Bivariate B-Splines From Centroid Triangulations

Yuanxin Liu, Jack Snoeyink
UNC Chapel Hill
Motivating Questions

Comput’l Geometry:
“PL surface meshes can be constructed from (irregular) points by triangulating.

What about smooth surfaces?”

CAGD:
“Smooth B-splines can be constructed over (irregular) points along the real line.

How do we make bivariate B-splines?”

A?
centroid triangulations, a generalization of higher order Voronoi duals.
Outline

• Context & Motivation

• Background concepts
  – B-splines
  – Simplex splines
  – Neamtu's B-splines from higher-order Delaunay configurations

• Generalizing to centroid triangulations
  – By generalizing the dual of D.T. Lee's construction of higher order Voronoi

• An application to blending
  – Reproducing box splines
Univariate B-splines

- **splines**: piecewise polynomials

- **B-spline space**: linear combination of basis functions

- A B-spline of deg. $k$ is defined for any $k+2$ knots.
Univariate B-splines

Properties

- *local support*
- *optimal smoothness*
- *partition of unity_*
  \[ \sum B_i = 1 \]
- *polynomial reproduction,*
  for any deg. \( k \) polynomial \( p \),
  with polar form \( P \),
  \[ p = \sum P(S_{i+1} .. S_{i+k}) B_i(., | S_i .. S_{i+k+1}) \]
What are multivariate splines?

Are they B-splines?

- tensor product
- subdivision
- box splines
What are multivariate B-splines?

(multivariate) B-splines should define basis functions with no restriction on knot positions and have these properties of the classic B-splines:

- local support
- optimal smoothness
- partition of unity \( \Sigma B_i = 1 \)
- polynomial reproduction:
  for any degree \( k \) polynomial \( p \), with polar form \( P \),
  \[
p = \Sigma P(S_{i+1} \ldots S_{i+k}) B_i(\cdot| S_i \ldots S_{i+k+1})
  \]
Simplex spline [dB76]

A degree $k$ polynomial defined on a set $X$ of $k+s+1$ points in $\mathbb{R}^s$.

Lift $X$ to $Y \in \mathbb{R}^{k+s}$ and take relative measure of the projection of this simplex:

$$M(x | X) := \frac{\text{vol} \{ y \mid y \in [Y] \text{ and projects to } x \}}{\text{vol} \{ [Y] \}}$$
Simplex spline

Properties

✓ local support: $M(\cdot | X)$ is non-zero only over the convex hull of $X$.

✓ optimally smooth, assuming $X$ is in general position.
What are multivariate B-splines?

Using simplex splines as basis:

The task of building multivariate B-splines becomes choosing the “right” configurations.

- All k-tuple configs [Dahmen & Micchelli 83]
- DMS-splines [Dahmen, Micchelli & Seidel 92]
- Delaunay configurations [Neamtu 01]
Neamtu's Delaunay configurations

A degree $k$ Delaunay configuration $(t, I)$ is defined by a circle through $t$ containing $I$ inside.

$$\Gamma^2_{\text{Del}} = \{ (aef, bc), (def, bc), (bef, cd) \ldots \}$$
Delaunay configurations

Spline space for $\Gamma^k_{\text{Del}}$:
$$\text{span}\{ d(t) M(\cdot | t \cup I) \} \quad (t,I) \in \Gamma^k_{\text{Del}}$$

Properties

✓ polynomial reproduction:

for any deg. $k$ polynomial $p$,
$$p = \sum (t,I) \in \Gamma^k_{\text{Del}} P(I) d(t) M(\cdot | t \cup U \cup I)$$
Voronoi/Delaunay diagrams
the classic (order 1)
Voronoi/Delaunay diagrams

order 2
An order k Voronoi diagram has two types of vertices, close and far, that correspond to circles with k-1 or k-2 points inside. [Lee 82, Aurenhammer 91] Delaunay triangles for these vertices may overlap, …

Delaunay triangles for these vertices may overlap, …
Voronoi/Delaunay diagrams

order 2

but transforming them gives a triangulation

Two ways to get the centroid triangulation:
- Project the lower hull of the centroids of all $k$-subsets of the lifted sites. [Aurenhammer 91]
- Map Delaunay configurations to centroid triangles. [Schmitt 95, Andrzejak 97]
Centroid triangulations

Dualizing Lee’s algorithm to compute order $k$ Voronoi diagrams

0. Begin w/ triangulation where every vertex has a unique label. Invariant: in our triangulation each vertex is the centroid of $k$ points that make its label, and each edge joins vertices whose union has $k+1$ labels.

1. For each edge, create the vertex that is its union.

2. For each triangle, join the three vertices created from its three edges.

3. Discard original triangles & vertices

4. Complete to triangulation:
Centroid triangulations

Dualizing Lee’s algorithm to compute order k Voronoi diagrams

Open problem: How do we guarantee that this algorithm works beyond order 3?
What are multivariate B-splines?

(multivariate) B-splines should define basis functions with no restriction on knot positions and have these properties of the classic B-splines:

- local support
- optimal smoothness
- partition of unity \( \sum B_i = 1 \)
- polynomial reproduction:
  for any degree k polynomial \( p \), with polar form \( P \),
  \[
p = \sum P(S_{i+1} \ldots S_{i+k}) B_i(.| S_i \ldots S_{i+k+1})
\]
Centroid triangulations

Delaunay configuration of degree $k$: $(t, I)$, s.t. the circle through $t$ contains exactly the $k$ points of $I$.

Centroid triangle of order $k$: $[A_{1..k}, B_{1..k}$ and $C_{1..k}]$, s.t. $\#(A \cap B) = \#(B \cap C) = \#(A \cap C) = k-1$.

Map Delaunay configurations of deg. $k-1$, $k-2$ to centroid triangles of order $k$:

$(abc, J) \leftrightarrow [JU\{a\}, JU\{b\}, JU\{c\}]$
- deg. $k-1$ type 1

$(abc, I) \leftrightarrow [IU\{b,c\}, IU\{a,c\}, IU\{b,c\}]$
- deg. $k-2$ type 2
Neamtu's use of Delaunay configs.

Key property for proof of polynomial repro. is boundary matching:
Centroid Triangulations
relations of conf. <-> triangle neighbors

**Obs** If $\Gamma^{k-1}$ and $\Gamma^k$ form a planar centroid triangulation, then they satisfy the boundary matching property.

**Problem** Given $\Gamma^{k-1}$ and $\Gamma^k$ that form $\Delta^k$, can we find $\Gamma^{k+1}$ so that $\Gamma^k$ and $\Gamma^{k+1}$ form $\Delta^{k+1}$?
Centroid Triangulations

For a set of sites $S$, a centroid triangle of order $k$ have vertices that are centroids of $k$-subsets of $S$, e.g. $A_{1..k}, B_{1..k}$ and $C_{1..k}$, and satisfy that

$$\#(A \cap B) = \#(B \cap C) = \#(A \cap C) = k-1.$$  

**Obs**  
$$\#(A \cap B \cap C) = k-1 ~ \text{or} ~ k-2.$$  

**Obs**  
There is a 1-1 mapping between the centroid triangles of order $k$ and conf of degree $k-1$ and $k-2$.

$$(abc, J) \leftrightarrow [JU\{a\}, JU\{b\}, JU\{c\}],$$

$$(abc, I) \leftrightarrow [IU\{b,c\}, IU\{a,c\}, IU\{b,c\}]$$
Theorem

Let $\Gamma^0..\Gamma^k$ be a sequence of configurations. If $\Gamma^0$ is a triangulation, and $\Gamma^{i-1}, \Gamma^i$ form a centroid triangulation for $0<i<k$, then the simplex splines assoc. with $\Gamma^k$ reproduce polynomials of deg. k.

- for any deg. k polynomial $p$, with polar form $P$
  
  $p = \sum P(I) d(t) M(.| t \cup I )$

  $(t,I) \in \Gamma^k$

classic B-spline

- for any deg. k polynomial $p$, with polar form $P$
  
  $p = \sum P(S_{i+1}..S_{i+k}) B_i(.| S_i..S_{i+k+1})$
Reproducing the Zwart-Powell element

• An example of our claim: *Box splines are special centroid triangulation splines.*

• Theory motivation: Evidence that 'ct's provide general basis for bivariate splines.

• Practical motivation: Smooth blending of box spline patches.
Polyhedral splines

**Polyhedron spline** \( M_{\Pi}(x | P) \)

- \( P \): a polyhedron in \( \mathbb{R}^n \)
- \( \Pi \): a projection matrix from \( \mathbb{R}^n \) to \( \mathbb{R}^m \)

is an \( n \)-variate, degree \((n-m)\) spline

\[
M_{\Pi}(x | P) := \frac{\text{vol} \{ y \mid y \in P, \Pi y = x \}}{\text{vol} P}
\]

**Box Spline**

\( M_{\Pi}(x) \)
Reproduction of Box splines

\[ \text{span} \{ M_{\Pi}(x + v) \}_{v \in \mathbb{N} \times \mathbb{N}} \subset \text{span} \{ M(x | X) \}_{X \in \Gamma^k} \]

ZP element

order 2 centroid triangulation

\[ \Pi = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix} = \{ \rightarrow \uparrow \swarrow \down\} \]
Reproduction of Box splines

Proof sketch (reproducing a single ZP element)

ZP-element
4-cube
(partition)
4-polytopes
(triangulate)
4-simplices
simplex splines
Reproduction of Box-splines

- Patch blending
  E.g., a partition of unity by blending regular splines.
Modeling sharp features

Data fitting with scattered data points and “break lines”
Modeling sharp features
Data fitting with scattered data points
and “break lines”
Modeling sharp features
Data fitting with scattered data points and “break lines”
Open Problems

- Prove no self-intersecting holes arise in the centroid triangulation algorithm
- Reproduce other box-splines by centroid triangulation
  - bilinear interpolation (quadratic)
  - loop subdivision (quartic)