p-ADIC ALGEBRAIC GEOMETRY
(SIMONS LECTURES AT STONY BROOK)

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1. Lecture 1: Overview

Fix a prime number p for the series.

Introduction

1.1. What are the p-adic numbers?

Construction 1.1 (Analytic construction)**.** There is a natural p-adic metric on $\mathbf{Q}$ determined by the norm

$$|\frac{a}{b}| = (1/p)^{\mathrm{val}(a)-\mathrm{val}(b)},$$

i.e., $|\frac{a}{b}|$ is small if the numerator is highly divisible by p. The completion of $\mathbf{Q}$ for this metric is the field $\mathbf{Q}_p$ of p-adic numbers. Thus, a typical $\alpha \in \mathbf{Q}_p$ is given by a series

$$\alpha := \sum_{i \geq -N} a_i p^i \quad \text{where} \quad 0 \leq a_i \leq p-1.$$

By construction, $\mathbf{Q}_p$ is a complete valued field.

Remark 1.2. The p-adic metric is nonarchimedean, i.e. $|a+b| \leq \max(|a|,|b|)$, $\rightsquigarrow$

$$\mathbf{Z}_p := \{a \in \mathbf{Q}_p | |a| \leq 1\}$$

is a subring of $\mathbf{Q}_p$. Note that $p \in \mathbf{Z}_p$ but $1/p \notin \mathbf{Z}_p$, so $\mathbf{Z}_p$ is not a field. In fact, we have $\mathbf{Z}_p[1/p] = \mathbf{Q}_p$.

Construction 1.3 (Algebraic construction)**.** One can show that

$$\mathbf{Z}_p = \varprojlim_n \mathbf{Z}/p^n\mathbf{Z} := \{(a_n)_{n\geq 1} \mid a_n \in \mathbf{Z}/p^n\mathbf{Z},\ a_{n+1} \equiv a_n \mod p^n\}.$$

We obtain the following picture:

$$\mathbf{Q}_p \xleftarrow{\text{invert } p} \mathbf{Z}_p \xrightarrow{\text{kill } p} \mathbf{Z}/p = \mathbf{F}_p.$$

Thus, $\mathbf{Z}_p$ relates the characteristic 0 field $\mathbf{Q}_p$ to the characteristic p field $\mathbf{F}_p$.

Variant 1.4 (The p-adic complex numbers)**.** One has a complete and algebraically closed extension $\mathbf{C}_p/\mathbf{Q}_p$ defined via

$$\mathbf{C}_p = \widehat{\overline{\mathbf{Q}_p}}.$$

As before, we obtain the following picture:

$$\mathbf{C}_p \xleftarrow{\text{invert } p} \mathcal{O}_{\mathbf{C}_p} := \{a \in \mathbf{C}_p \mid |a| \leq 1\} \xrightarrow{\text{kill } p^{1/n}\ \forall n} \overline{\mathbf{F}_p}.$$

Thus, $\mathcal{O}_{\mathbf{C}_p}$ relates algebraically closed fields of characteristic 0 and characteristic p.

Remark 1.5.
(1) One has $\mathbf{C}_p \simeq \mathbf{C}$ as abstract fields.
(2) The group $G_{\mathbf{Q}_p} := \mathrm{Gal}(\mathbf{C}_p/\mathbf{Q}_p)$ is *enormous*, unlike $\mathrm{Aut}(\mathbf{C}/\mathbf{R})$.

1.2. **How do the p-adic numbers arise in mathematics?**

(1) **Extrinsically.** The algebraic definition of completion makes sense with $\mathbf{Z}$ replaced by other abelian groups or fancier objects, e.g.,
- (Sullivan, Bousfeld-Kan) A topological space X admits a p-adic completion $\widehat{X}$ with each $\pi_i(X)$ being a $\mathbf{Z}_p$-module (and $\pi_i(\widehat{X}) = \pi_i(X)^\wedge$ under finiteness hypotheses).
- A complex M of abelian groups admits a p-adic completion $\widehat{M}$ with each $H_i(\widehat{M})$ being a $\mathbf{Z}_p$-module (and $H_i(\widehat{M}) = H_i(M)^\wedge$ under finiteness hypotheses).

(2) **Intrinsically.** There is a good notion of "analytic functions" over $\mathbf{Q}_p$ or $\mathbf{C}_p$, $\rightsquigarrow$ to a rich theory of p-adic analytic spaces, p-adic Hodge theory, etc.

Example 1.6. Tate showed (late 50s) that for any $q \in \mathbf{C}_p$ with $0 < |q| < 1$, the space

$$E_q := \mathbf{C}_p^*/q^{\mathbf{Z}}$$

is naturally an elliptic curve over $\mathbf{C}_p$.

(3) **As the glue between characteristic 0 and p.** A nice algebraic variety object $X/\mathcal{O}_{\mathbf{C}_p}$ (e.g., an algebraic variety) gives a very close relationship between the characteristic p variety $X_{\overline{\mathbf{F}_p}}$ and the (p-adic) complex variety $X_{\mathbf{C}_p}$

1.3. **What are some of the new techniques?**

(1) **Perfectoid spaces.**

These are "infinite sheeted covers of p-adic analytic spaces that are "infinitely ramified in characteristic p""

Example 1.7.
- Let $D = \{z \in \mathbf{C}_p \mid |z| \leq 1\}$ be the closed unit disc. Then the inverse limit of

$$....D \xrightarrow{z \mapsto z^p} D \xrightarrow{z \mapsto z^p} D \xrightarrow{z \mapsto z^p} D$$

is naturally a perfectoid space.
- Let E be an elliptic curve over $\mathbf{C}_p$. Then the inverse limit of

$$....E \xrightarrow{p} E \xrightarrow{p} E \xrightarrow{p} E$$

is naturally a perfectoid space.

Surprisingly, perfectoid spaces are simpler than p-adic analytic spaces in some important ways: they are completely controlled by certain objects that live in characteristic p and are thus easier to study (e.g., using the Frobenius endomorphism that acts on everything in characteristic p).

(2) **Prismatic cohomology.**

This is a new integral cohomology theory for geometric objects over $\mathbf{Z}_p$ that interpolates between all previous known p-adic cohomology theories available in this setting (e.g., de Rham, Hodge, crystalline, étale), leading to new relations between these theories.

A SAMPLING OF APPLICATIONS

1.4. **Number theory.**

Theorem 1.8 (Scholze's torsion Langlands theorem, 2013)**.** *For many number fields F, any $\mathbf{F}_p$-automorphic form on for $GL_{n,F}$ has an attached Galois representation* $\mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{F}_p})$.

Remark 1.9. (1) The key technical theorem above was:

Theorem 1.10. *Let $\mathcal{A}_g[p^\infty]$ be the space parametrizing abelian varieties $A/\mathbf{C}_p$ with a trivialization of $H_1(A, \mathbf{Z}_p)$. Then $\mathcal{A}_g[p^\infty]$ is a perfectoid space.*

(2) In 2018, the ten author[1] paper used the above to prove the Sato-Tate conjecture for elliptic curves over CM number fields.

1.5. Algebraic geometry.

Theorem 1.11 (Bhatt, 2020)**.** *Kodaira vanishing holds true, up to passage to finite covers, in mixed characteristic algebraic geometry.*

Remark 1.12. (1) The theorem has a *very* concrete consequence:

($*$) Let $R = \mathbf{Z}[x_1, ..., x_n]$ and let R^+ be the integral closure of R in $\overline{\mathrm{Frac}(R)}$. Then $(p, x_1, ..., x_n)$ is a regular sequence on R^+, i.e., x_i acts injectively on $R^+/(p, x_1, ..., x_{i-1})$ for $i \geq 1$.

($*$) is highly non-trivial even for $n = 2$.

(2) The proof of the theorem relies on prismatic cohomology as well as a p-adic Riemann-Hilbert correspondence for perverse $\mathbf{F}_p$-sheaves (Bhatt-Lurie) .

(3) ($*$) implies the "direct summand conjecture" and the "weakly functorial big Cohen-Macaulay module conjecture" of Hochster. These were recently shown by Y. André, and are known to imply most of the "homological conjectures" in commutative algebra.

(4) Theorem forms an essential ingredient of the following:

Theorem 1.13 (BMPSTWW and Yoshikawa-Takkamatsu, 2020)**.** *The minimal model program holds true in dimension ≤ 3 over $\mathbf{Z}_p$ for $p \geq 5$.*

1.6. Homotopy theory.

Write $K(X)$ for the complex K-theory of a topological space X. Recall the following basic result:

Theorem 1.14 (Bott, Atiyah-Hirzeburch)**.** *Given a nice topological space X, we can filter the K-theory $K(X)$ by singular cohomology, i.e., there exists a spectral sequence*

$$E_2^{i,j} : H^i(X, \mathbf{Z}(\frac{-j}{2})) \Rightarrow K^{i+j}(X)$$

that degenerates modulo torsion, where $\mathbf{Z}(\frac{-j}{2})$ vanishes if j is odd, and is $(2\pi i)^{-\frac{j}{2}}\mathbf{Z}$ for j even.

Theorem 1.15 (Bhatt-Morrow-Scholze and Clausen-Mathew-Morrow, 2018)**.** *Let R be a p-adically complete ring. Then we can filter the p-adic étale K-theory space $K_{et}(R)^\wedge$ of R in terms of syntomic cohomology $H^*(R, \mathbf{Z}_p(\frac{-j}{2}))$.*

Remark 1.16. (1) The complementary case where $p \in R^*$ was conjectured by Beilinson (mid 80s), and is classical (Thomason, Gabber, and Suslin (also 80s)).

(2) Syntomic cohomology is defined in terms of prismatic cohomology. In fact, the relevant cases of both were discovered in [BMS] in a quest to prove the above theorem.

(3) Theorem has led to new calculations in algebraic K-theory.

[1] Allen, Calegari, Caraiani, Gee, Helm, Le Hung, Newton, Scholze, Taylor, Thorne

2. Lecture 2: Prismatic cohomology

Unattributed results are joint with Morrow and Scholze

2.1. Understanding $\mathbf{F}_p$-cohomology geometrically.

Question 2.1. Let M be a compact Kähler manifold. Hodge theory describes $H^i(M, \mathbf{C})$ via differential forms. How to see $H^i(M, \mathbf{Z})_{tors}$ or $H^i(M, \mathbf{F}_p)$ geometrically?

Notation 2.2. We set $C := \mathbf{C}_p = \widehat{\overline{\mathbf{Q}_p}}$, giving rise to

$$C \xleftarrow{\text{invert } p} \mathcal{O}_C := \{a \in C \mid |a| \leq 1\} \xrightarrow{\text{kill } p^{1/n} \ \forall n} \overline{\mathbf{F}_p} =: k$$

as in the first talk.

Let $X/\mathcal{O}_C$ be a smooth projective variety,
$\rightsquigarrow$ smooth projective varieties X_C/C and X_k/k in characteristics 0 and p respectively.

Theorem 2.3. *$\mathbf{F}_p$-cohomology classes on X_C are obstructions to integration of forms on X_k. More precisely, we have*

(∗) $\dim_{\mathbf{F}_p} H^i(X_C, \mathbf{F}_p) \leq \dim_k H^i_{dR}(X_k)$.

Example 2.4. Say $p = 2$ and X_C is an Enriques surface, so $\pi_1(X_C) = \mathbf{F}_2$, whence $H^1(X_C, \mathbf{F}_2) \neq 0$. Then (∗) implies that $H^1_{dR}(X_k) \neq 0$ (W. Lang, Illusie).

Remark 2.5.
1. The inequality (∗) can be strict: there can be (topologically) distinct X_C's for the same X_k.
2. (∗) was previously known in some special cases where it is an equality (Faltings, Caruso).
3. (∗) has been extended to the "semistable" case (Cesnavicius-Koshikawa).

2.2. Fontaine's deformation (aka the prismatic cohomology of a point).

Observation 2.6. If R is a commutative ring of characteristic p, then there is a natural "Frobenius" endomorphism

$$\phi : R \to R, \quad \phi(f) = f^p.$$

By naturality, this acts on characteristic p algebraic geometry.

Can we do something similar in mixed characteristic?

Exercise 2.7. Show that there is no endomorphism $\phi : \mathcal{O}_C \to \mathcal{O}_C$ such that $\phi(f) = f^p \mod p$.

Nevertheless, Fontaine found a beautiful fix:

Construction 2.8 (Fontaine)**.**

$$A_{\text{inf}} := W\left(\varprojlim(\ldots \to \mathcal{O}_C/p \xrightarrow{\phi} \mathcal{O}_C/p \xrightarrow{\phi} \mathcal{O}_C/p)\right).$$

So what does this really mean??

- By functoriality of $W(-)$, the Frobenius on $\mathcal{O}_C/p$ gives an automorphism $\phi : A_{\text{inf}} \to A_{\text{inf}}$ such that
$$\phi(f) = f^p \mod pA_{\text{inf}}$$
for all $f \in A_{\text{inf}}$.
- There is an element $u \in A_{\text{inf}}$ such that $\phi(u) = u^p$ and $A_{\text{inf}}/(u - p) \simeq \mathcal{O}_C$.

The triple $(A_{\text{inf}}, \phi, (u - p))$ is an example of a *perfect prism.*

2.3. Prismatic cohomology in general.

Theorem 2.9. *There exists a cohomology theory $H^*_{\Delta}(X)$ valued in finitely generated A_{inf}-modules and equipped with a (non-bijective!) Frobenius action $\phi_X : H^*_{\Delta}(X) \to H^*_{\Delta}(X)$ with the following properties:*

1. *Extending scalars along $A_{\text{inf}} \to A_{\text{inf}}/(u - p)$ gives $H^*_{dR}(X)$.*
2. *Extending scalars along $A_{\text{inf}} \to A_{\text{inf}}[1/(u - p)]^\wedge$ gives $H^*(X_C, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} A_{\text{inf}}[1/(u - p)]^\wedge$.*

In particular, we obtain (∗) by semicontinuity.

2.4. **Where did prismatic cohomology come from?** Two rather distinct inspirations:

(1) *Abstract p-adic Hodge theory*: Say X is defined over $\mathbf{Z}_p$, so $G_{\mathbf{Q}_p}$ acts on $L := H^i(X_C, \mathbf{Z}_p)$. Kisin had attached (in 2006) certain A_{inf}-modules $T(L)$ equipped with Frobenius actions to the $G_{\mathbf{Q}_p}$-representation L/torsion with the property that $T(L)/(u-p)T(L)$ is closely related to $H^*_{dR}(X)$/torsion.

Question 2.10. Can one construct $T(L)$ geometrically via X in a fashion that sees torsion?

(2) *Hesselholt's Bott periodicity*: An important object in K-theory is the following spectrum attached to a ring R:

$$TP(R) = THH(R)^{tS^1} := (R \otimes_{R \otimes_{\mathbf{S}} R} R)^{tS^1}$$

Motivated by the Lichtenbaum-Quillen conjecture, Hesselholt had proved a periodicity theorem

$$\pi_* TP(\mathcal{O}_C) = A_{\mathrm{inf}}[u, u^{-1}] \quad \text{with} \quad \deg(u) = 2$$

and moreover observed that $\pi_* TP(\mathcal{O}_C)$ has a natural Frobenius action.
($\rightsquigarrow$ get purely p-adic proof of Bott periodicity (Hesselholt-Nikolaus).)

Question 2.11. Is there a version of this calculation for $TP(X)$?

Remark 2.12. By now, there are 3 constructions of prismatic cohomology, in increasing order of generality:

(1) p-adic Hodge theory — relies crucially on the Faltings almost purity theorem.
(2) Topological Hochschild homology — relies on quasi-syntomic descent.
(3) The prismatic site.

2.5. **Other applications and followups.**

(1) Prismatic cohomology is computed in local co-ordinates by q-deformations of de Rham complexes $\rightsquigarrow$ co-ordinate independence of q-de Rham cohomology (conjectured by Scholze).
(2) Syntomic cohomology and K-theory calculations (Liu-Wang, Bhatt-Clausen-Mathew,....).

Example 2.13 (Special case of odd vanishing)**.** $\pi_* K(\mathcal{O}_C/p^n)$ is concentrated in even degrees.

Example 2.14 (Weight 1 syntomic cohomology)**.** For any p-complete ring R, we get a fibre sequence

$$\mathbf{Z}_p(1)(R) = \mathrm{Pic}(R)^\wedge[-2] \to \mathrm{Fil}^1 {\Delta\!\!\!\!\Delta}_R \xrightarrow{\phi - 1} {\Delta\!\!\!\!\Delta}_R,$$

giving a (very weak) p-adic analog of the Lefschetz $(1,1)$-theorem.

(3) Perfections in mixed characteristic (discussed next time)
(4) Potential applications to the p-adic Langlands program (e.g., calculation of $H^*(\Omega, \mathbf{Z}_p)$ by Colmez-Dospinescu-Niziol),
(5) A good candidate for the notion of a "mod p crystalline Galois representation" (Drinfeld)

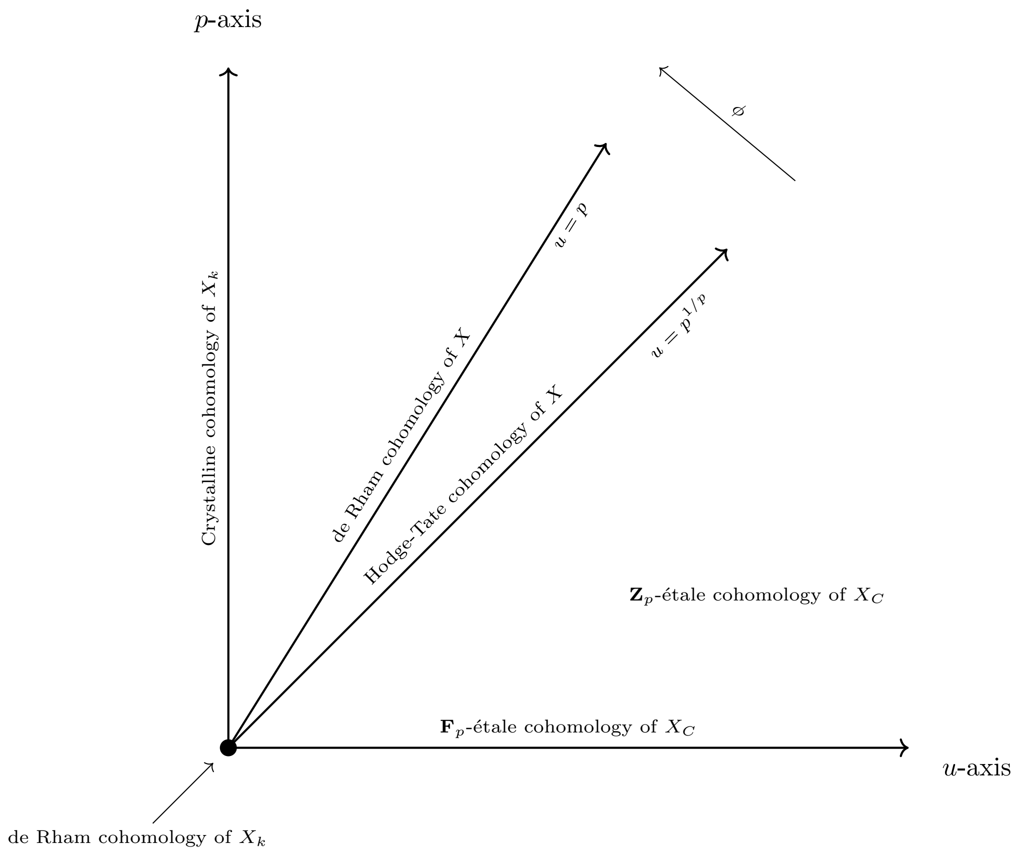

FIGURE 1. The "values" over $R\Gamma_{\Delta}(X)$ over $\mathrm{Spec}(A_{\mathrm{inf}})$ as provided by Theorem 2.9