Three Applications of Disk Packing with Four-Sided Gaps

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Circle Magic

The points of tangency of four disks, tangent in a cycle, ...
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The points of tangency of four disks, tangent in a cycle, ... Are always cocircular!
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Are always cocircular!

Proof by PowerPoint 😊
Outline

1) Disk packing of a polygon

1) Nonobtuse triangulation of a polygon

1) Origami magic trick

1) Origami embedding of Euclidean Piecewise-Linear 2-manifolds

Basic Technique

Applications
Disk Packing of a Polygon

[Bern – Scott Mitchell – Ruppert, 1994]

Requirements:

- Each gap has 3 or 4 sides
- A disk is centered on each vertex
- Each side of the polygon is a union of radii
Does such a packing always exist?

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- Each gap has 3 or 4 sides
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- Each side of the polygon is a union of radii

5-sided gap can be reduced with a disk at generalized Voronoi diagram vertex
Disk Packing Induces Decomposition

Connect the centers of each pair of tangent disks
Disk Packing Induces Decomposition

Decomposition into:

- Triangles
- Quadrangles of cross-ratio one
  $|ab||cd| = |bc||da|$
Disk Packing Induces Decomposition

Decomposition into:
- Triangles
- Quadrangles that act like triangles!
Outline

1) Disk packing of a polygon

1) Nonobtuse triangulation of a polygon

1) Origami magic trick

1) Origami embedding of Euclidean Piecewise-Linear 2-manifolds
Question: Can any n-sided polygon be triangulated with triangles with maximum angle $90^\circ$?
What makes the problem hard?

Naïve Algorithm: Start from any triangulation, cut obtuse angle with perpendicular to opposite edge.
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Naïve Algorithm: Start from any triangulation, cut obtuse angle with perpendicular to opposite edge.

No end in sight!
We might spiral around an interior vertex forever!
Question: Can any n-sided polygon be triangulated with triangles with maximum angle 90°?

[Gerver, 1984] used the Riemann mapping theorem to show that if all polygon angles exceed 36°, then there always exists a triangulation with maximum angle 72°.


[Bern – Eppstein, 1991] showed O(n²) triangles for simple polygons

[Bern – Scott Mitchell - Ruppert, 1994] showed O(n) for polygons with holes
Nonobtuse Triangulation

Question: Can any n-sided polygon be triangulated with triangles with maximum angle $90^\circ$?

Rumored Application: Such a triangular mesh gives an M-matrix for the Finite Element Method for solving elliptic PDEs.

Milder condition is actually sufficient
Nonobtuse Triangulation

Question: Can any n-sided polygon be triangulated with triangles with maximum angle $90^\circ$?
Nonobtuse Triangulation

Question: Can any n-sided polygon be triangulated with triangles with maximum angle 90°?

MATLAB program by Scott Mitchell
Nonobtuse Triangulation Algorithm

Cut each triangular piece into 6 right triangles by adding in-center and spokes
Nonobtuse Triangulation Algorithm

Triangulate each quadrangle into 16 right triangles by adding center, chords, and spokes of tangency circle
Complication – Badly shaped Quads

Problems:
1. Reflex Quadrangle
2. Circle center on wrong side of chord
Solution – Break up Bad Quads

Problems:
(1) Reflex Quadrangle
(2) Circle center on wrong side of chord

Either bad case can be solved by adding one more disk.
Outline

1) Disk packing of a polygon

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Origami Magic Trick

Question: Can any polygon be cut out of flat-folded paper with a single straight cut?
Origami Magic Trick

Question: Can any polygon be cut out of flat-folded paper with a single straight cut?

[Betsy Ross, ~1790] Five-pointed star

[Demaine – Demaine – Lubiw, 1998] Heuristic method that works if folding paths do not propagate forever


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Use the decomposition to form independently foldable “molecules”

Requirements:

- Triangles and quadrangles fold flat
- Molecule (and polygon) boundaries fold to a common line (for the cut)
- Folds exit molecules only at points of tangency (or else we can’t fold them independently)
Triangles fold in a known origami pattern

Mountain fold

Valley fold

Rabbit-Ear Molecule
Quadrangles magically work out, too!

Gusset Molecule

Gusset Molecule

Four-armed Starfish

Spine

Book of Flaps
How do folded molecules fit together?

- One book of flaps tucks into another book of flaps (as a new “chapter”)
- Spines collinear, boundaries collinear
Can we recover all the adjacencies?

(1) Cut along a spanning tree to give a tree of molecules

(2) Tuck book inside book in a walk up the tree of molecules

(3) “Tape” spanning tree cuts along bottom edges of pages

Required tapings nest like parentheses in a walk around molecule-tree boundary

Mountain / valley assignments
Mounted Marlin
Degenerate Solution

True solution uses an offset polygon and offset disk packing
Recent Implementation (last week)

Send us cool images. And if you are able to fold these 1000+ origamis, DONT CUT IT :).

Paulo Silveira, Rafael Cosentino, José Coelho, Deise Aoki.  U. São Paulo
Outline

1) Disk packing of a polygon

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1) Origami embedding of Euclidean Piecewise-Linear 2-manifolds
Question: [E. Demaine] Can any polyhedron be “crushed”? That is, can it be creased and folded to make a flat origami?

Example: Rectangular Parallelopiped can be folded flat using paper bag folds.

Note: We just want a flat embedding, not a continuous transformation.
Origami Embedding of PL 2-Manifolds

Theorem: [Bern – Hayes, 2006] Any orientable, metric, piecewise-linear 2-Manifold (Euclidean triangles glued together at edges) can be isometrically embedded in Euclidean 2-space “plus layers”, that is, as a flat origami.
Topological Disk

Magic trick algorithm flat-folds a polyhedral patch

Disks are now geodesic disks

$p$ = Boundary of patch
Topological Sphere

- Puncture the sphere by opening an edge $e$
- Fold disk
- Final taping closes edge $e$
For higher genus, we need a new trick: taping books of flaps at the top and bottom.

Joining to form a handle requires that tops are mirror-congruent.
Schematic of Construction

(1) Cut manifold to a disk with paired holes

(1) Paired holes will be taped over top of book of flaps
Beautiful Minds?

**Nash Embedding Theorem:** Any orientable Riemannian manifold embeds smoothly ($C^\infty$) and isometrically into some Euclidean space. (E.g., 2-manifold $\rightarrow$ 17 dimensions)

**Origami Embedding Theorem:** Any compact, orientable, metric PL 2-manifold embeds isometrically as a flat origami.
Beautiful Minds?

Nash Embedding Theorem: Any Riemannian manifold embeds smoothly \((C^\infty)\) and isometrically into some Euclidean space. (E.g., 2-manifold \(\to\) 17 dimensions)

Origami Embedding Theorem: Any compact, orientable, metric PL 2-manifold embeds isometrically as a flat origami.

[Zalgaller, 1958] Any 2- or 3-dimensional “polyhedral space” (orientable or not) can be immersed in Euclidean 2- or 3-space.


[Krat-Burago-Petrunin, 2006] Any compact, orientable, 2-dimensional polyhedral space embeds isometrically as a flat origami.
Open Problems

1) Bad examples for naïve nonobtuse triangulation algorithms.

2) Simultaneous inside/outside nonobtuse triangulation of a polygon with holes

3) Algorithm for quasiconformal mapping using disk packing with 4-sided gaps

4) Do the “quadrangles that think they’re triangles” (cross-ratio 1) have any good numerical-analysis properties?
Open Problems

1) Origami embedding of higher-dimensional PL manifolds?

2) Can any origami embedding of a PL 2-manifold be “opened up” to give an embedding in Euclidean 3-space?

3) Continuous deformation of polyhedron to a flat origami?

4) 3-sided gap disk packing : Conformal mapping :: 4-sided gap disk packing : ???