Episode 36: Presentation of functions as power series

$$\left\{\begin{array}{ll} \dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n, & |x|<1 \\ e^x = \displaystyle\sum_{n=0}^{\infty} \dfrac{x^n}{n!} & \text{for all } x \end{array}\right\} \text{ see Ep. 35}$$

Why is a presentation $\underbrace{f(x)}_{f\text{-}n} = \underbrace{\sum_{n=0}^{\infty} a_n x^n}_{p.s.}$ valuable?

- "infinite" pol. are easier to work with

$$e^x \approx 1 + x + \frac{x^2}{2}$$

- some f-ns are given as p.s. or integrals

$$\underbrace{Si(x)}_{\substack{\text{integral} \\ \text{sine}}} = \int_0^x \frac{\sin t}{t}\, dt \underset{\substack{\uparrow \\ \text{to be proven} \\ \text{later}}}{=} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)(2n-1)!} =$$

$$= x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \dots$$

<u>Ex. 1</u> Present the f-n $y = \ln(1+x)$ as the sum of a p.s.

<u>Sol.</u>

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1$$

$$x \rightsquigarrow -x$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x|<1$$

$$\Big\downarrow \text{ integrate}$$

$$\ln(1+x) \underset{\substack{\uparrow \\ \text{term-by-term} \\ \text{integration}}}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + C, \quad \underline{|x|<1}$$

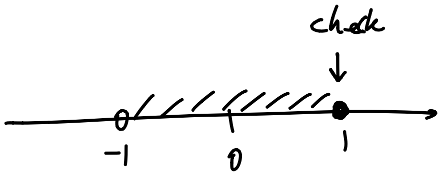
$C = ?$ Take $x = 0 \in (-1,1)$ and substitute

$$\underbrace{\ln(1+0)}_{0} = 0 + C \Rightarrow \underline{C = 0}$$

So $$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} \underset{m=n+1}{=} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} x^m =$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$|x| < 1$



For $x = 1$: p.s. is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot 1$ alt. harm. series it converges

Therefore,

$$\boxed{\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad \text{for } -1 < x \le 1}$$

Take $x = 1$:

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$ alt. harm series

(conv. very slowly)

Ex. 3 Find a p.s presentation for $y = \arctan x$.

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} + C, \quad |x| < 1$$

$\frac{d}{dx}$ ↓ ↑ integrate

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad |x| < 1$$

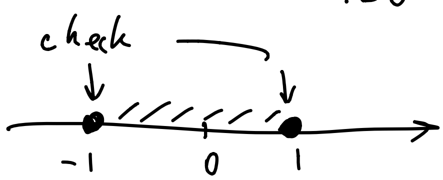
$$\frac{1}{1+x} = \sum_{n=0} (-1)^n x^n, \quad |x|<1$$

To determine C, set $x=0$:

$$\arctan 0 = 0 + C \Rightarrow C = 0$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$|x|<1$



For $x = \pm 1$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (\pm 1)}{2n+1}$$ <u>conv.</u> by alt. series test

Finally,

$$\boxed{\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad |x| \le 1}$$