A Theorem on Subdivision Approximation and Its Application to Obtaining PTAS’s for Geometric Optimization Problems

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A Dedication

Happy 60\textsuperscript{th} Birthday!!

\textbf{Michael Ian Shamos:}
PhD thesis "Computational Geometry", Yale University, 1978
Motivating Problem: TSP with Neighborhoods

Find shortest tour to visit a set of neighborhoods $P_1, P_2, \ldots, P_n$
TSPN for Disk Packing
TSPN in a Circle Packing

"Introduction to Circle Packing: the Theory of Discrete Analytic functions"
by Ken Stephenson
Another (Springtime) Motivation

Best method of mowing the lawn?

TSPN: Visit the disk centered at each blade of grass
Watchman Route Problem
Sensor Network Application: Cover Tour Problem

Min: Tour length + C * (sum of radii)

Alt, Arkin, Bronnimann, Erickson, Fekete, Knauer, Lenchner, Mitchell, Whittlesey, SoCG’06
Sensor Network Application: Minimizing # Relay Stations

Goal: Connect all subnetworks of sensors using min # of new relay stations

New result: PTAS

Efrat, Fekete, Mitchell 2007
Min-Weight Convex Subdivision

Steiner version

Special Case: Min-weight (Steiner) triangulation
c-approximation: cost at most $c$ times optimal, for a minimization problem ($c > 1$)

*Polynomial Time Approximation Scheme* (PTAS): method giving $(1 + \epsilon)$-approx to the optimal (minimum), in time polynomial in $n$, for any fixed $\epsilon > 0$.

Dependence on $\epsilon$ may be exponential in $(1/\epsilon)$; else *FPTAS*
Background on TSP

- $S =$ set of $n$ points in $\mathbb{R}^d$
- NP-hard
- $n^{O(n^{1-1/d})}$ exact (subexponential)
- Simple 2-approx: double the MST and shortcut (holds in metric spaces) [SmWo98]

- Christofides: 1.5-approx
  (use $\text{MST} \cup \text{min-weight matching}$ on odd-degree nodes of MST)
PTAS for Geometric TSP

- $O(n^{O(1/\epsilon)})$ in $\mathbb{R}^2$ [Ar96, Mi96]
- $O(n^{O(1)})$ in $\mathbb{R}^2$ [Mi97]
- $O(n(\log n)^{(O(\frac{d}{\epsilon}))^{d-1}})$ expected ($O(n^{d+1}{\text{polylog}})$ det.) [Ar97]
- $O(n \log n)$ deterministic [RaSm98]
  Idea: $t$-spanners and "$t$-banyons"
- NP-hard to get $(1 + \epsilon)$-approx in $\mathbb{R}^{O(\log n)}$, for some $\epsilon > 0$ [Tr97]
- MAX-SNP-hard in metric spaces
  No $c$-approx for $c < 129/128$ ( $c < 41/40$, asym.) [PV99]
TSPN Recent Result [SODA’07]

- TSPN has a PTAS for regions/neighborhoods that are “fat”, disjoint (or sufficiently disjoint) connected regions in the plane.
- Applies also to “MST with neighborhoods”, Steiner MSTN, and many related problems.

PTAS = Polynomial-Time Approximation Scheme = (1+\(\varepsilon\))-approx, any \(\varepsilon>0\)
Background on TSPN

Generalizes 2D Euclidean TSP (thus, NP-hard)

Introduced by [Arkin & Hassin, 1994]

• “obvious” heuristics do not work:
  ▪ TSP approx on centroids (as representative points)
  ▪ Greedy algorithms (Prim- or Kruskal-like)

• $O(1)$-approx, time $O(n + k \log k)$, for “nice” regions:
  ▪ Parallel unit segments
  ▪ Unit disks
  ▪ Translates of a polygon $P$

• Combination Lemma
General Connected Regions

$O(\log k)$-approx $[\text{Mata & M, SoCG'95}]$

Use guillotine rectangular subdivisions, DP

(non-disjoint: regions may overlap)

- $O(n^5)$ time $[\text{Mata & M, SoCG'95}]$
- $O(n^2 \log n)$ $[\text{Gudmundsson & Levcopoulos, 1999}]$

$k = \# \text{ regions}$

$n = \# \text{ vertices of all regions}$
O(1)-Approximations

- Unit disks, parallel unit segments, translates of $P$
  
  [Arkin & Hassin, 1993]

- Connected regions of comparable size
  
  [Dumitrescu & M, SODA'01]

- Disjoint fat regions of any size
  
  [de Berg, Gudmundsson, Katz, Levcopoulos, Overmars, van der Stappen, ESA'02]

- Discrete point sets within disjoint, fat, non-convex regions
  
  [Elbassioni, Fishkin, Mustafa, Sitters, ICALP'05]

- Non-disjoint, convex, fat, comparable size
  
  [Elbassioni, Fishkin, Sitters, ISAAC'06]
PTAS: $O(1 + \varepsilon)$-Approximations

- Disjoint (or nearly disjoint) fat regions of comparable size
  [Dumitrescu & M, SODA'01]
- Point clusters within disjoint fat regions of comparable size in $\mathbb{R}^d$
  [Feremans, Grigoriev, EWCG'05]

New: PTAS for disjoint (or nearly disjoint) fat regions of arbitrary sizes.

Def: $P$ is fat if $\text{area}(P) = \Omega(\text{diam}^2(P))$

Weaker notion than usual "fatness"
Related Work: APX-hardness

- General connected regions (overlapping):
  - No c-approx with $c < 391/390$, unless $P=NP$
    
    \[ \text{[de Berg, Gudmundsson, Katz, Levcopoulos, Overmars, van der Stappen, ESA’02]} \]
    
    (from MinVertexCover)

  - No c-approx with $c < 2$, unless $P=NP$
    
    \[ \text{TIME (} n^{O(\log \log n)} \text{)} \]
    
    \[ \text{[Safra, Schwartz, ESA’03]} \]
    
    (from Hypergraph VertexCover)

- Line segments, comparable length
  
  \[ \text{[Elbassioni, Fishkin, Sitters, ISAAC’06]} \]

- Pairs of points (disconnected)
  
  \[ \text{[Dror, Orlin, 2004]} \]
Exact Poly-Time Solutions

TSPN for a set of infinite lines in 2D:

Solved in $O(n^4 \log n)$ time using Watchman Route solution

[Dror, Efrat, Lubiw, 20, STOC’03]

Is this the only nontrivial case exactly solvable in poly-time?

What about visiting planes in 3D?
Recipe for PTAS

Use **dynamic programming** to compute shortest network with the required structure (connectivity, Eulerian subgraph, etc)

What should the special recursive structure be?
Network edge set $E$ is $m$-guillotine if it can be recursively partitioned by horiz/vertical cuts, each having small ($O(m)$) complexity wrt $E$

**Example:** 3-guillotine

Each cut intersects $E$ in at most 3 connected components
Desired Recursive Structure

Rectangular subproblem in dynamic program (recursion)

Constant $(O(m))$
information flow across boundary
m-Guillotine Structure Theorem

Any set $E$ of edges of length $L$ can be made to be $m$-guillotine by adding length $O(L/m)$ to $E$, for any positive integer $m$.

Proof is based on a simple charging scheme.

While this “scribble” may not be $m$-guillotine, it is “close” in that it can be made $m$-guillotine by adding only $(1/m)$th of its length.
Possible Vertical Cuts

\[ f(x) = \text{length of m-bridge} \]

\[ = \text{cost of construction} \]
Paying for the Bridge Construction: The Chargeable Length

\[ h(x) = \text{chargeable length} \]

Green portion: “m-dark”
Charging Scheme

- Let \( f(x) \) = length of \( m \)-span of vertical line through \( x \)
  Let \( g(y) \) = length of \( m \)-span of horizontal line through \( y \)

- Then,
  \[
  A_x = \int f(x) \, dx
  \]
  is simply the area of the “\( m \)-dark” (RED) region wrt horiz cuts
  Similarly,
  \[
  A_y = \int g(y) \, dy
  \]
  is the area of the “\( m \)-dark” (BLUE) region wrt vertical cuts

- Assume, WLOG, that \( A_x \geq A_y \)

- Thus, for \( h(y) \) = length of \( m \)-dark, for horiz line through \( y \),
  \[
  A_x = \int h(y) \, dy \geq \int g(y) \, dy = A_y > 0
  \]
  So, \( \exists y^* \) for which \( h(y^*) \geq g(y^*) \);
  i.e., \( \exists \) a horiz line through \( y^* \) whose \( m \)-dark portion \( \geq m \)-span.
  (If \( A_x \leq A_y \), then \( \exists \) a vertical favorable cut.)
Charging Scheme
Charging Scheme

Region of 2-dark points wrt horizontal cuts (RED)

\[ f(x) = \text{cost of construction of vert cut at } x \]

\[ h(y) = \text{chargeable length of horiz cut at } y \]

\[ \text{Red area} = \int f(x) \, dx = \int h(y) \, dy \]

\[ x \times f(x) = \text{cost of construction of vert cut at } x \]
Charging Scheme

Region of 2-dark points wrt vertical cuts (BLUE)

Blue area = $A_y = \int g(y) dy$
Charging Scheme

- Let $f(x) =$ length of $m$-span of vertical line through $x$
  Let $g(y) =$ length of $m$-span of horizontal line through $y$

- Then,
  $$A_x = \int f(x) \, dx$$

  is simply the area of the “$m$-dark” (RED) region wrt horiz cuts

  Similarly,
  $$A_y = \int g(y) \, dy$$

  is the area of the “$m$-dark” (BLUE) region wrt vertical cuts

- Assume, WLOG, that $A_x \geq A_y$

- Thus, for $h(y) =$ length of $m$-dark, for horiz line through $y$,
  $$A_x = \int h(y) \, dy \geq \int g(y) \, dy = A_y > 0$$

  So, $\exists y^*$ for which $h(y^*) \geq g(y^*)$;
  i.e., $\exists$ a horiz line through $y^*$ whose $m$-dark portion $\geq m$-span.

  (If $A_x \leq A_y$, then $\exists$ a vertical favorable cut.)
Subproblem

\[ R(i, i', j, j') \]
Dynamic Program: Min Steiner Tree

Sorted $x$-coord: $x_1 < x_2 < \cdots$ (for $P$, and grid lines)
Sorted $y$-coord: $y_1 < y_2 < \cdots$

**Subproblem:** $O(n^4 \cdot (2m)^4) = O(n^{8m+4})$ choices

**Input:**

1. a rectangle $R(i, i', j, j')$, defined by $x_i, x_{i'}, y_j, y_{j'}$  
   $O(n^4)$

2. four sets of “boundary information”, $\Sigma_l, \Sigma_r, \Sigma_b,$ and $\Sigma_t$, 
determined by $\leq 2m$ endpoints on each side  
   $O((n^{2m})^4)$

3. a partition, $\mathcal{P}$, of $\bigcup_{\alpha} \Sigma_\alpha$, giving required connectivity among 
   boundary pieces  
   $O(1)$

**Objective:** Find min-length $m$-guillotine subdivision, $S_G^*$ (edges $E_G^*$ interior to 
$R(i, i', j, j')$), such that $E_G^*$ covers $P$ and $E_G^*$ connects the boundary pieces, ac-

cording to partition $\mathcal{P}$. 
Difficulty in Applying TSP Methods to TSPN / MSTN

Consider a subproblem (rectangle):

Which regions must be visited inside R?
New Structure

- Build **region-bridges** in order to encode succinctly which regions are the “responsibility” of a subproblem.
- Cannot afford to build \(m\)-region-bridges for \(m = O(1/\varepsilon)\), constant wrt \(n\).
- But can afford to build \(M\)-region-bridges, with \(M=O((1/\varepsilon)\log n)\) and this is “just right”, since the remaining \(M\) bridges that are not part of the bridge can be specified in the subproblem: \(2^M = 2^{O(\log n)}\) is \(\text{poly}(n)\).
Subproblem: A Window into OPT

Bridges
\( m = 4 \)

Region-Bridges

\( M = 3 \)

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Subproblem Optimization

Specification of a subproblem:

1. Window W: \( n^4 \)
2. \( \leq 4 \) Bridges, \( \leq 2m \) segs/side of W
3. \( \leq 4 \) Region-Bridges, one bit per \( \leq 8M \) non-bridged crossing region: Is the subproblem responsible to visit? \( n^{O(1)} + 2^{O((1/\varepsilon) \log n)} = n^{O(m)} \)
4. Protruding regions not in \( R_{W_0} \): specified by \( \leq 2 \) sequences per side
5. Connection pattern among the \( O(m) \) segs crossing into W

Total # subproblems = \( n^{O(m)} \)

and \( n^{O(m)} \) choices for the best horiz/vertical cut, in DP optimization
(m,M)-Guillotine Structure

**Definition:** Network edge set $E$ is $(m,M)$-guillotine if it can be recursively partitioned by horiz/vertical cuts, each containing the “$m$-span” (bridge) of $E$ and the “$M$-region-span” (region-bridge) of the set of regions.
What the DP Computes

Minimum-length network that is

1. \((m,M)\)-guillotine wrt window \(W_0\) and regions \(R_{W_0}\)
2. Connected
3. Containing an Eulerian spanning subnetwork
4. Spanning (visits all regions)
Main Idea of PTAS

Use m-guillotine PTAS method, with new structure to address difficulty with TSPN

Structure Theorem

increasing length by $\leq (1+\varepsilon)$ factor

OPT $\rightarrow$ (m,M)-guillotine network with special structure

Use dynamic programming to compute shortest (m,M)-guillotine network with the required structure (connectivity, Eulerian subgraph, etc)

Optimal (m,M)-guillotine network with structure $\rightarrow$ TSPN tour
Theorem: Let $E$ be a connected set of edges of length $L$, spanning all regions. Then, for any positive integers $m$ and $M$, there is a superset, $E'$, of $E$, of length at most $L + (\sqrt{2}/m) L + (\sqrt{2}/M) \lambda(R_{w_0})$.

We pick $M = (1/\varepsilon) \log n$, and $m = 1/\varepsilon$.

Then, by the Key Lemma, we see that $T^*$ can be converted to be $(m,M)$-guillotine, adding length $O(T^*/\varepsilon)$.

Key Lemma: $L^* \geq C \lambda(R_{w_0}) / \log n$.
The sum of the perimeters of a set of n disjoint fat regions that are visited by a path of length L is at most $O(L \log n)$.

Uses PACKING argument

Ex: Bound is tight
**Key Lemma: Lower Bound on OPT**

**Key Lemma:** 
\[ L^* \geq C \frac{\lambda(R_{W_0})}{\log n} \]

Relates tour length of OPT, \( L^* \), to sum of diameters, \( \lambda(R_{W_0}) \).

**Ex: Tight**

\[ L^* = \Omega(\frac{\lambda(R_{W_0})}{\log n}) \]
Proof of Key Lemma

Cluster regions by size (diameter), into log \((n/\varepsilon)\) classes

There are \(n_i\) regions with diameter in range \((d_i/2,d_i)\)

Area (packing) argument:
Let \(A_i = \text{area}( (T^* \oplus B(d_i)) \cap W_0 )\)
Minkowski sum with ball of radius \(d_i\)

By fatness, \(A_i \geq C_0 d_i^2 n_i\), for some constant \(C_0\)

Thus, by \textbf{Claim} below, \(C_0 d_i^2 n_i \leq 2d_i L^*\), or \(L^* \geq (C_0/2) d_i n_i\)

Summing on \(i\), we get \(L^* \geq C \lambda(R_{W_0}) / \log n\)

\textbf{Claim}: \(A_i \leq 2d_i L^*\)
Bucketing Regions by Size

- Consider each minimal covering AAB, \( W_0 \), snapped to grid
- \( R_{W_0} \) = regions fully within \( W_0 \) (only \( O(n^4) \) of them)
- Partition \( R_{W_0} \) into \( K = O(\log(D/\delta)) = O(\log(n/\epsilon)) \) classes according to the diameters falling in ranges:
  - \((\delta, 2\delta)\)  \((2\delta, 4\delta)\)  \((4\delta, 8\delta)\)  ...  \((2^{i-1}\delta, 2^i\delta)\)  ...  \((2^{K-2}\delta, 2^{K-1}\delta)\)

\( \delta = \epsilon D/n \)

\( d_i = 2^i\delta \)

Can shrink to single grid points, by Grid Lemma
Proof of Claim

**Claim:** \( A_i = \text{area}( (T^* \oplus B(d_i)) \cap W_0 ) \leq 2d_i L^* \)

**Proof:**

\[
\text{area}( T^* \oplus B(d_i) ) \leq 2d_i L^* + \pi d_i^2
\]

That portion within \( W_0 \) does not include (at least) the area, \( \pi d_i^2 \)
**Main Result**

**Theorem:** TSPN for disjoint fat regions has a PTAS.

PTAS also for the case of non-disjoint regions, if there are disjoint disks $\beta_1, \ldots, \beta_n$, with $\beta_i \subset P_i$ and $\text{diam}(P_i)/\text{diam}(\beta_i) < C$.

Improve running time to $O(n^c)$, with $C$ independent of $1/\epsilon$: use grid-rounded guillotine subdivisions.
Generalizations/Extensions

- Disconnected regions: sets of points/regions that are within a “nice” set of regions
- $k$-TSPN
- Steiner MST with Neighborhoods
- MST with Neighborhoods (MSTN)
- $k$-MSTN
Approximation of 2D TSPN: Connected Regions

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<th>Fat Regions</th>
<th>non-Fat Regions</th>
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<td>O(log n)</td>
<td>APX-hard</td>
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Conjecture: PTAS for all

Conjecture: O(1) for all
Laundry List of Problems

- **Know PTAS**
  - TSP, k-TSP, Steiner
  - MST, k-MST
  - Red-blue separation
  - Min-weight convex subdivision
  - TSPN, fat regions
  - Orienteering problem
  - Lawnmowing problem

- **OPEN: PTAS?**
  - TSPN, disjoint regions in 2D
  - Vehicle routing; min-weight cover with k-tours
  - Deg-3, deg-4 spanning trees
  - Min-weight triangulation
  - Watchman route problem
  - Min-area triangulated surface; special case: terrain