Episode 40. Maclaurin series for trigonometric functions

Maclaurin series for $\underline{f(x) = \sin x}$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \underbrace{f(0)}_{0} + \frac{f'(0)}{1!} x + \underbrace{\frac{f''(0)}{2!} x^2}_{0} + \frac{f'''(0)}{3!} x^3 + \underbrace{\frac{f^{IV}(0)}{4!} x^4}_{0} + \dots$$

$$f^{(n)}(x) = \begin{cases} \sin x, & n = 0, 4, 8 \dots \\ \cos x, & n = 1, 5, 9 \dots \\ -\sin x, & n = 2, 6, 10 \dots \\ -\cos x, & n = 3, 7, 11 \dots \end{cases}$$

$$f^{(n)}(0) = \begin{cases} 0, & n \text{ is even} \\ 1, & n = 1, 5, 9, \dots \\ -1, & n = 3, 7, 11, \dots \end{cases}$$

M. series for $\sin x$ is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$|f^{(n)}(x)| \le 1$ for all x

so all derivatives are $\underline{\text{bounded}}$ $\Rightarrow$ $f(x) =$ its T.s. for all x

$$\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \underbrace{\underbrace{x}_{T_1(x)} - \frac{x^3}{3!}}_{T_3(x)} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad \text{for all } x}$$

Maclaurin pol. for $\sin x$:

$T_1(x) = x$

$T_3(x) = x - \frac{x^3}{3!}$

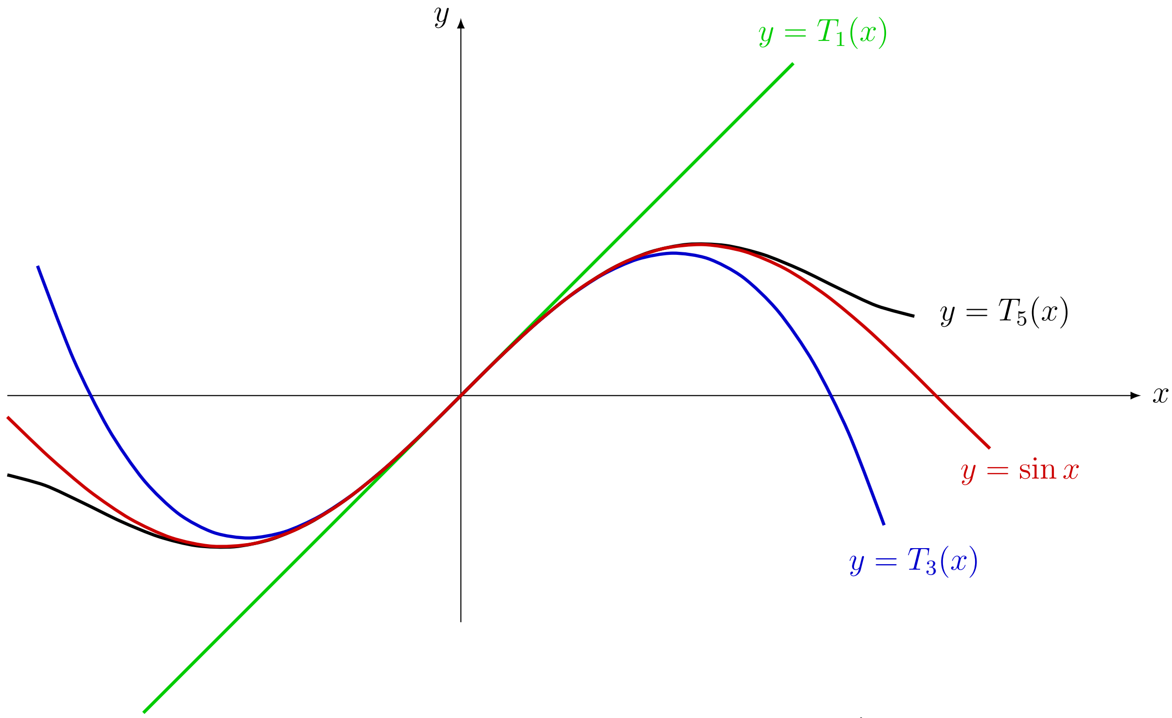
$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$\sin x \approx x$

$\sin x \approx x - \frac{x^3}{6}$

$\sin x \approx x - \frac{x^3}{6} + \frac{x^5}{120}$

for small x



M. series for $f(x) = \cos x$:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$\frac{d}{dx}$ $\downarrow$ term-by-term

$$\boxed{\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \underset{\sim}{1} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad \text{for all } x}$$

$T_1(x) = 1$

$T_2(x) = 1 - \frac{x^2}{2}$

$T_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$