MAT 132 HW 7-8

1. Problems

1. Determine whether the following improper integral converges or diverges. If it converges, compute the result:
$$\int_1^\infty \frac{1}{x^4}\,dx$$
2. Determine whether the following improper integral converges or diverges. If it converges, compute the result:
$$\int_{-2}^4 \frac{1}{(x+1)^2}\,dx$$
3. Determine whether the following improper integral converges or diverges. If it converges, compute the result:
$$\int_3^{12} \frac{1}{\sqrt{x-3}}\,dx$$
4. Determine whether the following improper integral converges or diverges. If it converges, compute the result:
$$\int_7^\infty \frac{1}{(x+2)^{3/2}}$$
5. Determine whether the following improper integral converges or diverges. If it converges, compute the result:
$$\int_0^\infty \cos(4x+5)\,dx$$

2. Answer Key

1. 1/3
2. The integral diverges
3. 6
4. 2/3
5. The integral diverges

3. Solutions

1. By definition, the improper integral is given by
$$\int_1^\infty \frac{1}{x^4}\,dx = \lim_{c\to\infty}\int_1^c \frac{1}{x^4}\,dx = \lim_{c\to\infty} -\frac{1}{3x^3}\bigg|_1^c = \lim_{c\to\infty}\left(-\frac{1}{3c^3}+\frac{1}{3}\right) = \frac{1}{3}.$$
Hence the integral converges and is equal to $1/3$.
2. The integrand has a singularity at $x=-1$, so by definition the improper integral is given by
$$\int_{-2}^4 \frac{1}{(x+1)^2}\,dx = \int_{-2}^{-1}\frac{1}{(x+1)^2}\,dx + \int_{-1}^4 \frac{1}{(x+1)^2}\,dx.$$
Our original integral converges if and only if both of these new integral converge, so we must calculate these separately. Let's start with the first one. By definition we have
$$\int_{-2}^{-1}\frac{1}{(x+1)^2}\,dx = \lim_{c\to -1^-}\int_{-2}^c \frac{1}{(x+1)^2}\,dx.$$
We can find the antiderivative using a u-substitution with $u=x+1$. Our bounds then become $u=-1$ and $u=c+1$ and we find that
$$\int_{-2}^{-1}\frac{1}{(x+1)^2}\,dx = \lim_{c\to-1^-}\int_{-1}^{c+1}\frac{1}{u^2}\,du = \lim_{c\to-1^-} -\frac{1}{u}\bigg|_{-1}^{c+1} = \lim_{c\to-1^-} -\frac{1}{c+1} - 1.$$
Now the limit we have obtained does not exist. Since one of the integrals does not converge, the original integral does not converge.
3. By definition, the improper integral is given by
$$\int_3^{12}\frac{1}{\sqrt{x-3}}\,dx = \lim_{c\to3^+}\int_c^{12}\frac{1}{\sqrt{x-3}}\,dx.$$
We can find the antiderivative using a u-substitution with $u=x-3$. Our bounds then become $u=c-3$ and $u=9$. We obtain
$$\lim_{c\to3^+}\int_c^{12}\frac{1}{\sqrt{x-3}}\,dx = \lim_{c\to3^+}\int_{c-3}^9\frac{1}{\sqrt{u}}\,dx = \lim_{c\to3^+} 2\sqrt{u}\bigg|_{c-3}^9 = \lim_{c\to3^+}\left(2\sqrt{9}-2\sqrt{c-3}\right) = 6.$$
Hence the integral converges and is equal to 6.
4. By definition, the improper integral is given by
$$\int_7^\infty \frac{1}{(x+2)^{3/2}}\,dx = \lim_{c\to\infty}\int_7^c \frac{1}{(x+2)^{3/2}}\,dx.$$
We can find the antiderivative using a u-substitution with $u=x+2$. Our bounds then become $u=9$ and $u=c+2$. We obtain
$$\lim_{c\to\infty}\int_7^c\frac{1}{(x+2)^{3/2}}\,dx = \lim_{c\to\infty}\int_9^{c+2}\frac{1}{u^{3/2}}\,dx = \lim_{c\to infty} -\frac{2}{\sqrt{u}}\bigg|_9^{c+2} = \lim_{c\to\infty}\left(-\frac{2}{\sqrt{c}}+\frac{2}{\sqrt{9}}\right) = \frac{2}{3}.$$
Hence the integral converges and is equal to $2/3$.
5. By definition, the improper integral is given by
$$\int_0^\infty \cos(4x+5)\,dx = \lim_{c\to\infty}\int_0^c \cos(4x+5)\,dx.$$

The antiderivative of the integrand is $\frac{1}{4}\sin(4x+5)$, so this then becomes

$$\int_0^\infty \cos(4x+5)\;dx = \lim_{c\to\infty}\int_0^c \cos(4x+5)\;dx = \lim_{c\to\infty}\frac{1}{5}\left(\sin(4c+5)-\sin(4\cdot 0+5)\right).$$

Now, since $\sin(4\cdot 0+5)=\sin 5$ is a real number and $\lim_{c\to\infty}\sin(4c+5)$ does not exist due to the oscillatory nature of sine, this limit does not exist. Hence the integral diverges.