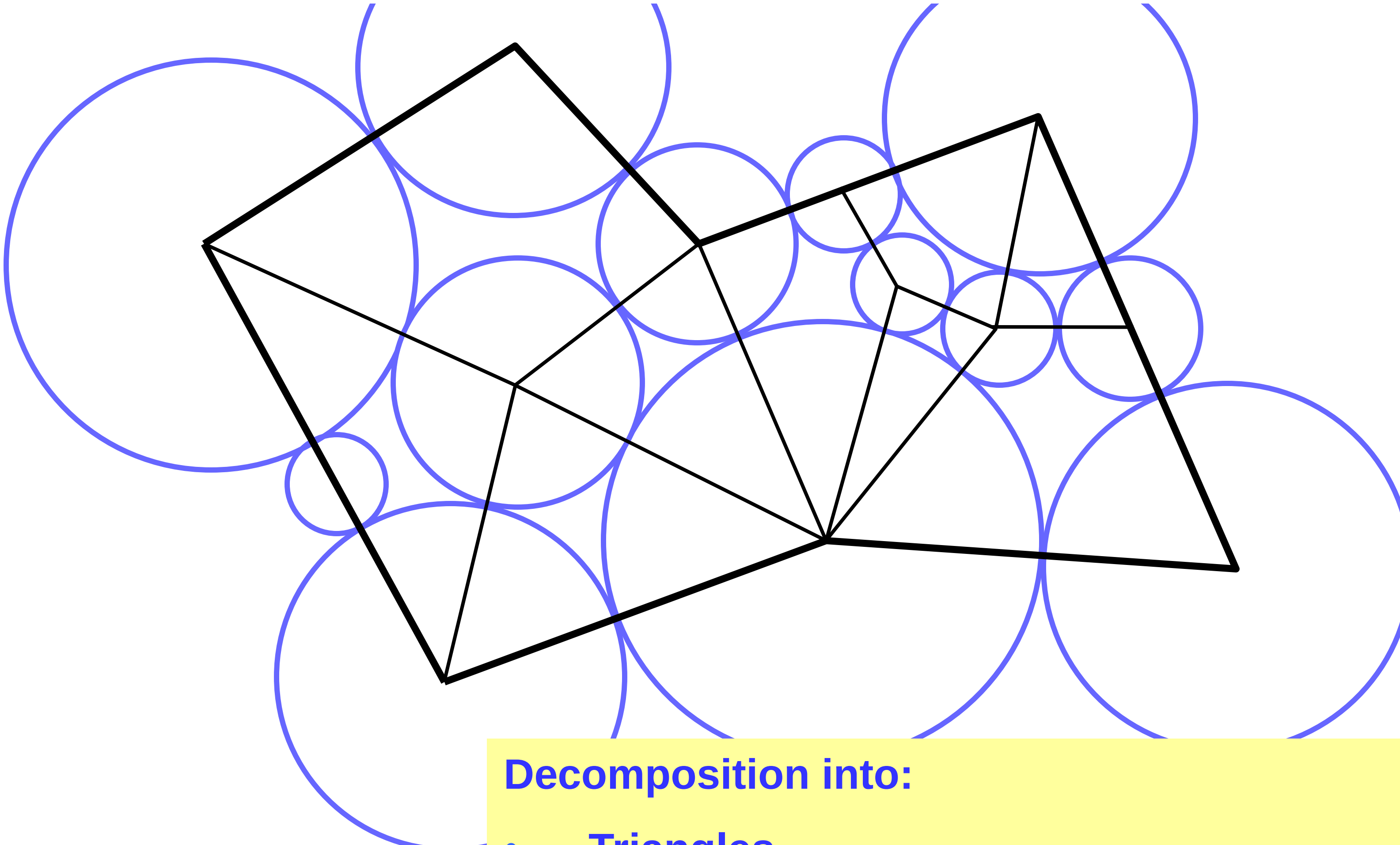
Disk Packing Induces Decomposition



Decomposition into:

- **Triangles**
- **Quadrangles of cross-ratio one**
  |ab||cd| = |bc||da|

Disk Packing Induces Decomposition



Decomposition into:

- **Triangles**
- **Quadrangles that act like triangles!**

Outline

1) Disk packing of a polygon

1) Nonobtuse triangulation of a polygon

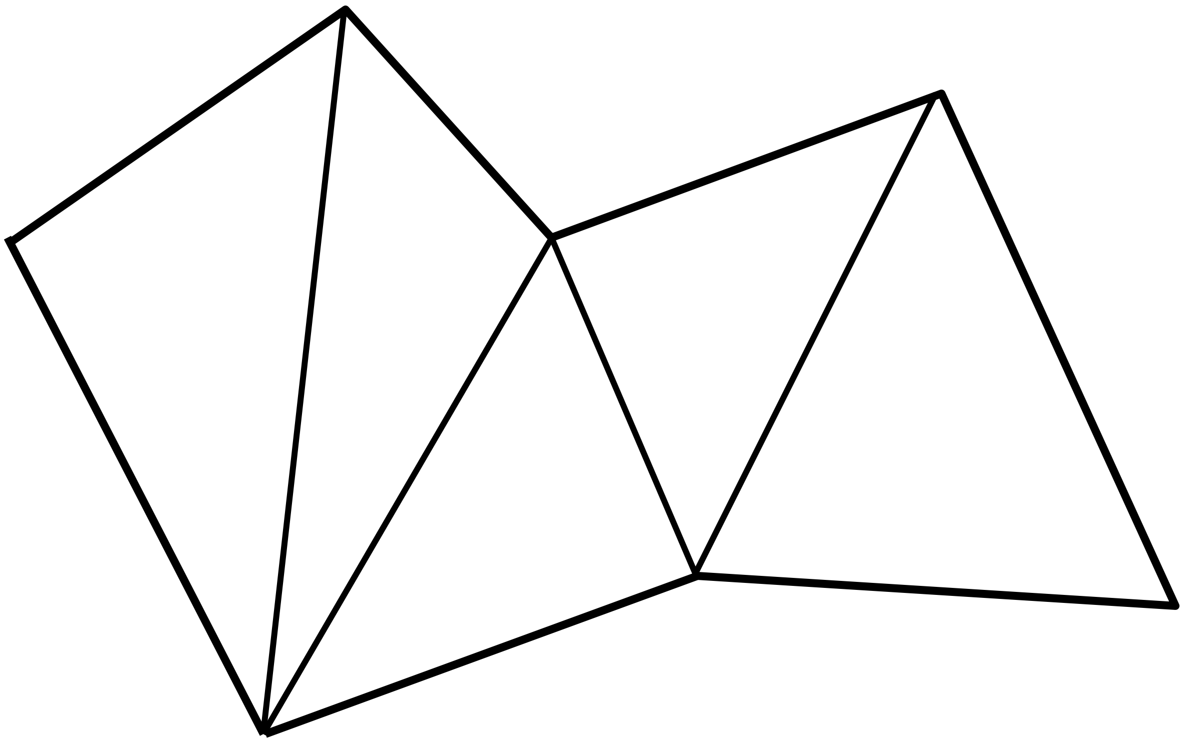
1) Origami magic trick

1) Origami embedding of Euclidean Piecewise-Linear 2-manifolds

Nonobtuse Triangulation

Question: **Can any n-sided polygon be triangulated with triangles with maximum angle 90^{o} ?**

What makes the problem hard?



Naïve Algorithm: **Start from any triangulation,**

Cut obtuse angle with perpendicular to opposite edge

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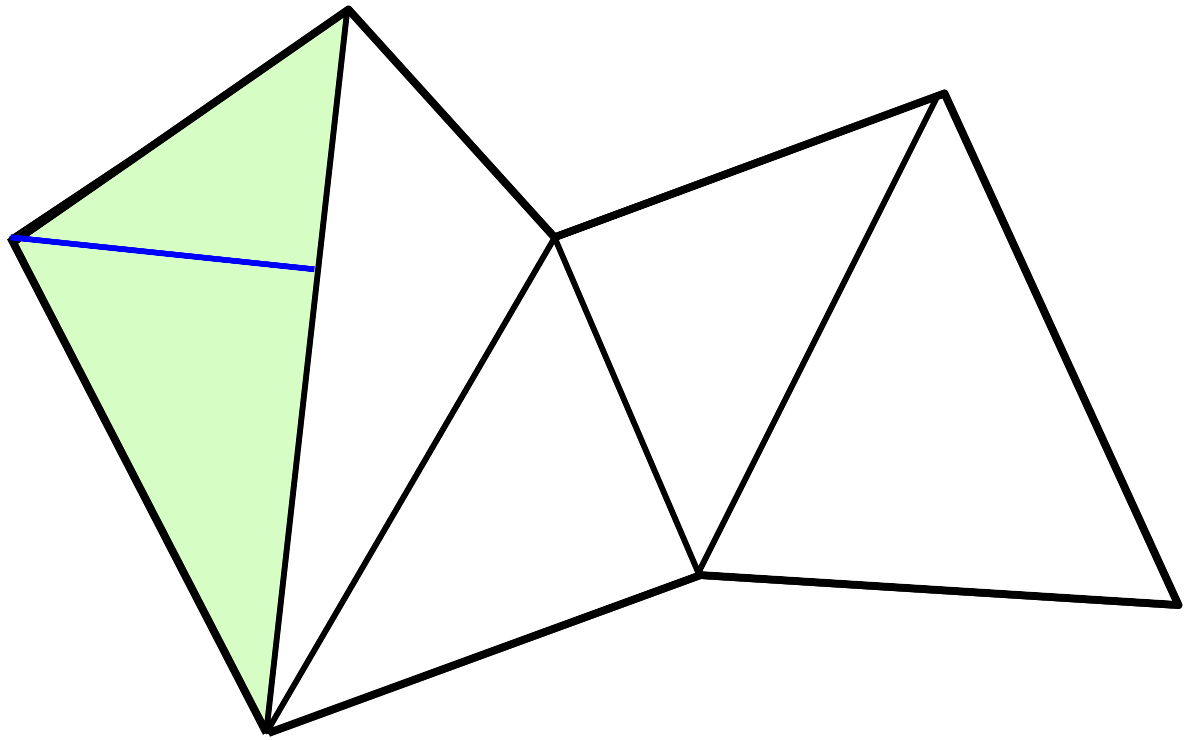

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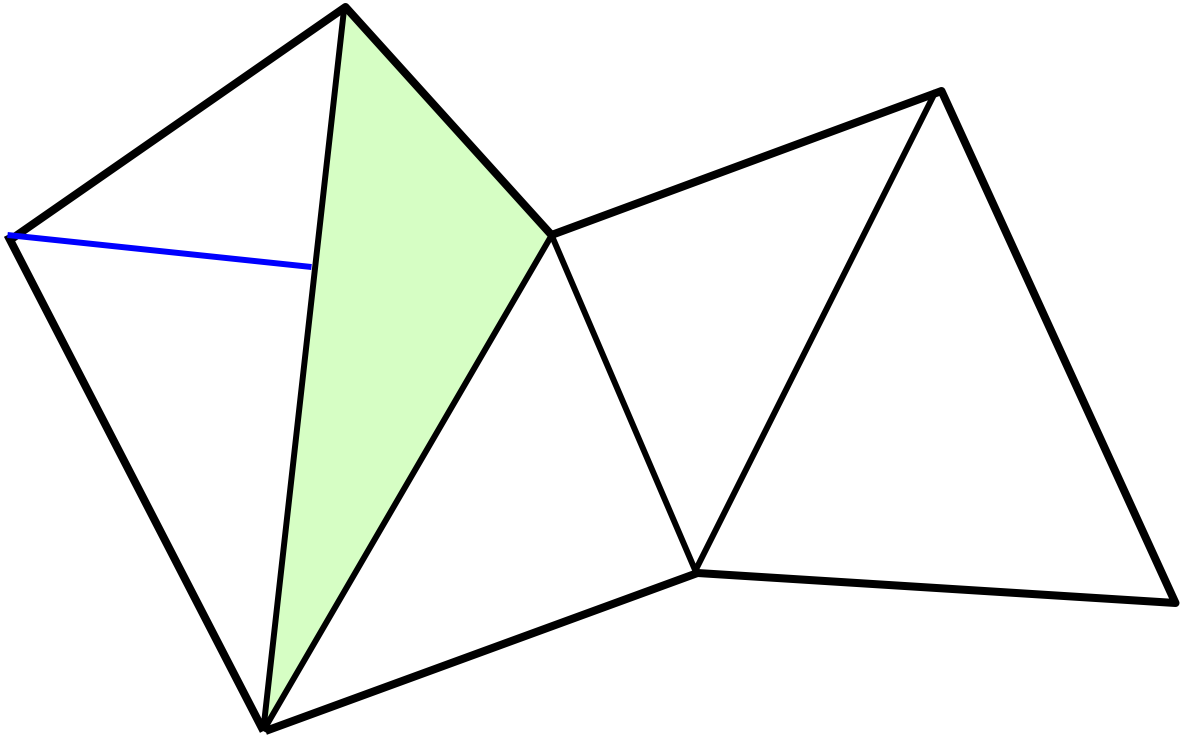

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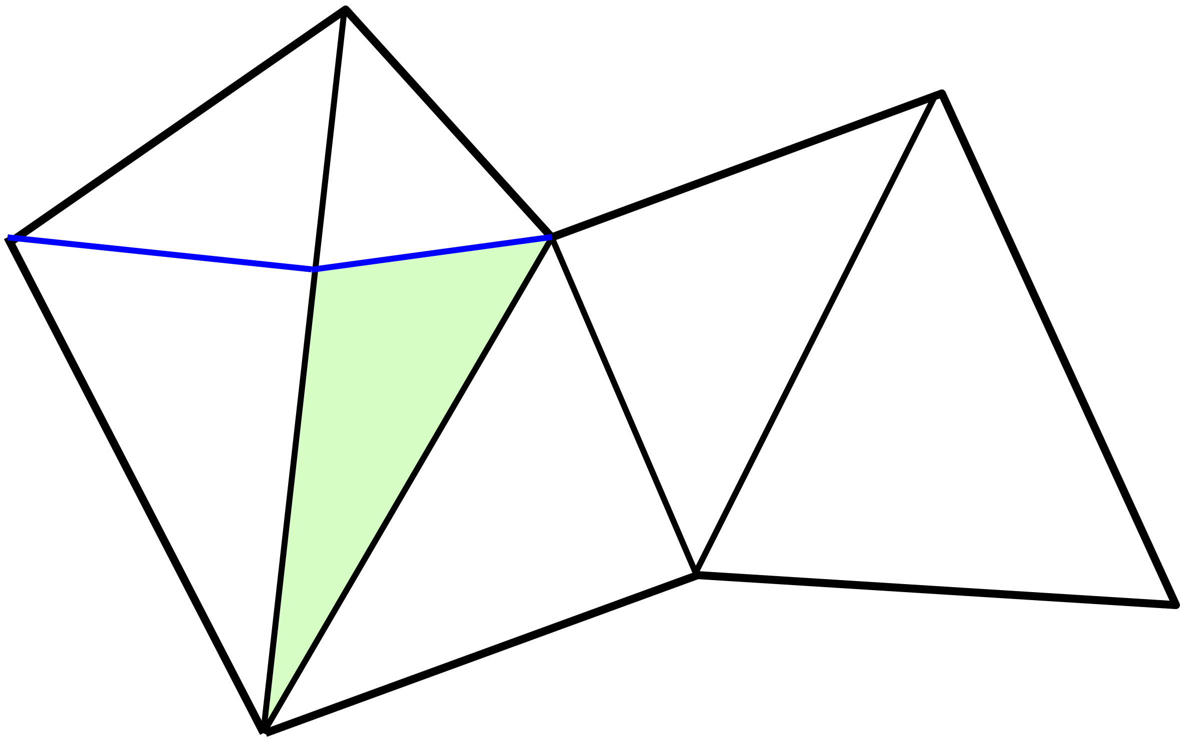

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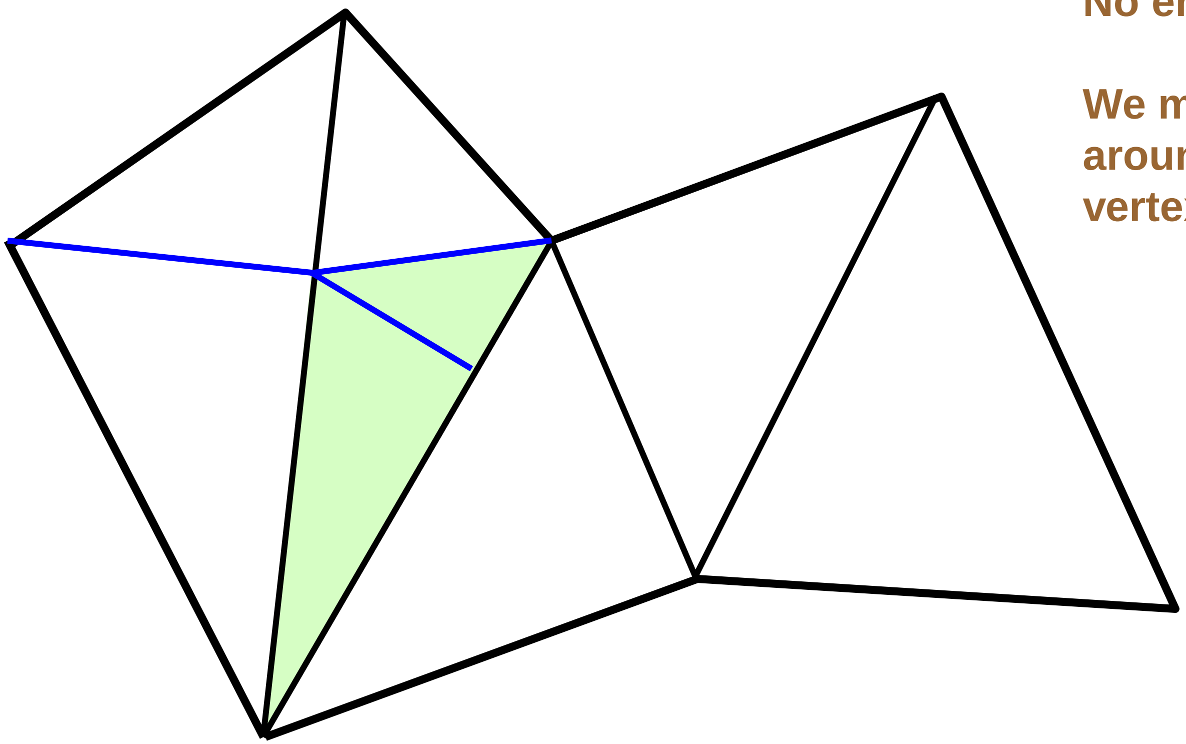


No end in sight!

We might spiral around an interior vertex forever!

Naïve Algorithm: **Start from any triangulation,**

Cut obtuse angle with perpendicular to opposite edge

Nonobtuse Triangulation

Question: Can any n-sided polygon be triangulated with triangles with maximum angle 90^{o} ?

[Gerver, 1984] used the Riemann mapping theorem to show that if all polygon angles exceed 36^{o}, then there always exists a triangulation with maximum angle 72^{o}.

[Baker – Grosse – Rafferty, 1988] showed there always exists a nonobtuse triangulation (no bound on the number of triangles).

[Bern – Eppstein, 1991] showed O(n^2) triangles for simple polygons

[Bern – Scott Mitchell - Ruppert, 1994] showed O(n) for polygons with holes

Nonobtuse Triangulation

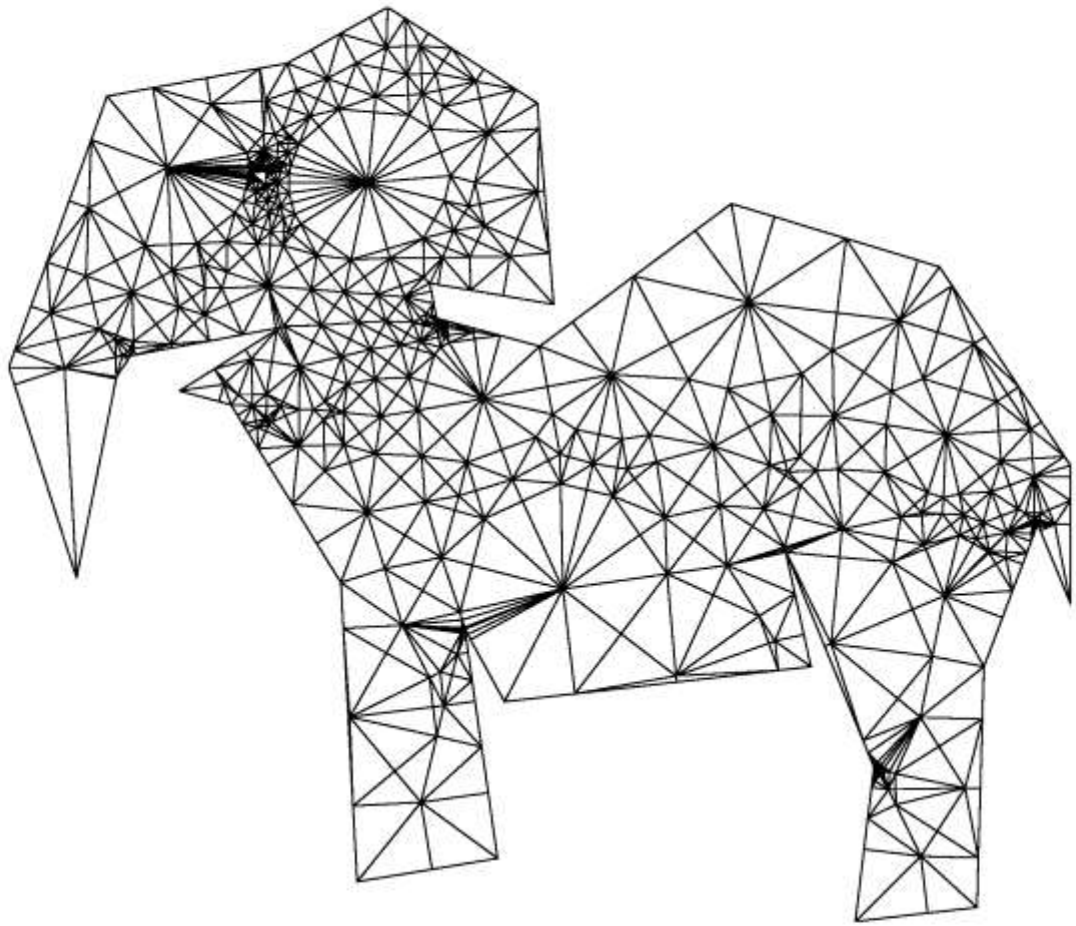
Question: **Can any n-sided polygon be triangulated with triangles with maximum angle 90^o ?**

Rumored Application: **Such a triangular mesh gives an M-matrix for the Finite Element Method for solving elliptic PDEs.**

Milder condition is actually sufficient

Nonobtuse Triangulation

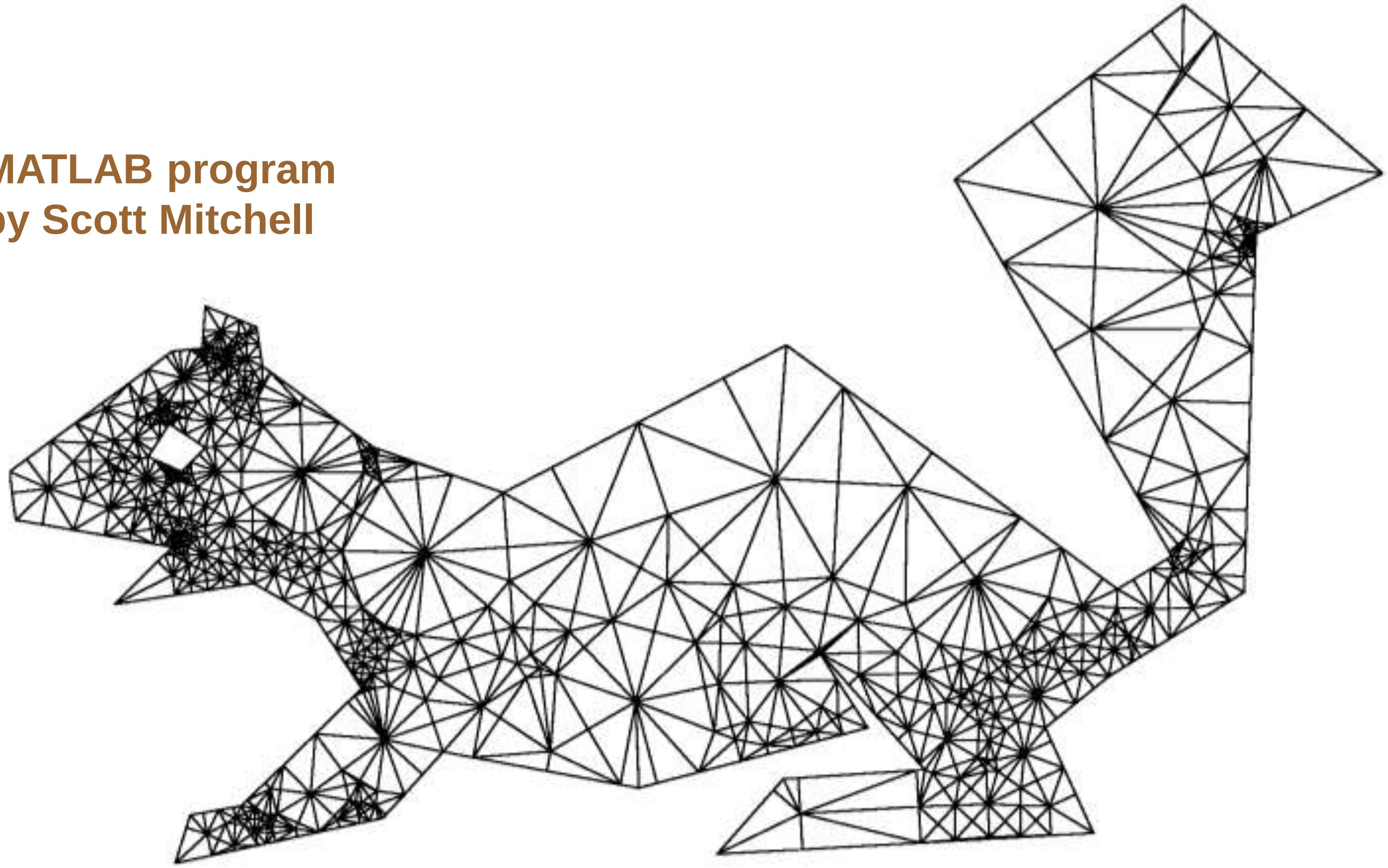
Question: **Can any n-sided polygon be triangulated with triangles with maximum angle 90^o ?**



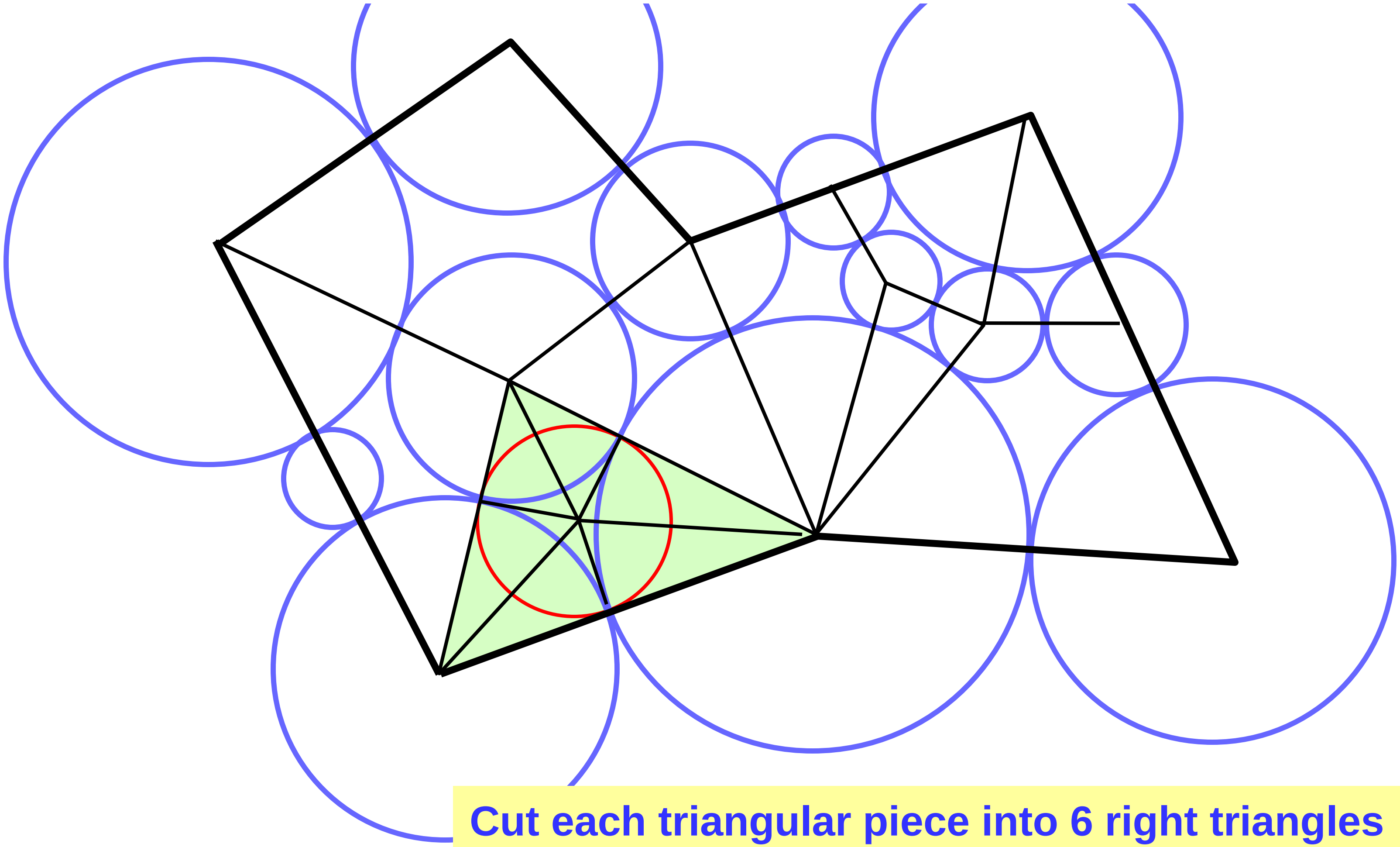
Nonobtuse Triangulation

Question: **Can any n-sided polygon be triangulated with triangles with maximum angle 90^o ?**

MATLAB program by Scott Mitchell

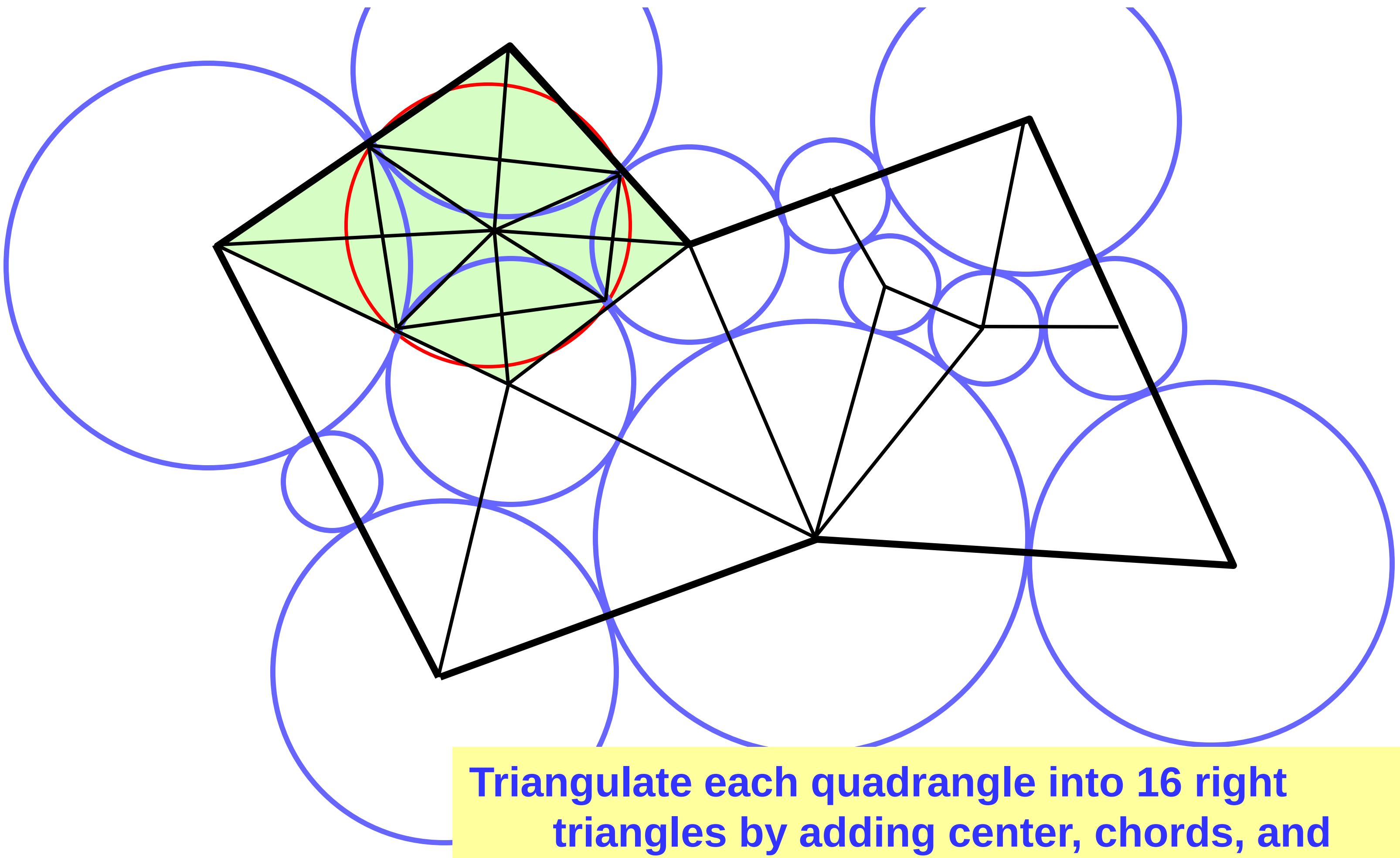


Nonobtuse Triangulation Algorithm



Cut each triangular piece into 6 right triangles by adding in-center and spokes

Nonobtuse Triangulation Algorithm

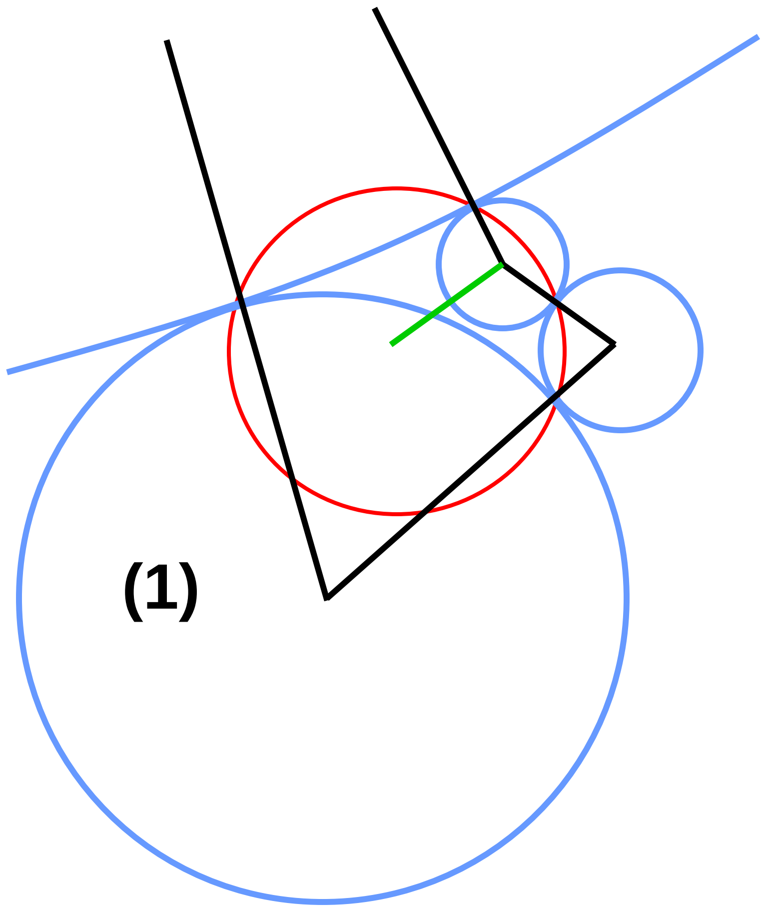


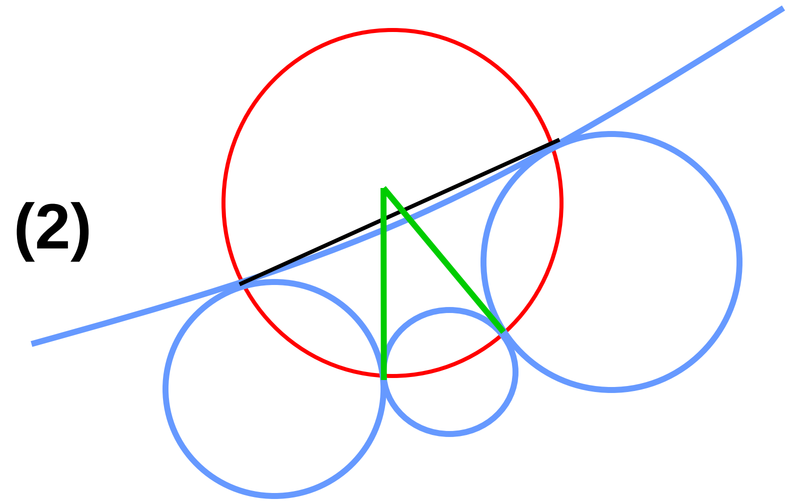
Triangulate each quadrangle into 16 right triangles by adding center, chords, and spokes of tangency circle

Complication – Badly shaped Quads

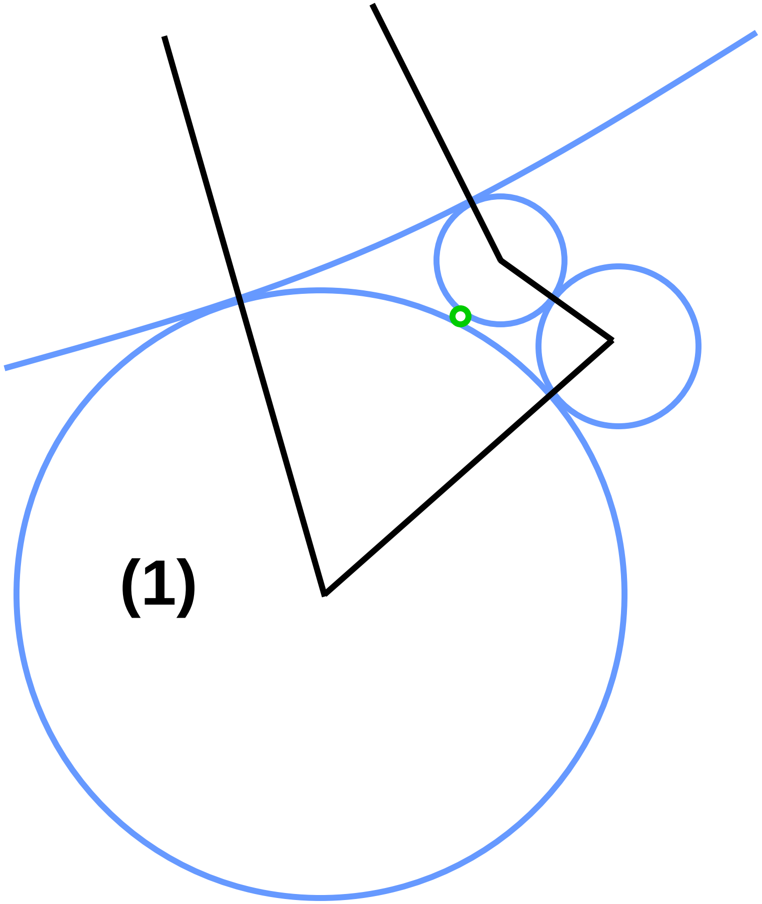
Problems:

- **(1) Reflex Quadrangle**
- **(2) Circle center on wrong side of chord**





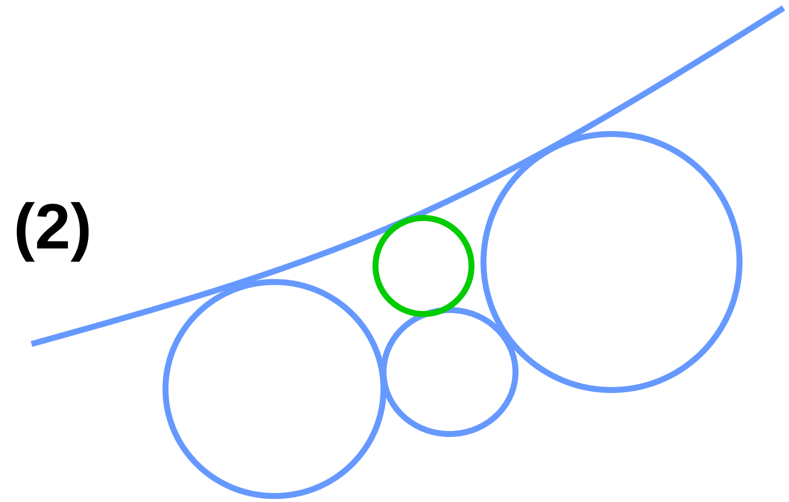
Solution – Break up Bad Quads



Problems:

(1) Reflex Quadrangle

(2) Circle center on wrong side of chord



Either bad case can be solved by adding one more disk.

Outline

1) Disk packing of a polygon

1) Nonobtuse triangulation of a polygon

1) Origami magic trick

1) Origami embedding of Euclidean Piecewise-Linear 2-manifolds

Origami Magic Trick

Question: **Can any polygon be cut out of flat-folded paper with a single straight cut ?**

Origami Magic Trick

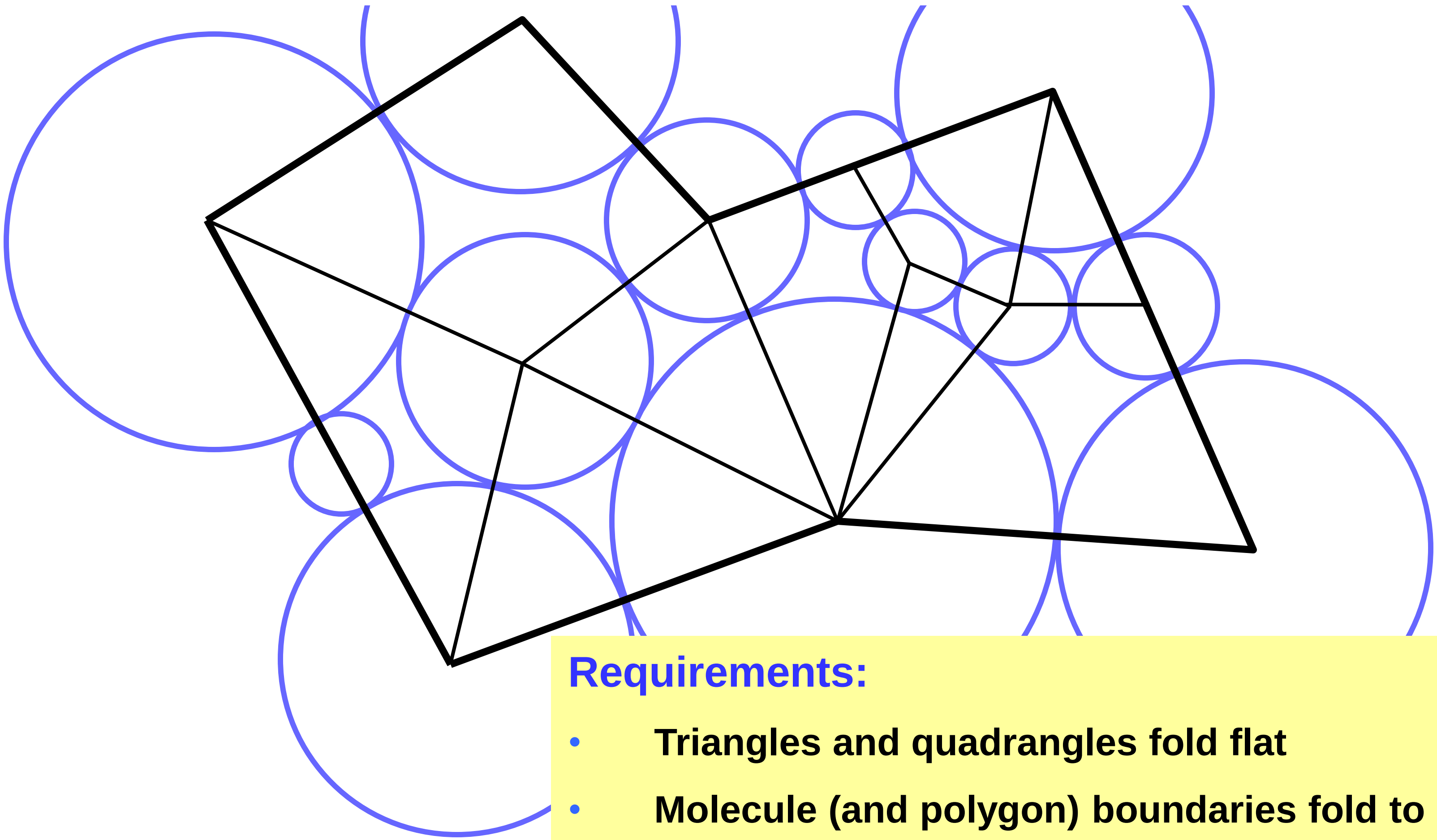
Question: **Can any polygon be cut out of flat-folded paper with a single straight cut ?**

[Betsy Ross, ~1790] **Five-pointed star**

[Demaine – Demaine – Lubiw, 1998]
Heuristic method that works if folding paths do not propagate forever

[Bern – Demaine – Eppstein – Hayes, 1998]
Solution for any polygon with holes

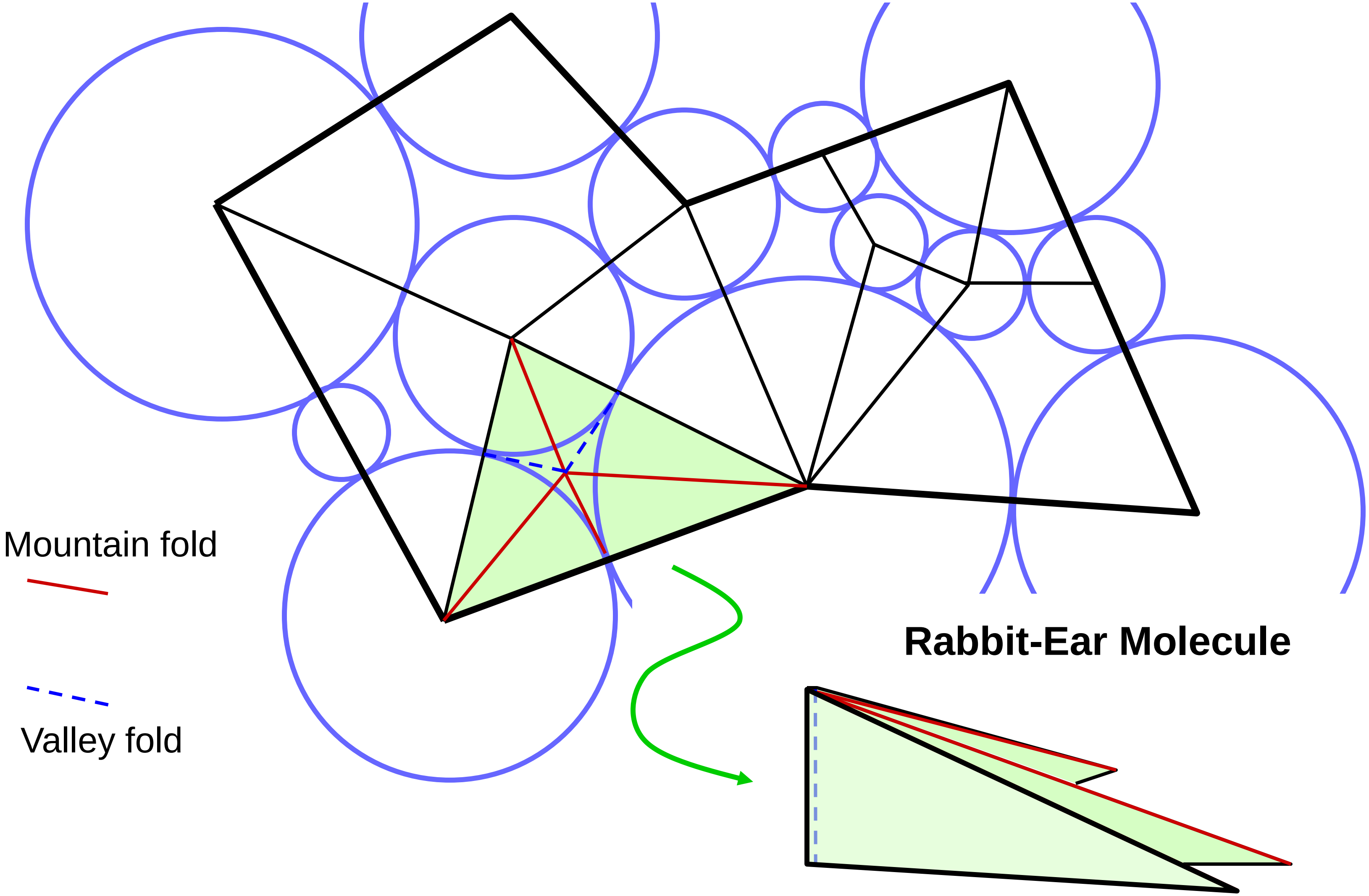
Use the decomposition to form independently foldable “molecules”



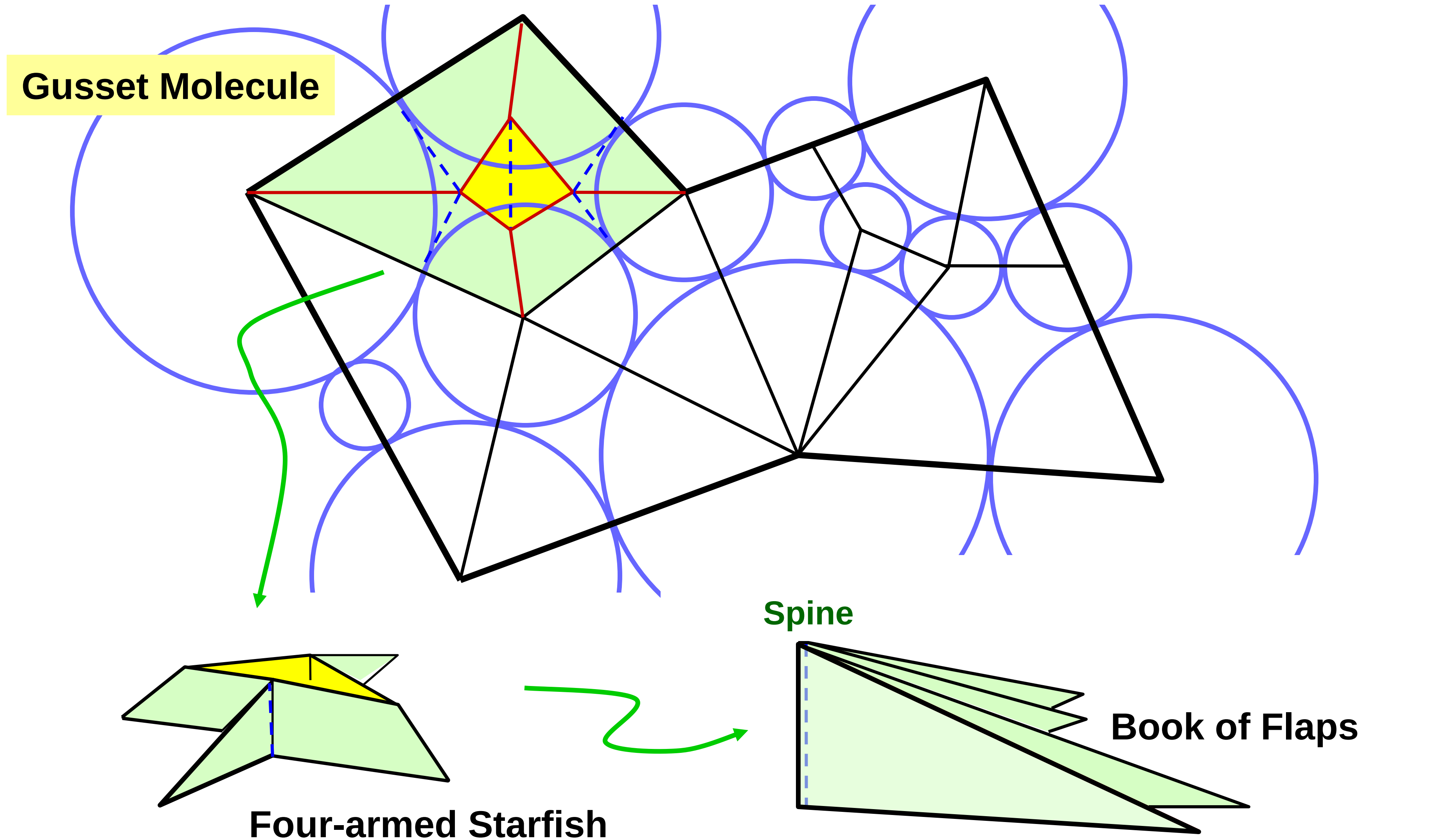
Requirements:

- **Triangles and quadrangles fold flat**
- **Molecule (and polygon) boundaries fold to a common line (for the cut)**
- **Folds exit molecules only at points of tangency (or else we can’t fold them independently)**

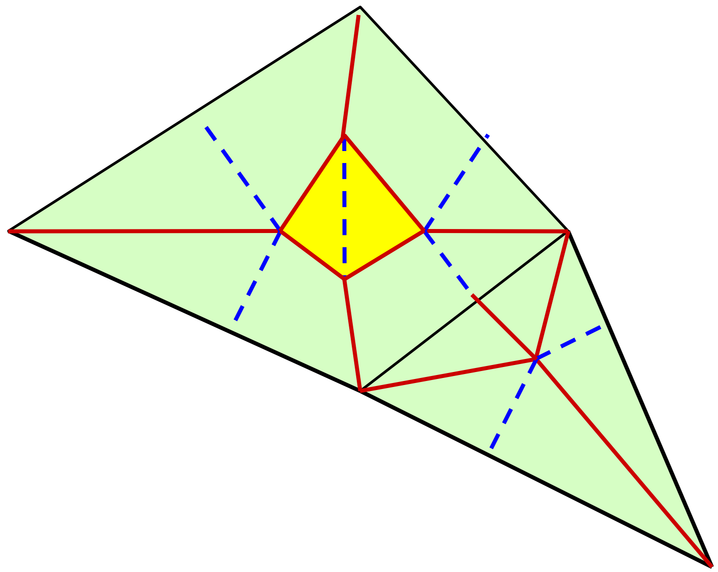
Triangles fold in a known origami pattern



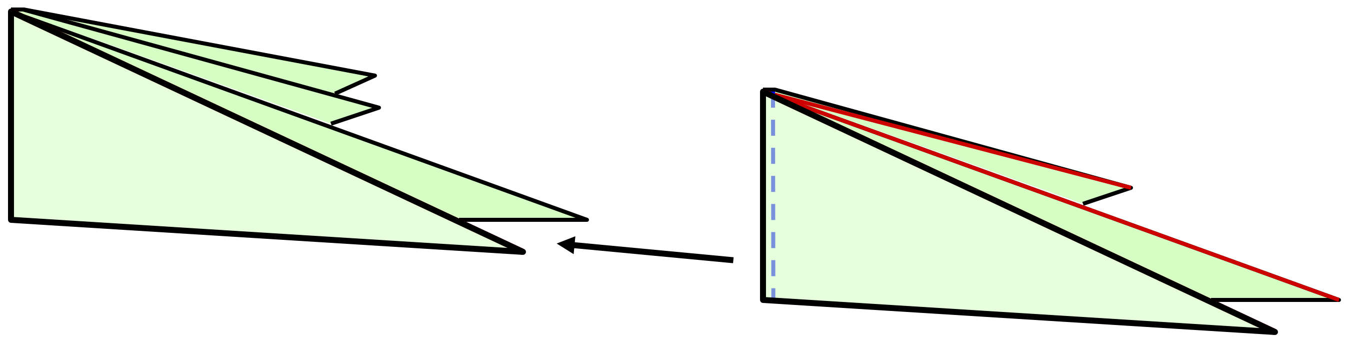
Quadrangles magically work out, too!



How do folded molecules fit together?

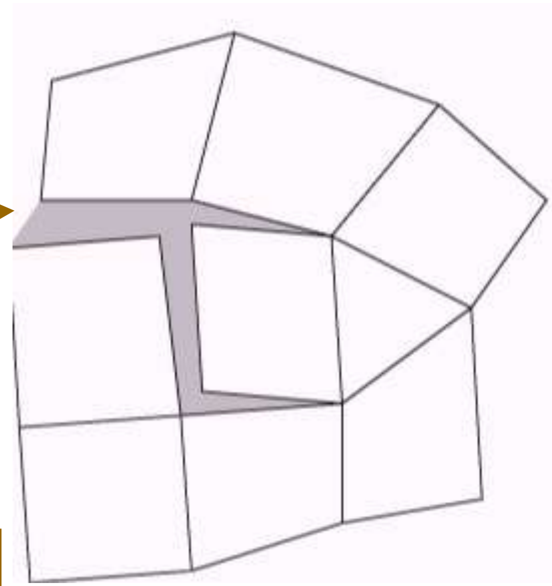


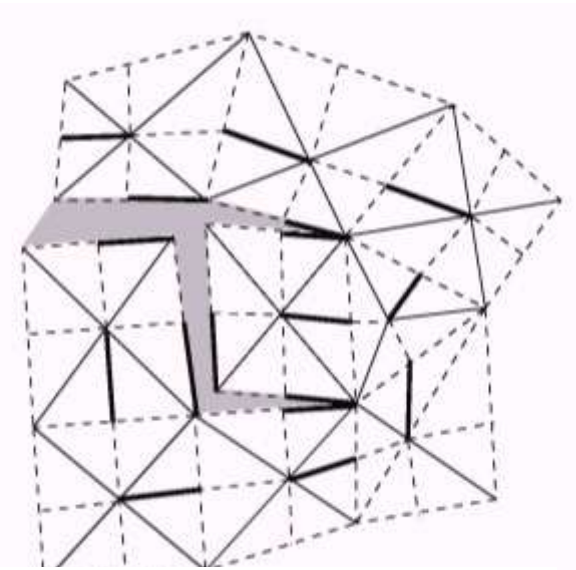
- One book of flaps tucks into another book of flaps (as a new “chapter”)
- Spines collinear, boundaries collinear



Can we recover all the adjacencies?

(1) Cut along a spanning tree to give a tree of molecules



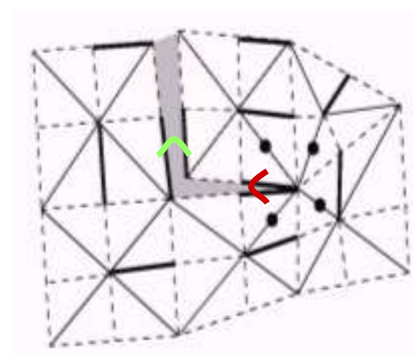


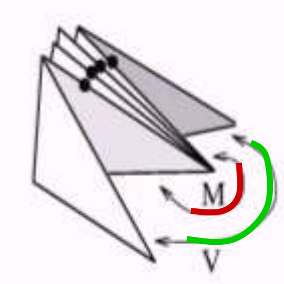
Mountain / valley assignments

(2) Tuck book inside book in a walk up the tree of molecules

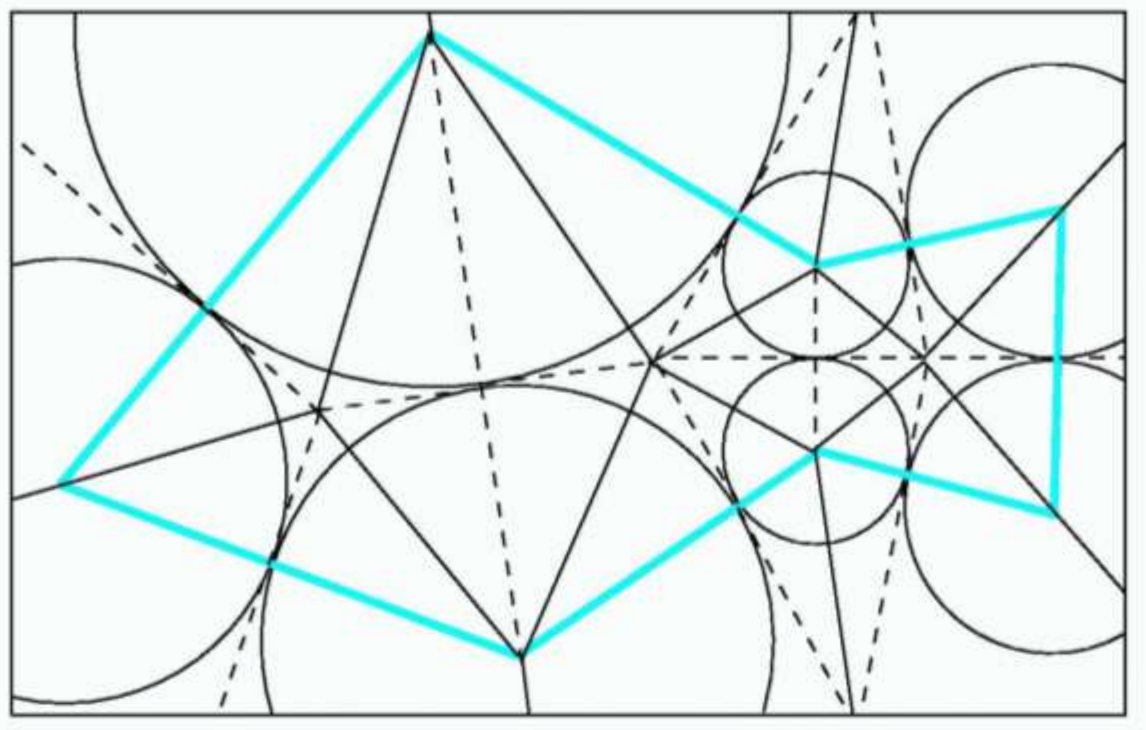
(3) “Tape” spanning tree cuts along bottom edges of pages

Required tapings nest like parentheses in a walk around molecule-tree boundary



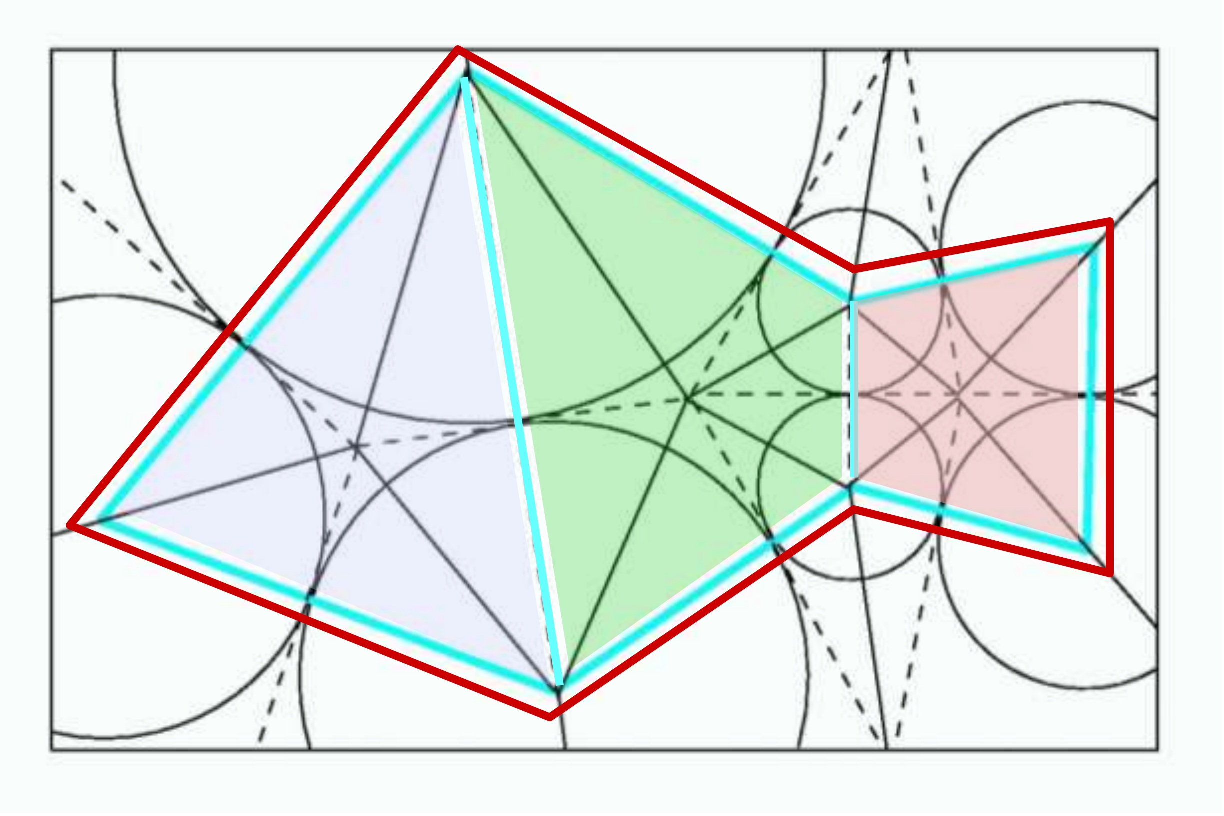


Mounted Marlin



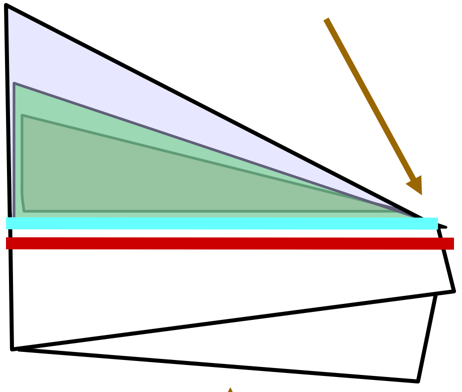
Degenerate Solution

True solution uses an offset polygon and offset disk packing



Cut along red

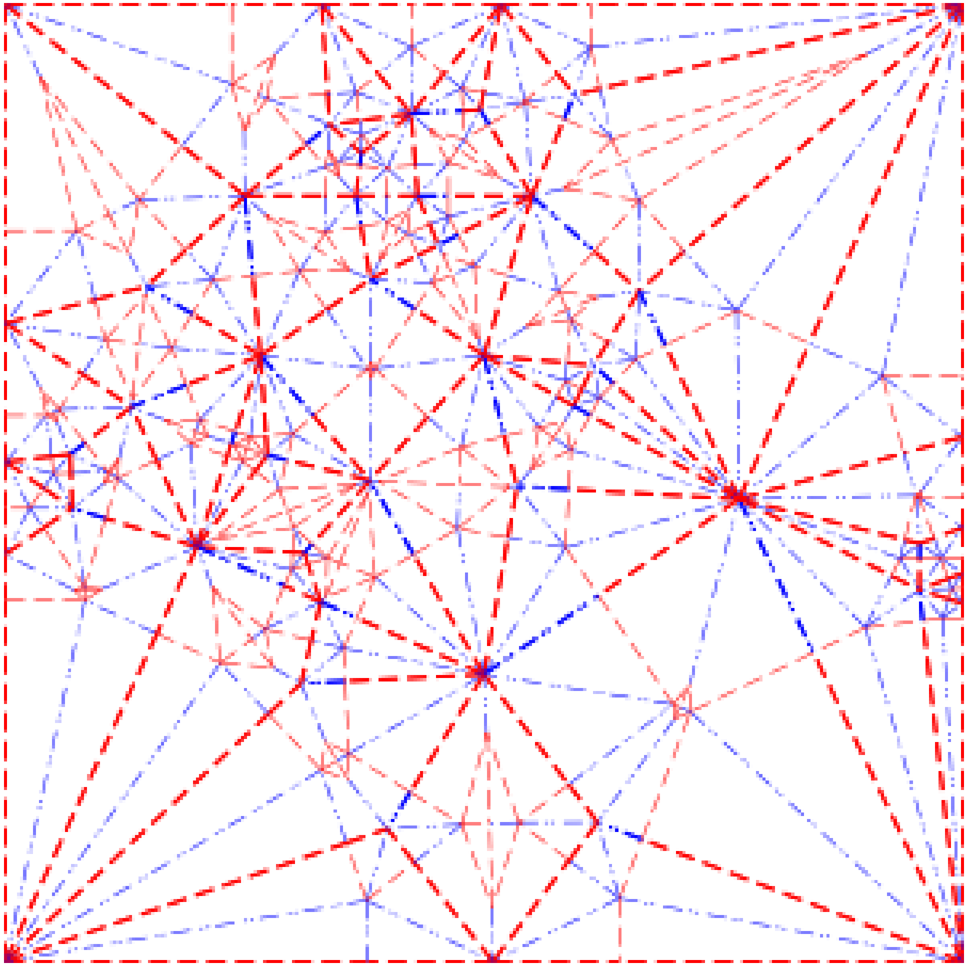
Exterior to *P*



Recent Implementation (last week)

Send us cool images. And if you are able to fold these 1000+ origamis, DONT CUT IT :).

Paulo Silveira, Rafael Cosentino, José Coelho, Deise Aoki. U. São Paulo



Outline

1) Disk packing of a polygon

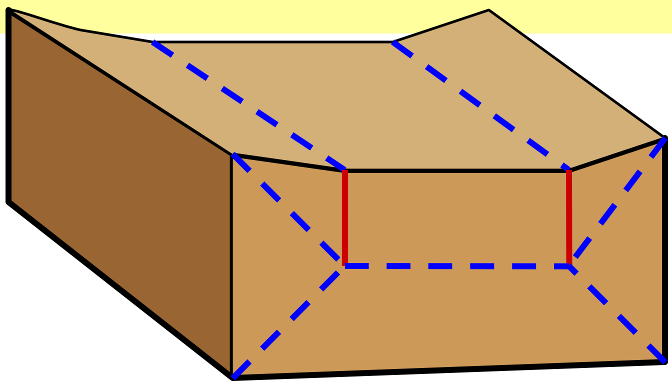
1) Nonobtuse triangulation of a polygon

1) Origami magic trick

1) Origami embedding of Euclidean Piecewise-Linear 2-manifolds

Origami Embedding of PL 2-Manifolds

Question: [E. Demaine] **Can any polyhedron be "crushed"?**
That is, can it be creased and folded to make a flat origami?



Example: **Rectangular Parallelopiped can be folded flat using paper bag folds.**

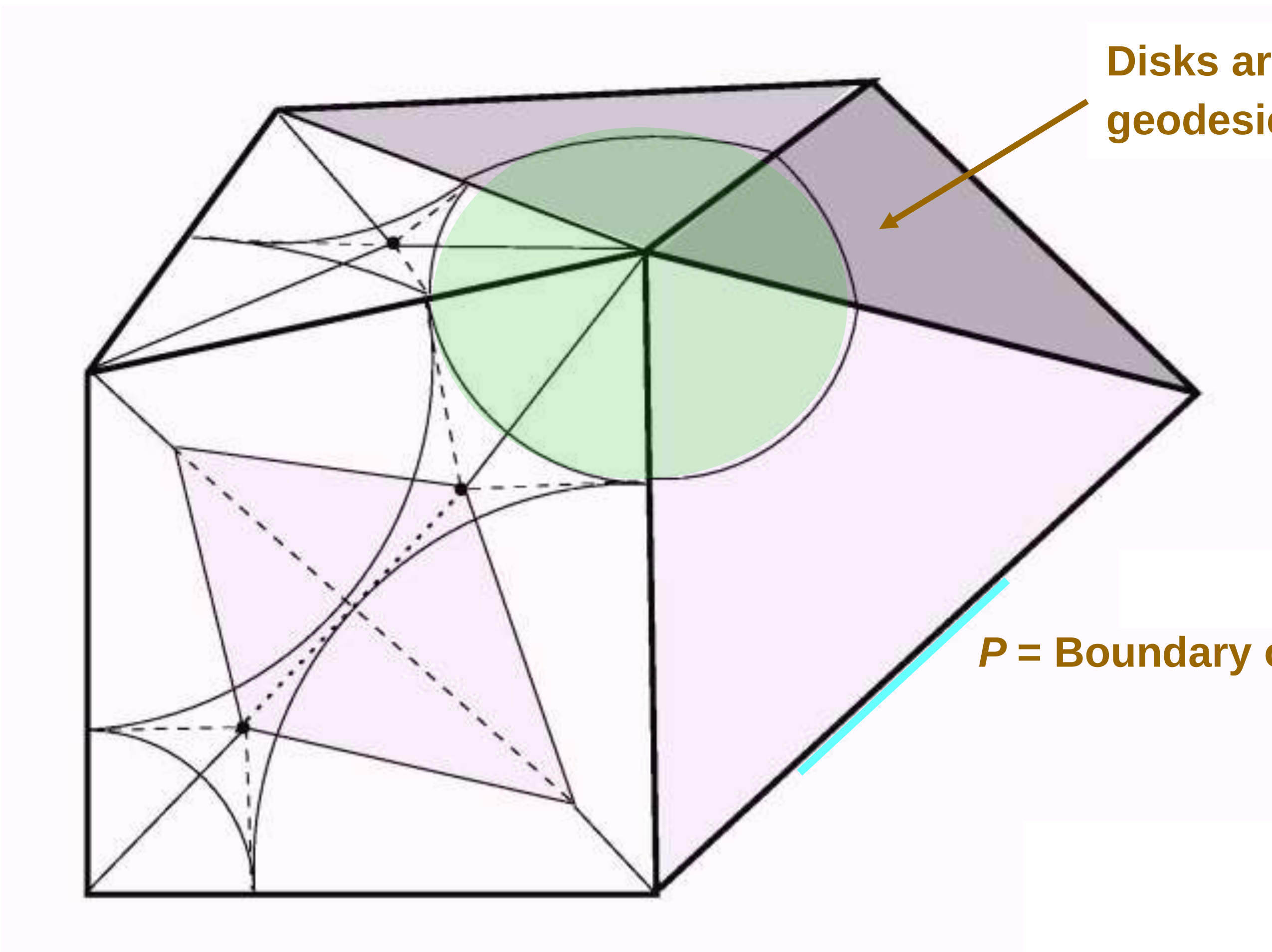
Note: **We just want a flat embedding, not a continuous transformation.**

Origami Embedding of PL 2-Manifolds

Theorem: **[Bern – Hayes, 2006]** **Any orientable, metric, piecewise-linear 2-Manifold (Euclidean triangles glued together at edges) can be isometrically embedded in Euclidean 2-space "plus layers", that is, as a flat origami.**

Topological Disk

Magic trick algorithm flat-folds a polyhedral patch

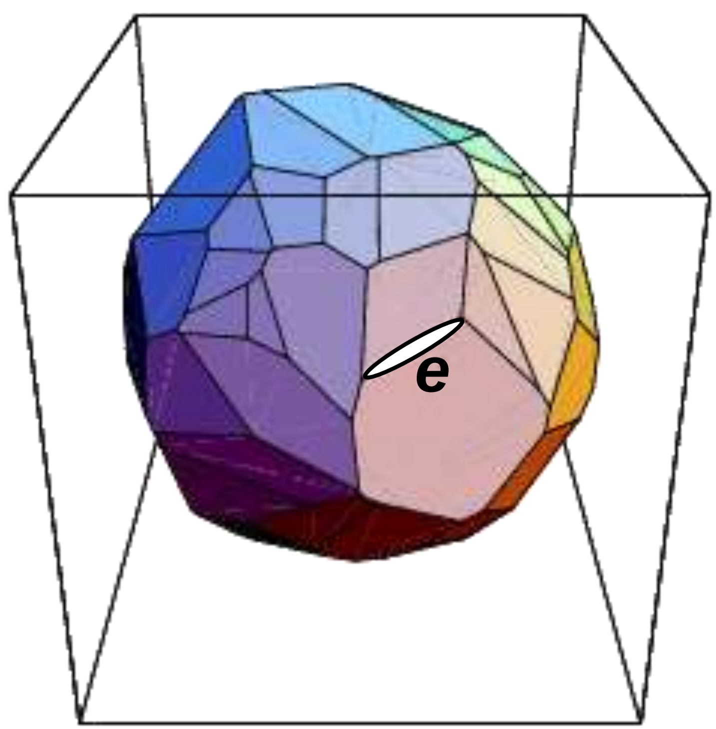


Disks are now geodesic disks

P = Boundary of patch

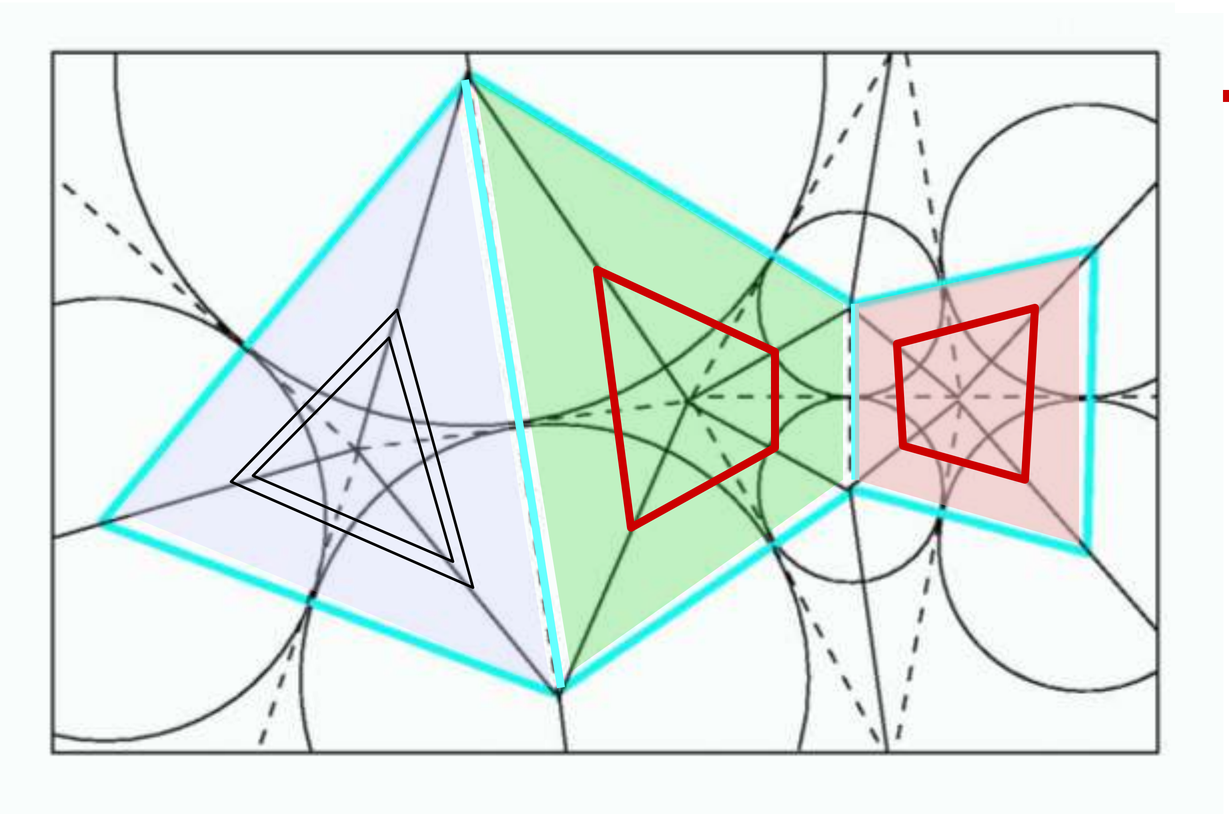
Topological Sphere

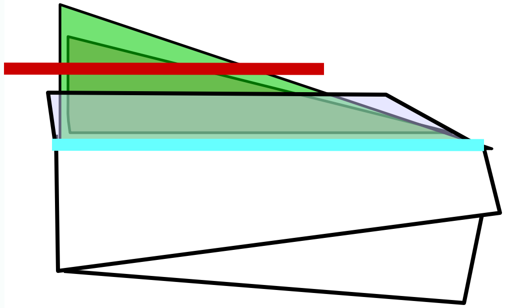
- **Puncture the sphere by opening an edge *e***
- **Fold disk**
- **Final taping closes edge *e***



For higher genus, we need a new trick: taping books of flaps at the top and bottom

Joining to form a handle requires that tops are mirror-congruent

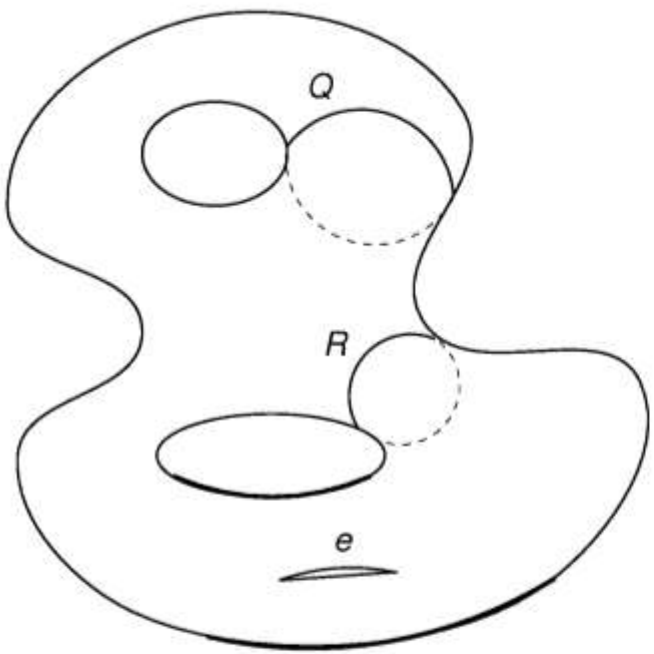


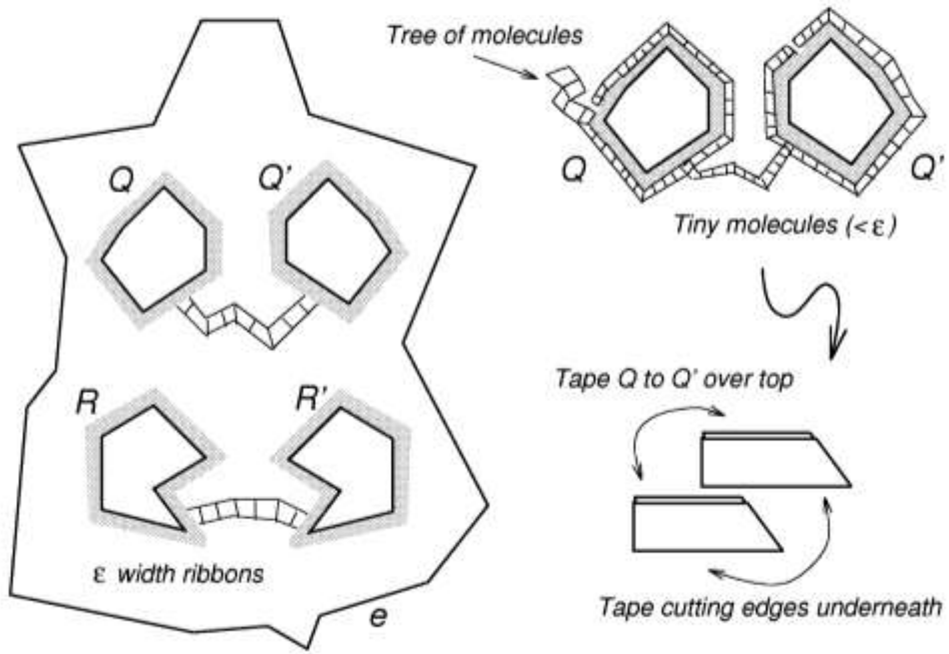


Schematic of Construction

(1) **Cut manifold to a disk with paired holes**

(1) **Paired holes will be taped over top of book of flaps**





Beautiful Minds?

Nash Embedding Theorem: **Any orientable Riemannian manifold embeds smoothly (C^∞) and isometrically into some Euclidean space. (E.g., 2-manifold → 17 dimensions)**

Origami Embedding Theorem: **Any compact, orientable, metric PL 2-manifold embeds isometrically as a flat origami.**

Beautiful Minds?

Nash Embedding Theorem: Any Riemannian manifold embeds smoothly (C^∞) and isometrically into some Euclidean space. (E.g., 2-manifold → 17 dimensions)

Origami Embedding Theorem: Any compact, orientable, metric PL 2-manifold embeds isometrically as a flat origami.

[Zalgaller, 1958] **Any 2- or 3-dimensional "polyhedral space" (orientable or not) can be immersed in Euclidean 2- or 3-space.**

[Burago – Zalgaller, 1960, 1996] **Any orientable PL 2-manifold can be isometrically embedded in Euclidean 3-space.**

[Krat-Burago-Petrunin, 2006] **Any compact, orientable, 2-dimensional polyhedral space embeds isometrically as a flat origami.**

Open Problems

1) Bad examples for naïve nonobtuse triangulation algorithms.

2) Simultaneous inside/outside nonobtuse triangulation of a polygon with holes

3) Algorithm for quasiconformal mapping using disk packing with 4-sided gaps

4) Do the "quadrangles that think they're triangles" (cross-ratio 1) have any good numerical-analysis properties?

Cat that thinks he's a dog

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Open Problems

1) Origami embedding of higher-dimensional PL manifolds?

2) Can any origami embedding of a PL 2-manifold be "opened up" to give an embedding in Euclidean 3-space?

3) Continuous deformation of polyhedron to a flat origami?

4) 3-sided gap disk packing : Conformal mapping ::

   4-sided gap disk packing : ???

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