p-ADIC ALGEBRAIC GEOMETRY (SIMONS LECTURES AT STONY BROOK)

BHARGAV BHATT

1. Lecture 1: Overview

Fix a prime number p for the series.

Introduction

1.1. What are the p-adic numbers?

Construction 1.1 (Analytic construction)**.** There is a natural p-adic metric on $\mathbf{Q}$ determined by the norm

$$|\frac{a}{b}| = (1/p)^{\mathrm{val}(a)-\mathrm{val}(b)},$$

i.e., $|\frac{a}{b}|$ is small if the numerator is highly divisible by p. The completion of $\mathbf{Q}$ for this metric is the field $\mathbf{Q}_p$ of p-adic numbers. Thus, a typical $\alpha \in \mathbf{Q}_p$ is given by a series

$$\alpha := \sum_{i \geq -N} a_i p^i \quad \text{where} \quad 0 \leq a_i \leq p-1.$$

By construction, $\mathbf{Q}_p$ is a complete valued field.

Remark 1.2. The p-adic metric is nonarchimedean, i.e. $|a+b| \leq \max(|a|,|b|)$, $\rightsquigarrow$

$$\mathbf{Z}_p := \{a \in \mathbf{Q}_p || a| \leq 1\}$$

is a subring of $\mathbf{Q}_p$. Note that $p \in \mathbf{Z}_p$ but $1/p \notin \mathbf{Z}_p$, so $\mathbf{Z}_p$ is not a field. In fact, we have $\mathbf{Z}_p[1/p] = \mathbf{Q}_p$.

Construction 1.3 (Algebraic construction)**.** One can show that

$$\mathbf{Z}_p = \varprojlim_n \mathbf{Z}/p^n\mathbf{Z} := \{(a_n)_{n\geq 1} \mid a_n \in \mathbf{Z}/p^n\mathbf{Z},\ a_{n+1} \equiv a_n \mod p^n\}.$$

We obtain the following picture:

$$\mathbf{Q}_p \xleftarrow{\text{invert } p} \mathbf{Z}_p \xrightarrow{\text{kill } p} \mathbf{Z}/p = \mathbf{F}_p.$$

Thus, $\mathbf{Z}_p$ relates the characteristic 0 field $\mathbf{Q}_p$ to the characteristic p field $\mathbf{F}_p$.

Variant 1.4 (The p-adic complex numbers)**.** One has a complete and algebraically closed extension $\mathbf{C}_p/\mathbf{Q}_p$ defined via

$$\mathbf{C}_p = \widehat{\overline{\mathbf{Q}_p}}.$$

As before, we obtain the following picture:

$$\mathbf{C}_p \xleftarrow{\text{invert } p} \mathcal{O}_{\mathbf{C}_p} := \{a \in \mathbf{C}_p \mid |a| \leq 1\} \xrightarrow{\text{kill } p^{1/n}\ \forall n} \overline{\mathbf{F}_p}.$$

Thus, $\mathcal{O}_{\mathbf{C}_p}$ relates algebraically closed fields of characteristic 0 and characteristic p.

Remark 1.5. (1) One has $\mathbf{C}_p \simeq \mathbf{C}$ as abstract fields.

(2) The group $G_{\mathbf{Q}_p} := \mathrm{Gal}(\mathbf{C}_p/\mathbf{Q}_p)$ is *enormous*, unlike $\mathrm{Aut}(\mathbf{C}/\mathbf{R})$.

1.2. How do the p-adic numbers arise in mathematics?

(1) **Extrinsically.** The algebraic definition of completion makes sense with $\mathbf{Z}$ replaced by other abelian groups or fancier objects, e.g.,
   - (Sullivan, Bousfeld-Kan) A topological space X admits a p-adic completion $\widehat{X}$ with each $\pi_i(X)$ being a $\mathbf{Z}_p$-module (and $\pi_i(\widehat{X}) = \pi_i(X)^\wedge$ under finiteness hypotheses).
   - A complex M of abelian groups admits a p-adic completion $\widehat{M}$ with each $H_i(\widehat{M})$ being a $\mathbf{Z}_p$-module (and $H_i(\widehat{M}) = H_i(M)^\wedge$ under finiteness hypotheses).

(2) **Intrinsically.** There is a good notion of "analytic functions" over $\mathbf{Q}_p$ or $\mathbf{C}_p$, $\rightsquigarrow$ to a rich theory of p-adic analytic spaces, p-adic Hodge theory, etc.

Example 1.6. Tate showed (late 50s) that for any $q \in \mathbf{C}_p$ with $0 < |q| < 1$, the space

$$E_q := \mathbf{C}_p^*/q^{\mathbf{Z}}$$

is naturally an elliptic curve over $\mathbf{C}_p$.

(3) **As the glue between characteristic 0 and p.** A nice algebraic variety object $X/\mathcal{O}_{\mathbf{C}_p}$ (e.g., an algebraic variety) gives a very close relationship between the characteristic p variety $X_{\overline{\mathbf{F}_p}}$ and the (p-adic) complex variety $X_{\mathbf{C}_p}$

1.3. What are some of the new techniques?

(1) **Perfectoid spaces.**

These are "infinite sheeted covers of p-adic analytic spaces that are "infinitely ramified in characteristic p""

Example 1.7.
   - Let $D = \{z \in \mathbf{C}_p \mid |z| \leq 1\}$ be the closed unit disc. Then the inverse limit of

$$....D \xrightarrow{z\mapsto z^p} D \xrightarrow{z\mapsto z^p} D \xrightarrow{z\mapsto z^p} D$$

   is naturally a perfectoid space.
   - Let E be an elliptic curve over $\mathbf{C}_p$. Then the inverse limit of

$$....E \xrightarrow{p} E \xrightarrow{p} E \xrightarrow{p} E$$

   is naturally a perfectoid space.

Surprisingly, perfectoid spaces are simpler than p-adic analytic spaces in some important ways: they are completely controlled by certain objects that live in characteristic p and are thus easier to study (e.g., using the Frobenius endomorphism that acts on everything in characteristic p).

(2) **Prismatic cohomology.**

This is a new integral cohomology theory for geometric objects over $\mathbf{Z}_p$ that interpolates between all previous known p-adic cohomology theories available in this setting (e.g., de Rham, Hodge, crystalline, étale), leading to new relations between these theories.

A sampling of applications

1.4. Number theory.

Theorem 1.8 (Scholze's torsion Langlands theorem, 2013). *For many number fields F, any $\mathbf{F}_p$-automorphic form on for $GL_{n,F}$ has an attached Galois representation* $\mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{F}_p})$.

Remark 1.9. (1) The key technical theorem above was:

Theorem 1.10. *Let $\mathcal{A}_g[p^\infty]$ be the space parametrizing abelian varieties $A/\mathbf{C}_p$ with a trivialization of $H_1(A, \mathbf{Z}_p)$. Then $\mathcal{A}_g[p^\infty]$ is a perfectoid space.*

(2) In 2018, the ten author[1] paper used the above to prove the Sato-Tate conjecture for elliptic curves over CM number fields.

1.5. Algebraic geometry.

Theorem 1.11 (Bhatt, 2020)**.** *Kodaira vanishing holds true, up to passage to finite covers, in mixed characteristic algebraic geometry.*

Remark 1.12. (1) The theorem has a *very* concrete consequence:

($*$) Let $R = \mathbf{Z}[x_1, ..., x_n]$ and let R^+ be the integral closure of R in $\overline{\mathrm{Frac}(R)}$. Then $(p, x_1, ..., x_n)$ is a regular sequence on R^+, i.e., x_i acts injectively on $R^+/(p, x_1, ..., x_{i-1})$ for $i \geq 1$.

($*$) is highly non-trivial even for $n = 2$.

(2) The proof of the theorem relies on prismatic cohomology as well as a p-adic Riemann-Hilbert correspondence for perverse $\mathbf{F}_p$-sheaves (Bhatt-Lurie) .

(3) ($*$) implies the "direct summand conjecture" and the "weakly functorial big Cohen-Macaulay module conjecture" of Hochster. These were recently shown by Y. André, and are known to imply most of the "homological conjectures" in commutative algebra.

(4) Theorem forms an essential ingredient of the following:

Theorem 1.13 (BMPSTWW and Yoshikawa-Takkamatsu, 2020)**.** *The minimal model program holds true in dimension ≤ 3 over $\mathbf{Z}_p$ for $p \geq 5$.*

1.6. Homotopy theory.

Write $K(X)$ for the complex K-theory of a topological space X. Recall the following basic result:

Theorem 1.14 (Bott, Atiyah-Hirzeburch)**.** *Given a nice topological space X, we can filter the K-theory $K(X)$ by singular cohomology, i.e., there exists a spectral sequence*

$$E_2^{i,j} : H^i(X, \mathbf{Z}(\frac{-j}{2})) \Rightarrow K^{i+j}(X)$$

that degenerates modulo torsion, where $\mathbf{Z}(\frac{-j}{2})$ vanishes if j is odd, and is $(2\pi i)^{-\frac{j}{2}}\mathbf{Z}$ for j even.

Theorem 1.15 (Bhatt-Morrow-Scholze and Clausen-Mathew-Morrow, 2018)**.** *Let R be a p-adically complete ring. Then we can filter the p-adic étale K-theory space $K_{et}(R)^\wedge$ of R in terms of syntomic cohomology $H^*(R, \mathbf{Z}_p(\frac{-j}{2}))$.*

Remark 1.16. (1) The complementary case where $p \in R^*$ was conjectured by Beilinson (mid 80s), and is classical (Thomason, Gabber, and Suslin (also 80s)).

(2) Syntomic cohomology is defined in terms of prismatic cohomology. In fact, the relevant cases of both were discovered in [BMS] in a quest to prove the above theorem.

(3) Theorem has led to new calculations in algebraic K-theory.

[1] Allen, Calegari, Caraiani, Gee, Helm, Le Hung, Newton, Scholze, Taylor, Thorne