Episode 39. Taylor polynomials

Let $f(x)$ be ∞ diff-able f-n (has derivatives of all orders).
Then Taylor series for $f(x)$ around $x=a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n =$$

$$= \underbrace{f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\substack{\text{Taylor polynomial for } f(x) \text{ at } a \\ T_n(x)}} + \underbrace{\sum_{k=n+1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k}_{\text{tail}}$$

If all derivatives of f are bounded near a,
then

$$\underbrace{f(x)}_{\text{f-n}} = \underbrace{\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\text{its T. S.}}$$

and

$$f(x) \approx T_n(x) \quad \text{near } a.$$

Ex. 1 Find T. polynomials of deg 1, 2 and 3
for $f(x) = e^x$ around $x=0$ (Maclaurin pol.)
Calculate app value of $\sqrt{e}$.

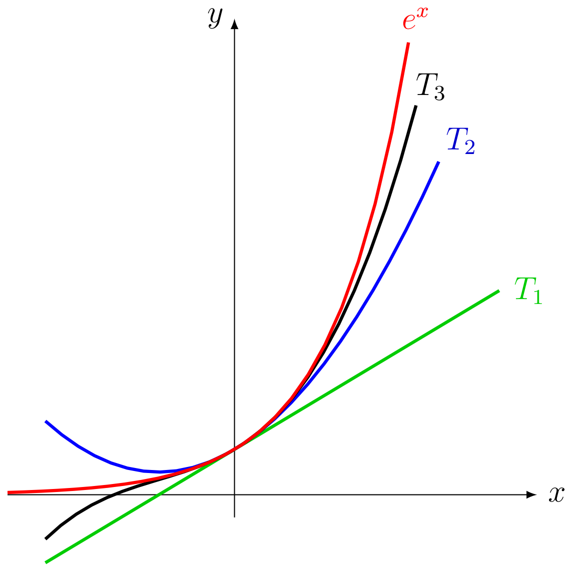
Sol.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \underbrace{1 + \frac{x}{1!}}_{T_1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$T_1(x) = 1 + x$ $\qquad e^x \approx T_1(x)$

$T_2(x) = 1 + x + \frac{x^2}{2}$ $\qquad e^x \approx T_2(x)$

$T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ $\qquad e^x \approx T_3(x)$



$$\sqrt{e} = e^{\frac{1}{2}} = e^x\Big|_{x=\frac{1}{2}} \approx T_3(x)\Big|_{x=\frac{1}{2}} = 1 + \frac{1}{2} + \frac{\left(\frac{1}{2}\right)^2}{2} + \frac{\left(\frac{1}{2}\right)^3}{6} =$$

$$= \frac{79}{48} \approx 1.6458$$

Actual value : $\sqrt{e} = 1.64872\ldots$

Ex. 2 Find T.s. for $f(x) = \underbrace{3x^2 - 4x + 2}_{\text{pol. of deg. 2}}$ at $\underbrace{x=1}_{\text{center}}$

T.s. is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n =$

$$= f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \ldots$$

	$x=1$
$f(x) = 3x^2 - 4x + 2$	$f(1) = 3 - 4 + 2 = 1$
$f'(x) = 6x - 4$	$f'(1) = 6 - 4 = 2$
$f''(x) = 6$	$f''(1) = 6$
$f^{(n)}(x) = 0$	$f^{(n)}(1) = 0$ for all $n = 3, 4, 5, \ldots$

T.s. for f is

$$f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 =$$

$$= 1 + 2(x-1) + \frac{6}{2}(x-1)^2 =$$

$$\boxed{1 + 2(x-1) + 3(x-1)^2}$$

T.s. for f at $x=1$

$(= T_2(x))$