Episode 38. Taylor series

As we know (see Episodes 35-36),

geom. $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \qquad |x| < 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \qquad \text{for all } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \qquad -1 < x \le 1$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \qquad -1 \le x \le 1$$

Each of these equalities

$$f(x) = \sum_{n=0}^{\infty} c_n x^n, \qquad |x| < R$$

has a **dual** nature. It represents

- a function expanded in a power series
- a power series converging to a function

Given a function, how to find its expansion into a power series, that is, to find a power series converging to this function?

> **Theorem 1 (from the power series to a function).**
> Let a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges to a function $f(x)$ for $|x-a| < R$.
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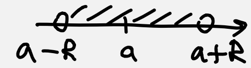
> 
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>
> Then f has derivatives of all orders and
>
> $$c_n = \frac{f^{(n)}(a)}{n!}, \qquad \text{so} \qquad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \text{ for } |x-a| < R.$$

Proof. Take $a = 0$ for simplicity of calculations. The case of an arbitrary a is handled similarly.

Since the power series $\sum_{n=0}^{\infty} c_n x^n$ converges to the function $f(x)$, we have

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots + c_n x^n + \ldots.$$

Substitute $x = 0$: $\quad f(0) = c_0$.

Differentiate the power series for $f(x)$:

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots + nc_n x^{n-1} + \ldots$$

Substitute $x = 0$: $\quad f'(0) = c_1$.

Calculate the second derivative:

$$f''(x) = 2c_2 + 3 \cdot 2c_3\, x + 4 \cdot 3c_4\, x^2 + \cdots + n \cdot (n-1)c_n x^{n-2} + \ldots$$

Substitute $x = 0$: $\quad f''(0) = 2c_2$.

Calculate the third derivative:

$$f'''(x) = 3 \cdot 2c_3 + 4 \cdot 3 \cdot 2c_4\, x + \cdots + n \cdot (n-1)(n-2)c_n x^{n-3} + \ldots$$

Substitute $x = 0$: $\quad f'''(0) = 3 \cdot 2c_3$.

And so on. After n differentiations and substituting $x = 0$, we get

$$f^{(n)}(0) = \boxed{n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1 \cdot c_n} = n!\, c_n.$$

$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$

Therefore, $c_n = \dfrac{f^{(n)}(0)}{n!}$, and $f(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(0)}{n!} x^n$, as required.

Definition.

If a function $f(x)$ has derivatives of all orders, then the power series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is called the *Taylor series for $f(x)$ centered at a*.

Taylor series centered at 0, that is the series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$, is called the *Maclaurin series*.

Given an infinitely differentiable function, we may construct its Taylor series. Does this series converge? If yes, then to which function?

Theorem 2 (from the function to a power series).

If all derivatives of a function f are **bounded** near a, then the Taylor series of f converges to f:

$$f(x) = \underbrace{\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\text{T.s. for } f} \quad \text{for} \quad |x-a| < R.$$

Remark.

There are functions which Taylor series converge, but not to the function itself.

Ex.) Find Maclaurin series for $f(x) = e^x$.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n =$$

for $f(x) = e^x$, $\quad f^{(n)}(x) = e^x$

$f^{(n)}(0) = e^0 = 1$ for all n

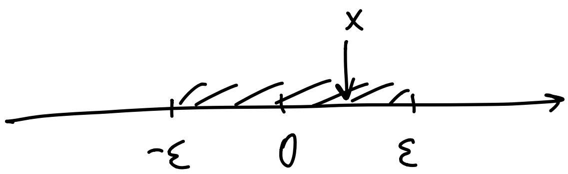
$$= \underbrace{\sum_{n=0}^{\infty} \frac{x^n}{n!}}_{\text{M.s. for } f(x)=e^x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots \qquad R = \infty$$

conv. for all x

$$e^x \stackrel{?}{=} \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$|f^{(n)}(x)| = |e^x| \le e^{\varepsilon} \quad \text{for all} \quad |x| < \varepsilon$$

(near 0)



So all derivatives of $f(x)=e^x$ are $\underline{\text{bounded}}$ near 0 $\underset{\text{Th 2}}{\Longrightarrow}$ M.S. for e^x conv. to e^x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

By product: $\underbrace{}_{\text{conv.}} \Longrightarrow \boxed{\underbrace{\frac{x^n}{n!}}_{\text{general term}} \xrightarrow[n\to\infty]{} 0 \quad \text{for any } x}$

<u>Ex. 2</u> Find T.S. for $f(x) = e^x$ at $\underline{x=2}$

center of expansion

T.S. for $f(x) = e^x$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \;=$$

$a = 2$

$$f^{(n)}(x) = \frac{d^n}{dx^n}\left(e^x\right) = e^x$$

$$f^{(n)}(2) = e^2$$

$$= \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n = e^2\left(1 + \frac{x-2}{1!} + \frac{(x-2)^2}{2!} + \dots\right)$$

p.s. in $(x-2)$

$$0! = 1$$