Homework

1. Find the orthogonal family to the family of curves $y = Ce^x$.

2. Find the orthogonal family to the family of curves $y = \dfrac{-1}{x^2 + C}$.

3. Find the orthogonal family to the family of curves $y = Cx^3$.

4. Use Euler's method with step size $h = 0.1$ to estimate $y(0.3)$ where $y(x)$ is a solution to the differential equation $y' + 3y = x^2$ passing through the point $(0, 2)$.

5. For the initial value problem with differential equation $yy' = x$ and initial condition $y(1) = 1$, estimate $y(1.4)$ using Euler's method with step size $h = 0.1$.

Solutions

1. Find the orthogonal family to the family of curves $y = Ce^x$.

   Solution:

   (a) Isolate C: $C = ye^{-x}$.

   (b) Differentiate: Taking derivatives of both sides, we get $0 = y'e^{-x} - ye^{-x}$.

   (c) Solve for y': Solving for y', we get $y'e^{-x} = ye^{-x}$ which implies $y' = y$.

   (d) Compose DE for orthogonal family: We write the DE setting y' equal to the negative reciprocal of the original. This gives us $y' = -1/y$ for our orthogonal family.

   (e) Solve the new DE: This is a separable equation. Separating, we get $yy' = -1$. So we want to take the integrals of $\int y dy = \int -dx$. This gives us $y^2/2 = -x + C_1$. We can rewrite this as $y = \pm\sqrt{-2x + C}$. And we are done.

2. Find the orthogonal family to the family of curves $y = \dfrac{-1}{x^2 + C}$.

   Solution:

   (a) Isolate C: To isolate C, we first get $y(x^2 + C) = -1$. Thus, $yx^2 = -1 - yC$. So then $yC = -yx^2 - 1$ and so finally, $C = -x^2 - 1/y$.

   (b) Differentiate: Taking derivatives of both sides, we get $0 = -2x + (1/y^2)y'$.

   (c) Solve for y': Solving for y' we get $y' = 2xy^2$.

   (d) Compose DE for orthogonal family: We write the DE setting y' equal to the negative reciprocal of the original. This gives us $y' = \dfrac{-1}{2xy^2}$.

   (e) Solve the new DE: This is a separable equation. Separating, we get $y^2y' = \dfrac{-1}{2x}$. So we want to take the integrals of $\int y^2 dy = \int \dfrac{-1}{2x} dx$. This gives us $y^3/3 = -\ln|2x| + C_1$. We can rewrite this as $y = (-3\ln|2x| + C)^{1/3}$. And we are done.

3. Find the orthogonal family to the family of curves $y = Cx^3$.

   Solution:

   (a) Isolate C: To isolate C, we get $C = y/x^3$.

   (b) Differentiate: Taking derivatives of both sides (using the quotient rule), we get $0 = (x^3y' - 3yx^2)/x^6$. Multiplying both sides by x^6, we get $0 = x^3y' - 3yx^2$.

   (c) Solve for y': Solving for y' we get $y' = 3yx^2/x^3 = 3y/x$.

(d) Compose DE for orthogonal family: We write the DE setting y' equal to the negative reciprocal of the original. This gives us $y' = \dfrac{-x}{3y}$.

(e) Solve the new DE: This is a separable equation. Separating, we get $3yy' = -x$. So we want to compute $\int 3y dy = \int -x dx$. This gives us $3y^2/2 = -x^2/2 + C_1$. We can rewrite this as $x^2 + 3y^2 = C^2$. And we are done.

4. Use Euler's method with step size $h = 0.1$ to estimate $y(0.3)$ where $y(x)$ is a solution to the differential equation $y' + 3y = x^2$ passing through the point $(0, 2)$.

Solution: To estimate $y(0.3)$ with a step size of $h = 0.1$ starting at 0 requires taking 3 steps. We first express y' as a function $f(x, y)$. Since $y' + 3y = x^2$, we have $y' = x^2 - 3y = f(x, y)$. To begin the Euler's method algorithm, we begin with $x_0 = 0, y_0 = 2$.

$$x_1 = X_0 + h = 0 + 0.1 = 0.1$$
$$y_1 = y_0 + h * f(x_0, y_0) = 2 + 0.1 * (0 - 3 * 2) = 1.4$$
$$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$$
$$y_2 = y_1 + h * f(x_1, y_1) = 1.4 + 0.1 * (0.1^2 - 3 * 1.4) = .981$$
$$x_3 = x_2 + h = 0.2 + 0.1 = 0.3$$
$$y_3 = y_2 + h * f(x_2, y_2) = .981 + 0.1 * (0.2^2 - 3 * .981) = .6907$$

We have thus found $y(0.3) \simeq y_3 = .6907$ and we are done.

5. For the initial value problem with differential equation $yy' = x$ and initial condition $y(1) = 1$, estimate $y(1.4)$ using Euler's method with step size $h = 0.1$.

To estimate $y(1.4)$ with a step size of $h = 0.1$ starting at 1 requires taking 4 steps. We first express y' as a function $f(x, y)$. Since $yy' = x$, we have $y' = x/y = f(x, y)$. To begin the Euler's method algorithm, we begin with $x_0 = 1, y_0 = -3$.

$$x_1 = X_0 + h = 1 + 0.1 = 1.1$$
$$y_1 = y_0 + h * f(x_0, y_0) = -3 + 0.1 * (1/-3) \simeq -3.03333$$
$$x_2 = x_1 + h = 1.1 + 0.1 = 1.2$$
$$y_2 = y_1 + h * f(x_1, y_1) \simeq -3.03333 + 0.1 * (1.1/-3.03333) \simeq -3.0696$$
$$x_3 = x_2 + h = 1.2 + 0.1 = 1.3$$
$$y_3 = y_2 + h * f(x_2, y_2) \simeq -3.0696 + 0.1 * (1.2/-3.0696) \simeq -3.1087$$
$$x_4 = x_3 + h = 1.3 + 0.1 = 1.4$$
$$y_4 = y_3 + h * f(x_2, y_2) \simeq -3.1087 + 0.1 * (1.3/-3.1087) \simeq -3.1505$$

We have thus found $y(1.4) \simeq y_4 \simeq -3.1505$ and we are done.

Answer Key

1. Find the orthogonal family to the family of curves $y = Ce^x$.

   $y = \pm\sqrt{-2x + C}$

2. Find the orthogonal family to the family of curves $y = \dfrac{-1}{x^2 + C}$.

   $y = (-3 \ln |2x| + C)^{1/3}$

3. Find the orthogonal family to the family of curves $y = Cx^3$.

   $x^2 + 3y^2 = C^2$

4. Use Euler's method with step size $h = 0.1$ to estimate $y(0.3)$ where $y(x)$ is a solution to the differential equation $y' + 3y = x^2$ passing through the point $(0, 2)$.

   $y(0.3) \simeq y_3 = .6907$

5. For the initial value problem with differential equation $yy' = x$ and initial condition $y(1) = 1$, estimate $y(1.4)$ using Euler's method with step size $h = 0.1$.

   $y(1.4) \simeq -3.1505$