MAT 132 HW 9-12

1. Problems

1. Find the area of the bounded region contained between the curves $f(x) = x^3$ and $g(x) = \sqrt{x}$.
2. Find the area of the bounded region contained between the curves $f(x) = \sin(\frac{1}{2}x)$ and $g(x) = (\frac{1}{2}x - 1)^2$ and between the lines $x = \pi/2$ and $x = \pi$.
3. Find the area of the region contained in the first quadrant and bounded by the polar curve $r(\theta) = \sin\theta + \cos\theta$.
4. Find the area of the region bounded by the y-axis and the parametric curve given by $x(t) = t^3 - 9t$ and $y(t) = t^2$.
5. Find the length of the curve $y = \frac{2}{3}(x^2 + 1)^{3/2}$ over the interval $[1, 4]$.

2. Answer Key

1. The area is $5/12$
2. The area is
$$-\frac{2}{3}\left(\frac{\pi}{2}-1\right)^3+\sqrt{2}+\frac{2}{3}\left(\frac{\pi}{4}-1\right)^3$$
3. The area is $(\pi+2)/4$
4. The area is $324/5$
5. 45

3. Solutions

1. The curves meet at $x = 0$ and $x = 1$, with the graph of $\sqrt{x}$ lying above that of x^3. So the area we are searching for is
$$\int_0^1 \sqrt{x} - x^3 \, dx = \frac{2}{3}x^{3/2} - \frac{1}{4}x^4 \Big|_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}.$$
2. Graphing the functions on a plane, one sees that $f(x)$ lies above $g(x)$ on the specified interval $[\pi/2, \pi]$, so the area we are searching for is
$$\begin{aligned}\int_{\frac{\pi}{2}}^{\pi} \sin\left(\frac{1}{2}x\right) - \left(\frac{1}{2}x - 1\right)^2 \, dx &= -2\cos\left(\frac{1}{2}x\right) - \frac{2}{3}\left(\frac{1}{2}x - 1\right)^3 \Big|_{\frac{\pi}{2}}^{\pi} \\ &= -\frac{2}{3}\left(\frac{\pi}{2} - 1\right)^3 + \sqrt{2} + \frac{2}{3}\left(\frac{\pi}{4} - 1\right)^3\end{aligned}$$
3. The first quadrant is described by $0 \le \theta \le \pi/2$ so we apply this to the formula for area bounded by a polar curve and we get
$$\begin{aligned}\frac{1}{2}\int_0^{\pi/2} r(\theta)^2 \, d\theta &= \frac{1}{2}\int_0^{\pi/2} (\sin\theta + \cos\theta)^2 \, d\theta \\ &= \frac{1}{2}\int_0^{\pi/2} \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta \, d\theta.\end{aligned}$$
Now we use the identities $\sin^2\theta + \cos^\theta = 1$ and $2\sin\theta\cos\theta = \sin(2\theta)$ to get that this equals
$$\begin{aligned}\frac{1}{2}\int_0^{\pi/2} \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta \, d\theta &= \frac{1}{2}\int_0^{\pi/2} 1 + \sin(2\theta) \, d\theta \\ &= \frac{1}{2}\theta - \frac{1}{2}\cos(2\theta) \Big|_0^{\pi/2} = \frac{\pi + 2}{4}.\end{aligned}$$
4. Graphing the curve, we see that it $x(t)$ is negative and $y(t)$ seems to start at the origin and goes up until the curve intersects the y-axis. The point at which this happens is found by setting $x(t) = 0$, and we find that $t = 0$ and $t = 3$. Indeed we can check that x is negative for all t between 0 and 3, and $dy/dt = 2t$ is positive for all such t. Hence we can say that the area is equal to
$$-\int_0^3 x \, dy = -\int_0^3 (t^3 - 9t)(2t) \, dt = -\int_0^3 2t^4 - 18t^2 \, dt = -\left(\frac{2}{5}t^5 - 6t^3\right)\Big|_0^3 = \frac{324}{5}.$$
5. Differentiating gives
$$\frac{dy}{dx} = 2x\sqrt{x^2 + 1}.$$
It follows that $1 + (dy/dx)^2 = 1 + 4x^2(x^2 + 1) = 4x^4 + 4x^2 + 1 = (2x^2 + 1)^2$. Hence the arclength is
$$\int_1^4 \sqrt{1 + (dy/dx)^2} \, dx = \int_1^4 \sqrt{(2x^2 + 1)^2} \, dx = \int_1^4 2x^2 + 1 \, dx = \frac{2x^3}{3} + x \Big|_1^4 = 45.$$