What is a (compact) Riemann surface?

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Wednesday, March 5, 2014

Math CLUB talk.
To help you make a decision about topics to study (in graduate school)?

A WARNING and A REQUEST

YOU WILL NOT BE FAMILIAR WITH ALL CONCEPTS.
INTERRUPT AND ASK QUESTIONS. (IT DOES NOT MATTER IF WE DO NOT COVER ALL THE TOPICS.)
There are many aspects to the theory. Concentrate on:
analytic, algebraic, and algebraic-geometric aspects.
Ignoring mostly;
topological, differential-geometric and PDE aspects.

WHAT (most of) YOU KNOW?

Calculus.
ODEs (and perhaps some PDEs).
Complex numbers.
Vector spaces and linear transformations.
Some abstract algebra.
Perhaps a little bit of complex analysis.

A step backwards.
Two power series (from calculus).
THE FIRST ONE

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + ...$$

Converges for $|x| < 1$. WHY?
THE SECOND ONE

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + ...$$

Converges for $|x| < 1$. WHY?

CALCULUS DONE RIGHT!

Complex analysis course: Study of *holomorphic* functions; i.e., solutions to the PDE (complex notation)

$$f_{\bar{z}} = 0$$

in subsets of $\mathbb{C}$.
Above is a short hand for (real notation/concepts, except for $\imath = \sqrt{-1}$)

$$z = x + \imath y,\ f = u + \imath v,\ u_x = v_y,\ u_y = -v_x.$$

The two series define holomorphic functions (replace x by z).
COMPLEX ANALYSIS IS CALCULUS (in a more appropriate setting).

Definition (Informal)

A *Riemann surface* is a one dimensional complex manifold. The correct space for the study holomorphic functions. Locally an open set in $\mathbb{C} \cong \mathbb{R}^2$.

FORMAL DEFINITION

Definition (Riemann surface)

A *Riemann surface* is a (second countable) topological space X together with a (maximal) cover of open *coordinate charts* $X = \cup_i U_i$, where for each i we have a (usually not surjective) homeomorphism $z_i : U_i \to \mathbb{C}$ such that for each pair i, j,

$$z_i \circ z_j^{-1} : z_j(U_j \cap U_i) \to z_i(U_i \cup U_j)$$

is a holomorphic function.

In particular, THE DEFINITION MAKES SENSE and every domain in $\mathbb{C}$ is a Riemann surface; so is the *Riemann sphere* $\mathbb{C} \cup \{\infty\}$. In the last example, we take $U_1 = \mathbb{C}$, $z_1(z) = z$, $U_2 = (\mathbb{C} - \{0\}) \cup \{\infty\}$, $z_2(z) = \frac{1}{z}$. Thus

$$z_1 \circ z_2^{-1}(z) = \frac{1}{z} = z_2 \circ z_1^{-1}(z),$$

which certainly is a holomorphic function on $\mathbb{C} - \{0\}$.

DICHOTOMIES

- closed = compact. (MOST OF) THE SUBJECT OF THIS TALK.
- open = not compact.
- with or without (throughout this talk) boundary.

Classifications

- Topological. Genus of compact surface: 0, 1, 2, ... One surface for each genus up to topological equivalence (homeomorphisms).
- Complex analytic – up to bi-holomorphic (called *conformal*) maps.
  - holomorphic or analytic.
  - bi–holomorphic (natural equivalence).
- One Riemann surface of genus 0. Higher genera have *moduli*.

Uniformization

There are only three (up to equivalence) simply connected Riemann surfaces:

- The Riemann sphere $\mathbb{C} \cup \{\infty\}$.
- The complex plane $\mathbb{C}$.
- The upper half plane $\mathbf{H}^2$; points in the complex plane whose imaginary part is positive.

Natural motions

A matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathrm{SL}(2, \mathbb{C})$ acts on the Riemann sphere by

$$z \mapsto \frac{az + b}{cz + d}.$$

The group of natural motions for

- the Riemann sphere: $\mathrm{PSL}(2, \mathbb{C})$.
- the complex plane: (the affine group) upper triangular matrices in $\mathrm{PSL}(2, \mathbb{C})$.
- the upper half plane: $\mathrm{PSL}(2, \mathbb{R})$.

Topology (covering spaces theory) now tells us "in some sense" what each surface "looks like."

Holomorphic functions

Holomorphic maps from the Riemann surface X into $\mathbb{C}$.
An integral domain.
Only constants if X is compact.

Meromorphic functions

Definition

Locally the ratio of two holomorhic functions.

Not globally for compact case, of course.
Existence is big issue.
Holomorphic maps of X into $\mathbb{C} \cup \{\infty\}$ excluding the constant map that sends all of X to ∞.
A field.

The function field for compact surfaces

The field $K(X)$ of meromorphic functions on the compact surface X is isomorphic to a quotient field of

$$\mathbb{C}(z)[w].$$

We regard z and w as meromorphic functions on X. They satisfy an *irreducible* polynomial equation in two variables

$$P(z, w) = 0;$$

Thus

$$K(X) \cong \mathbb{C}(z)[w]/(P).$$

Algebraic/geometric point of view

Start With $P \in \mathbb{C}[z][w]$.

- Affine point of view.

$$P(z,w) = \sum a_{ij} z^i w^j.$$

$$S = \{(z,w) \in \mathbb{C}^2; P(z,w) = 0\}.$$

Almost a Riemann surface. Must de-singularize some points. Must add points at infinity.

- Projective point of view.
Projective P; that is, change it appropriately. Define the degree of P by

$$n = \max\{i+j\},$$

and its *projective version* by

$$\mathbf{P}(z,w,x) = \sum a_{ij} z^i w^j x^{n-i-j}.$$

$$\mathbf{S} = \{(z,w,x) \in \mathbf{P}\mathbb{C}^2; \mathbf{P}(z,w,x) = 0\}.$$

Almost a Riemann surface. Must de-singularize some points.

The function field

As before, let $K(X)$ be the field of meromorphic functions on the Riemann surface X.
X can be recovered from $K(X)$ as the set of discrete valuations of rank 1 (PURE ALGEBRA!)

- CLOSED CASE. Classical (19th century).
  Because $K(X)$ is algebraic.
- OPEN CASE. Modern (20th century).
  $H(X)$ the integral domain of holomorphic functions on the surface X is big: X can be recovered as the set of maximal principal ideals in $H(X)$. $K(X)$ is the field of fractions of $H(X)$. Both theorems connect ANALYSIS and ALGEBRA
  $H(X)$ can be characterized using algebra and TOPOLOGY.

Geometric connection

- Spherical.
- Euclidean.
- Hyperbolic.

The action is in the last case.
Not surprising is the connections to 2-dimensional hyperbolic manifolds.
BUT ROLE IN THE STUDY OF 3-DIMENSIONAL HYPERBOLIC MANIFOLDS IS A SURPRISE!

Moduli problems

Genus dependent.