① Calculation of sums

Ex. 1 Find the sum

$$\underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}}_{\text{what's that}} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{6}\right)^{2n}}{(2n)!} \textcircled{=}$$

looks like

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos x$$

$$\textcircled{=} \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\right]_{x=\frac{\pi}{6}} = \cos x \Big|_{x=\frac{\pi}{6}} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

② Limit calculations

Ex. 2

$$\lim_{x\to 0} \frac{(e^x - 1 - x)^2}{x^2 - \ln(1+x^2)} = \left[\frac{0}{0}\right] \textcircled{=}$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

$$\textcircled{=} \lim_{x\to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots - 1 - x\right)^2}{x^2 - \left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \dots\right)} =$$

$$= \lim_{x\to 0} \frac{\left(\frac{x^2}{2} + \frac{x^3}{6} + \dots\right)^2}{\frac{x^4}{2} - \frac{x^6}{3} + \dots} =$$

$$= \lim_{x\to 0} \frac{\frac{x^4}{4} + \frac{x^5}{6} + \dots}{\frac{x^4}{2} - \frac{x^6}{3} + \dots} =$$

div. by x^4

$$= \lim_{x\to 0} \frac{\frac{1}{4} + \frac{\cancel{x}}{6} + \dots}{\frac{1}{2} - \frac{x^2}{3} + \dots} = \frac{1/4}{1/2} = \boxed{\frac{1}{2}}$$

③ Integration

Ex. 3 Evaluate the integral in terms of p.s.

$$\underbrace{Si(x)}_{\text{the integral sign}} = \int_0^x \frac{\sin t}{t}\, dt$$

doesn't have elem. antideriv.

$$\frac{\sin t}{t} = \frac{1}{t} \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!}$$

$$Si(x) = \int_0^x \frac{\sin t}{t}\, dt = \int_0^x \left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!} \right) dt =$$

term-by-term integr.

$$= \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n t^{2n}}{(2n+1)!}\, dt =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{t^{2n+1}}{2n+1} \Bigg|_{t=0}^{t=x} \right) =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \quad x^{2n+1}}{(2n+1)! \ (2n+1)} =$$

$$= x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \dots$$