Episode 35: Operations on power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

Operations on p.s.

Algebraic operations

1. Multiplication by a number

$$\sum_{n=0}^{\infty} (\underset{\text{a const}}{c}\, a_n) x^n = c \sum_{n=0}^{\infty} a_n x^n$$ (rad. of conv. is the same)

2. Addition / subtraction

$$\sum_{n=0}^{\infty} a_n x^n \pm \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$$ rad. of conv. $\geq \min(R_a, R_b)$

3. Multiplication

$$\sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} c_n x^n$$ rad. of conv. $\geq \min(R_a, R_b)$

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}$$

Analytic operation

4. Differentiation
5. Integration

Th (differentiation & integration of p.s)

Let p.s. $\sum_{n=0}^{\infty} a_n x^n$ conv. to the sum $f(x)$ for $|x|<R$.

Then

1) $f(x)$ is differentiable for $|x|<R$ and

$$f'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n x^n\right) \underset{\substack{\uparrow \\ \text{term-by-term} \\ \text{differentiation}}}{=} \sum_{n=0}^{\infty} \frac{d}{dx}(a_n x^n) = \underbrace{\sum_{n=1}^{\infty} n a_n x^{n-1}}_{\text{conv. for } |x|<R}$$

$$\frac{d}{dx}(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots) = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

2) $f(x)$ is integrable for $|x|<R$ and

$$\int f(x)\,dx = \int\left(\sum_{n=0}^{\infty} a_n x^n\right)dx \underset{\substack{\uparrow \\ \text{term-by-term} \\ \text{integration}}}{=} \sum_{n=0}^{\infty}\int a_n x^n\,dx =$$

$$= \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} + C$$

$|x|<R$

$$\int (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots)\,dx = a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \dots + C$$

<u>Remark</u> Similar formulas are valid for a p.s. centered at any $x=a$.

<u>Ex. 1</u> p.s. $\displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \overset{|x|<1}{=} \frac{1}{1-x} = f(x)$

geom. series with ratio x

p.s $\displaystyle\sum_{n=0}^{\infty} x^n$ conv. to its sum, which is $\dfrac{1}{1-x}$ for all $|x|<1$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x|<1$$

$$\Big\downarrow \frac{d}{dx}$$

$$\boxed{\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}, \quad |x|<1}$$

$$\sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \dots = \frac{1}{(1-x)^2}, \quad |x|<1$$

∞

Extra Find the sum $\underbrace{\sum_{n=1}^{\infty} \frac{n}{3^n}}_{\text{a series}}$.

$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \sum_{n=1}^{\infty} n \cdot \left(\frac{1}{3}\right)^{n-1} \cdot \frac{1}{3} = \frac{1}{3} \boxed{\sum_{n=1}^{\infty} n \left(\frac{1}{3}\right)^{n-1}} =$$

$$= \frac{1}{3} \left(\sum_{n=1}^{\infty} n x^{n-1} \right) \Bigg|_{\underbrace{x=\frac{1}{3}}_{\substack{\text{within the} \\ \text{interval of conv.} \\ (-1,\,1)}}} = \frac{1}{3} \left(\frac{1}{(1-x)^2} \right) \Bigg|_{x=\frac{1}{3}} = \frac{1}{3} \frac{1}{\left(1-\frac{1}{3}\right)^2} =$$

$$= \boxed{\frac{3}{4}}$$

So $\displaystyle \sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{3}{4}$

Ex. 2 p.s. $\displaystyle \sum_{n=0}^{\infty} \frac{x^n}{n!}$ conv. for all x (see Episode 34)

What is its sum?

Let $f(x)$ be its sum;

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\Big\downarrow \frac{d}{dx}$$

$$f'(x) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \underset{m=n-1}{=} \sum_{m=0}^{\infty} \frac{x^m}{m!} =$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$$

So, $\quad f'(x) = f(x) \qquad$ (Find $f(x)$)

$y=f(x) \qquad \frac{dy}{dx} = y$

$$\frac{dy}{y} = dx$$

$$\ln|y| = x + C_1$$

$$y = Ce^x$$

$$f(x) = Ce^x \qquad C = ?$$

$$\underbrace{Ce^x}_{f(x)} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \underset{n=0}{1} + \underset{n=1}{\frac{x}{1!}} + \frac{x^2}{2!} + \dots \qquad \underline{\underline{\text{for all } x}}$$

Substitute $x=0$:

$$C\underbrace{e^0}_{1} = 1 + 0 + 0 + \dots = 1 \Rightarrow C = 1$$

Finally, $\quad f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad \text{for all } x}$$