INDEX THEORY
REVISITED

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\[ T : V_+ \rightarrow V_ - \]
\[ \text{IND } T = \dim \ker T - \dim \text{cok } T \]
\[ \text{cok}(T) = V_ - /_{T(V_+)} = \ker T^* \]
FREDHOLM ALTERNATIVE

If \[ T^* T(v_+) = \lambda v_+ \], then
\[ T T^*(T v_+) = \lambda (T v_+) \].
\[ \lambda \neq 0 \]

\[ T^* T \text{ and } TT^* \text{ have same non-zero eigenvalues with mult.} \]

If \( f \) is a cont. function with \( f(0) = 1 \), then
\[ \text{IND } T = \text{tr } f(T^* T) - \text{tr } f(T T^*). \]
FOR \( f(x) = e^{-tx} \), \( t > 0 \)

\[ \text{IND } T = \text{tr} (e^{-t\Delta^+}) - \text{tr} (e^{-t\Delta_-}) \]

where \( \Delta^+ = T^*T \) and \( TT^* = \Delta_- \).

\[ \mathbb{Z}_2 \text{ grading:} \]
\[
\begin{pmatrix}
0 & T^* \\
T & 0
\end{pmatrix}
\begin{pmatrix}
\psi_+ \\
\psi_-
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\psi_+ \\
\psi_-
\end{pmatrix}
\]

with square 
\[
\begin{pmatrix}
\Delta^+ & 0 \\
0 & \Delta_-
\end{pmatrix} = \Delta
\]

let \( \gamma_5 = \begin{pmatrix} +I & 0 \\ 0 & -I \end{pmatrix} \), then

\[ \text{IND } T = \text{tr} (e^{-t\Delta} \gamma_5) \]

The heat kernel \( k_\pm (t, x, y) \)
for the operator \( e^{-t\Delta^\pm} \) has an
asymptotic expansion for \( t \to \infty \)

\[ k_\pm (t, x, y) \sim \frac{1}{(4\pi t)^{d/2}} (\chi^{\pm}_0(x) + \chi^{\pm}_x(x)) t + \ldots \]

For \( d \) even, assuming \( T \) is Dirac-like,

\[ \text{IND} (T) = \int_M \chi^{\pm}_0(x) - \chi^{\pm}_x(x) \]

But \( \chi^{\pm}_x(x) \) can be expressed

in terms of the metric \( g_{ij} \) and
its derivatives and hence in terms of \( R_{ij} \) and its derivatives.

and \( T = \int_M \text{poly}(R, DR, D^2 R, \ldots) \)
Levi method:

Integral equation for
$k(t, x, y)$ in terms of
$k_0(t, x, y)$ for constant coeff $\Delta_0$

\[
e^{-s\Delta} e^{-(t-s)\Delta_0} - e^{-t\Delta_0} = e^{-t\Delta} - e^{-t\Delta_0} =
\]

\[
\int_0^t ds \frac{d}{ds} (e^{-s\Delta} e^{-(t-s)\Delta_0}) =
\]

\[
\int_0^t ds \, e^{-s\Delta} (\Delta_0 - \Delta) e^{-(t-s)\Delta_0}
\]

The Atiyah-Singer index theorem:

\[
IND T = \int_M \text{poly}(R)
\]

McKean-Singer (above viewpoint) asked why $\int_M \text{poly}(R, DR, DR^*)$?

Explained by Gel'fand's lemma.

Nice exposition by A.-B.-P.

\[
e^{-t\Delta} \xrightarrow{t \to \infty} \begin{pmatrix} \text{ker } TT & 0 \\ 0 & \text{ker } TT \end{pmatrix}
\]

$s(t, x, y) \xrightarrow{t \to \infty} \text{local computable formula}$
QM APPROACH

WIENER: \( \Phi(t,x,y) \) is an integral over all paths \( \gamma: [0,t] \to M, \gamma(0) = x, \gamma(t) = y \)

\[ \Phi(t,x,y) = \int_{\gamma(0) = x \atop \gamma(t) = y} \partial_t e^{-\frac{\gamma^*}{2}} \]

Feynman-Kac: For the Schrödinger operator \( \Delta + V \),

\[ \Phi(t,x,y) = \int_{\gamma(0) = x \atop \gamma(t) = y} \partial_t e^{-\frac{\gamma^*}{2}} e^{-\int_{0}^{t} V(\gamma(s)) \, ds} \]

Suppose \( E \) is a vector bundle with covariant differential \( D \), what is the path integral for kernel of \( e^{-tD^*D} \)?

Ito-Malliavin-STROOCK:

\[ \Phi(t,x,y) = \int_{E} \partial_t e^{-\frac{x^2}{2}} \text{ (parallel transport along } \gamma) \]

For the spinor vector bundle \( S_{\pm} \) with Riemannian spin connection:

\[ \text{tr}(e^{-tD^*_+ D_+}) - \text{tr}(e^{-tD^*_- D_-}) = \int_{E} \partial_t e^{-\frac{x^2}{2}} \left( \text{tr}(P_{\pm}^+) - \text{tr}(P_{\pm}^-) \right) \]

where \( P_{\pm}^\pm \) is parallel transport around \( \gamma \) to \( S_{\pm} \)

For the Dirac operator \( D : S_{\pm} \to S_{\mp} \)

\[ \mathcal{D} \Phi = D^*_+ D_+ + \sigma \quad \text{and} \quad \mathcal{D} \Phi = D^*_- D_- + \sigma \]

where \( \sigma \) is multiplication by scalar curvature
Conclusion: \( \text{index } \Phi = \int \partial \gamma e^{-\frac{\partial}{\partial t} S(\gamma, \Psi)} \psi \phi \gamma(t)\psi \)\]

Note for a spin manifold, \( \Phi(P^+ - \Phi(P^-) = \sqrt{\text{det}(I - U)} \) where \( U \) is parallel transport of vectors around \( \gamma \), e.g. \( U \in SO(d) \). So

\[ \text{index } \Phi = \int \partial \gamma e^{-\frac{\partial}{\partial t} S(\gamma, \Psi)} \sqrt{\text{det}(I - U)} \] \]

A correct, well defined path integral formula, but not easy to compute with.

SSQM

Physics says (Witten, Alvarez-Gaumé, Freedman-Windecy) to compute \( \Phi(e^{-tA} \gamma) \) one should integrate over super-curves with appropriate superaction and supermeasure: \( \text{ind } \Phi = \int \partial \gamma \partial \Psi e^{-\frac{\partial}{\partial t} S(\gamma, \Psi)} \) where \( S(\gamma, \Psi) \) is the superfunction \( \gamma^* + \Psi D/\partial \Psi \)

\[ \frac{D}{\partial \Theta} = \frac{d}{d\Theta} + \Gamma^i_\alpha \] = covariant derivative of tangent vector fields along \( \gamma \) and \( \Psi \) is a vector field along \( \gamma \).
supercurve is a vector field along the curve \( \gamma \), i.e., on \( T(L(M), \gamma) \)

\[
\frac{D}{d\theta} \psi \quad \text{the two-form } \psi_{1,2} \rightarrow \langle \psi_{1}, \frac{D}{d\theta} \psi_{2} \rangle
\]

\[
\int_{\gamma} \frac{D}{d\theta} \psi \quad \text{on } T(L(M), \gamma)
\]

\[
\int e^{\frac{D}{d\theta}} \psi \quad \text{means Pfaffian of the skew adjoint operator } \frac{D}{d\theta}, \text{ well defined on } L(M) \text{ if } M \text{ is spin.}
\]

In finite dim, the Berezin integral

\[
\int e^{a_j \psi} \psi \quad \text{means the top part of the form exp } (a_j \psi) = \text{Pfaffian } (a_j),
\]

\( a_j \) is a skew matrix.

\[\text{Lemma: Pfaffian } \frac{D}{d\theta} = \sqrt{\det(I - U)} \]

because...

So physics formal path integral and math path integral are the same except for \( e^{-\frac{t}{2}} \text{ yields harmless as } t \to 0 \).
The point of the superlagrangian $S(\phi, \psi) = \phi^2 + \psi D\phi \psi$ is that it has supersymmetry, a super vector field $Q$ with $QS = 0$,

$$Q: S - \psi \quad \Rightarrow \quad Q = d + i\frac{\partial}{\partial \psi}$$

$$d\psi = 0 \quad \Rightarrow \quad \text{derivative}$$

$$Q^2 = d i\psi + i\frac{\partial}{\partial \psi} d \phi \quad = \frac{\partial}{\partial \phi}$$

Super Noether gives a super current and supercharge. Quantization sends $Q \to \bar{\psi}$ and the super path integral gives $\psi^\dagger$.

Because the index is independent of $t$, one can expect the semi-classical approximation to be exact - taking $t \to 0$.

Use Hessian of $S(\phi, \psi)$ in normal direction, which is quadratic.
Fixing $m_0 \in M$, $\varphi_1, \varphi_2 \in T(M, m_0)$ and letting $f, \eta : S \to T(M, m_0)$, with $\int_S f = \int_S \eta$, the L.C. gives
\[
\int \Theta + \Theta e^{-<f, f + R \varphi_1, \varphi_2> e^{-\eta i}} \frac{df}{d\theta} = \frac{pf + \frac{1}{2} \frac{df}{d\theta}}{\det \left( \frac{1}{2} \left( \frac{df}{d\theta} + R \right) \right)} = \det^{-\frac{1}{2} \frac{1}{2} \text{cosh} \frac{R}{2}} \frac{1}{R^{1/2}},
\]

the $\hat{A}$-genus.

The semiclassical limit is indeed the index (Stroock).

And going back to the heat kernel, one learns that it is better to apply the Levi iteration scheme starting with the heat kernel for the Schrödinger operator on $\Lambda T(M, m)$:
\[\Delta + R \varphi_1 \text{ex} \quad \text{(Getzler)}\]
Mathematically, neither the Dirac-Ramond nor the "measure of the normal bundle to \( M \) exists, but their semiclassical approximations do.

\[
\int_{\mathbb{T}^2} \int_0^{2\pi} \int_0^{2\pi} \partial_X \psi \bar{\psi} \, d\theta \, d\phi
\]

\( \psi \in \bigwedge^{1,2,0} \otimes X^* T(M) \quad \bigwedge^{1,2,0} \approx \sqrt{K} \)

\( \bar{\psi} \) is \( \bar{\partial} \wedge \chi T(M) \) and is skew-adjoint as a map \( \bigwedge^{1,2,0} \otimes X^* T(M) \rightarrow \bigwedge^{1,2,0} \otimes \left(X^* T(M) \right)^* \)

\( (Alvarez, Kellengbeck, Mangano, Windey) \)

\( \text{circle acts on } L(M) \).
The s.c. DR operator $\Phi_R$ has an equivariant $S^1$ index $\Sigma \text{Tr} \, Z^m$, with $m$ the index of $\Phi_R$ restricted to the $m^{th}$ representation space of $S^1$.

The path integral sc can be easily computed and gives the elliptic genus formula for $\text{ind}_{S^1}(\Phi_R)$.

Modular invariance is free of charge because the path integral depends only on the complex structure of the torus.

If $Z_0(\tau) = \int_{\mathcal{M}} \mathcal{D}X \mathcal{D}Y \, e^{-\frac{1}{2} \int \frac{S}{2}(X, Y)}$

its semiclassical approx is $Z_{sc}(\tau) = \text{Pf}(-i \tau + R)$, when $p_1(M) = 0$ one gets, setting $\theta = e^{-2\pi i \tau}$

$$\sum_{g \in \mathbb{Z}} \exp \left( \frac{1}{2 \pi i} \left( \frac{\mathcal{K} \iota}{\mathcal{K}} \right) \tilde{G}_R(\bar{w}) \left( \frac{1}{\eta(\tau)} \right) \right)$$

with $G_R(\bar{w})$. The Eisenstein series $\sum_{w = \text{even}} \frac{1}{w^{2k}}$. 

\[\text{(17)}\]
We can extend the action: \( S(x, t, \phi) = \int \frac{1}{2} x \wedge \overline{\partial} x + 4 \bar{\partial} \phi + \chi x x \)

with \( x \in C^\infty(A^{0, \frac{1}{2}} \otimes \mathbb{X}^* T(M)) \)

Now have two symmetries \((1, 0)\) and \((1, 1)\). For them to anti-commute need to add a term to the action

\( R + 4 \chi x \)