Homework

1. Solve for the general solution to the differential equation $y'' - 6y' + 5 = 0$.

2. Solve for the general solution to the differential equation $y'' + 4y = 0$

3. Solve for the general solution to the differential equation $16y''+8y'+1 = 0$.

4. Solve for the general solution to the differential equation $2y'' + 7y' - 4 = 0$ and also for the particular solution the same differential equation with initial conditions $y(0) = 8$, $y'(0) = 1$.

5. Solve for the general solution to the differential equation $y'' - 6y' + 9y = 0$ and also for the particular solution to the same differential equation with initial conditions $y(0) = -2$, $y'(0) = 1$.

6. Solve for the general solution to the differential equation $y'' - y' + y = 0$ and also for the particular solution to the same differential equation with initial conditions $y(0) = 2$, $y'(0) = 0$.

Solutions

1. Solve for the general solution to the differential equation $y'' - 6y' + 5 = 0$.

   Solution: Since this is a second order linear differential equation with constant coefficients, we may solve it by using the characteristic equation: $r^2 - 6r + 5 = 0$. The general solution to this equation has the form $y = C_1e^{r_1x} + C_2e^{r_2x}$ where C_1, C_2 are arbitrary constants and r_1, r_2 are roots of the characteristic equation, satisfying $r_1 \neq r_2$. Factoring the characteristic equation, we get $(r-5)(r-1) = 0$. Hence, the roots are $r_1 = 5$, $r_2 = 1$. Hence, the general solution is the function $y = C_1e^{5x} + C_2e^x$.

2. Solve for the general solution to the differential equation $y'' + 4y = 0$

   Solution: Since this is a second order linear differential equation with constant coefficients, we may solve it by using the characteristic equation: $r^2 + 4 = 0$. The roots of this equation are $r = \pm 2i$. For complex roots of the form $a \pm bi$, the general solution to this equation is of the form $y = e^{ax}(A\cos(bx) + B\sin(bx))$ where A, B are arbitrary constants. Since our roots are purely imaginary, $a = 0$. Following the same notation, we have that $b = 2$. Hence, the general solution is $y = A\cos(2x) + B\sin(2x)$.

3. Solve for the general solution to the differential equation $16y'' + 8y' + 1 = 0$.

   Solution: Since this is a second order linear differential equation with constant coefficients, we may solve it by using the characteristic equation: $16r^2 + 8r + 1 = 0$. However, this may be rewritten as $(4r+1)^2 = 0$. Since $r = -1/4$ is a double root (and the only root) to this equation, the general solution to this equation has the form $y = C_1e^{rx} + C_2xe^{rx}$. Plugging in our value for r, we get that the general solution is $y = C_1e^{-x/4} + C_2xe^{-x/4}$.

4. Solve for the general solution to the differential equation $2y'' + 7y' - 4 = 0$ and also for the particular solution the same differential equation with initial conditions $y(0) = 8$, $y'(0) = 1$.

   Solution: Since this is a second order linear differential equation with constant coefficients, we may solve it by using the characteristic equation: $2r^2 + 7r - 4 = 0$. The general solution to this equation has the form $y = C_1e^{r_1x} + C_2e^{r_2x}$ where C_1, C_2 are arbitrary constants and r_1, r_2 are roots of the characteristic equation, satisfying $r_1 \neq r_2$. Factoring the characteristic equation, we get $(2r-1)(r+4) = 0$. Hence, the roots are $r_1 = 1/2$, $r_2 = -4$. Hence, the general solution is the function $y = C_1e^{x/2} + C_2e^{-4x}$. Plugging in the initial condition $y(0) = 8$, we see that $8 = C_1e^0 + C_2e^0 = C_1 + C_2$. In order to utilize the second initial condition, we must first compute y'. $y' = C_1/2e^{x/2} - 4C_2e^{-4x}$. Plugging in the initial condition $y'(0) = 1$, we see that $1 = (C_1/2)e^0 - 4C_2e^0 = C_1/2 - 4C_2$. Solving this system of equations, we get $C_1 = 22/3$, $C_2 = 2/3$. Hence, the particular solution is $y = (22/3)e^{x/2} + 2/3e^{-4x}$.

5. Solve for the general solution to the differential equation $y'' - 6y' + 9y = 0$ and also for the particular solution to the same differential equation with initial conditions $y(0) = -2$, $y'(0) = 1$.

   Solution: The characteristic equation is $r^2 - 6r + 9 = 0$. Factoring, we get $(r-3)(r-3) = 0$. Since both roots are at $r = 3$, the general solution is $y = C_1 e^{3x} + C_2 x e^{3x}$. Plugging in the initial condition $y(0) = -2$, we have $-2 = C_1 e^0 + C_2(0)e^0 = C_1$. Hence, $C_1 = -2$. To utilize the other initial condition, $y'(0) = 0$, we must first compute y'. $y' = 3C_1 e^{3x} + (C_2 e^{3x} + 3C_2 x e^{3x})$. Plugging in the point $(0, 1)$ and $C_1 = -2$, we have

   $$1 = 3(-2)e^0 + C_2 e^0 + 3C_2(0)e^0 \qquad = -6 + C_2$$

   Hence, $C_2 = 7$. So the particular solution is $y = -2e^{3x} + 7xe^{3x}$.

6. Solve for the general solution to the differential equation $y'' - y' + y = 0$ and also for the particular solution to the same differential equation with initial conditions $y(0) = 2$, $y'(0) = 0$.

   Solution: The characteristic equation is $r^2 - r + 1 = 0$. Using the quadratic formula to find the roots, we have $r = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}i}{2}$. Since these are complex roots and we only care about a real solution, the general solution is of the form $y = e^{x/2}(A\cos(\sqrt{3}x) + B\sin(\sqrt{3}x))$.

   Utilizing the initial conditions, we solve for the particular solution. Since $y(0) = 2$, we have $2 = e^0(A\cos(0) + B\sin(0)) = A$. So $A = 2$. To find B, we must first compute y'. Using product rule, we have

   $$y' = \frac{1}{2}e^{x/2}(A\cos(\sqrt{3}x)+B\sin(\sqrt{3}x))+e^{x/2}(-\sqrt{3}A\sin(\sqrt{3}x)+\sqrt{3}B\cos(\sqrt{3}x))$$

   Plugging in the initial condition $y'(0) = 0$ gives us

   $$0 = \frac{1}{2}e^0(A\cos(0)+B\sin(0))+e^0(-\sqrt{3}A\sin(0)+\sqrt{3}B\cos(0)) = \frac{1}{2}(A)+1(\sqrt{3}B).$$

   Hence, $B = (-\frac{1}{2}A)/\sqrt{3} = -1/\sqrt{3}$. Plugging A, B into the general solution gives us the particular solution $y = e^{x/2}(2\cos(\sqrt{3}x) - \frac{1}{\sqrt{3}}\sin(\sqrt{3}x))$.

Answer Key

1. Solve for the general solution to the differential equation $y'' - 6y' + 5 = 0$.

   $y = C_1 e^{5x} + C_2 e^{x}$

2. Solve for the general solution to the differential equation $y'' + 4y = 0$

   $y = A\cos(2x) + B\sin(2x)$.

3. Solve for the general solution to the differential equation $16y'' + 8y' + 1 = 0$.

   $y = C_1 e^{-x/4} + C_2 x e^{-x/4}$.

4. Solve for the general solution to the differential equation $2y'' + 7y' - 4 = 0$ and also for the particular solution the same differential equation with initial conditions $y(0) = 8$, $y'(0) = 1$.

   General solution: $y = C_1 e^{x/2} + C_2 e^{-4x}$.

   Particular solution: $y = (22/3)e^{x/2} + 2/3e^{-4x}$.

5. Solve for the general solution to the differential equation $y'' - 6y' + 9y = 0$ and also for the particular solution to the same differential equation with initial conditions $y(0) = -2$, $y'(0) = 1$.

   General solution: $y = C_1 e^{3x} + C_2 x e^{3x}$.

   Particular solution: $y = -2e^{3x} + 7xe^{3x}$.

6. Solve for the general solution to the differential equation $y'' - y' + y = 0$ and also for the particular solution to the same differential equation with initial conditions $y(0) = 2$, $y'(0) = 0$.

   General solution: $y = e^{x/2}(A\cos(\sqrt{3}x) + B\sin(\sqrt{3}x))$.

   Particular solution: $y = e^{x/2}(2\cos(\sqrt{3}x) - \frac{1}{\sqrt{3}}\sin(\sqrt{3}x))$.