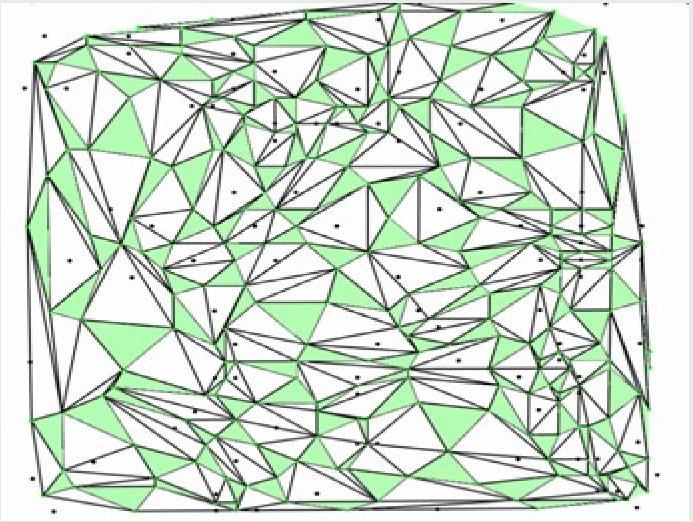
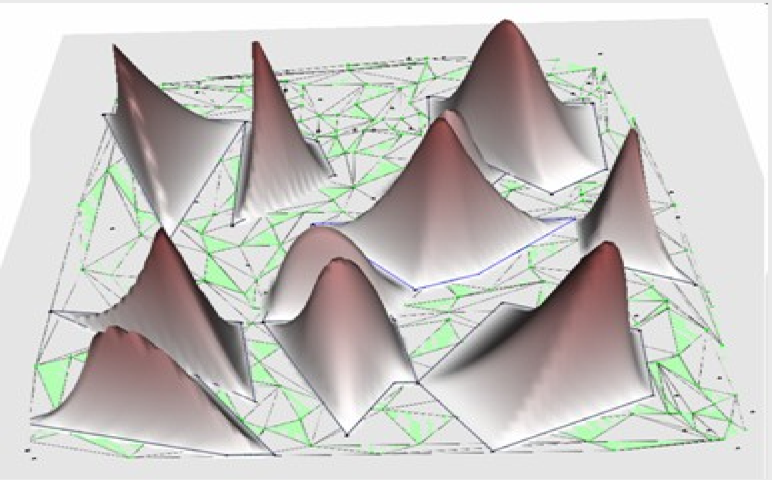
Bivariate B-Splines From Centroid Triangulations





Yuanxin Liu, Jack Snoeyink
UNC Chapel Hill

Q Motivating Questions

Comput'l Geomety:

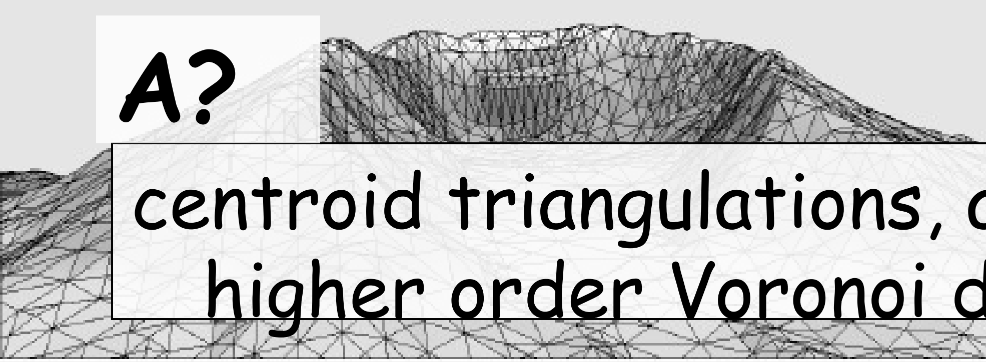
"PL surface meshes can be constructed from (irregular) points by triangulating.

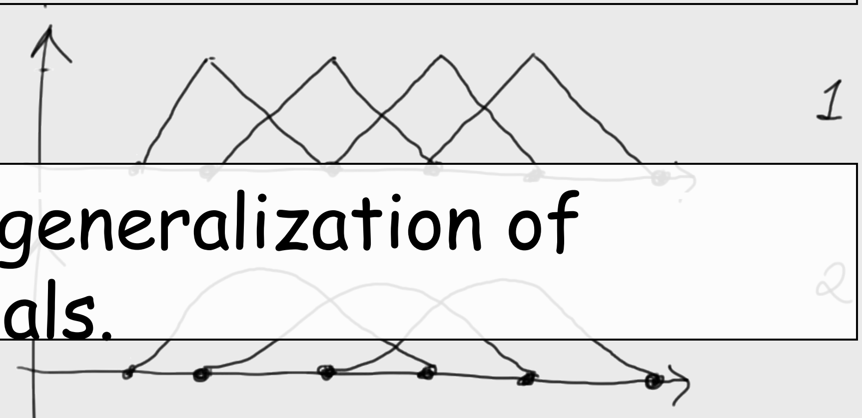
What about smooth surfaces?"

CAGD:

"Smooth B-splines can be constructed over (irregular) points along the real line.

How do we make bivariate B-splines?"

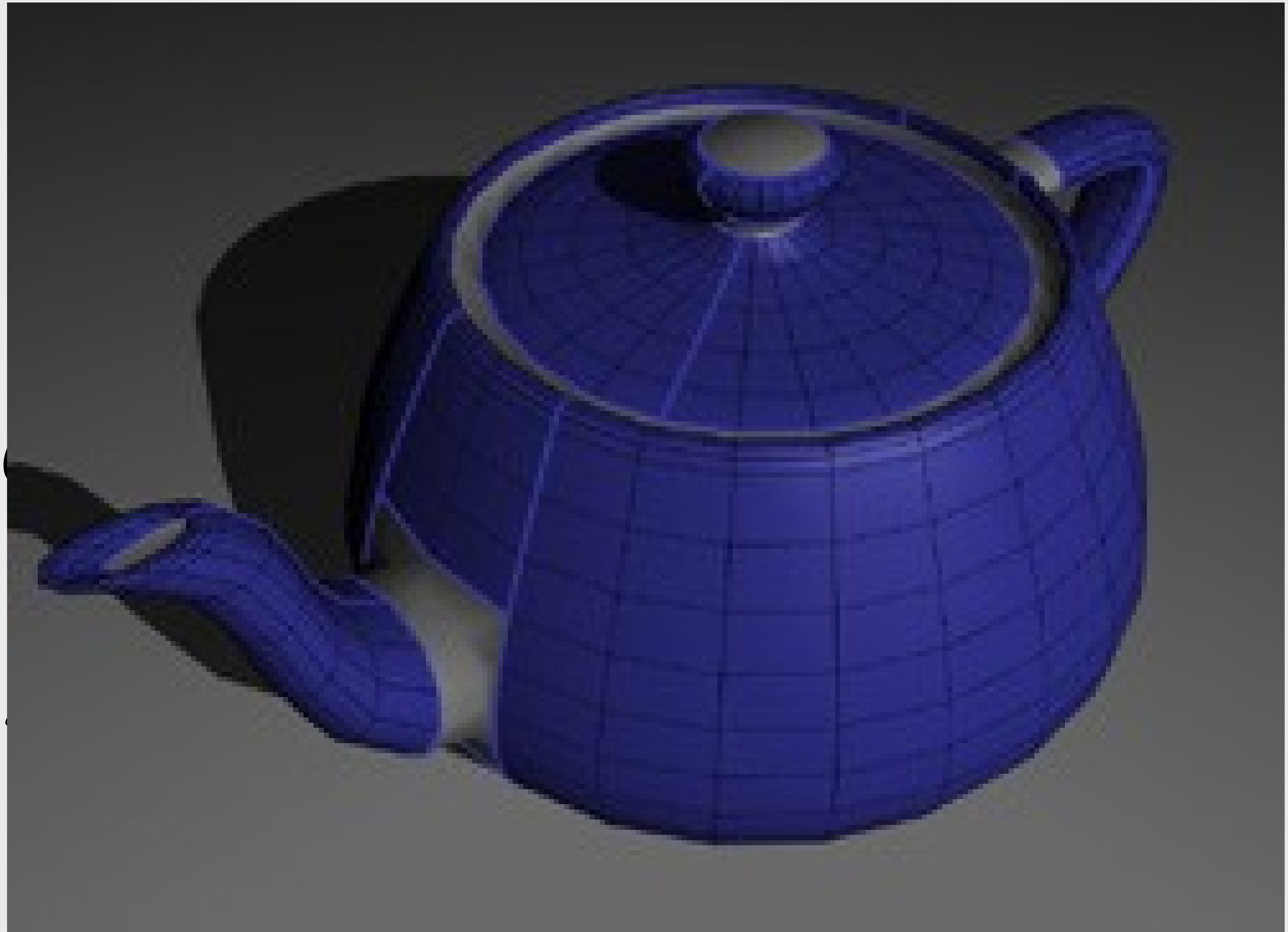




A?

centroid triangulations, a generalization of higher order Voronoi duals.

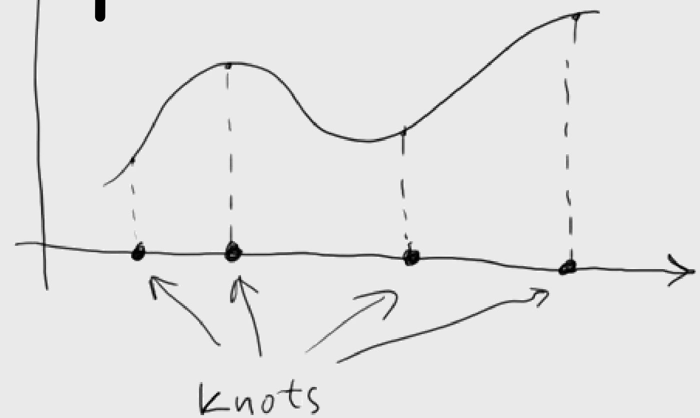
Outline

- Context & Motivation
-


-
-

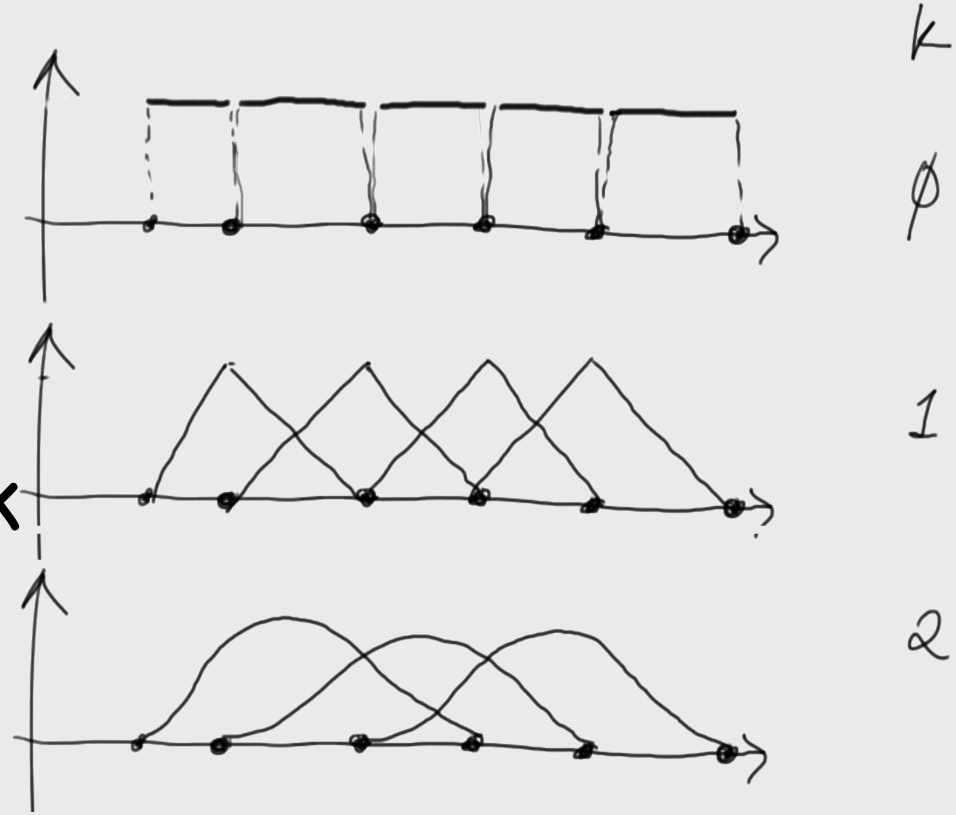
Univariate B-splines

- *splines* :
  piecewise polynomials

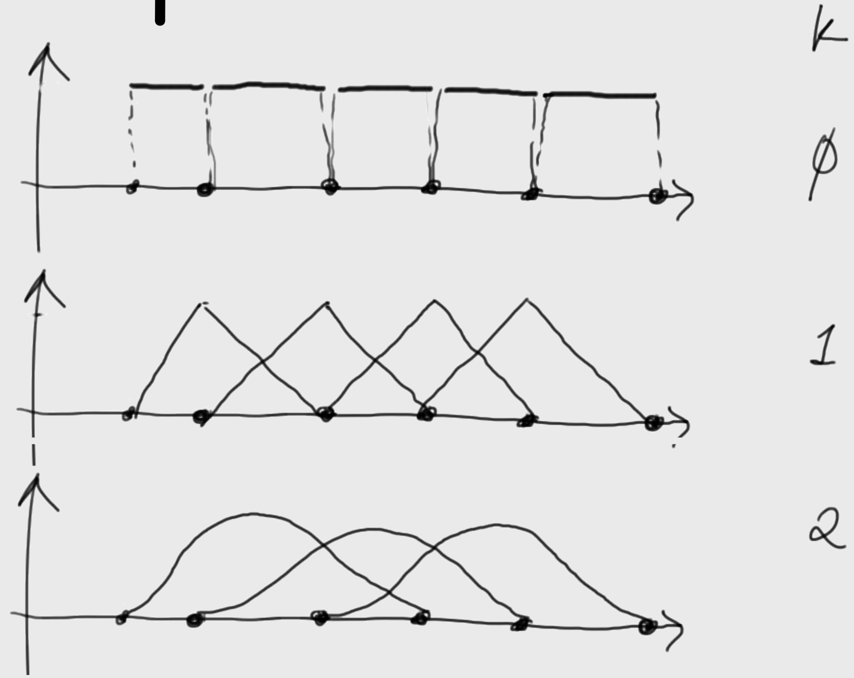


- *B-spline space*:
  linear combination
  of basis functions

- *A B-spline* of deg. k
  is defined for
  any k+2 knots.



Univariate B-splines



<u>Properties</u>

- *local support*
- *optimal smoothness*
- *partition of unity*

  $\Sigma B_i = 1$

- *polynomial reproduction,*

  for any deg. k polynomial p,

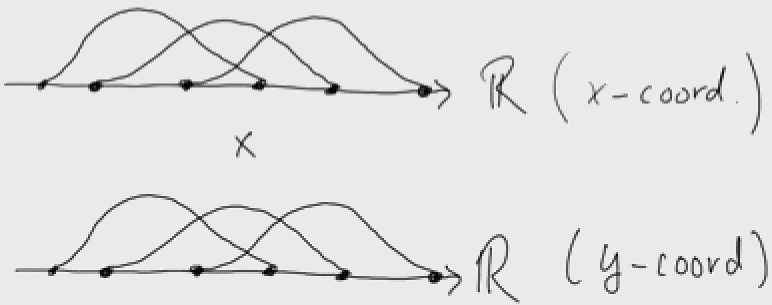
  with polar form P,

  $$p = \Sigma P(S_{i+1} .. S_{i+k})\, B_i(.|\, S_i .. S_{i+k+1})$$

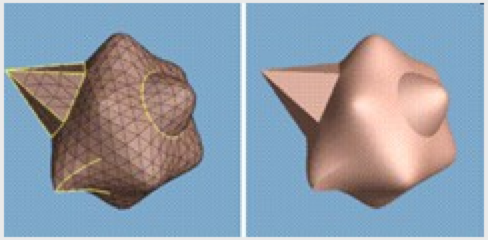
What are multivariate splines?

Are they B-splines?

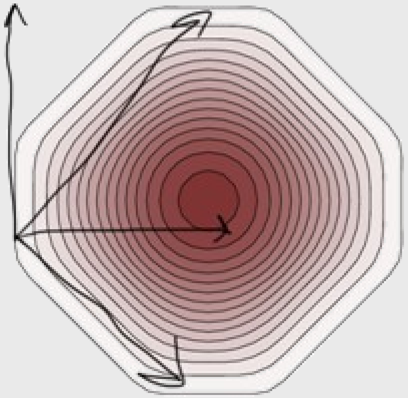
- tensor product



- subdivision



- box splines



What are multivariate B-splines?

(multivariate) B-splines should define basis functions with no restriction on knot positions and have these properties of the classic B-splines:

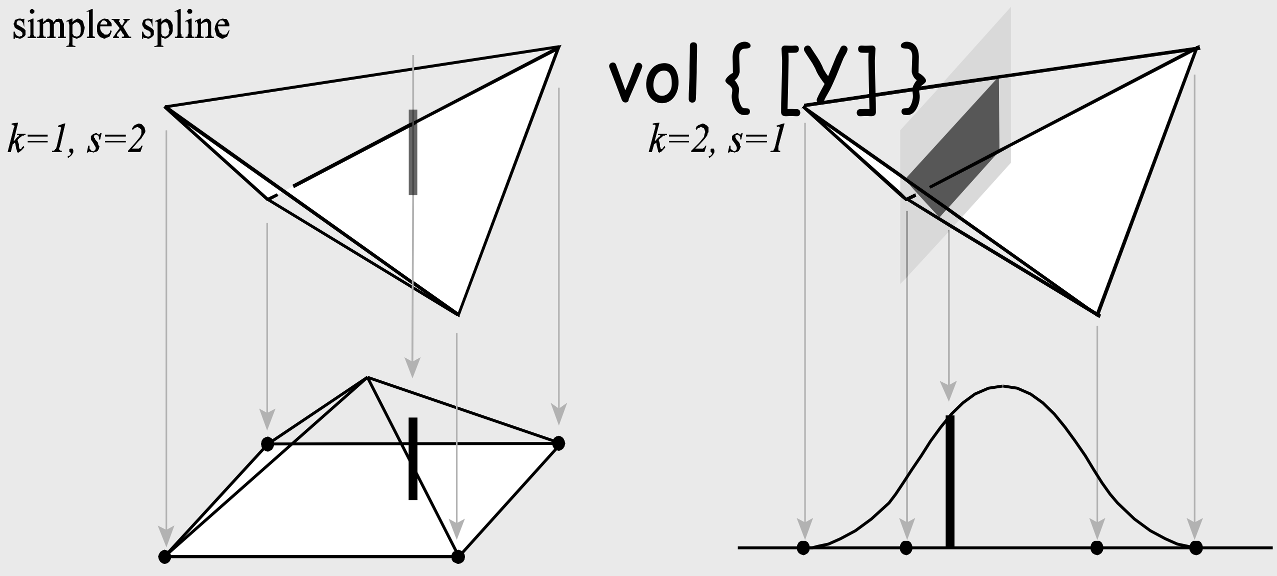
- local support
- optimal smoothness
- partition of unity $\Sigma B_i = 1$
- polynomial reproduction:
  for any degree k polynomial p,
  with polar form P,
  $p = \Sigma P(S_{i+1} .. S_{i+k}) B_i(. | S_i .. S_{i+k+1})$

Simplex spline [dB76]

A degree k polynomial defined on a set X of k+s+1 points in R^s.

Lift X to $Y \in R^{k+s}$ and take relative measure of the projection of this simplex:

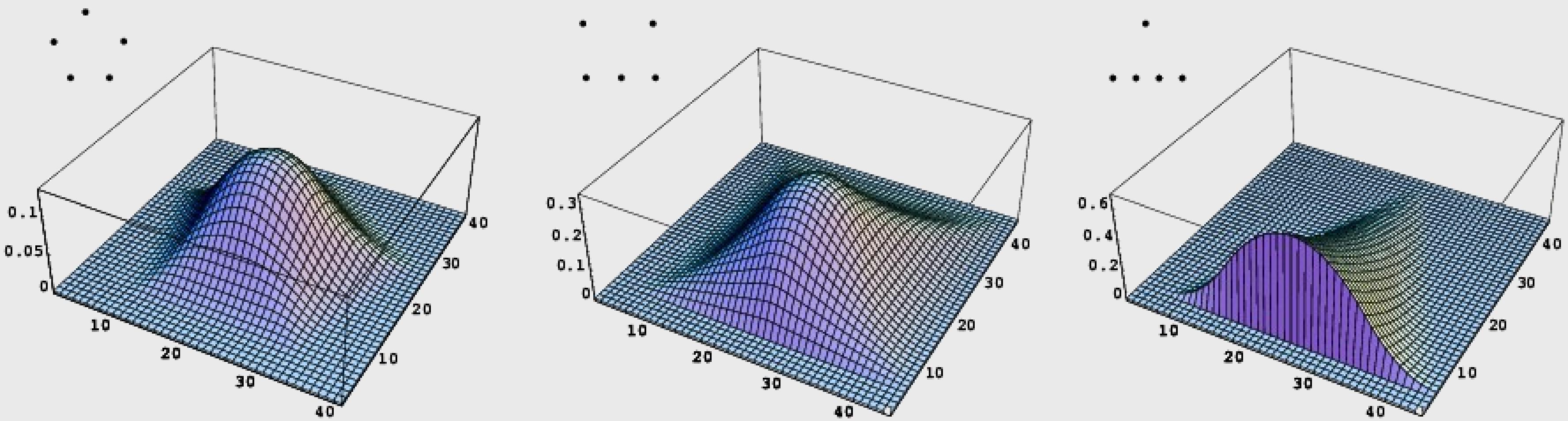
$M(\ x \mid X\ ) :=$ <u>vol { y | y ∈ [Y] and projects to x}</u>



Simplex spline

Properties

- ✓ local support: $M(\cdot|X)$ is non-zero only over the convex hull of X.
- ✓ optimally smooth, assuming X is in general position.



What are multivariate B-splines?

Using simplex splines as basis:

The task of building multivariate B-splines becomes choosing the “right” configurations.

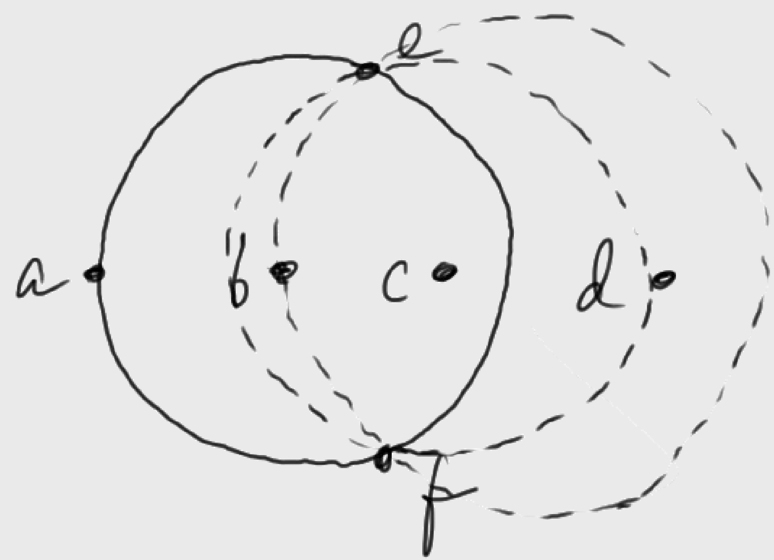
- All k-tuple configs [Dahmen & Micchelli 83]
- DMS-splines [Dahmen, Micchelli & Seidel 92]
- Delaunay configurations [Neamtu 01]

Neamtu's Delaunay configurations

A degree k Delaunay configuration (t, I) is defined by a circle through t containing I inside.

Γ^2_{Del}

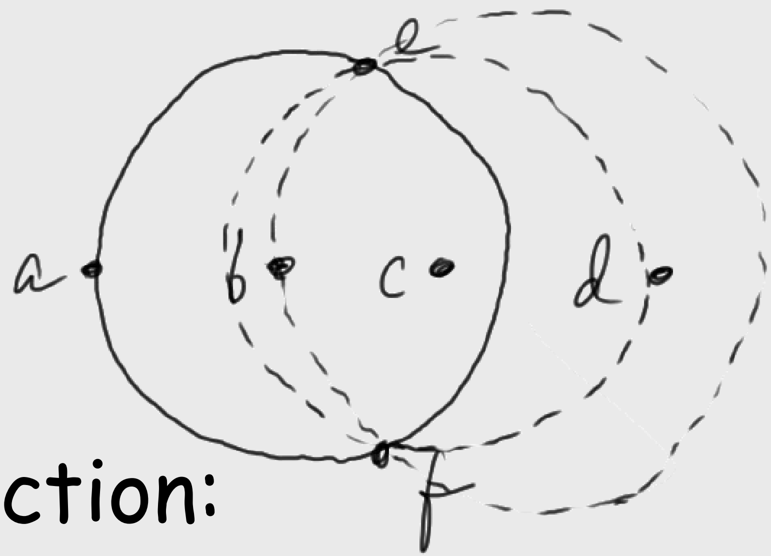
= { (aef, bc),
(def, bc),
(bef, cd)... }



Delaunay configurations

Spline space for Γ^k_{Del}:

$$\mathrm{span}\{\, d(t)\, M(\,\cdot \mid t \cup I)\,\}_{(t,I) \in \Gamma^k_{Del}}$$



Properties

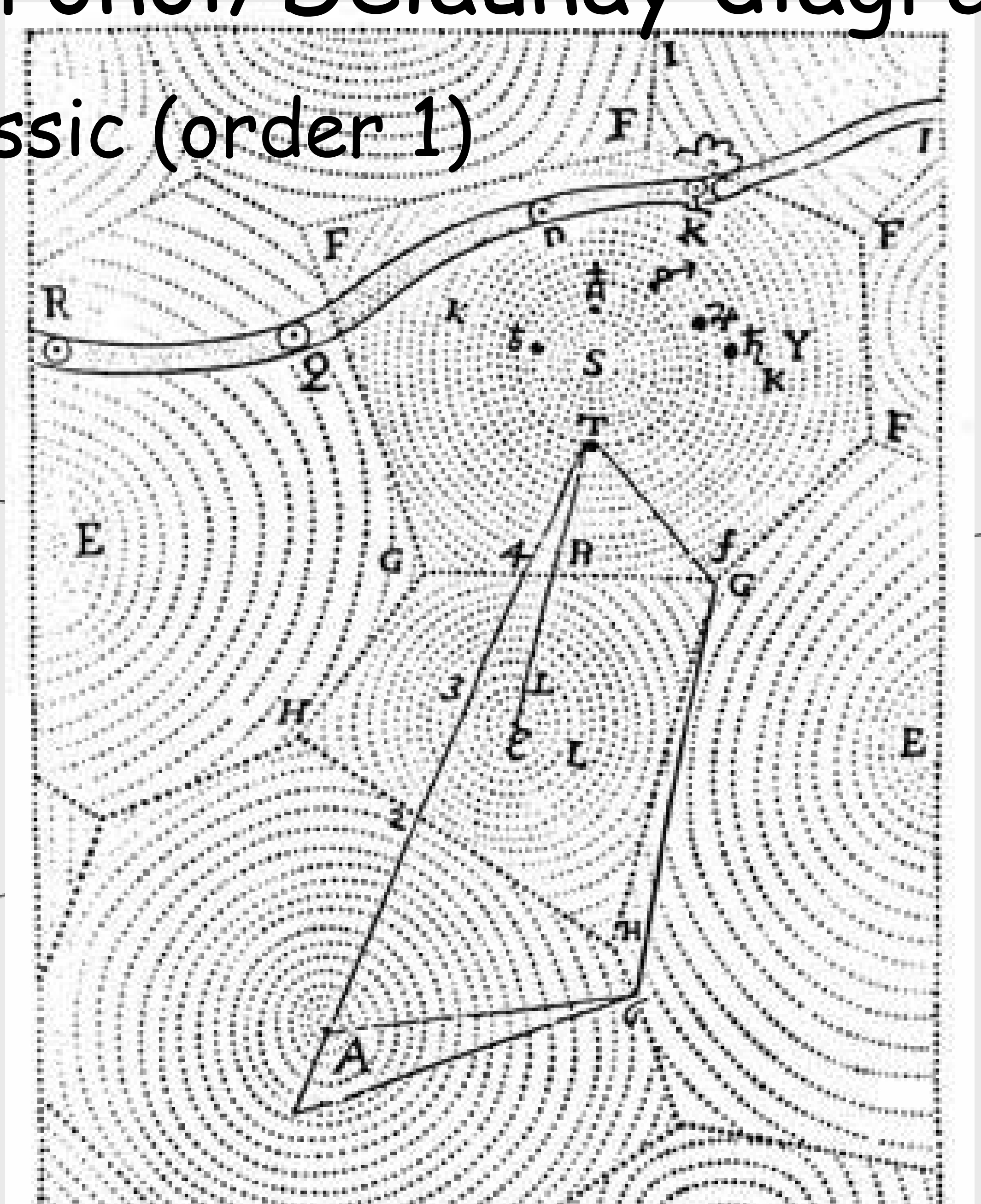
✓ polynomial reproduction:

for any deg. k polynomial p,

$$p = \sum_{(t,I)\in \Gamma^k_{Del}} P(I)\, d(t)\, M(.\mid t \cup I\,)$$

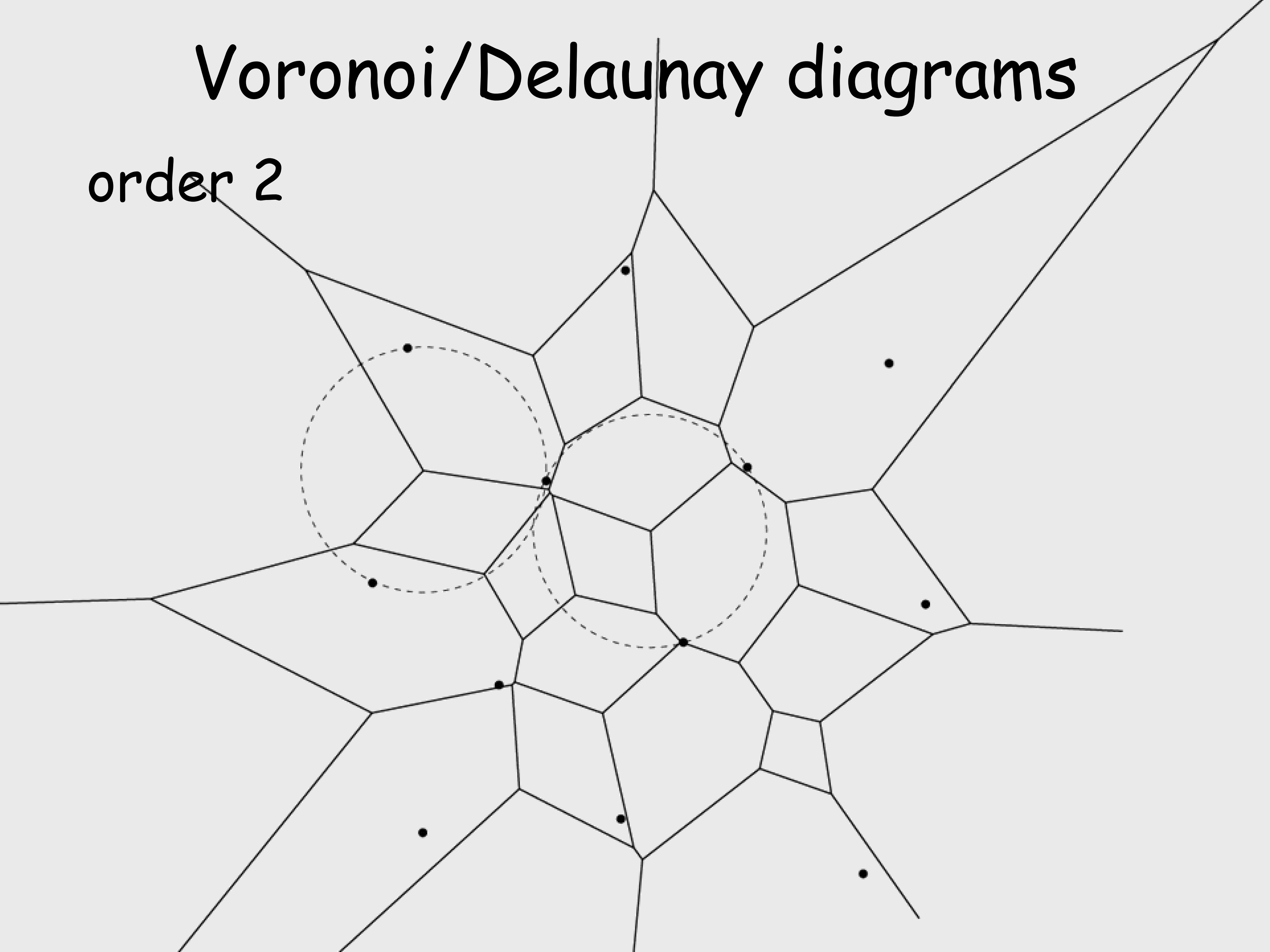
Voronoi/Delaunay diagrams

the classic (order 1)



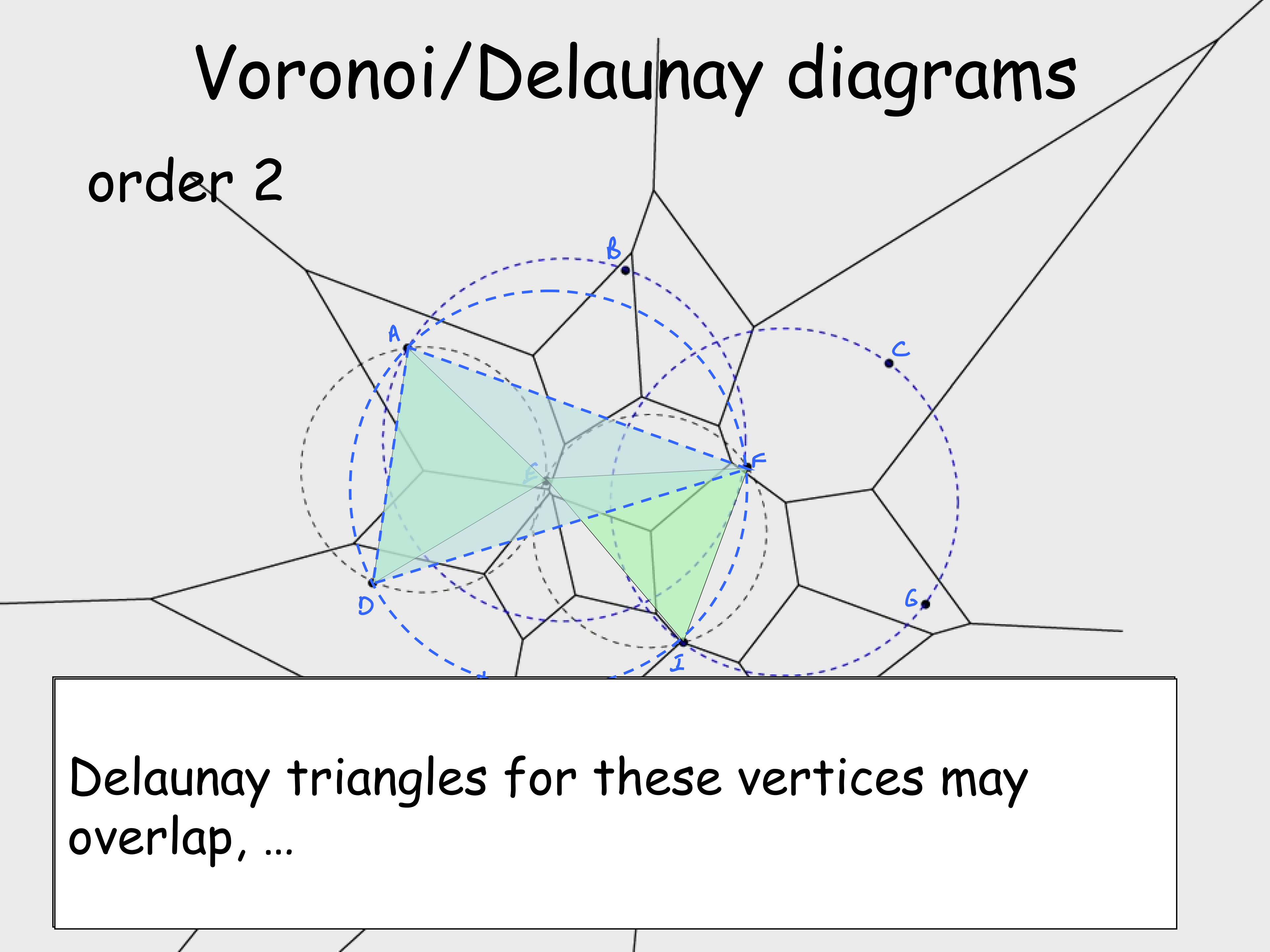
Voronoi/Delaunay diagrams

order 2



Voronoi/Delaunay diagrams

order 2

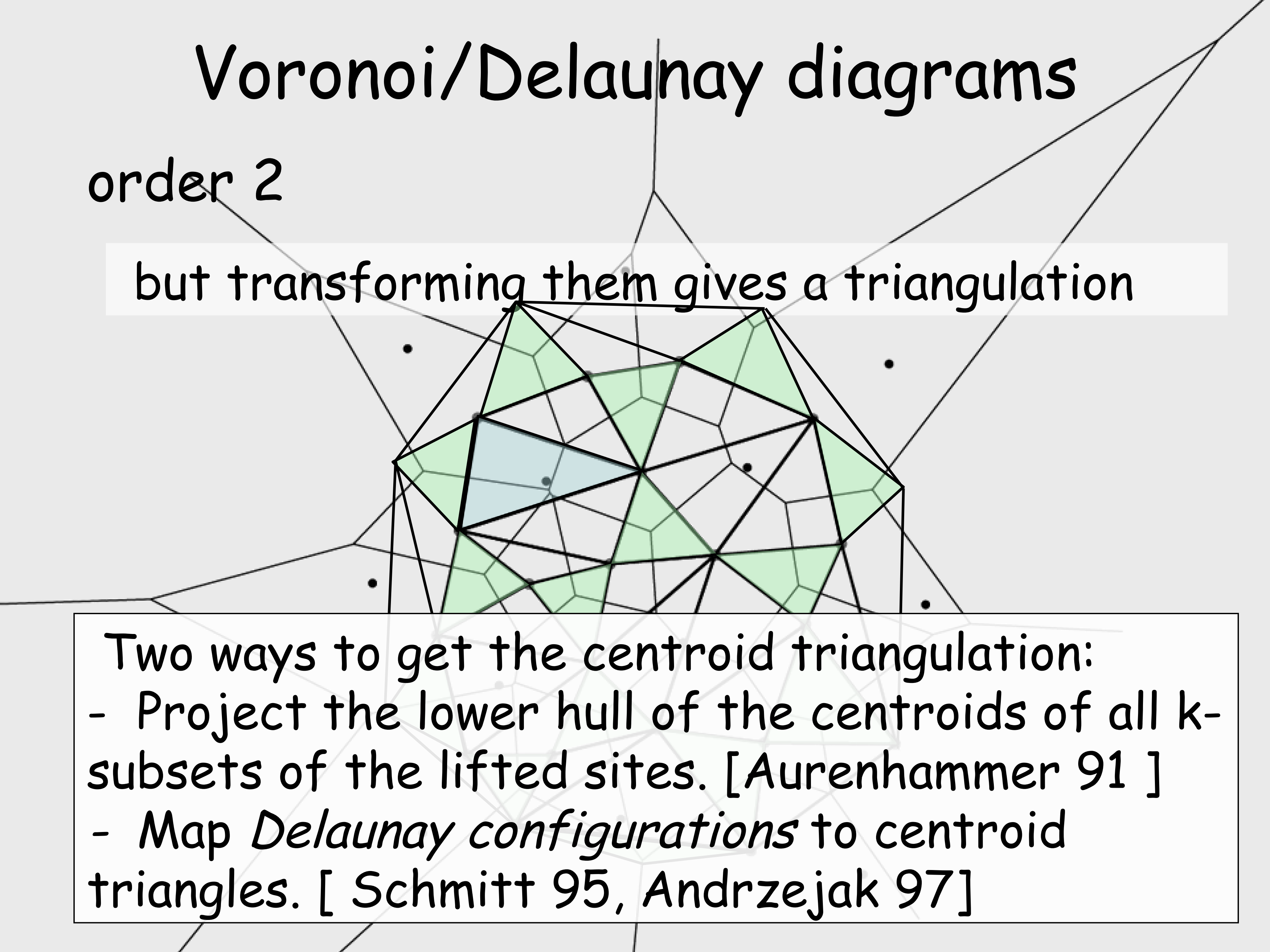


Delaunay triangles for these vertices may overlap, ...

Voronoi/Delaunay diagrams

order 2

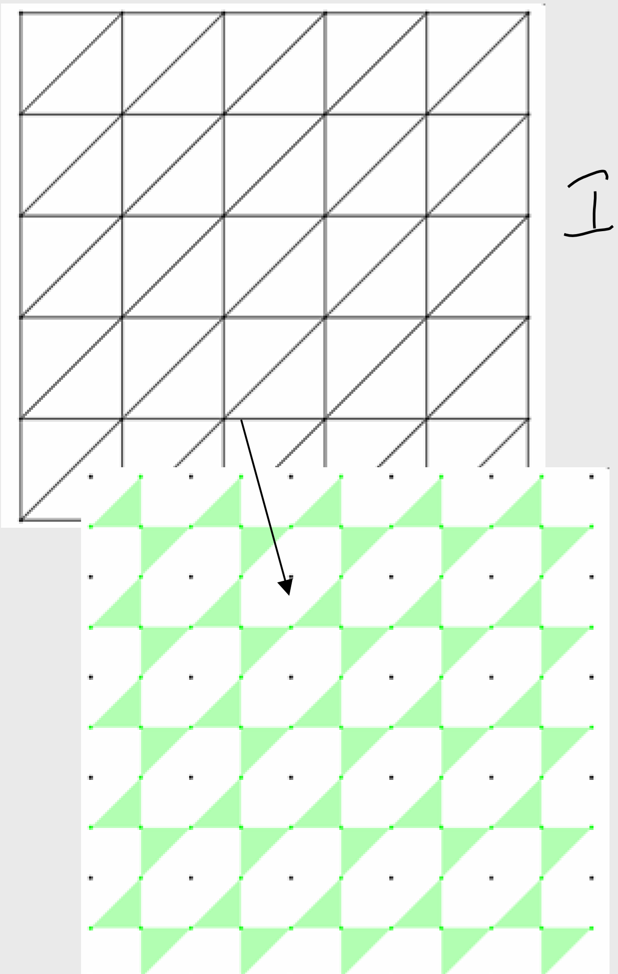
but transforming them gives a triangulation



Two ways to get the centroid triangulation:
- Project the lower hull of the centroids of all k-subsets of the lifted sites. [Aurenhammer 91]
- Map *Delaunay configurations* to centroid triangles. [Schmitt 95, Andrzejak 97]

Centroid triangulations

Dualizing Lee's algorithm to compute order k Voronoi diagrams



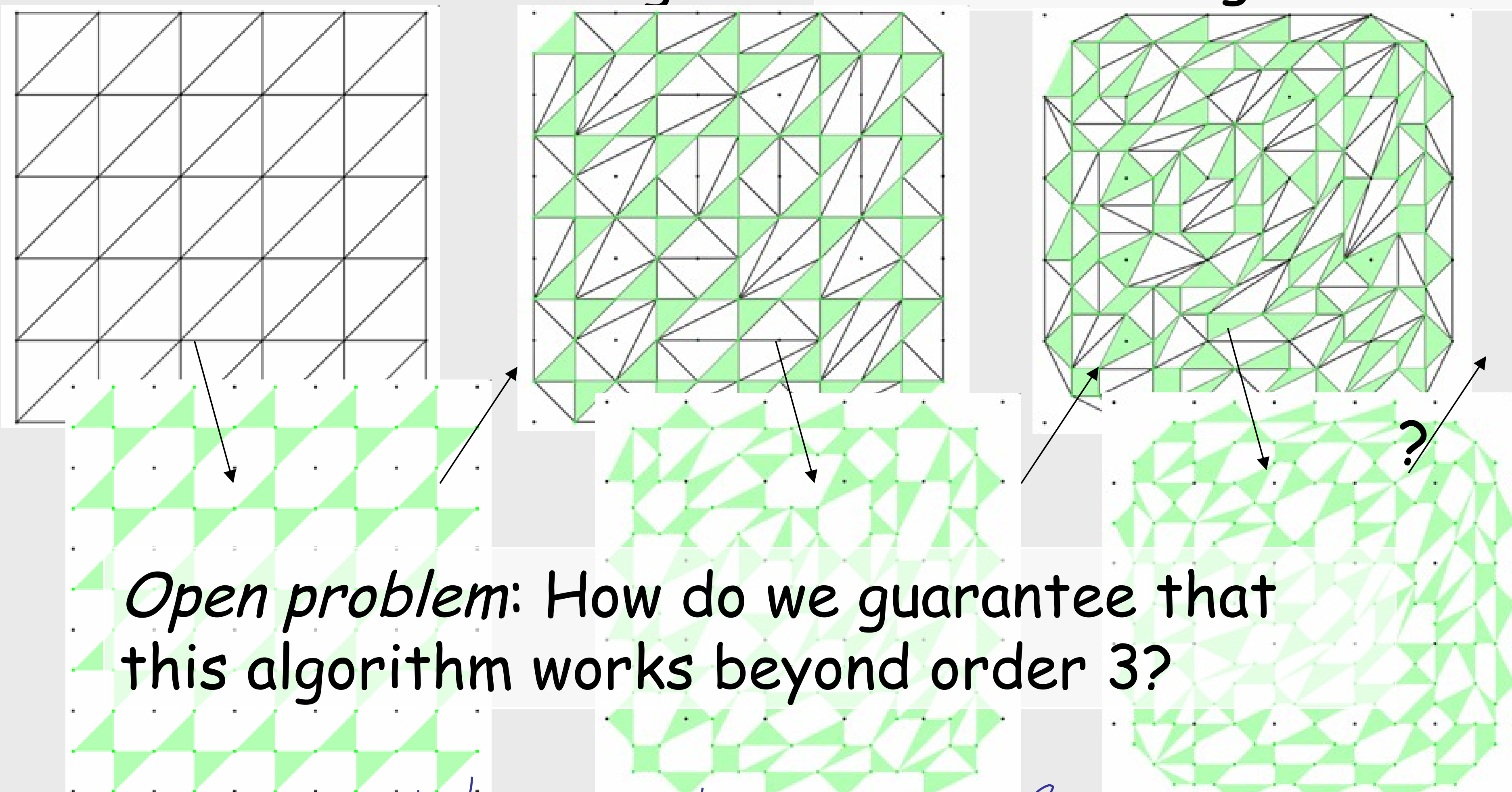
0. Begin w/ triangulation where every vertex has a unique label.

Invariant: in our triangulation each vertex is the centroid of k points that make its label, and each edge joins vertices whose union has k+1 labels.

1. For each edge, create the vertex that is its union.
2. For each triangle, join the three vertices created from its three edges
3. Discard original triangles & vertices
4. Complete to triangulation:

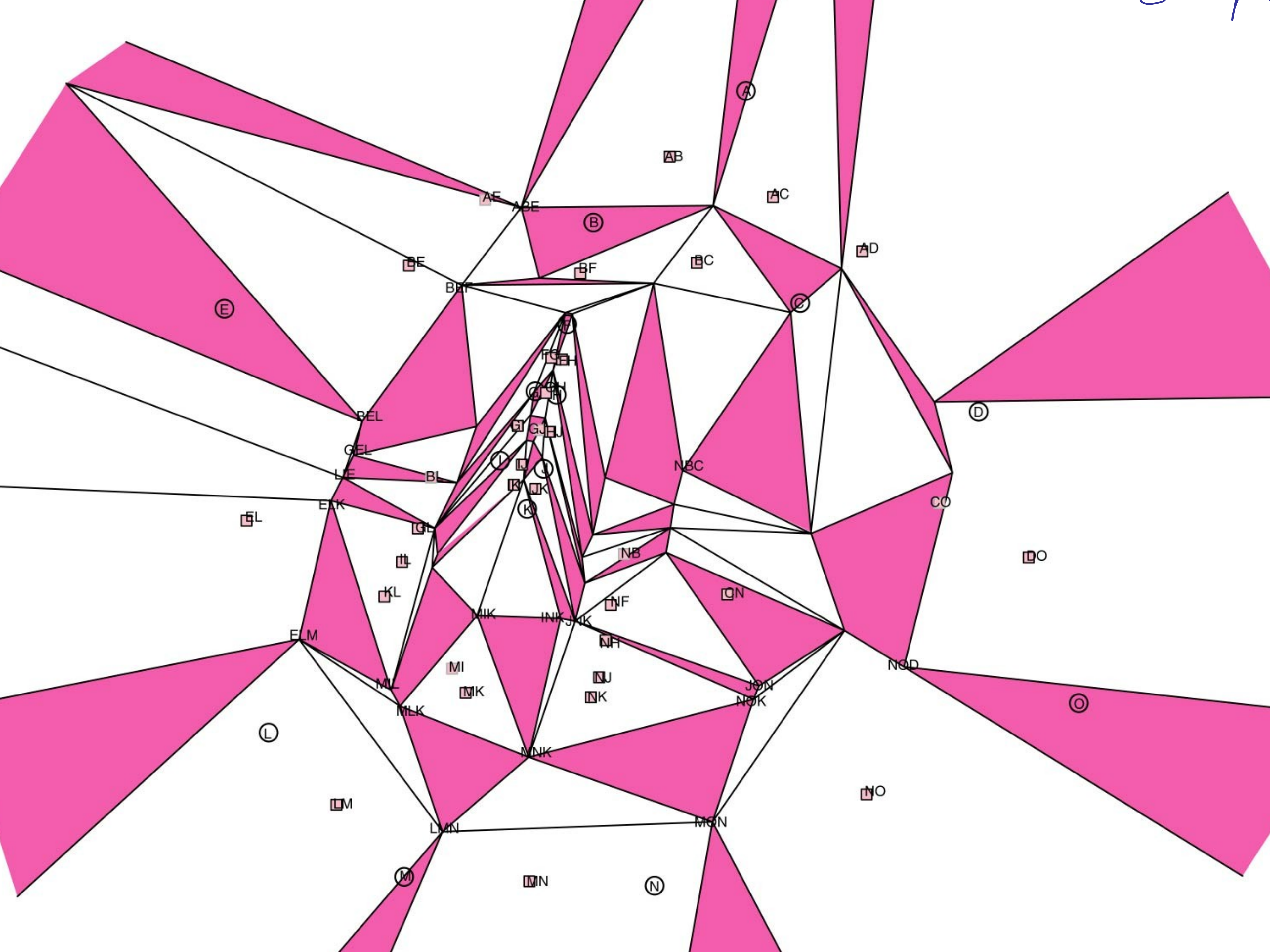
Centroid triangulations

Dualizing Lee's algorithm to compute order k ~~Voronoi diagrams~~ centroid triangulations



Open problem: How do we guarantee that this algorithm works beyond order 3?

are holes simple polygons? (no self intersection)



A
AB
AC
AD
AE
ABE
B
BE
BF
BC
BEF
C
E
F
FG
FH
G
GH
H
GI
GJ
HJ
I
IJ
J
IK
JK
K
BEL
GEL
LE
BL
ELK
EL
GL
IL
KL
NBC
D
CO
DO
NB
CN
NF
MIK
INK
JNK
NH
NJ
NK
ELM
ML
MLK
MI
MK
JON
NOK
NOD
O
L
MNK
LM
LMN
MON
NO
M
MN
N

What are multivariate B-splines?

(multivariate) B-splines should define basis functions with no restriction on knot positions and have these properties of the classic B-splines:

- local support
- optimal smoothness
- partition of unity $\Sigma B_i = 1$
- polynomial reproduction:
  for any degree k polynomial p,
  with polar form P,
  $p = \Sigma P(S_{i+1} .. S_{i+k})\, B_i(.|\, S_i .. S_{i+k+1})$

Centroid triangulations

Delaunay configuration of degree k: (t, I), s.t. the circle through t contains exactly the k points of I.

Centroid triangle of order k: [$A_{1..k}$,$B_{1..k}$ and $C_{1..k}$], s.t. $\#(A \cap B) = \#(B \cap C) = \#(A \cap C) = k\text{-}1$.

Map Delaunay configurations of deg. k-1, k-2 to centroid triangles of order k:

(abc, J) <-> [JU{a}, JU{b}, JU{c}]

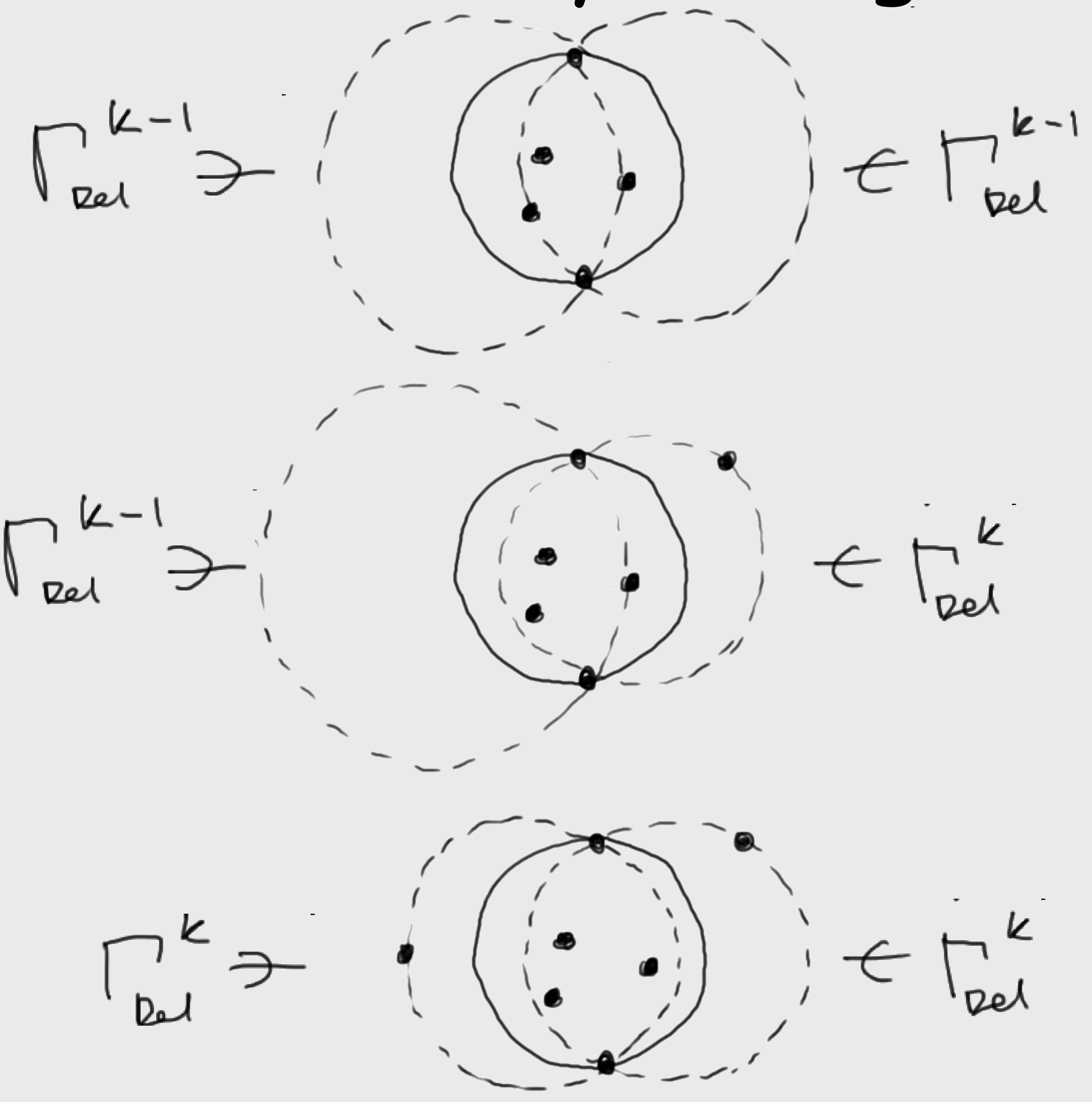
deg. k-1 type 1

(abc, I) <-> [IU{b,c}, IU{a,c}, IU{b,c}]

deg. k-2 type 2

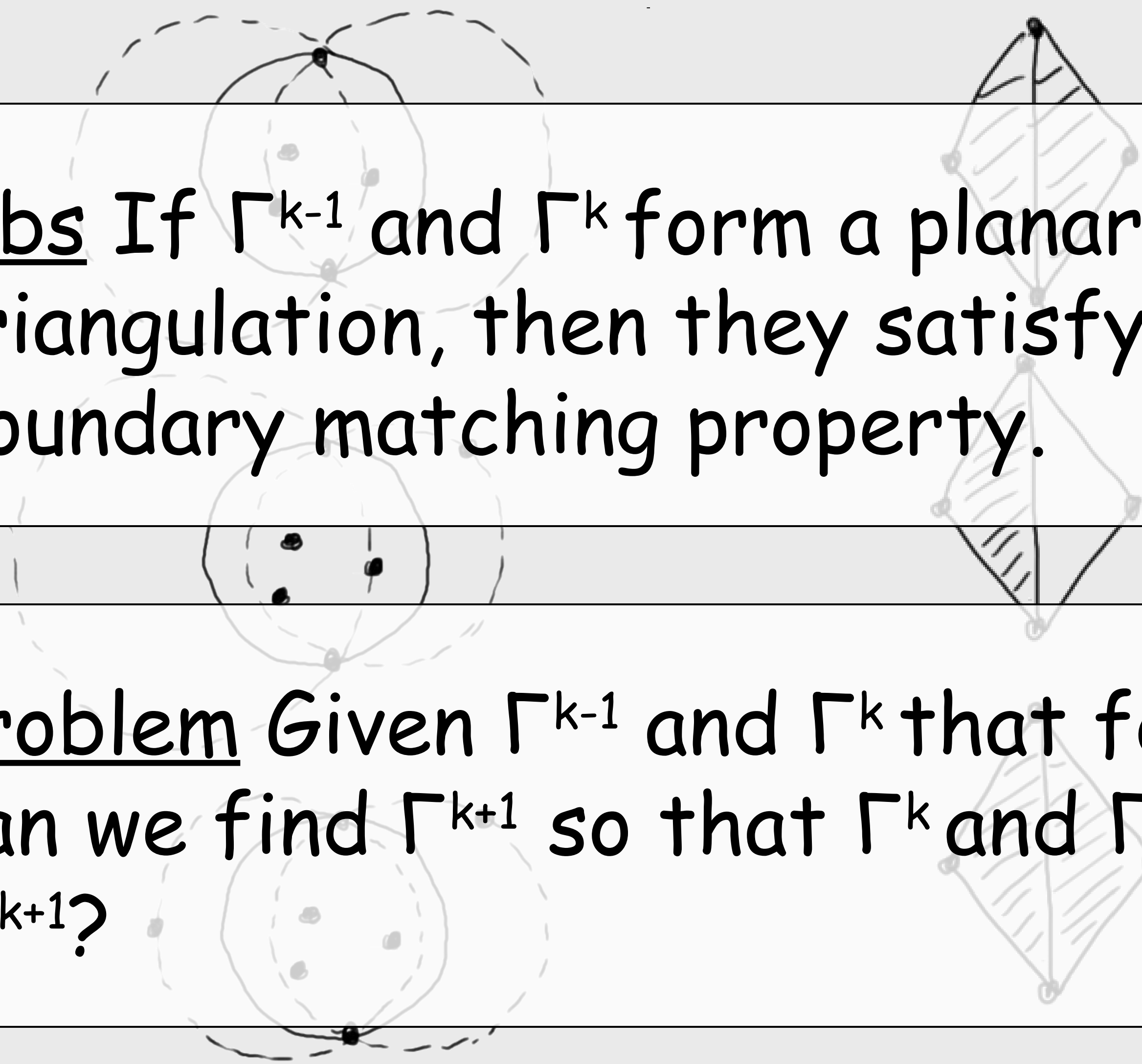
Neamtu's use of Delaunay configs.

Key property for proof of polynomial repro. is *boundary matching*:



Centroid Triangulations

relations of conf. <-> triangle neighbors



<u>Obs</u> If Γ^{k-1} and Γ^k form a planar centroid triangulation, then they satisfy the boundary matching property.

<u>Problem</u> Given Γ^{k-1} and Γ^k that form Δ^k, can we find Γ^{k+1} so that Γ^k and Γ^{k+1} form Δ^{k+1}?

Centroid Triangulations

For a set of sites S, a *centroid triangle* of order k have vertices that are centroids of k-subsets of S, e.g. $A_{1..k}$,$B_{1..k}$ and $C_{1..k}$, and satisfy that

$$\#(A \cap B) = \#(B \cap C) = \#(A \cap C) = k-1.$$

<u>Obs</u> $\#(A \cap B \cap C) = k-1$ or $k-2$.

type 1 type 2

<u>Obs</u> There is a 1-1 mapping between the centroid triangles of order k and conf of degree k-1 and k-2.

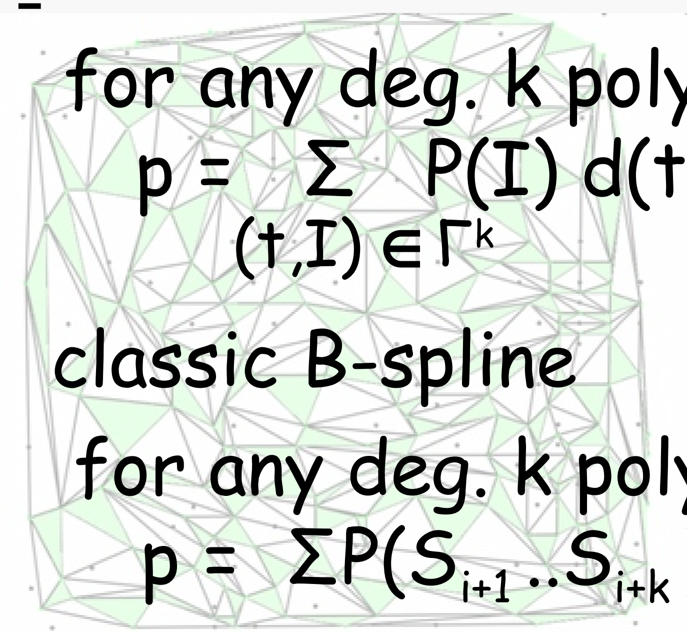
(abc, J) <-> [JU{a}, JU{b}, JU{c}],

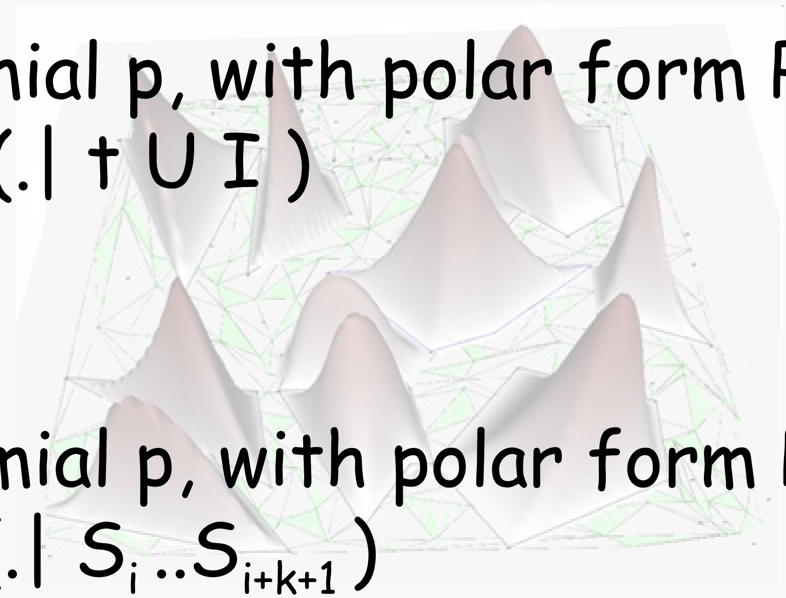
(abc, I) <-> [IU{b,c}, IU{a,c}, IU{b,c}]

Centroid triangulations -> B-splines

Theorem

Let Γ^0 .. Γ^k be a sequence of configurations. If Γ^0 is a triangulation, and Γ^{i-1}, Γ^i form a centroid triangulation for $0<i<k$, then the simplex splines assoc. with Γ^k reproduce polynomials of deg. k.





-

for any deg. k polynomial p, with polar form P

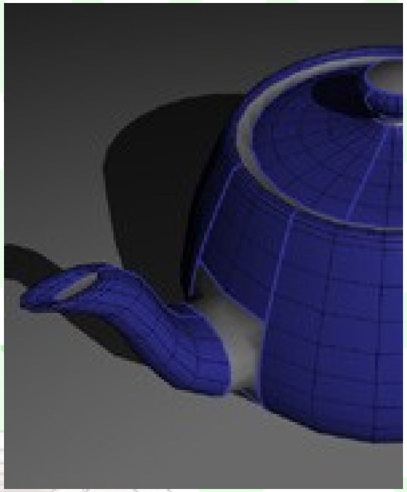
$$p = \sum_{(t,I) \in \Gamma^k} P(I)\, d(t)\, M(.| \, t \cup I)$$

classic B-spline

for any deg. k polynomial p, with polar form P

$$p = \sum P(S_{i+1} .. S_{i+k})\, B_i(.| \, S_i .. S_{i+k+1})$$

Reproducing the Zwart-Powell element



- An example of our claim:
  Box splines are special centroid triangulation splines.
- Theory motivation: Evidence that 'ct's provide general basis for bivariate splines.
- Practical motivation: Smooth blending of box spline patches.

Polyhedral splines

Polyhedron spline $M_\Pi(x \mid P)$

P : a polyhedron in R^n

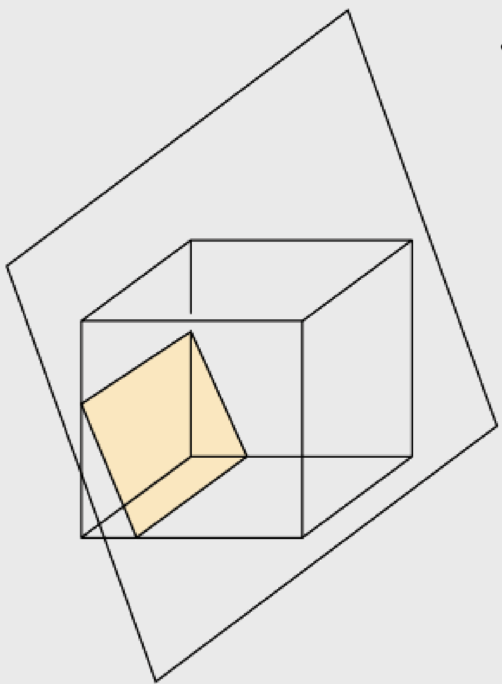
Π: a projection matrix from R^n to R^m

is an n-variate, degree (n-m) spline

$$M_\Pi(x \mid P) := \frac{\text{vol}\,\{\, y \mid y \in P,\ \Pi y = x \,\}}{\text{vol}\, P}$$

Box Spline

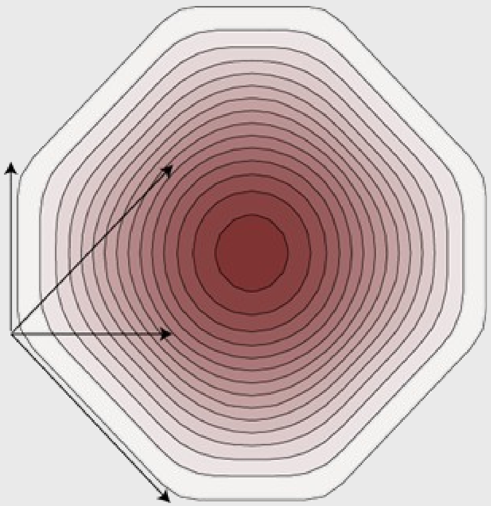
$M_\Pi(x)$



Reproduction of Box splines

$$\operatorname*{span}_{v \in N \times N} \{M_\Pi(x + v)\} \subset \operatorname*{span}_{X \in \Gamma^k} \{ M(x \mid X) \}$$

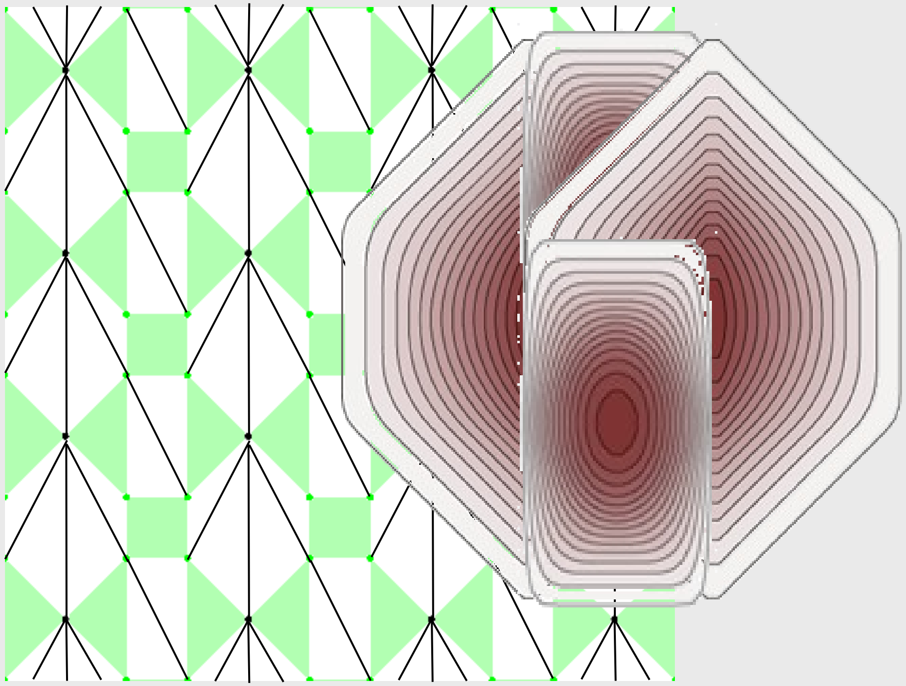
ZP element



$$\Pi = \begin{matrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{matrix}$$

$= \{ \rightarrow \uparrow \nearrow \searrow \}$

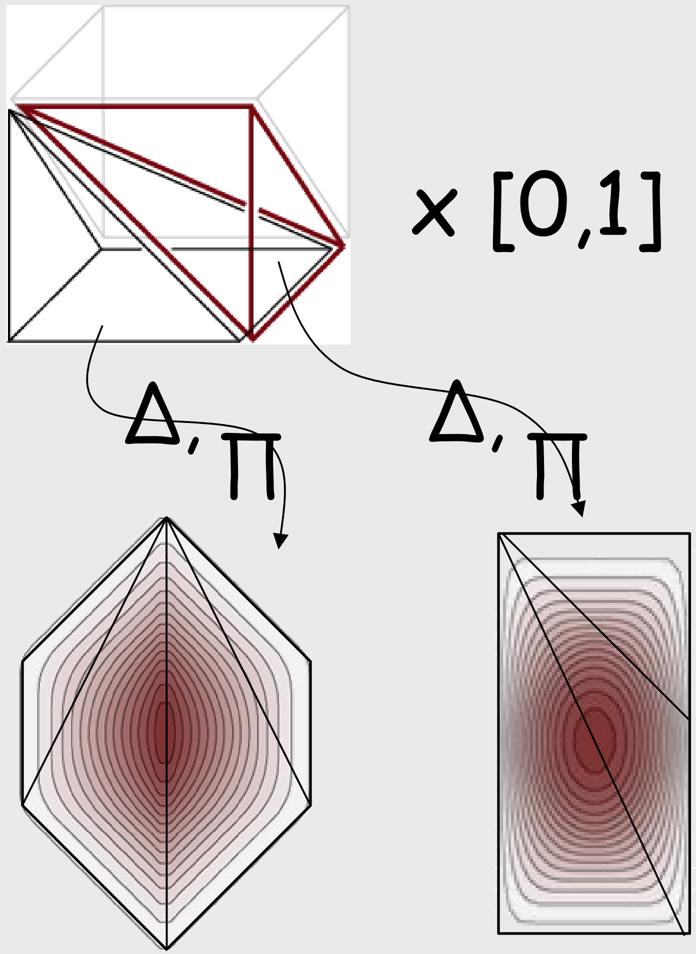
order 2 centroid triangulation



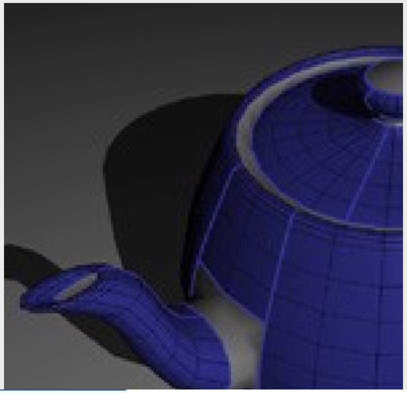
Reproduction of Box splines

Proof sketch (reproducing a single ZP element)

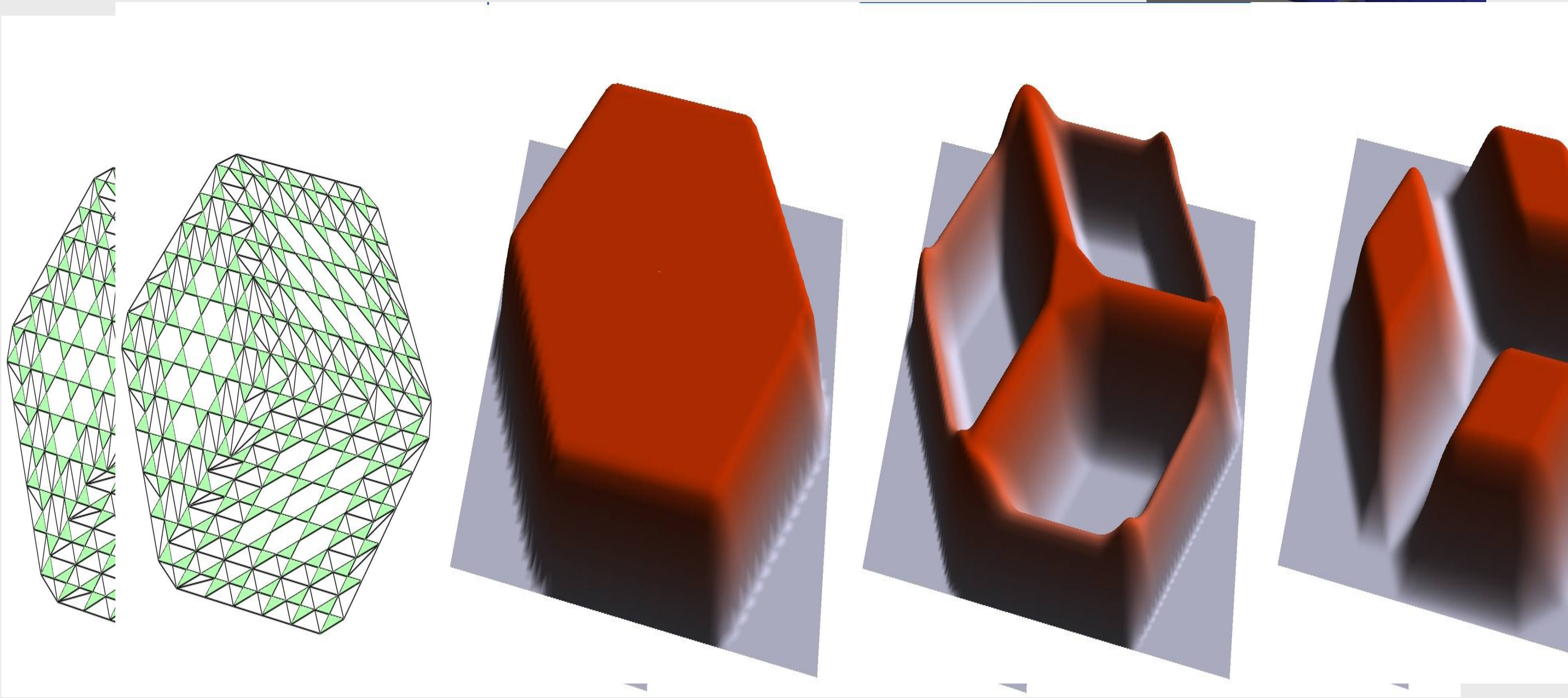
ZP-element
= 4-cube
(partition)
4-polytopes
(triangulate)
4-simplices
= simplex splines



Reproduction of Box-splines

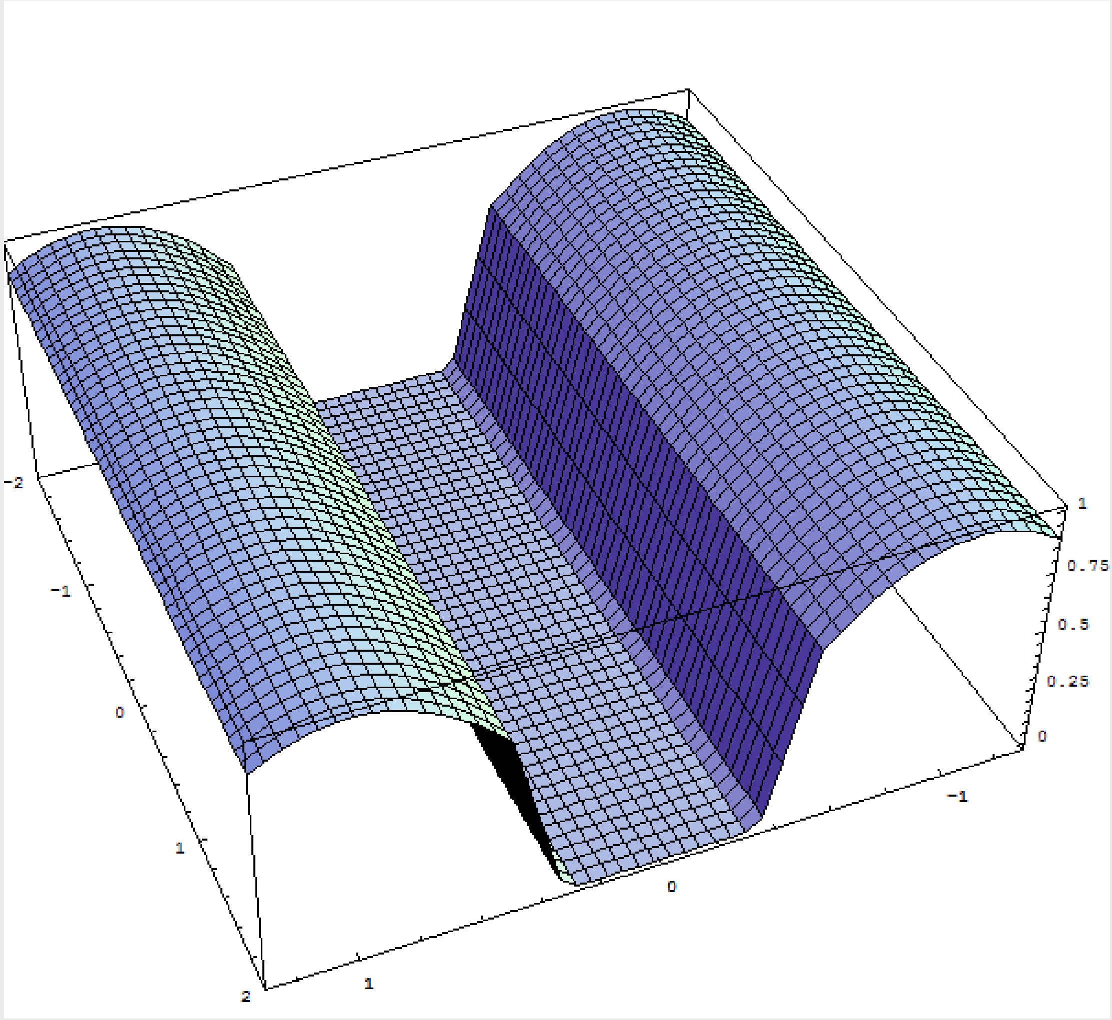


- Patch blending
  E.g., a partition of unity by
  blending regular splines.



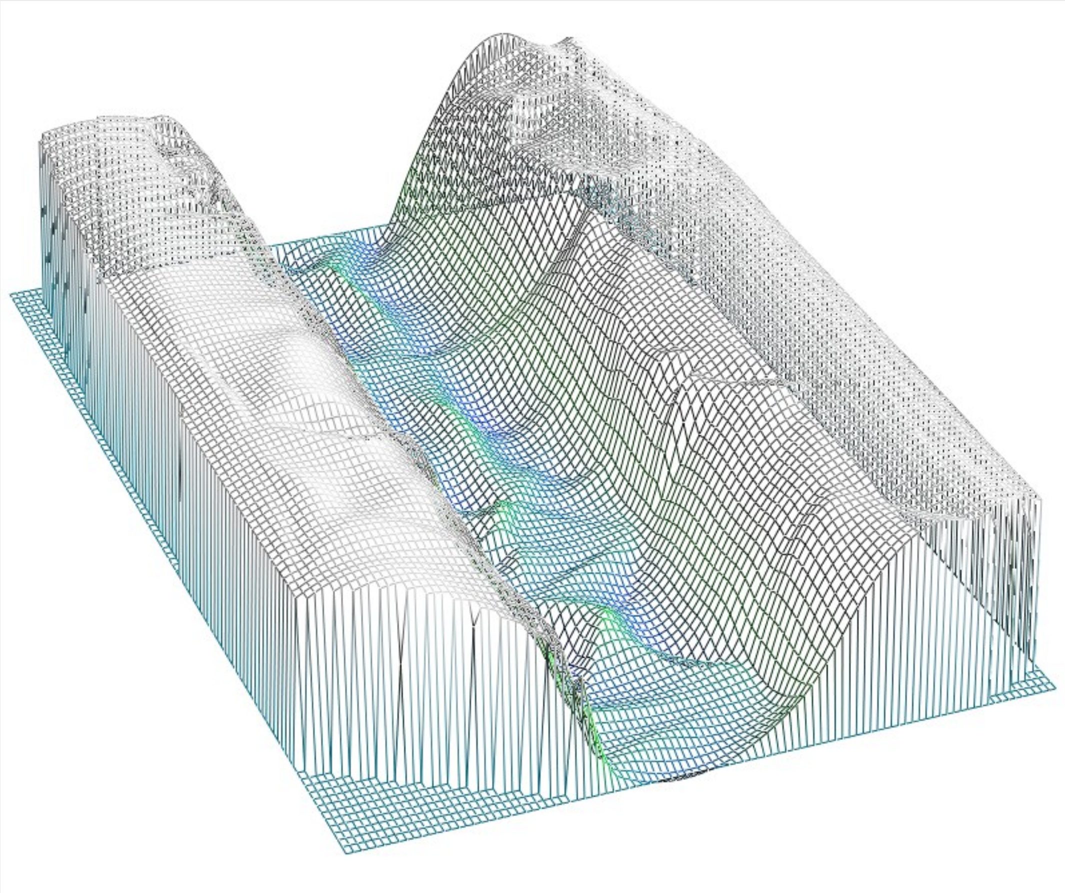
Modeling sharp features

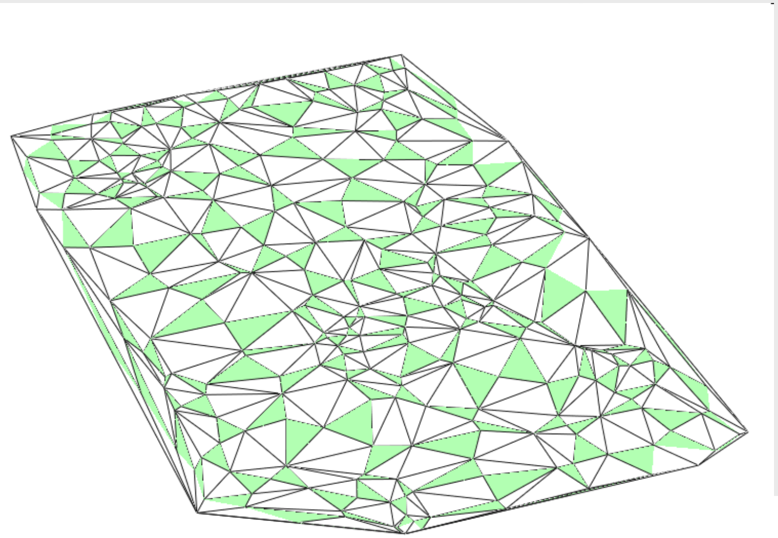
Data fitting with scattered data points and "break lines"

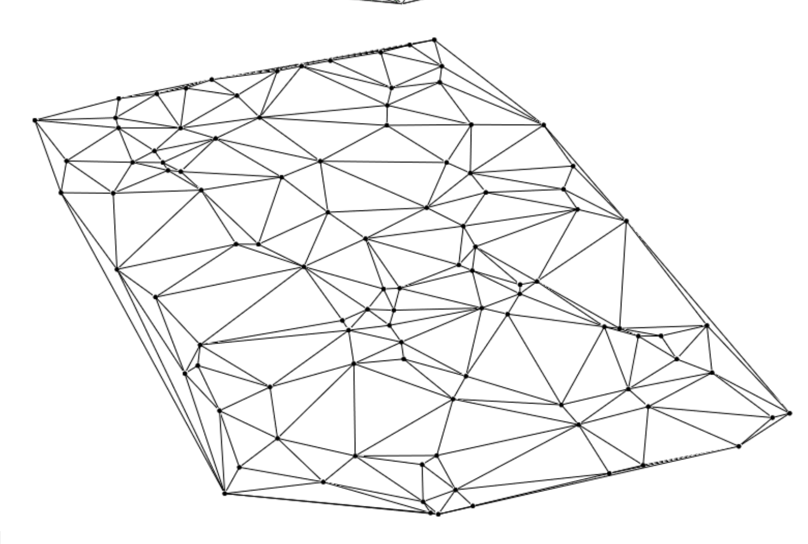


Modeling sharp features

Data fitting with scattered data points and "break lines"

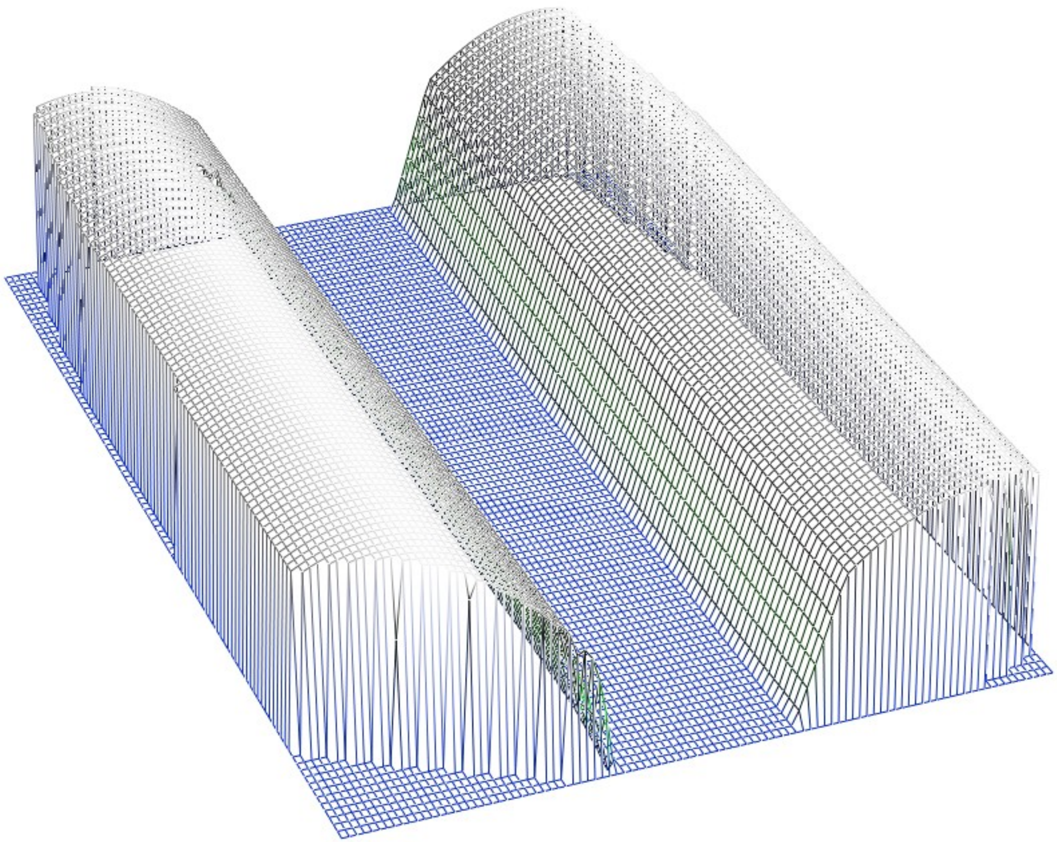


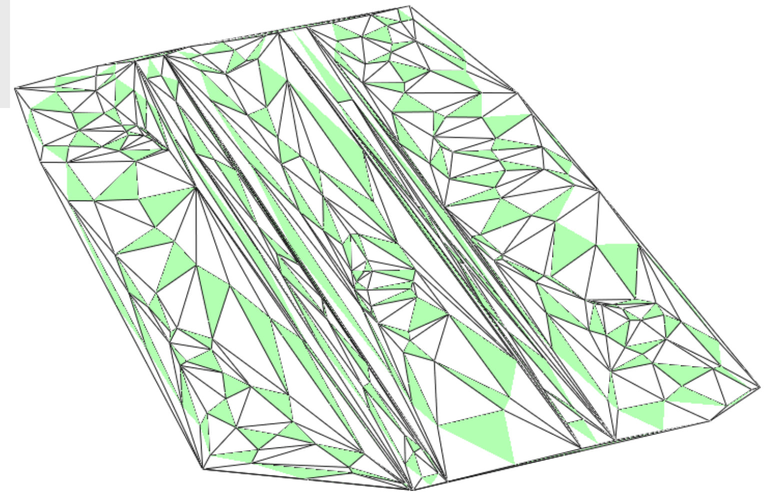


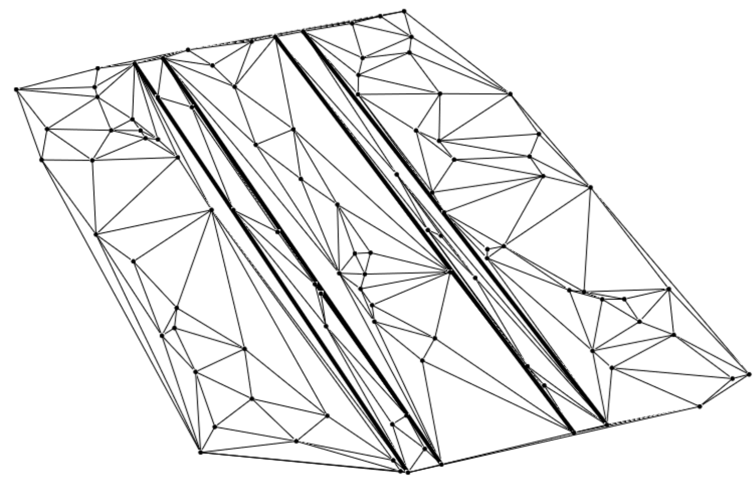


Modeling sharp features

Data fitting with scattered data points and "break lines"

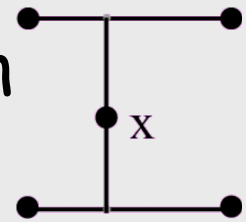






Open Problems

- Prove no self-intersecting holes arise in the centroid triangulation algorithm
- Reproduce other box-splines by centroid triangulation
  - bilinear interpolation (quadratic)



$\Pi = \{\rightarrow \rightarrow \uparrow \uparrow\}$

  - loop subdivision (quartic)

$\Pi = \{\rightarrow \rightarrow \uparrow \uparrow \nearrow \nearrow\}$