1. Sum calculations

Ex. 1 Calculate the $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\dots$

$$\sum_{n=1}^{\infty}\frac{1}{n\cdot 2^n}=\left(\sum_{n=1}^{\infty}\frac{1}{n}\cdot x^n\right)\Bigg|_{x=\frac{1}{2}}$$

$C=0$ (plug in $x=0$)

$$-\ln(1-x)+C=\sum_{n=1}^{\infty}\frac{x^n}{n}$$

integrate $\uparrow$ $\qquad$ $\uparrow$ integrate

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=\sum_{n=1}^{\infty}x^{n-1},\qquad |x|<1$$

geom. series

$$-\ln(1-x)=\sum_{n=1}^{\infty}\frac{x^n}{n},\qquad |x|<1$$

$x=\frac{1}{2}\in(-1,1)$:

$$\underbrace{-\ln\left(1-\tfrac{1}{2}\right)}_{=\,-\ln\frac{1}{2}\,=\,\ln 2}=\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)^n}{n}=\sum_{n=1}^{\infty}\frac{1}{n\cdot 2^n}$$

$$\sum_{n=1}^{\infty}\frac{1}{n\cdot 2^n}=\ln 2 \quad\leftarrow \text{Answer}$$

2. Working with non-elem. f-ns given as integrals

Ex. 2 Gauss error f-n

x

$$Erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

has no elem. antideriv.

Express this f-n as a p.s. and find $Erf(1)$.

Sol. For any x,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$$

$$\int_0^x e^{-t^2}\,dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}\,dt \underset{\text{integrate term-by-term}}{=} \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n t^{2n}}{n!}\,dt =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left. \frac{t^{2n+1}}{2n+1} \right|_{t=0}^{t=x} =$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1}$$

So $$Erf(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\cdot n!}$$ for any x

$$Erf(1) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\cdot n!} = \frac{2}{\sqrt{\pi}} \left(1 - \frac{1}{3\cdot 1!} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \dots\right)$$

③ Limit calculations

Ex. 3

$$\lim_{x\to 0} \frac{x - \ln(1+x)}{x^2} = \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{x - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right)}{x^2} =$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$-1 < x \le 1$$

$$= \lim_{x\to 0} \frac{\frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \dots}{x^2} =$$

$$= \lim_{x\to 0} \left(\frac{1}{2} - \underbrace{\frac{x}{3} + \frac{x^2}{4} - \dots}_{0} \right) = \boxed{\frac{1}{2}}$$