Convex Hull Boundaries and Conformal Maps

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The Riemann mapping and the first derivative

One of the classical problems in Geometric function theory is to estimate the derivative $|f'|$ of the Riemann map, and the corresponding estimates on harmonic measure.

What is the upper bound on harmonic measure of a small disc in Ω?

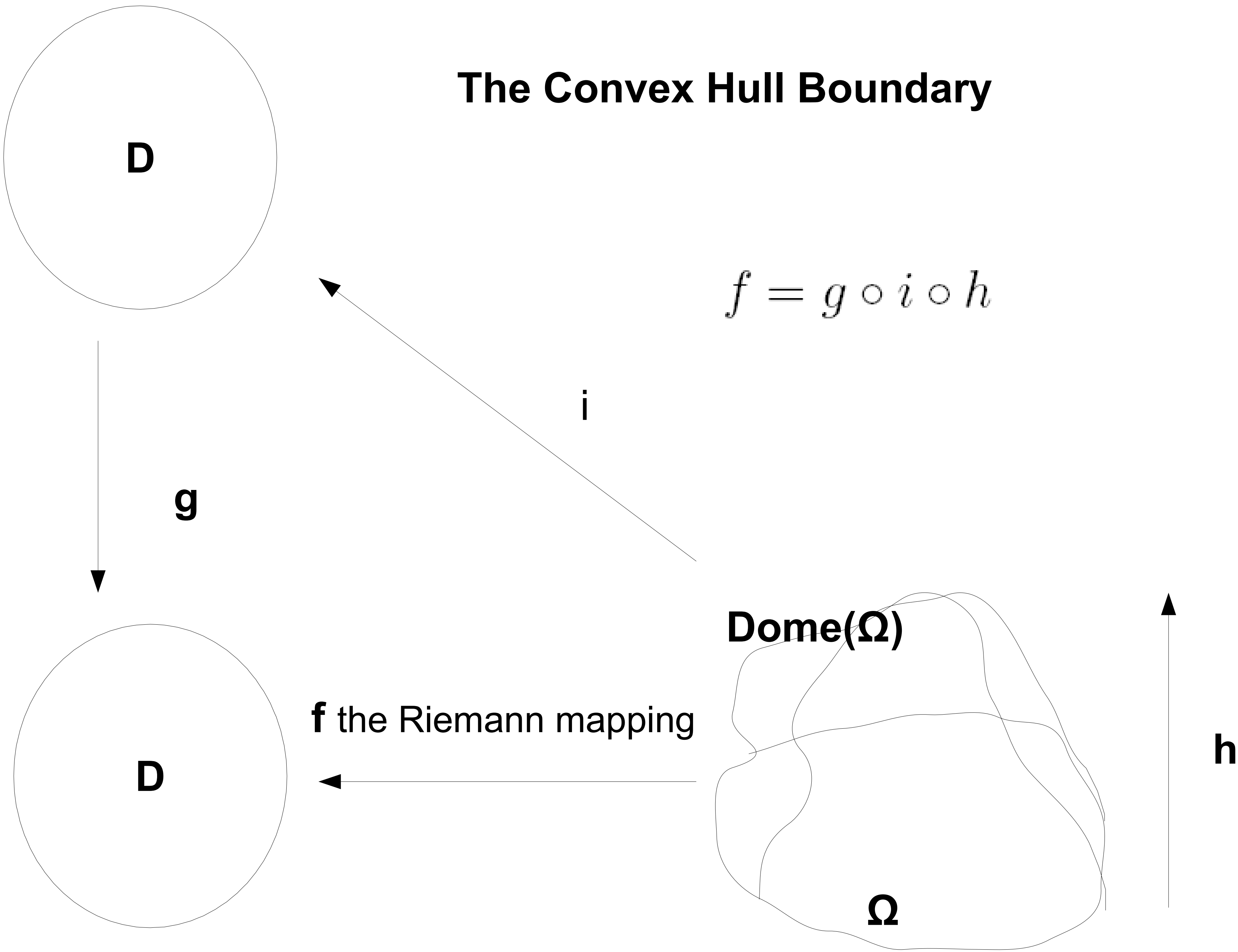
For which values of $p \in \mathbf{R}$ does the integral

$$\int_\Omega |f'|^p \, dx dy$$

converge.

The Convex Hull Boundary

$$f = g \circ i \circ h$$



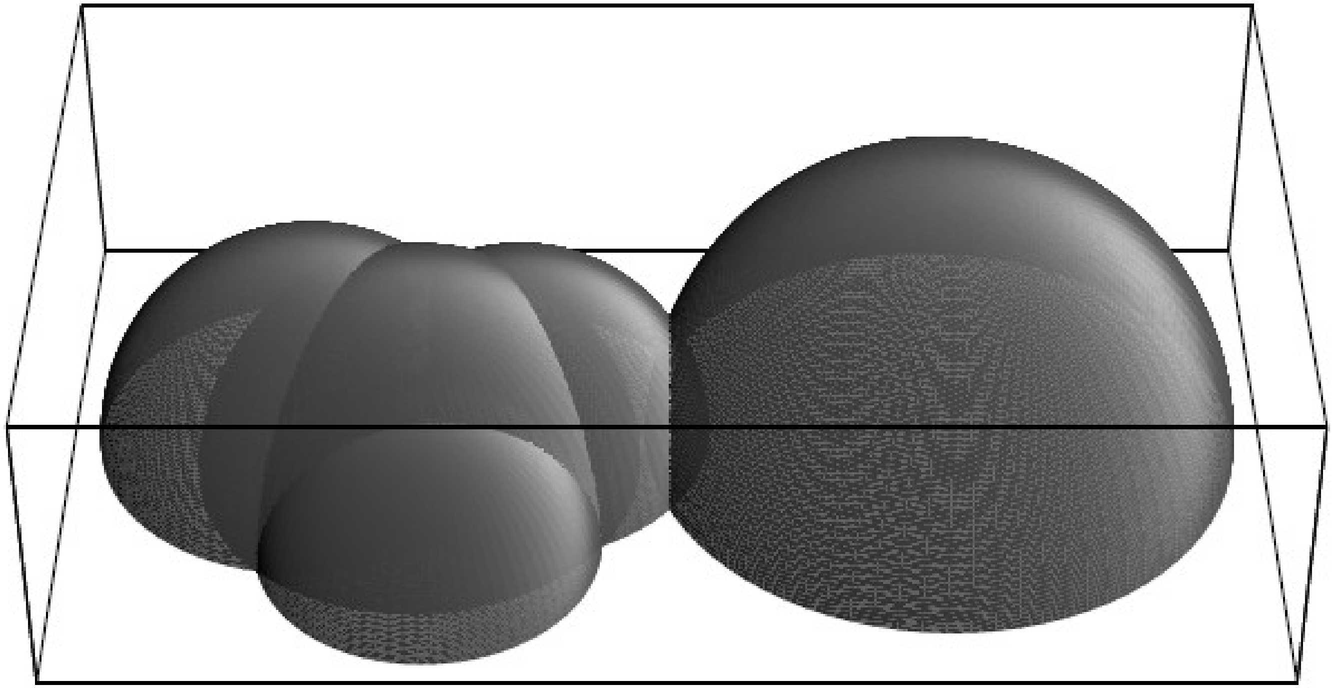


Figure 5. $\mathrm{Dome}(\Omega)$, where Ω is shown in Figure 4. The dome is placed in the upper halfspace model, and is viewed from inside the convex hull of the complement of Ω, using Euclidean perspective. The space under the dome lies between Ω and $\mathrm{Dome}(\Omega)$. Since the upper halfspace model is conformal, the angle between disks in Figure 4 is equal to the angle between flat pieces shown in Figure 5.

The map h is chosen to be the identity on the boundary of Ω. This implies that the restriction of the map g on the unit circle does not depend on the particular choice of h.

$r : \Omega \to Dome(\Omega)$ denotes the nearest point retraction map.

Theorem 1 (Bishop). *The map $i \circ r : \Omega \to \mathbf{D}$ is Lipschitz with respect to the Euclidean metric.*

By deforming slightly the nearest point retraction map one can construct a quasiconformal homeomorphism $h : \Omega \to \mathbf{D}$ so that the map $i \circ h : \Omega \to \mathbf{D}$ is Lipschitz (one can arrange that the corresponding map $g : \mathbf{D} \to \mathbf{D}$ is real analytic). This implies that

$$|f'(z)| \leq const\, |g'((i \circ h)(z))|.$$

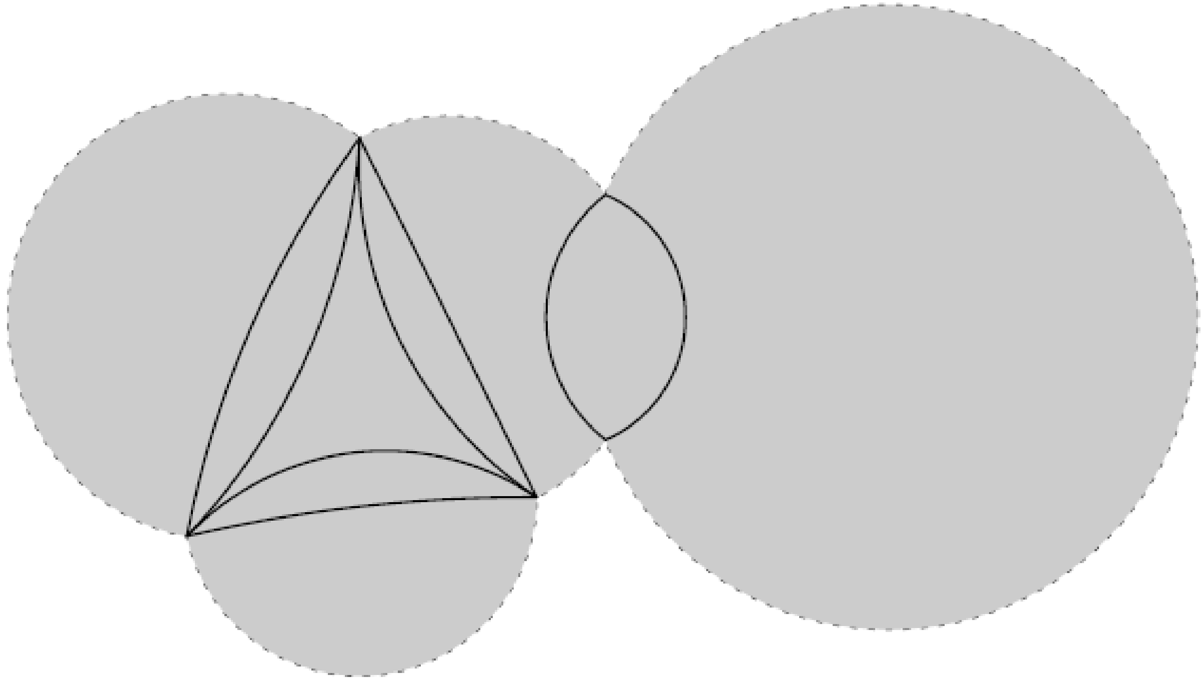


Figure 4. Ω is the union of four disks. $\mathrm{Dome}(\Omega)$ is the union of five flat pieces as can be seen in Figure 5. The fifth piece is a hyperbolic triangle in the hyperbolic plane represented by a circle lying in the union of three of the original disks. The dome has four bending lines, as shown in Figure 5. The crescents shown are the inverse images of the bending lines under the nearest point retraction. Notice that each boundary component of a crescent is orthogonal to the appropriate circle.

Define

$$K(\Omega) = \{K(h) : h : \Omega \to Dome(\Omega); h|_{\partial\Omega} = id\}$$

$$K = sup_{\Omega} K(\Omega)$$

$$L = \{L(r) : r : \Omega \to Dome(\Omega)\},$$

where $r : \Omega \to Dome(\Omega)$ is the nearest point retraction.

Theorem 2 (Sullivan, Epstein-Marden). *The numbers K and L are well defined. Moreover $2 \leq K < 70$ and $2 \leq L < 4$.*

Let $D(\rho) \subset \Omega$ be a disc of radius ρ, and let $\omega(D)$ denote its harmonic measure. It can be shown that for small values of $\rho > 0$

$$\omega(D(\rho)) \leq const\, \rho^{\frac{1}{L}}.$$

Bishop has shown (by applying Astala's theorem) that the integral

$$\int_{\Omega} |f'|^p\, dxdy$$

converges for $p < \frac{2K}{K+1}$.

It is conjectured (the Brennan conjecture) that the integral

$$\int_{\Omega} |f'|^p dx dy,$$

converges for $\frac{4}{3} < p < 4$.

Conjecture 1. *We have $K = L = 2$.*

In the case when Ω is the slit region in the complex plane we have $K(\Omega) = L(\Omega) = 2$.

Theorem 3. *If Ω is (Euclidean) convex domain in the complex plane then $K(\Omega) \leq 2$.*

Theorem 4. *L=2.*

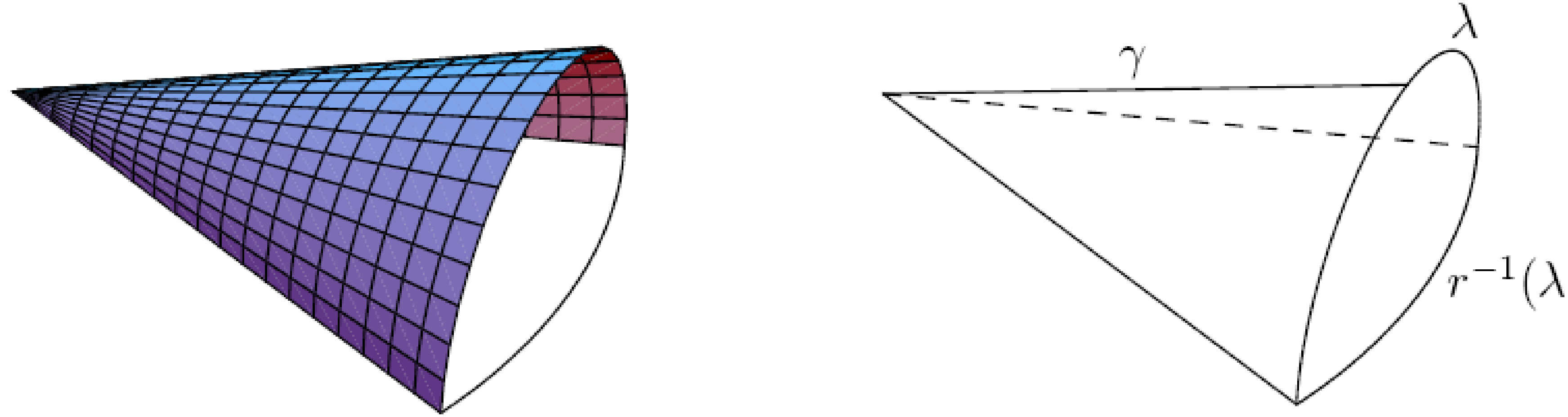


Figure 2.0.i. The dome of a wedge is a cone. In the lefthand picture, the mesh shown consists, on the one hand, of semicircles orthogonal to the plane $\{z = 0\}$ and, on the other hand, rays through the origin. The semicircles are hyperbolic geodesics in $\mathbb{H}^3$. The rays are not geodesics in $\mathbb{H}^3$. However, the ray γ, running along the highest points of the semicircles, is a geodesic for the induced Riemannian metric on the dome. One of the semicircles is labelled λ, in agreement with the surrounding text, and its inverse image $r^{-1}(\lambda)$ under the nearest point retraction r is also labelled.

Bending measure

For every domain Ω, the $Dome(\Omega)$ is determined by a measured lamination (Λ, μ) on $\mathbf{D}$ (here μ is a positive measure on intervals that are transverse to Λ). The $Dome(\Omega)$ is obtained by bending along (Λ, μ). One can apply the complex bending (called the complex earthquake) that is determined by $(\Lambda, t\mu)$, where $t \in \mathbf{C}$ and $Im(t) \geq 0$. This determines a map:

$$F : \{Im(z) \geq 0\} \to Teich(\mathbf{D}).$$

The map F is holomorphic and it can be holomorphically extended to a portion of the lower half plane (this was proved by McMullen). Let D denote the domain of F. The image $F(D) \subset Teich(\mathbf{D})$ is called an earthquake disc.

By considering domains Ω that are regular sets of quasi-Fuchsian once-punctured torus groups we obtain a biholomorphic map

$$F : D \to Teich(T),$$

where $Teich(T)$ is the Teichmuller space of once-punctured torus.

Theorem 5. *There exists a domain Ω (which is a regular domain of some once-punctured torus quasi-Fuchsian group) so that $K_{eq}(\Omega) > 2$.*

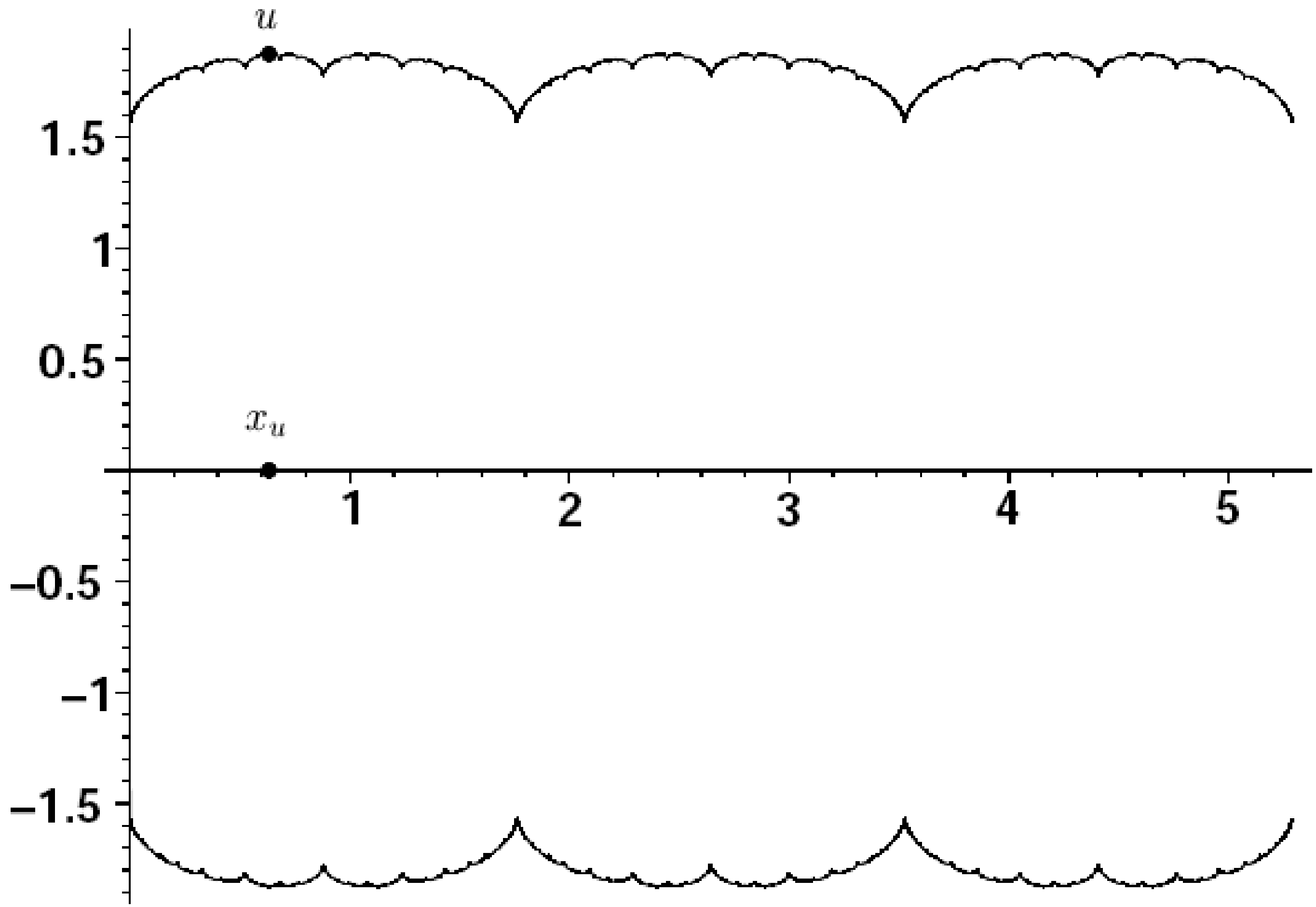


Figure 2. The values of z for which φ_z is injective and G_z is a discrete group of isometries is the region lying between the upper and lower curves. The whole picture is invariant by translation by $\operatorname{arccosh}(3)$, which is the length of α in the punctured square torus. The Teichmüller space of T is holomorphically equivalent to the subset of $\mathbb{C}$ above the lower curve. The point marked u is a highest point on the upper curve, and x_u is its x-coordinate. This picture was drawn by David Wright.

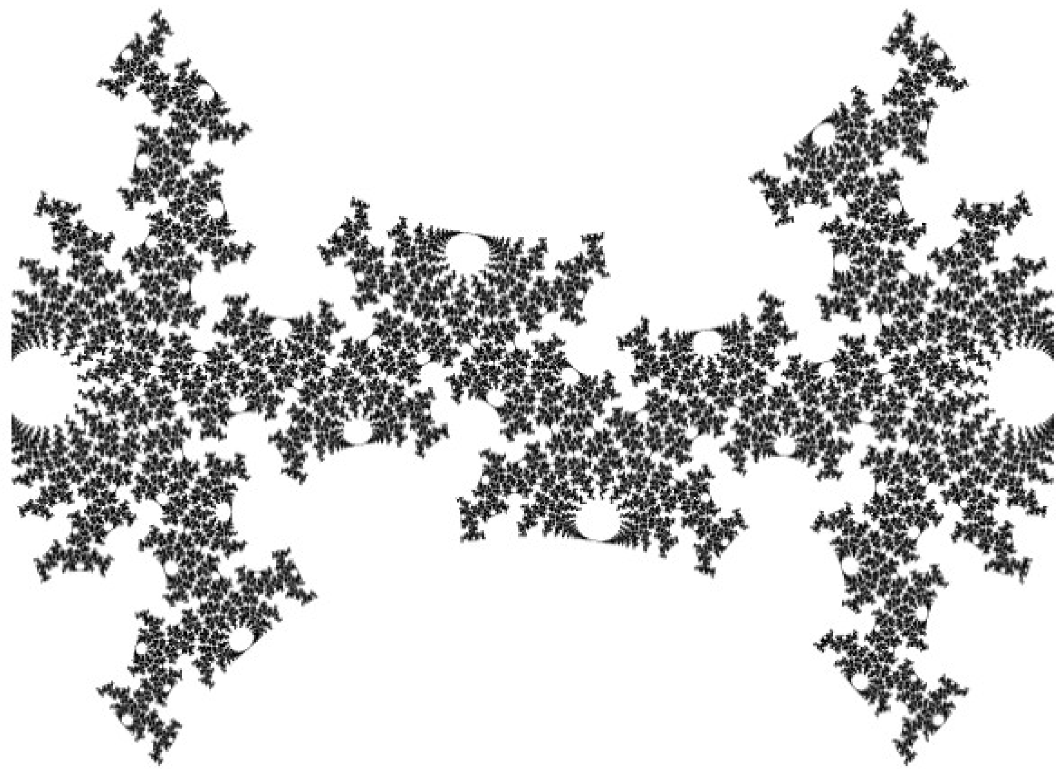


Figure 3. The complement in $\mathbb{S}^1$ of the limit set shown here is a counterexample to the equivariant $K = 2$ conjecture. The picture shows the limit set of G_u, where u is a highest point in $\mathcal{QF} \subset \mathcal{T} \subset \mathbb{C}$. This seems to be a one-sided degeneration of a quasifuchsian punctured torus group. This would mean that, mathematically, the white part of the picture is dense. However, according to Bishop and Jones (see [8]), the limit set of such a group must have Hausdorff dimension two, so the blackness of the nowhere dense limit set is not surprising. In fact, the small white round almost-disks should have a great deal of limit set in them—this detail is absent because of intrinsic computational difficulties. This picture was drawn by David Wright.

Let $c > 0$ and let Ω_c be the complement of the logarithmic spiral

$$x \to exp(x(c+i)).$$

Set $K(c) = K(\Omega_c)$.

Theorem 6. *There exists $0 < c_0 < 1$, so that for $c_0 < c < \infty$ we have $K(c) > 2$. In particular, for $c = 1.08968$ we get $K(c) = 2.111$.*

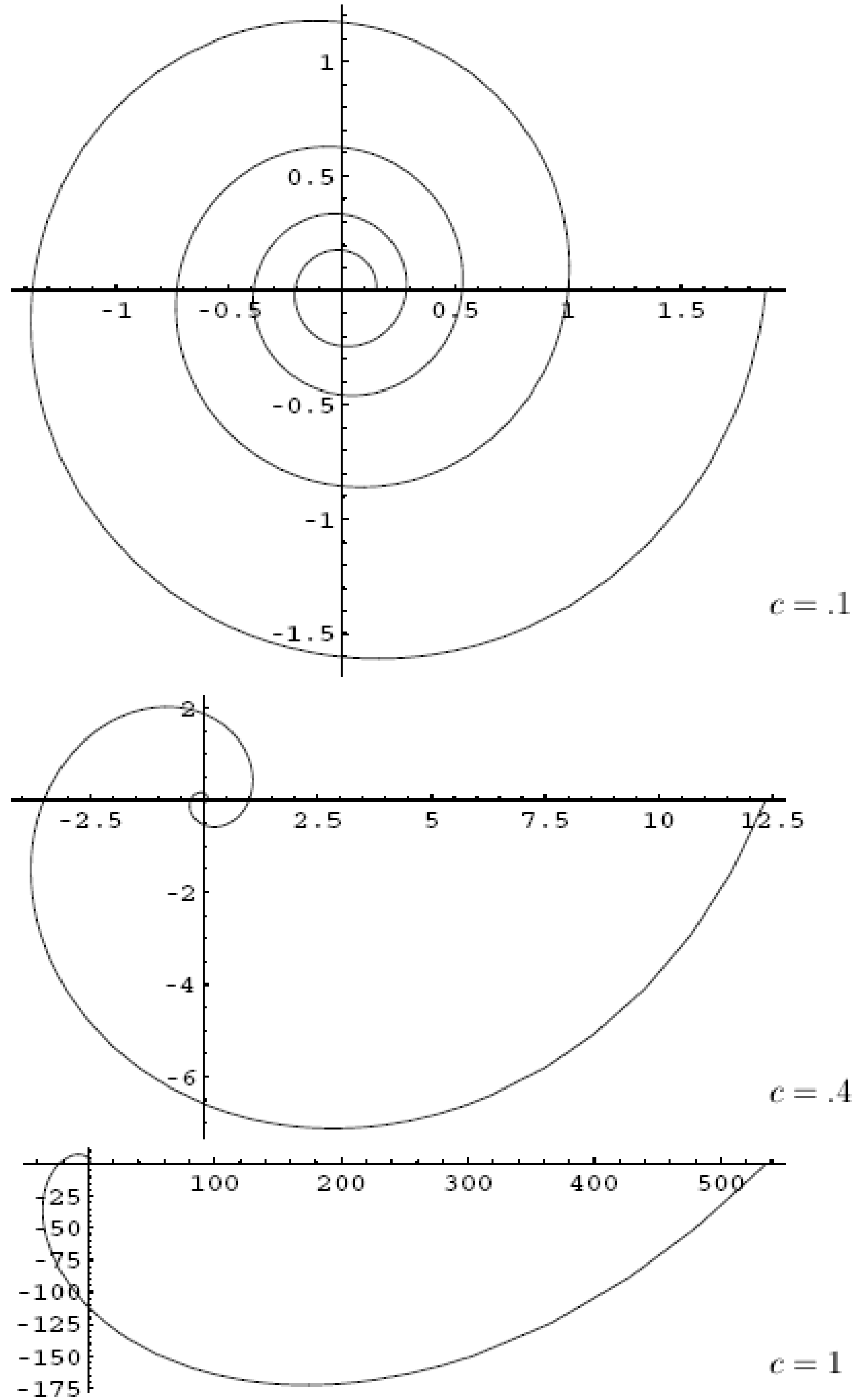


Figure 3. Some logarithmic spirals

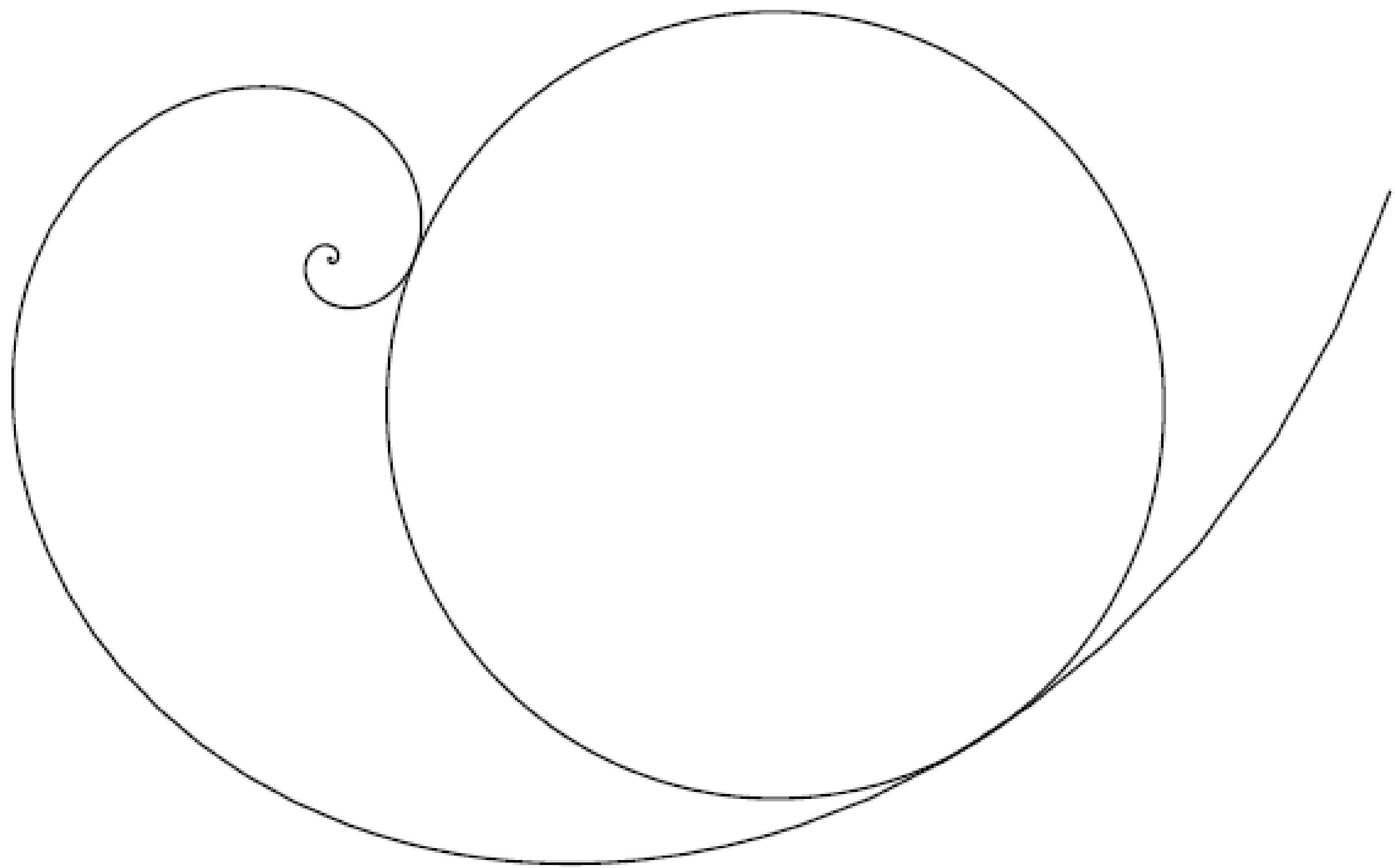


Figure 5. This shows the logarithmic spiral with $c = 0.4$ and the maximal circle which is tangent at $z = 1$. The parameter values at which the circle touches are $t = 0$ and $s_0 = 0.900104$. We find s_0 using Newton's method following Lemma 5.1.

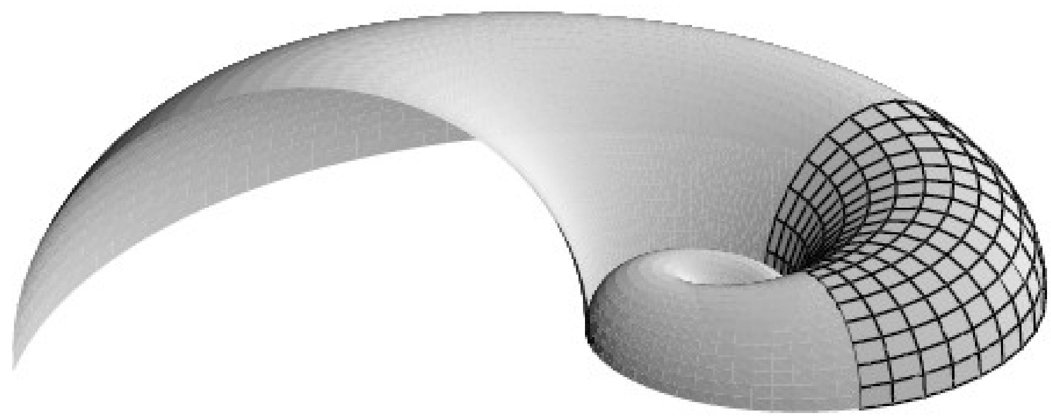


Figure 6. This shows $\mathrm{Dome}(\Omega) \subset \mathbb{U}^3$. The picture is drawn for $c = 0.4$. Part of the spiral is covered with a mesh of bending lines and constant distance curves. These are curves which are at constant distance from the main geodesic with respect to the induced Riemannian metric on the dome.

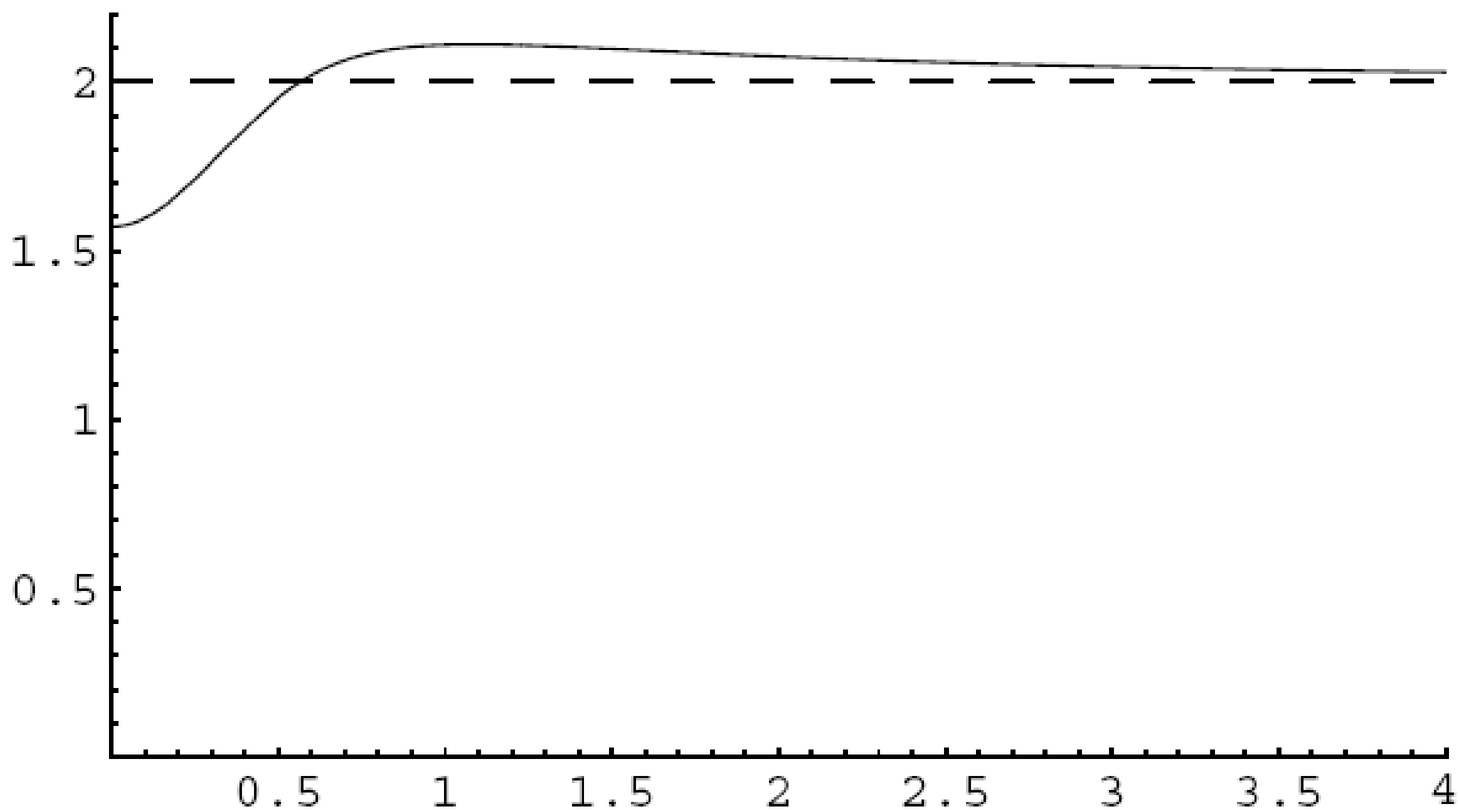


Figure 9. The graph of the dilatation $K(c)$ against c.

Let Ω be any domain (non necessarily simply connected). One can define the constants $K(\Omega)$ and $L(\Omega)$ accordingly.

Theorem 7. *$L(\Omega) < \infty$ if and only if Ω is uniformly perfect set. If Ω is uniformly perfect then $K(\Omega) < \infty$.*